PHYSICAL REVIEW D **81,** 084050 (2010)

Galactic rotation curves in modified gravity with nonminimal coupling between matter and geometry

T. Harko*

Department of Physics and Center for Theoretical and Computational Physics, The University of Hong Kong, Pok Fu Lam Road, Hong Kong, People's Republic of China

(Received 7 February 2010; published 28 April 2010)

We investigate the possibility that the behavior of the rotational velocities of test particles gravitating around galaxies can be explained in the framework of modified gravity models with nonminimal matter-geometry coupling. Generally, the dynamics of test particles around galaxies, as well as the corresponding mass deficit, is explained by postulating the existence of dark matter. The extra terms in the gravitational field equations with geometry-matter coupling modify the equations of motion of test particles and induce a supplementary gravitational interaction. Starting from the variational principle describing the particle motion in the presence of the nonminimal coupling, the expression of the tangential velocity of a test particle, moving in the vacuum on a stable circular orbit in a spherically symmetric geometry, is derived. The tangential velocity depends on the metric tensor components, as well as on the coupling function between matter and geometry. The Doppler velocity shifts are also obtained in terms of the coupling function. If the tangential velocity profile is known, the coupling term between matter and geometry can be obtained explicitly in an analytical form. The functional form of this function is obtained in two cases, for a constant tangential velocity and for an empirical velocity profile obtained from astronomical observations, respectively. All the physical and geometrical quantities in the modified gravity model with nonminimal coupling between matter and geometry can be expressed in terms of observable/measurable parameters, like the tangential velocity, the baryonic mass of the galaxy, and the Doppler frequency shifts. Therefore, these results open the possibility of directly testing the modified gravity models with nonminimal coupling between matter and geometry by using direct astronomical and astrophysical observations at the galactic or extragalactic scale.

DOI: 10.1103/PhysRevD.81.084050 PACS numbers: 04.50.Kd, 04.20.Cv, 95.35.+d

I. INTRODUCTION

The rotation curves for galaxies or galaxy clusters should, according to Newton's gravitation theory, show a Keplerian decrease with distance r of the orbital rotational speed v_{tg} at the rim of the luminous matter, $v_{tg}^2 \propto M(r)/r$, where $M(r)$ is the dynamical mass. However, one observes instead rather flat rotation curves [1,2]. Observations show that the rotational velocities increase near the center of the galaxy and then remain nearly constant at a value of $v_{tg\infty} \sim 200$–300 km/s. This leads to a general mass profile $M(r) \approx r v_{tg\infty}^2/G$ [1,2]. Consequently, the mass within a distance r from the center of the galaxy increases linearly with r, even at large distances, where very little luminous matter can be detected.

This behavior of the galactic rotation curves is explained by postulating the existence of some dark (invisible) matter, distributed in a spherical halo around the galaxies. The dark matter is assumed to be a cold, pressureless medium. There are many possible candidates for dark matter, the most popular ones being the weekly interacting massive particles (for a recent review of the particle physics aspects of dark matter see [3]). Their interaction cross section with normal baryonic matter, while extremely small, are expected to be nonzero, and we may expect to detect them directly. Scalar fields, Bose-Einstein condensates, or long range coherent fields coupled to gravity have also been used to model galactic dark matter [4].

However, despite more than 20 years of intense experimental and observational effort, up to now no *nongravitational* evidence for dark matter has ever been found: no direct evidence of it, and no annihilation radiation from it.

Therefore, it seems that the possibility that Einstein's (and the Newtonian) gravity breaks down at the scale of galaxies cannot be excluded *a priori*. Several theoretical models, based on a modification of Newton's law or of general relativity, have been proposed to explain the behavior of the galactic rotation curves [5].

A very promising way to explain the recent observational data [6,7] on the acceleration of the Universe and on dark matter is to assume that at large scales the Einstein gravity model of general relativity breaks down, and a more general action describes the gravitational field. Theoretical models in which the standard Einstein-Hilbert action is replaced by an arbitrary function of the Ricci scalar R, first proposed in [8], have been extensively investigated lately. For a review of $f(R)$ generalized gravity models and their physical implications see [9]. The possibility that the galactic dynamic of massive test particles can be understood without the need for dark matter

*harko@hkucc.hku.hk

was also considered in the framework of $f(R)$ gravity models [10–14].

A generalization of the $f(R)$ gravity theories was proposed in [15] by including in the theory an explicit coupling of an arbitrary function of the Ricci scalar R with the matter Lagrangian density $L_{\rm mat}$. As a result of the coupling, the motion of the massive particles is nongeodesic, and an extra force, orthogonal to the four-velocity, arises. The connections with modified orbital Newtonian dynamics and the Pioneer anomaly were also explored, and it was suggested that the matter-geometry coupling may be responsible for the observed behavior of the galactic rotation curves. The model was extended to the case of the arbitrary couplings in both geometry and matter in [16]. The implications of the nonminimal coupling on the stellar equilibrium were investigated in [17], where constraints on the coupling were also obtained. An inequality that expresses a necessary and sufficient condition to avoid the Dolgov-Kawasaki instability for the model was derived in [18]. The relation between the model with geometry-matter coupling and ordinary scalar-tensor gravity, or scalar-tensor theories that include nonstandard couplings between the scalar and matter, was studied in [19]. In the specific case where both the action and the coupling are linear in R the action leads to a theory of gravity that includes higher order derivatives of the matter fields without introducing more dynamics in the gravity sector [20]. The equivalence between a scalar theory and the model with the nonminimal coupling of the scalar curvature and matter was considered in [21]. This equivalence allows for the calculation of the parametrized post-Newtonian parameters β and γ, which may lead to a better understanding of the weak-field limit of $f(R)$ theories. The equations of motion of test bodies in the nonminimal coupling model by means of a multipole method were derived in [22]. The energy conditions and the stability of the model under the Dolgov-Kawasaki criterion were studied in [23]. The perturbation equation of matter on subhorizon scales in models with arbitrary matter-geometry coupling, as well as the effective gravitational constant $G_{\rm eff}$ and two parameters Σ and η, which along with the perturbation equation of the matter density are useful to constrain the theory from growth factor and weak lensing observations, were derived in [24]. The age of the oldest star clusters and the primordial nucleosynthesis bounds were used in order to constrain the parameters of a toy model. The problem of the correct definition of the matter Lagrangian of the theory and of the definition of the energy-momentum tensor, considered in [25,26], was solved in [27]. For a review of modified $f(R)$ gravity with geometry-matter coupling see [28].

It is the purpose of the present paper to investigate the possibility that the observed properties of the galactic rotation curves could be explained in the framework of the modified gravity theory with nonminimal coupling between matter and geometry, without postulating the existence of dark matter. As a first step in this study, starting from the variational formulation of the equations of motion, we obtain the expression of the tangential velocity of test particles in stable circular orbits. Not only is the tangential velocity determined by the metric, as in standard general relativity, but it also depends on the explicit form of the coupling function between matter and geometry, as well as on its derivative with respect to the radial coordinate. In the asymptotic limit of a flat space-time, far away from the matter sources, the tangential velocity becomes a function of the coupling function only, and it does not decay to zero. Therefore the behavior of the neutral hydrogen gas clouds outside the galaxies, and their flat rotation curves, can be explained in terms of a nonminimal coupling between matter and geometry. Since the observations on the galactic rotation curves are obtained from the Doppler frequency shifts, we generalize the expression of the frequency shifts by including the effect of the matter-geometry coupling. Thus, at least in principle, the coupling function can be obtained directly from astronomical observations. Since the tangential velocity directly depends on the geometry-matter coupling, its knowledge allows the complete determination of the coupling function from the observational data. The form of the coupling function can be immediately obtained in the flat rotation curves region. By adopting an empirical law for the general form of the rotation curves, and by adopting some reasonable assumptions on the metric, the coupling function can be obtained exactly for the entire galactic space-time. Therefore, all the parameters of the modified gravity model with linear coupling between matter and geometry can be either obtained directly or severely constrained by astronomical observations.

The present paper is organized as follows. The equations of motion in modified gravity with linear coupling between matter and geometry, the variational principle for the equations of motion, as well as the matter Lagrangian, are presented in Sec. II. The tangential velocity of test particles in stable circular orbits, and the corresponding Doppler frequency shifts are derived in Sec. III. From the study of the galactic rotation curves the explicit form of the matter-geometry coupling is obtained in Sec. IV. We discuss and conclude our results in Sec. V. In the present paper we use the Landau-Lifshitz [29] sign conventions and definitions, and the natural system of units with $c = 1$.

II. EQUATIONS OF MOTION IN MODIFIED GRAVITY WITH LINEAR COUPLING BETWEEN MATTER AND GEOMETRY

By assuming a linear coupling between matter and geometry, the action for the modified theory of gravity takes the form [15]

$$S = \int\left\{\frac{1}{2}f_1(R) + [1 + \zeta f_2(R)]L_{\rm mat}\right\}\sqrt{-g}d^4x, \quad (1)$$

where $f_i(R)$ (with $i = 1, 2$) are arbitrary functions of the Ricci scalar R, and L_{mat} is the Lagrangian density corresponding to matter. The strength of the interaction between $f_2(R)$ and the matter Lagrangian L_{mat} is characterized by a coupling constant ζ.

By assuming that the Lagrangian density L_{mat} of the matter depends on the metric tensor components $g_{\mu\nu}$ only, and not on its derivatives, the energy-momentum tensor of the matter is given by $T_{\mu\nu} = L_{\mathrm{mat}} g_{\mu\nu} - 2\partial L_{\mathrm{mat}}/\partial g^{\mu\nu}$. With the help of the field equations one obtains for the covariant divergence of the energy-momentum tensor the equation [15,27]

$$\nabla^{\mu} T_{\mu\nu} = 2\{\nabla^{\mu} \ln[1 + \zeta f_2(R)]\} \frac{\partial L_{\mathrm{mat}}}{\partial g^{\mu\nu}}. \tag{2}$$

In the following we will restrict our analysis to the case in which the matter, assumed to be a perfect thermodynamic fluid, obeys a barotropic equation of state, with the thermodynamic pressure p being a function of the rest mass density of the matter (for short: matter density) ρ only, so that $p = p(\rho)$. In this case, the matter Lagrangian density becomes an arbitrary function of ρ, so that $L_{\mathrm{mat}} = L_{\mathrm{mat}}(\rho)$. Then the energy-momentum tensor of the matter is given by [16,27]

$$T^{\mu\nu} = \rho \frac{dL_{\mathrm{mat}}}{d\rho} u^{\mu} u^{\nu} + \left(L_{\mathrm{mat}} - \rho \frac{dL_{\mathrm{mat}}}{d\rho}\right) g^{\mu\nu}, \tag{3}$$

where the four-velocity $u^{\mu} = dx^{\mu}/ds$ satisfies the condition $g^{\mu\nu} u_{\mu} u_{\nu} = 1$. To obtain Eq. (3), we have imposed the condition of the conservation of the matter current, $\nabla_{\nu}(\rho u^{\nu}) = 0$, and we have used the relation $\delta\rho = (1/2)\rho(g_{\mu\nu} - u_{\mu}u_{\nu})\delta g^{\mu\nu}$. With the use of the identity $u^{\nu}\nabla_{\nu}u^{\mu} = d^2x^{\mu}/ds^2 + \Gamma^{\mu}_{\nu\lambda}u^{\nu}u^{\lambda}$, from Eqs. (2) and (3) we obtain the equation of motion of a test particle in the modified gravity model with linear coupling between matter and geometry as

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\lambda} u^{\nu} u^{\lambda} = f^{\mu}, \tag{4}$$

where

$$f^{\mu} = -\nabla_{\nu} \ln\left\{[1 + \zeta f_2(R)] \frac{dL_{\mathrm{mat}}(\rho)}{d\rho}\right\}(u^{\mu}u^{\nu} - g^{\mu\nu}). \tag{5}$$

The extra force f^{μ}, generated due to the presence of the coupling between matter and geometry, is perpendicular to the four-velocity, $f^{\mu}u_{\mu} = 0$. The equation of motion Eq. (4) can be obtained from the variational principle [16,27]

$$\delta S_p = \delta \int L_p ds = \delta \int \sqrt{Q}\sqrt{g_{\mu\nu}u^{\mu}u^{\nu}} ds = 0, \tag{6}$$

where S_p and $L_p = \sqrt{Q}\sqrt{g_{\mu\nu}u^{\mu}u^{\nu}}$ are the action and the Lagrangian density for test particles, respectively, and

$$\sqrt{Q} = [1 + \zeta f_2(R)] \frac{dL_{\mathrm{mat}}(\rho)}{d\rho}. \tag{7}$$

The matter Lagrangian can be expressed as [27]

$$L_{\mathrm{mat}}(\rho) = \rho[1 + \Pi(\rho)] - \int_{p_0}^{p} dp, \tag{8}$$

where $\Pi(\rho) = \int_{p_0}^{p} dp/\rho$, and p_0 is an integration constant, or a limiting pressure. The corresponding energy-momentum tensor of the matter is given by [27]

$$T^{\mu\nu} = \{\rho[1 + \Phi(\rho)] + p(\rho)\} u^{\mu}u^{\nu} - p(\rho) g^{\mu\nu}, \tag{9}$$

respectively, where

$$\Phi(\rho) = \int_{\rho_0}^{\rho} \frac{p}{\rho^2} d\rho = \Pi(\rho) - \frac{p(\rho)}{\rho}, \tag{10}$$

and with all the constant terms included in the definition of p.

III. STABLE CIRCULAR ORBITS AND FREQUENCY SHIFTS IN MODIFIED GRAVITY WITH LINEAR COUPLING BETWEEN MATTER AND GEOMETRY

The galactic rotation curves provide the most direct method of analyzing the gravitational field inside a spiral galaxy. The rotation curves are obtained by measuring the frequency shifts z of the 21-cm radiation emission from the neutral hydrogen gas clouds. Usually the astronomers report the resulting z in terms of a velocity field v_{tg} [1].

The starting point in the analysis of the motion of the hydrogen gas clouds in modified gravity with linear coupling between matter and geometry is to assume that gas clouds behave like test particles, moving in a static and spherically symmetric space-time. Next, we consider two observers O_E and O_{∞}, with four-velocities u^{μ}_E and u^{μ}_{∞}, respectively. Observer O_E corresponds to the light emitter (i.e., to gas clouds placed at a point P_E of the space-time), and O_{∞} represent the detector at point P_{∞}, located far from the emitter, and that can be idealized to correspond to "spatial infinity."

Without loss of generality, we can assume that the gas clouds move in the galactic plane $\theta = \pi/2$, so that $u^{\mu}_E = (\dot{t}, \dot{r}, 0, \dot{\phi})_E$, where the dot stands for derivation with respect to the affine parameter s. On the other hand, we suppose that the detector is static (i.e., O_{∞}'s four-velocity is tangent to the static Killing field $\partial/\partial t$), and in the chosen coordinate system its four-velocity is $u^{\mu}_{\infty} = (\dot{t}, 0, 0, 0)_{\infty}$ [30].

The static spherically symmetric metric outside the galactic baryonic mass distribution is given by

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{11}$$

where the metric coefficients are functions of the radial coordinate r only. The motion of a test particle in the gravitational field in modified gravity with linear coupling between matter and geometry can be described by the Lagrangian

$$L = Q\left[e^{\nu(r)}\left(\frac{dt}{ds}\right)^2 - e^{\lambda(r)}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\Omega}{ds}\right)^2\right], \tag{12}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. For $\theta = \pi/2$, $d\Omega^2 = d\phi^2$. From the Lagrange equations it follows that we have two constants of motion, the energy E,

$$E = Qe^{\nu(r)}\dot{t}, \tag{13}$$

and the angular momentum l, given by

$$l = Qr^2\dot{\phi}, \tag{14}$$

where a dot denotes the derivative with respect to the affine parameter s. The condition $u^\mu u_\mu = 1$ gives $1 = e^{\nu(r)}\dot{t}^2 - e^{\lambda(r)}\dot{r}^2 - r^2\dot{\phi}^2$, from which, with the use of the constants of motion, we obtain

$$E^2 = Q^2 e^{\nu+\lambda}\dot{r}^2 + e^{\nu}\left(\frac{l^2}{r^2} + Q^2\right). \tag{15}$$

This equation shows that the radial motion of the particles is the same as that of a particle in ordinary Newtonian mechanics, with velocity $\dot{r}$, position dependent mass $m_{\rm eff} = 2Q^2e^{\nu+\lambda}$, and energy E^2, respectively, moving in the effective potential

$$V_{\rm eff}(r) = e^{\nu(r)}\left(\frac{l^2}{r^2} + Q^2\right). \tag{16}$$

The conditions for circular orbits $\partial V_{\rm eff}/\partial r = 0$ and $\dot{r} = 0$ lead to

$$l^2 = \frac{1}{2}\frac{r^3 Q(\nu' Q + 2Q')}{1 - r\nu'/2} \tag{17}$$

and

$$E^2 = \frac{e^{\nu}Q(rQ' + Q)}{1 - r\nu'/2}, \tag{18}$$

respectively.

The line element given by Eq. (11) can be rewritten in terms of the spatial components of the velocity, normalized with the speed of light, measured by an inertial observer far from the source, as $ds^2 = dt^2(1 - v^2)$, where

$$v^2 = e^{-\nu}\left[e^{\lambda}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\Omega}{dt}\right)^2\right]. \tag{19}$$

For a stable circular orbit $dr/dt = 0$, and the tangential velocity of the test particle can be expressed as

$$v_{tg}^2 = e^{-\nu}r^2\left(\frac{d\Omega}{dt}\right)^2. \tag{20}$$

In terms of the conserved quantities the angular velocity is given, for $\theta = \pi/2$, by

$$v_{tg}^2 = \frac{e^{\nu}}{r^2}\frac{l^2}{E^2}. \tag{21}$$

With the use of Eqs. (17) and (18) we obtain

$$v_{tg}^2 = \frac{1}{2}\frac{r(\nu' Q + 2Q')}{rQ' + Q}. \tag{22}$$

Thus, the rotational velocity of the test body in modified gravity with linear coupling between matter and geometry is determined by the metric coefficient $\exp(\nu)$, and by the function Q and its derivative with respect to the radial coordinate r. In the standard general relativistic limit $\zeta = 0$, $Q = 1$, and we obtain $v_{tg}^2 = r\nu'/2$.

The rotation curves of spiral galaxies are inferred from the red- and blueshifts of the emitted radiation by gas clouds moving in circular orbits on both sides of the central region. The light signal travels on null geodesics with tangent k^μ. We may restrict k^μ to lie in the equatorial plane $\theta = \pi/2$, and evaluate the frequency shift for a light signal emitted from O_E in circular orbit and detected by O_∞. The frequency shift associated with the emission and detection of the light signal is given by

$$z = 1 - \frac{\omega_E}{\omega_\infty}, \tag{23}$$

where $\omega_I = -k_\mu u_I^\mu$, and the index I refers to emission ($I = E$) or detection ($I = \infty$) at the corresponding space-time point [30,31]. Two frequency shifts, corresponding to maximum and minimum values, are associated with light propagation in the same and opposite directions of motion of the emitter, respectively. Such shifts are frequency shifts of a receding or approaching gas cloud, respectively. In terms of the tetrads $e_{(0)} = e^{-\nu/2}\partial/\partial t$, $e_{(1)} = e^{-\lambda/2}\partial/\partial r$, $e_{(2)} = r^{-1}\partial/\partial\theta$, and $e_{(3)} = (r\sin\theta)^{-1}\partial/\partial\phi$, the frequency shifts take the form [30]

$$z_\pm = 1 - e^{[\nu_\infty - \nu(r)]/2}(1 \mp v)\Gamma, \tag{24}$$

where $v = [\sum_{i=1}^3(u_{(i)}/u_{(0)})^2]^{1/2}$, with $u_{(i)}$ the components of the particle's four-velocity along the tetrad (i.e., the velocity measured by an Eulerian observer whose world line is tangent to the static Killing field). $\Gamma = (1 - v^2)^{-1/2}$ is the usual Lorentz factor, and $\exp(\nu_\infty)$ is the value of $\exp[\nu((r))]$ for $r \to \infty$. In the case of circular orbits in the $\theta = \pi/2$ plane, we obtain

$$z_\pm = 1 - e^{[\nu_\infty - \nu(r)]/2}\frac{1 \mp \sqrt{r(\nu' Q + 2Q')/2(rQ' + Q)}}{\sqrt{1 - r(\nu' Q + 2Q')/2(rQ' + Q)}}. \tag{25}$$

It is convenient to define two other quantities, $z_D = (z_+ - z_-)/2$ and $z_A = (z_+ + z_-)/2$, respectively [30]. In the modified gravity model with linear coupling between matter and geometry these redshift factors are given by

$$z_D(r) = e^{[\nu_\infty - \nu(r)]/2}\frac{\sqrt{r(\nu' Q + 2Q')/2(rQ' + Q)}}{\sqrt{1 - r(\nu' Q + 2Q')/2(rQ' + Q)}} \tag{26}$$

and

$$z_A(r) = 1 - \frac{e^{[\nu_\infty - \nu(r)]/2}}{\sqrt{1 - r(\nu' Q + 2Q')/2(rQ' + Q)}}, \tag{27}$$

respectively, which can easily be connected to the observations [30]. z_A and z_D satisfy the relation $(z_A - 1)^2 - z_D^2 = \exp[2(\nu_\infty - \nu(r))]$, and thus in principle, by assuming that the metric tensor component $\exp[\nu((r))]$ is known, Q and Q' can be obtained directly from the astronomical observations. This could provide a direct observational test of the galactic geometry and, implicitly, of the modified gravity models with linear coupling between matter and geometry.

IV. GALACTIC ROTATION CURVES AND THE MATTER-GEOMETRY COUPLING

The tangential velocity v_{tg} of gas clouds moving like test particles around the center of a galaxy is not directly measurable, but can be inferred from the redshift z_∞ observed at spatial infinity, for which $1 + z_\infty = \exp[(\nu_\infty - \nu)/2](1 \pm v_{tg})/\sqrt{1 - v_{tg}^2}$. Because of the nonrelativistic velocities of the gas clouds, with $v_{tg} \leq (4/3) \times 10^{-3}$, we observe $v_{tg} \approx z_\infty$ (as the first part of a geometric series), with the consequence that the lapse function $\exp(\nu)$ necessarily tends at infinity to unity, i.e., $e^\nu \approx e^{\nu_\infty}/(1 - v_{tg}^2) \approx e^{\nu_\infty} \to 1$. The observations show that at distances large enough from the Galactic center $v_{tg} \approx$ constant [1].

In the following we use this observational constraint to reconstruct the coupling term between matter and geometry in the "dark matter" dominated region, far away from the baryonic matter distribution. By assuming that $v_{tg} =$ constant, Eq. (22) can be written as

$$v_{tg}^2 \frac{1}{rQ}\frac{d}{dr}(rQ) = \frac{\nu'}{2} + \frac{Q'}{Q}, \tag{28}$$

and can immediately be integrated to give

$$Q(r) = \left(\frac{r}{r_0}\right)^{v_{tg}^2/(1-v_{tg}^2)} \exp\left[-\frac{\nu}{2(1 - v_{tg}^2)}\right], \tag{29}$$

where r_0 is an arbitrary constant of integration. By assuming that the hydrogen clouds are a pressureless dust ($p = 0$) that can be characterized by their density ρ only, the Lagrangian of the matter (gas cloud) is given by $L_{\mathrm{mat}}(\rho) = \rho$. Therefore, from Eqs. (7) and (29) we obtain

$$\begin{aligned}\zeta f_2(R) &= \left(\frac{r}{r_0}\right)^{v_{tg}^2/2(1-v_{tg}^2)} \exp\left[-\frac{\nu}{4(1 - v_{tg}^2)}\right] - 1 \\ &\approx \frac{v_{tg}^2}{2(1 - v_{tg}^2)}\ln\frac{r}{r_0} - \frac{\nu}{4(1 - v_{tg}^2)} \\ &\quad - \frac{v_{tg}^2}{8(1 - 2v_{tg}^2)}\nu \ln\frac{r}{r_0}.\end{aligned} \tag{30}$$

For a general velocity profile $v_{tg} = v_{tg}(r)$, the general solution of Eq. (28) is given by

$$\begin{aligned}\sqrt{Q(r)} &= 1 + \zeta f_2(R) \\ &= \sqrt{Q_0}\exp\left[\frac{1}{2}\int \frac{v_{tg}^2(r)/r - \nu'/2}{1 - v_{tg}^2(r)} dr\right],\end{aligned} \tag{31}$$

where Q_0 is an arbitrary constant of integration.

For v_{tg}^2 we assume the simple empirical dark halo rotational velocity law [32]

$$v_{tg}^2 = \frac{v_0^2 x^2}{a^2 + x^2}, \tag{32}$$

where $x = r/r_{\mathrm{opt}}$ and r_{opt} is the optical radius containing 83% of the galactic luminosity. The parameters a, the ratio of the halo core radius and r_{opt}, and the terminal velocity v_0 are functions of the galactic luminosity. For spiral galaxies $a = 1.5(L/L_*)^{1/5}$ and $v_0^2 = v_{\mathrm{opt}}^2(1 - \beta_*)(1 + a^2)$, where $v_{\mathrm{opt}} = v(r_{\mathrm{opt}})$ and $\beta_* = 0.72 + 0.44\log_{10}(L/L_*)$, with $L_* = 10^{10.4} L_{\mathrm{Sun}}$.

By assuming that the coupling between the neutral hydrogen clouds and the geometry is small, $\zeta f_2(R) L_{\mathrm{mat}} \ll 1$, and consequently does not significantly modify the galactic geometry, the vacuum metric outside the baryonic mass distribution with mass M_B, corresponding to $L_{\mathrm{mat}} \approx 0$, is given by the static spherically symmetric solution of the $f(R)$ modified gravity. By assuming, for simplicity, that the galactic metric is given by the Schwarzschild metric (which is also a solution of the vacuum field equations of the $f(R)$ modified gravity [33]), written as

$$e^\nu = e^{-\lambda} = 1 - \frac{2R_0}{x}, \tag{33}$$

where $R_0 = GM_B/r_{\mathrm{opt}}$, from Eq. (31) we obtain

$$\begin{aligned}1 + \zeta f_2(R) &= \exp\left[\alpha \times \operatorname{arctanh}\left(\frac{\sqrt{1 - v_0^2}}{a}x\right)\right] \\ &\quad \times \left(1 - \frac{2R_0}{x}\right)^\beta \frac{[(1 - v_0^2)x^2 + a^2]^\gamma}{Q_0^{-1/2} x^{1/4}},\end{aligned} \tag{34}$$

where

$$\alpha = -\frac{aR_0 v_0^2}{2\sqrt{1 - v_0^2}[a^2 + 4(1 - v_0^2)R_0^2]}, \tag{35}$$

$$\beta = \frac{a^2 - 4R_0^2}{4[a^2 + 4(1 - v_0^2)R_0^2]}, \tag{36}$$

and

$$\gamma = -\frac{v_0^2[a^2 + 6(1 - v_0^2)R_0^2]}{4(1 - v_0^2)[a^2 + 4(1 - v_0^2)R_0^2]}, \tag{37}$$

respectively. Thus the geometric part of the coupling between matter and geometry can be completely reconstructed from the observational data on the galactic rotation curves.

V. DISCUSSIONS AND FINAL REMARKS

The galactic rotation curves and the mass distribution in clusters of galaxies continue to pose a challenge to present day physics. One would like to have a better understanding of some of the intriguing phenomena associated with them, like their universality, and the very good correlation between the amount of dark matter and the luminous matter in the galaxy. To explain these observations, the most commonly considered models are based on particle physics in the framework of Newtonian gravity, or of some extensions of general relativity.

In the present paper we have considered, and further developed, an alternative view to the dark matter problem [15], namely, the possibility that the galactic rotation curves and the mass discrepancy in galaxies and clusters of galaxies can naturally be explained in gravitational models in which there is a nonminimal coupling between matter and geometry. The extra terms in the gravitational field equations modify the equations of motion of test particles and induce a supplementary gravitational interaction, which can account for the observed behavior of the galactic rotation curves. As one can see from Eq. (22), in the limit of large r, when $\nu' \rightarrow 0$ (in the case of the Schwarzschild metric $\nu' \approx 2M_B/r^2$), the tangential velocity of test particles at infinity is given by

$$v_{tg\infty}^2 = \frac{r}{r + Q/Q'}, \tag{38}$$

which, due to the presence of the matter-gravity coupling, does not decay to zero at large distances from the Galactic center, a behavior that is perfectly consistent with the observational data [1] and is usually attributed to the existence of the dark matter.

By using the simple observational fact of the constancy of the galactic rotation curves, the matter-geometry coupling function can be completely reconstructed, without any supplementary assumption.

If, for simplicity, we consider that the metric in the vacuum outside the galaxy can be approximated by the Schwarzschild metric, with $\exp(\nu) = 1 - 2GM_B/r$, where M_B is the mass of the baryonic matter of the galaxy, then, in the limit of large r, we have $\nu \rightarrow 0$. Therefore from Eq. (30) we obtain

$$\zeta \lim_{r\rightarrow\infty} f_2(R) \approx \frac{v_{tg}^2}{2(1 - v_{tg}^2)} \ln\frac{r}{r_0}. \tag{39}$$

If the galactic rotation velocity profiles and the galactic metric are known, the coupling function can be reconstructed exactly over the entire mass distribution of the galaxy.

One can formally associate an approximate dark matter mass profile $M_{\rm DM}(r)$ with the tangential velocity profile, which is determined by the nonminimal coupling between matter and geometry, and is given by

$$M_{\rm DM}(r) \approx \frac{1}{2G}\frac{r^2(\nu' Q + 2Q')}{rQ' + Q}. \tag{40}$$

The corresponding dark matter density profile $\rho_{\rm DM}(r)$ can be obtained as

$$\begin{aligned}\rho_{\rm DM}(r) &= \frac{1}{4\pi r^2}\frac{dM}{dr}\\ &= \frac{1}{4\pi G}\left[\frac{\nu' Q + 2Q'}{r(rQ' + Q)} + \frac{\nu'' Q + \nu Q' + 2Q''}{2(rQ' + Q)}\right.\\ &\quad \left. - \frac{(\nu' Q + 2Q')(rQ'' + 2Q')}{2(rQ' + Q)^2}\right].\end{aligned} \tag{41}$$

In order to observationally constrain $M_{\rm DM}$ and $\rho_{\rm DM}$, we assume that each galaxy consists of a single, pressure-supported stellar population that is in dynamic equilibrium and traces an underlying gravitational potential resulting from the nonminimal coupling between matter and geometry. Further assuming spherical symmetry, the equivalent mass profile induced by the geometry-matter coupling (the mass profile of the "dark matter" halo) relates to the moments of the stellar distribution function via the Jeans equation [2]

$$\frac{d}{dr}[\rho_s\langle v_r^2\rangle] + \frac{2\rho_s(r)\beta}{r} = -\frac{G\rho_s M_{\rm DM}(r)}{r^2}, \tag{42}$$

where $\rho_s(r)$, $\langle v_r^2\rangle$, and $\beta(r) = 1 - \langle v_\theta^2\rangle/\langle v_r^2\rangle$ describe the three-dimensional density, radial velocity dispersion, and orbital anisotropy of the stellar component, where $\langle v_\theta^2\rangle$ is the tangential velocity dispersion. By assuming that the anisotropy is a constant, the Jeans equation has the solution [34]

$$\rho_s\langle v_r^2\rangle = Gr^{-2\beta}\int_r^\infty s^{2(1-\beta)}\rho_s(s)M_{\rm DM}(s)ds. \tag{43}$$

With the use of Eq. (40) we obtain for the stellar velocity dispersion the equation

$$\rho_s\langle v_r^2\rangle \approx \frac{1}{2}r^{-2\beta}\int_r^\infty s^{2(2-\beta)}\rho_s(s)\frac{\nu'(s)Q(s) + 2Q'(s)}{sQ'(s) + Q(s)}ds. \tag{44}$$

Projecting along the line of sight, the dark matter mass profile relates to two observable profiles, the projected stellar density $I(R)$ and the stellar velocity dispersion

$\sigma_p(R)$ according to the relation [2]

$$\sigma_P^2(R) = \frac{2}{I(R)} \int_R^\infty \left(1 - \beta \frac{R^2}{r^2}\right) \frac{\rho_s \langle v_r^2 \rangle r}{\sqrt{r^2 - R^2}} dr. \tag{45}$$

Given a projected stellar density model $I(R)$, one recovers the three-dimensional stellar density from $\rho_s(r) = -(1/\pi) \int_r^\infty (dI/dR)(R^2 - r^2)^{-1/2} dR$ [2]. Therefore, once the stellar density profile $I(R)$, the stellar velocity dispersion $\langle v_r^2 \rangle$, and the galactic metric are known, with the use of the integral equation Eq. (44) one can obtain the explicit form of the geometry-matter coupling function Q, and the equivalent mass profile induced by the nonminimal coupling between matter and geometry. The simplest analytic projected density profile is the Plummer profile [2], given by $I(R) = \mathcal{L}(\pi r_{\rm half}^2)^{-1}(1 + R^2/r_{\rm half}^2)^{-2}$, where $\mathcal{L}$ is the total luminosity, and $r_{\rm half}$ is the projected half-light radius (the radius of the cylinder that encloses half of the total luminosity).

An important physical requirement for the circular orbits of the test particle around galaxies is that they must be stable. Let r_0 be a circular orbit and consider a perturbation of it of the form $r = r_0 + \delta$, where $\delta \ll r_0$ [31]. Taking expansions of $V_{\rm eff}(r)$, $\exp(\nu + \lambda)$, and $Q^2(r)$ about $r = r_0$, it follows from Eq. (15) that

$$\ddot{\delta} + \frac{1}{2} Q^2(r_0) e^{\nu(r_0) + \lambda(r_0)} V_{\rm eff}''(r_0) \delta = 0. \tag{46}$$

The condition for stability of the simple circular orbits requires $V_{\rm eff}''(r_0) > 0$ [31]. This gives for the coupling function Q the constraint

$$\frac{d^2 Q}{dr^2}\bigg|_{r=r_0} < \left[\nu''\left(\frac{l^2}{r^2} + Q^2\right) + \nu'\left(-\frac{2l^2}{r^2} + 2QQ'\right) - \frac{6l^2}{r^4}\right]\bigg|_{r=r_0}, \tag{47}$$

a condition that must be satisfied at any point r_0 of the galactic space-time.

From the observational as well as from the theoretical point of view an important problem is to estimate an upper bound for the cutoff of the constancy of the tangential velocities. If in the large r limit the coupling function satisfies the condition $Q'/Q \to 0$, then $Q/Q' \to \infty$, and in this limit the tangential velocity decays to zero. If the exact functional form of Q is known, the value of r at which the rotational velocity becomes zero can be accurately estimated.

In the present model all the relevant physical quantities, including the dark mass associated with the coupling, and which plays the role of the dark matter, its corresponding density profile, as well as the matter-geometry coupling function, are expressed in terms of observable parameters—the tangential velocity, the baryonic (luminous) mass, and the Doppler frequency shifts of test particles moving around the galaxy. Therefore, this opens the possibility of directly testing the modified gravity models with nonminimal coupling between matter and geometry by using direct astronomical and astrophysical observations at the galactic or extragalactic scale. In this paper we have provided some basic theoretical tools necessary for the in-depth comparison of the predictions of this model and of the observational results.

ACKNOWLEDGMENTS

This work is supported by an RGC grant of the government of the Hong Kong SAR.

[1] V. C. Rubin, W. K. Ford, and N. Thonnard, Astrophys. J. **238**, 471 (1980); M. Persic, P. Salucci, and F. Stel, Mon. Not. R. Astron. Soc. **281**, 27 (1996); A. Boriello and P. Salucci, Mon. Not. R. Astron. Soc. **323**, 285 (2001).

[2] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, NJ, 2008).

[3] J. M. Overduin and P. S. Wesson, Phys. Rep. **402**, 267 (2004).

[4] E. W. Mielke and F. E. Schunk, Phys. Rev. D **66**, 023503 (2002); J. P. Mbelek, Astron. Astrophys. **424**, 761 (2004); B. Fuchs and E. W. Mielke, Mon. Not. R. Astron. Soc. **350**, 707 (2004); X. Hernández, T. Matos, R. A. Sussman, and Y. Verbin, Phys. Rev. D **70**, 043537 (2004); C. G. Boehmer and T. Harko, J. Cosmol. Astropart. Phys. 06 (2007) 025; D. Bertacca, N. Bartolo, and S. Matarrese, J. Cosmol. Astropart. Phys. 05 (2008) 005; J.-W. Lee, Phys. Lett. B **681**, 118 (2009); J. L. Cervantes-Cota, M. A. Rodriguez-Meza, and D. Nunez, Phys. Rev. D **79**, 064011 (2009); J.-W. Lee and S. Lim, J. Cosmol. Astropart. Phys. 01 (2010) 007.

[5] M. Milgrom, Astrophys. J. **270**, 365 (1983); P. D. Mannheim, Astrophys. J. **479**, 659 (1997); M. Milgrom, Astrophys. J. **599**, L25 (2003); M. K. Mak and T. Harko, Phys. Rev. D **70**, 024010 (2004); J. D. Bekenstein, Phys. Rev. D **70**, 083509 (2004); T. Harko and K. S. Cheng, Astrophys. J. **636**, 8 (2006); J. R. Brownstein and J. W. Moffat, Astrophys. J. **636**, 721 (2006); T. Harko and K. S. Cheng, Phys. Rev. D **76**, 044013 (2007); J. R. Brownstein and J. W. Moffat, Mon. Not. R. Astron. Soc. **382**, 29 (2007); M. Milgrom, Phys. Rev. D **80**, 123536 (2009).

[6] A. G. Riess *et al.*, Astron. J. **116**, 1009 (1998); S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999); P. de

Bernardis *et al.*, Nature (London) **404**, 955 (2000); S. Hanany *et al.*, Astrophys. J. **545**, L5 (2000).
[7] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003); T. Padmanabhan, Phys. Rep. **380**, 235 (2003).
[8] H. A. Buchdahl, Mon. Not. R. Astron. Soc. **150**, 1 (1970); R. Kerner, Gen. Relativ. Gravit. **14**, 453 (1982); J. P. Duruisseau, R. Kerner, and P. Eysseric, Gen. Relativ. Gravit. **15**, 797 (1983); J. D. Barrow and A. C. Ottewill, J. Phys. A **16**, 2757 (1983).
[9] T. P. Sotiriou and V. Faraoni, arXiv:0805.1726.
[10] S. Capozziello, V. F. Cardone, and A. Troisi, J. Cosmol. Astropart. Phys. 08 (2006) 001; Mon. Not. R. Astron. Soc. **375**, 1423 (2007).
[11] A. Borowiec, W. Godlowski, and M. Szydlowski, Int. J. Geom. Methods Mod. Phys. **4**, 183 (2007).
[12] C. F. Martins and P. Salucci, Mon. Not. R. Astron. Soc. **381**, 1103 (2007).
[13] C. G. Boehmer, T. Harko, and F. S. N. Lobo, Astropart. Phys. **29**, 386 (2008).
[14] C. G. Boehmer, T. Harko, and F. S. N. Lobo, J. Cosmol. Astropart. Phys. 03 (2008) 024.
[15] O. Bertolami, C. G. Boehmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **75**, 104016 (2007).
[16] T. Harko, Phys. Lett. B **669**, 376 (2008).
[17] O. Bertolami and J. Paramos, Phys. Rev. D **77**, 084018 (2008).
[18] V. Faraoni, Phys. Rev. D **76**, 127501 (2007).
[19] T. P. Sotiriou and V. Faraoni, Classical Quantum Gravity **25**, 205002 (2008).
[20] T. P. Sotiriou, Phys. Lett. B **664**, 225 (2008).
[21] O. Bertolami and J. Paramos, Classical Quantum Gravity **25**, 245017 (2008).
[22] D. Puetzfeld and Y. N. Obukhov, Phys. Rev. D **78**, 121501 (R) (2008).
[23] O. Bertolami and M. C. Sequeira, Phys. Rev. D **79**, 104010 (2009).
[24] S. Nesseris, Phys. Rev. D **79**, 044015 (2009).
[25] O. Bertolami, F. S. N. Lobo, and J. Paramos, Phys. Rev. D **78**, 064036 (2008).
[26] V. Faraoni, Phys. Rev. D **80**, 124040 (2009).
[27] T. Harko, Phys. Rev. D **81**, 044021 (2010).
[28] O. Bertolami, T. Harko, F. S. N. Lobo, and J. Paramos, arXiv:0811.2876.
[29] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth-Heinemann, Oxford, 1998).
[30] U. Nucamendi, M. Salgado, and D. Sudarsky, Phys. Rev. D **63**, 125016 (2001).
[31] K. Lake, Phys. Rev. Lett. **92**, 051101 (2004).
[32] P. Salucci and M. Persic, in *Dark and Visible Matter in Galaxies*, edited by M. Persic and P. Salucci, ASP Conference Series No. 117 (Astronomical Society of the Pacific, San Francisco, 1997), p. 1.
[33] C. S. J. Pun, Z. Kovacs, and T. Harko, Phys. Rev. D **78**, 024043 (2008).
[34] G. A. Mamon and E. L. Lokas, Mon. Not. R. Astron. Soc. **363**, 705 (2005).