NOTE ON WEIGHTED CARLEMAN-TYPE INEQUALITY

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A double inequality involving the constant e is proved by using an inequality between the logarithmic mean and arithmetic mean. As an application, we generalize the weighted Carleman-type inequality.

1. Introduction

Let $p > 1$ and $a_n \geq 0$ with $0 < \sum_{n=1}^{\infty} a_n^p < \infty$. Then

$$\sum_{n=1}^{\infty}\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.1}$$

The constant $(p/(p-1))^p$ is the best possible.

Inequality (1.1) is due to Hardy [6, page 239].

Replacing a_n in (1.1) by $a_n^{1/p}$ for $n \in \mathbb{N}$, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{a_1^{1/p} + a_2^{1/p} + \cdots + a_n^{1/p}}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n. \tag{1.2}$$

In (1.2), letting $p \to \infty$, then the following Carleman inequality [6, page 249] is deduced:

$$\sum_{n=1}^{\infty}\left(a_1 a_2 \cdots a_n\right)^{1/n} < e\sum_{n=1}^{\infty} a_n, \tag{1.3}$$

where $a_n \geq 0$ for $n \in \mathbb{N}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. The constant e is the best possible.

Carleman's inequality (1.3) was generalized in [6, page 256] by Hardy as follows. Let $a_n \geq 0$, $\lambda_n > 0$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ for $n \in \mathbb{N}$, and $0 < \sum_{n=1}^{\infty} \lambda_n a_n < \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e\sum_{n=1}^{\infty} \lambda_n a_n. \tag{1.4}$$

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Note that inequality (1.4) is usually referred to as a Carleman-type inequality or weighted Carleman-type inequality. In his original paper [5], Hardy himself said that it was Pölya who pointed out this inequality to him.

In several recent papers [2, 4, 11, 12, 13, 14, 15], some strengthened and generalized results of (1.3) and (1.4) have been given by estimating the weight coefficient $(1+1/n)^n$.

For information about the history of both Hardy's inequality and Carleman-type inequalities, please refer to [7, 9].

In this note, we will give a generalization of (1.4) as follows.

Theorem 1.1. *Let* $0<\lambda_{n+1}\le\lambda_n$ *with* $\Lambda_n=\sum_{m=1}^{n}\lambda_m\ge 1$ *and* $\lim_{n\to\infty}\Lambda_n=\infty$, *and let* $a_n\ge 0$ *for* $n\in\mathbb{N}$ *satisfying* $0<\sum_{n=1}^{\infty}\lambda_n a_n<\infty$. *Then for* $0<p\le 1$,

$$\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} \le \frac{1}{p}\sum_{n=1}^{\infty}\left[\left(1+\frac{1}{\Lambda_n/\lambda_n}\right)^{p\Lambda_n/\lambda_n}\lambda_n a_n^p\Lambda_n^{p-1}\left(\sum_{k=1}^{n}\lambda_k\left(c_k a_k\right)^p\right)^{(1-p)/p}\right], \tag{1.5}$$

in particular,

$$\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < \frac{e^p}{p}\sum_{n=1}^{\infty}\left[\left(1-\frac{1-2/e}{\Lambda_n/\lambda_n}\right)^{p}\lambda_n a_n^p\Lambda_n^{p-1}\left(\sum_{k=1}^{n}\lambda_k\left(c_k a_k\right)^p\right)^{(1-p)/p}\right], \tag{1.6}$$

where

$$c_k^{\lambda_k}=\frac{\left(\Lambda_{k+1}\right)^{\Lambda_k}}{\left(\Lambda_k\right)^{\Lambda_{k-1}}}. \tag{1.7}$$

Remark 1.2. In particular, taking in (1.6) $p=1$, we obtain the following strengthened Hardy's inequality:

$$\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e\sum_{n=1}^{\infty}\left(1-\frac{1-2/e}{\Lambda_n/\lambda_n}\right)\lambda_n a_n. \tag{1.8}$$

Taking in (1.8) $\lambda_n\equiv 1$, we obtain the following strengthened Carleman's inequality:

$$\sum_{n=1}^{\infty}\left(a_1a_2\cdots a_n\right)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{1-2/e}{n}\right)a_n. \tag{1.9}$$

2. Lemma

The well-known arithmetic mean $A(a,b)$ and logarithmic mean $L(a,b)$ of two positive numbers a and b are defined, respectively, for $a=b$ by $A(a,b)=L(a,b)=a$ and for $a\ne b$

by

$$A(a,b) = \frac{a+b}{2}, \qquad L(a,b) = \frac{b-a}{\ln b - \ln a}. \tag{2.1}$$

For $a \neq b$, we have

$$L(a,b) < A(a,b). \tag{2.2}$$

See [1] and the references therein.

LEMMA 2.1. *Let $x \geq 1$ be a real number. Then*

$$e\left(1 - \frac{1/2}{x}\right) < \left(1 + \frac{1}{x}\right)^x \leq e\left(1 - \frac{1-2/e}{x}\right). \tag{2.3}$$

The constants $1/2$ and $1-2/e$ are best possible.

Proof. Inequality (2.3) is equivalent to

$$1 - \frac{2}{e} \leq x\left[1 - \frac{1}{e}\left(1 + \frac{1}{x}\right)^x\right] < \frac{1}{2}. \tag{2.4}$$

Define a function f for $x > 0$ by

$$f(x) = x\left[1 - \frac{1}{e}\left(1 + \frac{1}{x}\right)^x\right]. \tag{2.5}$$

In order to prove (2.4), it is sufficient to show that the function f is strictly increasing on $[1,\infty)$ and with

$$f(1) = 1 - \frac{2}{e}, \qquad \lim_{x\to\infty} f(x) = \frac{1}{2}. \tag{2.6}$$

The following proof shows that in fact $f'(x) > 0$ holds on $(0,\infty)$.

Easy computation yields

$$ef'(x) = e - [1 + xg(x)]\left(1 + \frac{1}{x}\right)^x, \tag{2.7}$$

where

$$g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1} = \frac{1}{L(x,x+1)} - \frac{1}{x+1}. \tag{2.8}$$

Now we are in a position to prove $f'(x) > 0$, which is equivalent to

$$h(x) = [1 + xg(x)]\left(1 + \frac{1}{x}\right)^x < e. \tag{2.9}$$

Differentiation yields

$$h'(x) = \left[xg^2(x) + 2g(x) - \frac{1}{(x+1)^2}\right]\left(1+\frac{1}{x}\right)^x. \tag{2.10}$$

In the following we show $h'(x) > 0$. Clearly, the equation

$$xt^2 + 2t - \frac{1}{(x+1)^2} = 0 \tag{2.11}$$

has two roots

$$t_{1,2} = \frac{-(x+1) \pm \sqrt{(x+1)^2 + x}}{x(x+1)}. \tag{2.12}$$

To prove $h'(x) > 0$, it is sufficient to show that

$$\frac{-(x+1) + \sqrt{(x+1)^2 + x}}{x(x+1)} = t_2 < g(x) = \frac{1}{L(x,x+1)} - \frac{1}{x+1}, \tag{2.13}$$

which is equivalent to

$$\frac{\sqrt{(x+1)^2 + x} - 1}{x(x+1)} < \frac{1}{L(x,x+1)}. \tag{2.14}$$

Inequality (2.14) holds based on the following fact:

$$\frac{\sqrt{(x+1)^2 + x} - 1}{x(x+1)} < \frac{2}{2x+1} = \frac{1}{A(x,x+1)} < \frac{1}{L(x,x+1)}. \tag{2.15}$$

Hence, the function h is increasing on $(0,\infty)$, and then $h(x) < \lim_{x\to\infty} h(x) = e$. This means $f'(x) > 0$, and then

$$1 - \frac{2}{e} = f(1) < \lim_{x\to\infty} f(x). \tag{2.16}$$

Using Maclaurin formula

$$(1+t)^{1/t} = e - \frac{e}{2}t + o(t), \tag{2.17}$$

we have

$$\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x) = \lim_{t\to 0+} f\left(\frac{1}{t}\right) = \lim_{t\to 0+} \frac{(et)/2 + o(t)}{et} = \frac{1}{2}. \tag{2.18}$$

The proof of Lemma 2.1 is complete. □

Remark 2.2. There are other very sharp estimates of the crucial factor $(1+1/n)^n$ in [8] and the references therein.

3. Proof of Theorem 1.1

By the power mean inequality, we have

$$\prod_{m=1}^{n} \alpha_m^{q_m} \leq \left(\sum_{m=1}^{n} q_m \alpha_m^p \right)^{1/p}, \tag{3.1}$$

where $p \geq 0$, $\alpha_m \geq 0$, and $q_m > 0$ for $m \in \mathbb{N}$ with $\sum_{m=1}^{n} q_m = 1$.

Let $c_m > 0$, $\alpha_m = c_m a_m$, and $q_m = \lambda_m / \Lambda_m$, then we obtain

$$(c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n} \leq \left(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \right)^{1/p}. \tag{3.2}$$

Further, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} \\ &\quad = \sum_{n=1}^{\infty} \lambda_{n+1} \frac{(c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n}}{\left(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} \right)^{1/\Lambda_n}} \\ &\quad \leq \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\left(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} \right)^{1/\Lambda_n}} \left(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \right)^{1/p}. \end{aligned} \tag{3.3}$$

By the following inequality (see [3, 10])

$$\left(\sum_{m=1}^{n} z_m \right)^t \leq t \sum_{m=1}^{n} z_m \left(\sum_{k=1}^{m} z_k \right)^{t-1}, \tag{3.4}$$

where $t \geq 1$ is constant and $z_m \geq 0$ for $m \in \mathbb{N}$, it is easy to see that

$$\begin{aligned} \left(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \right)^{1/p} &\leq \frac{1}{\Lambda_n} \left(\sum_{m=1}^{n} \lambda_m (c_m a_m)^p \right)^{1/p} \\ &\leq \frac{1}{p\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \left(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p}, \end{aligned} \tag{3.5}$$

where $\Lambda_n \geq 1$ and $0 < p \leq 1$. Thus, we obtain from (3.3) and (3.5) that

$$\begin{aligned} &\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \right)^{1/\Lambda_n} \\ &\quad \leq \frac{1}{p} \sum_{n=1}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n \left(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} \right)^{1/\Lambda_n}} \sum_{m=1}^{n} \lambda_m (c_m a_m)^p \left(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\ &\quad = \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \left(\frac{\lambda_{n+1}}{\Lambda_n \left(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} \right)^{1/\Lambda_n}} \right) \left(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p}. \end{aligned} \tag{3.6}$$

Choosing $c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n}$ for $n \in \mathbb{N}$ and setting $\Lambda_0 = 0$, we get from $0 < \lambda_{n+1} \le \lambda_n$ that

$$c_n = \left[\frac{(\Lambda_{n+1})^{\Lambda_n}}{(\Lambda_n)^{\Lambda_{n-1}}} \right]^{1/\lambda_n} = \left(1 + \frac{\lambda_{n+1}}{\Lambda_n}\right)^{\Lambda_n/\lambda_n} \Lambda_n \le \left(1 + \frac{\lambda_n}{\Lambda_n}\right)^{\Lambda_n/\lambda_n} \Lambda_n. \tag{3.7}$$

This implies that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} \\
&\quad \le \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{\Lambda_n \Lambda_{n+1}} \left(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\
&\quad = \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \sum_{n=m}^{\infty} \left(\frac{1}{\Lambda_n} - \frac{1}{\Lambda_{n+1}} \right) \left(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\
&\quad = \frac{1}{p} \sum_{m=1}^{\infty} \lambda_m (c_m a_m)^p \frac{1}{\Lambda_m} \left(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p} \\
&\quad \le \frac{1}{p} \sum_{m=1}^{\infty} \left(1 + \frac{1}{\Lambda_m/\lambda_m}\right)^{p\Lambda_m/\lambda_m} \lambda_m a_m^p \Lambda_m^{p-1} \left(\sum_{k=1}^{m} \lambda_k (c_k a_k)^p \right)^{(1-p)/p}.
\end{aligned} \tag{3.8}$$

Hence, we obtain from the above inequality and Lemma 2.1 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} \\
&\quad < \frac{e^p}{p} \sum_{n=1}^{\infty} \left(1 - \frac{1 - 2/e}{\Lambda_n/\lambda_n}\right)^p \lambda_n a_n^p \Lambda_n^{p-1} \left(\sum_{k=1}^{n} \lambda_k (c_k a_k)^p \right)^{(1-p)/p}.
\end{aligned} \tag{3.9}$$

The last inequality holds strictly since the right-hand inequality of (2.3) is valid if and only if $n = 1$. The proof is complete.

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