Test polynomials, retracts, and the Jacobian conjecture

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and

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Abstract. Let $K[x, y]$ be the algebra of two-variable polynomials over a field $K$. A polynomial $p = p(x, y)$ is called a test polynomial for automorphisms if, whenever $\varphi(p) = p$ for a mapping $\varphi$ of $K[x, y]$, this $\varphi$ must be an automorphism. Here we show that $p \in \mathbb{C}[x, y]$ is a test polynomial if and only if $p$ does not belong to any proper retract of $\mathbb{C}[x, y]$. This has the following corollary that may have application to the Jacobian conjecture: if a mapping $\varphi$ of $\mathbb{C}[x, y]$ with invertible Jacobian matrix is “invertible on one particular polynomial”, then it is an automorphism. More formally: if there is a non-constant polynomial $p$ and an injective mapping $\psi$ of $\mathbb{C}[x, y]$ such that $\psi(\varphi(p)) = p$, then $\varphi$ is an automorphism.

1 Introduction

Let $K[x, y]$ be the algebra of two-variable polynomials over a field $K$ of characteristic 0. A subalgebra $R$ of $K[x, y]$ is called a retract if there is an idempotent homomorphism $\pi$ of $K[x, y]$ (called a retraction or a projection) such that $\pi(K[x, y]) = R$.

There are several equivalent descriptions of retracts of $K[x, y]$ known by now:

(i) $K[x, y] = R \oplus I$ for some ideal $I$ of $K[x, y]$;

(ii) $K[x, y]$ is a projective extension of $R$ in the category of $K$-algebras;

(iii) By a theorem of Costa [2], every proper retract of $K[x, y]$ (i.e., one different from $K[x, y]$ and $K$) is of the form $K[p]$ for some $p = p(x, y) \in K[x, y]$. The authors earlier proved [9] that there exists an automorphism of $K[x, y]$ which takes $p(x, y)$ to $x + y \cdot q(x, y)$ for some $q(x, y) \in K[x, y]$, and every polynomial of the form $x + y \cdot q(x, y)$ generates a proper retract of $K[x, y]$.

(iv) (see [9]) $p(x, y)$ generates a retract of $K[x, y]$ if and only if there is an endomorphism of $K[x, y]$ which takes $p(x, y)$ to $x$.

(v) (see [3]) $p(x, y)$ belongs to a proper retract of $\mathbb{C}[x, y]$ if and only if $p(x, y)$ is fixed by some endomorphism of $\mathbb{C}[x, y]$ with nontrivial kernel.

Recently retracts have found another application in a general setup of arbitrary free algebras and groups in relation with test elements, introduced in [8]. In general,
an element $g$ of a group or an algebra $F$ is a test element if any endomorphism of $F$
fixing $g$ is actually an automorphism. It is easy to see that a test element does not belong to any proper retract of $F$; a remarkable result of Turner \cite{10} says that, if $F$
is a free group, then the converse is also true. Thus, an element of a free group $F$ is a test element if and only it does not belong to any proper retract of $F$.

Here we establish a similar characterization of test polynomials in $\mathbb{C}[x,y]$:

**Theorem 1.** A polynomial $p \in \mathbb{C}[x,y]$ is a test polynomial if and only if $p$ does not belong to any proper retract of $\mathbb{C}[x,y]$.

Our proof uses several recent results, in particular, a result of Drensky and Yu \cite{3} mentioned in the item (v) above. Crucial for our proof is the following result of independent interest.

**Theorem 2.** Let $\varphi$ be an injective endomorphism of $\mathbb{C}[x,y]$ which is not an automorphism. Suppose that $\varphi(p) = p$ for some non-constant polynomial $p \in \mathbb{C}[x,y]$. Then $p \in \mathbb{C}[q]$, where $q$ is a coordinate polynomial of $\mathbb{C}[x,y]$. In particular, $p$ belongs to a proper retract of $\mathbb{C}[x,y]$.

Recall that $q = q(x,y)$ is a coordinate polynomial of $\mathbb{C}[x,y]$ if it can be taken to $x$ by an automorphism of $\mathbb{C}[x,y]$.

We also use results of Shestakov and Umirbaev \cite{7} on estimating degrees of polynomials in two-generated subalgebras of $K[x,y]$. Another ingredient is a result of Kraft \cite{5} concerning the subalgebra $\varphi^\infty(\mathbb{C}[x,y]) = \bigcap_{k=1}^\infty \varphi^k(\mathbb{C}[x,y])$.

**Theorems 1, 2** have the following corollary:

**Corollary.** Let $\varphi$ be an endomorphism of $\mathbb{C}[x,y]$ with invertible Jacobian matrix. If there is a non-constant polynomial $p \in \mathbb{C}[x,y]$ and an injective mapping $\psi$ such that $\psi(\varphi(p)) = p$, then $\varphi$ is an automorphism of $\mathbb{C}[x,y]$.

This strengthens our earlier result \cite[Corollary 1.7]{9}, where we showed that, if $\varphi$ has invertible Jacobian matrix, then $\varphi(p) = p$ implies that $\varphi$ is an automorphism of $\mathbb{C}[x,y]$.

To conclude the Introduction, we raise a problem motivated by results of this paper:

**Problem.** Suppose $p \in \mathbb{C}[x,y]$ is a test polynomial and $\varphi$ is an injective mapping of $\mathbb{C}[x,y]$. Is $\varphi(p)$ necessarily a test polynomial?

It is interesting to note that, by a result of Jelonek \cite{4}, a “generic” polynomial of degree $\geq 4$ is a test polynomial.

2 Proof of Theorem 2

We consider the following two principal cases.

**Case I.** There is a coordinate polynomial in $\varphi(\mathbb{C}[x,y])$. 

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Case II. There are no coordinate polynomials in \( \varphi(\mathbb{C}[x,y]) \).

In Case I, consider two subcases:

1. \( \varphi \) is not birational, i.e., does not induce an automorphism of the field of fractions.

   Then, by a result of Kraft [5, Lemma 1.3], \( \varphi^\infty(\mathbb{C}[x,y]) \) is either \( \mathbb{C} \) or \( \mathbb{C}[f] \), where \( f = f(x,y) \) is some polynomial. Obviously, if \( \varphi(p) = p \), then \( p \in \varphi^\infty(\mathbb{C}[x,y]) \). We are therefore going to focus on the case \( \varphi^\infty(\mathbb{C}[x,y]) = \mathbb{C}[f] \) and show that, if \( \varphi \) is injective, then \( p \in \mathbb{C}[q] \), where \( q \) is a coordinate polynomial.

   Now suppose \( r = r(x,y) \) is a coordinate polynomial in \( \varphi(\mathbb{C}[x,y]) \), and let \( r = \varphi(s(x,y)) \). Then, since \( \varphi \) is injective, the polynomial \( s = s(x,y) \) must be coordinate, too, by the result of [1]. Therefore, upon changing generating set of \( C[x,y] \) if necessary, we may assume that \( r = \varphi(x) \). Furthermore, we can replace \( \varphi \) with its conjugate by an arbitrary automorphism, say \( \alpha \), i.e., with \( \psi = \alpha \varphi \alpha^{-1} \), and at the same time replace \( p \) with \( p_1 = \alpha(p) \). Then we have:

   \[
   \psi(p_1) = \alpha \varphi \alpha^{-1}(\alpha(p)) = \alpha(p) = p_1.
   \]

   Therefore, the pair \((\psi, p_1)\) has the same properties that the pair \((\varphi, p)\) does, namely, \( \psi \) is injective but not birational, and \( \psi(p_1) = p_1 \); in particular, \( p_1 \in \psi^\infty(\mathbb{C}[x,y]) \). By choosing \( \alpha \) appropriately, we can also have \( \psi(x) = x \), thus getting \( x \in \psi^\infty(\mathbb{C}[x,y]) \).

   Then, if \( p \) (and therefore \( p_1 \)) does not belong to \( \mathbb{C}[q] \) for any coordinate polynomial \( q \), \( \psi^\infty(\mathbb{C}[x,y]) \) cannot be of the form \( \mathbb{C}[f] \), which is in contradiction with the result of Kraft mentioned above. This completes case (1).

2. \( \varphi \) is birational, i.e., induces an automorphism of the field of fractions. Again, as in the case (1) above, we deduce from [1] that \( \varphi \) must take some coordinate polynomial to coordinate. Thus, upon changing generating set of \( \mathbb{C}[x,y] \) if necessary, we may assume that \( \varphi \) takes \( x \) to \( u \), and \( y \) to \( v \cdot f(u) \), where \( \mathbb{C}[u,v] = \mathbb{C}[x,y] \), and \( f(u) \) is a non-constant polynomial (otherwise, \( \varphi \) would be an automorphism).

   Now let

   \[
   p = p(x,y) = \sum_{i,j} c_{ij} x^i y^j.
   \]

   Then

   \[
   \varphi(p) = \sum_{i,j} c_{ij} u^i v^j (f(u))^j.
   \]

   Let \( x^r y^s \) be the highest term of \( p(x,y) \) in the “pure lex” order with \( y > x \). Then in \( \varphi(p) \), the highest term is that of \( \varphi(x^r y^s) \) because the \( y \)-degree of \( \varphi(y) \) is not lower than that of \( \varphi(x) \). Furthermore, the highest term of \( \varphi(x^r y^s) \) must have the \( y \)-degree at least \( s \) since otherwise, one would have both \( u \) and \( v \) of \( y \)-degree equal to 0, which is impossible.

   If the \( y \)-degree of \( \varphi(x^r y^s) \) is \( > s \), this gives a contradiction with \( \varphi(p) = p \). Now suppose the \( y \)-degree of \( \varphi(x^r y^s) \) is exactly \( s \). This is only possible if the \( y \)-degree of \( v
is 1 and the $y$-degree of $u$ is 0. Then, arguing as in the case (1) above, we may assume that $\varphi(x) = x$. Therefore, $\varphi(y) = (y + g(x)) \cdot f(x)$. Then from $\varphi(p) = p$ we get:

$$\sum_{i,j} c_{ij} x^i y^j = \sum_{i,j} c_{ij} (y + g(x))^j (f(x))^j.$$ 

Again we use the "pure lex" order with $y > x$ to focus on the monomial of highest degree on either side, but this time we compare the $x$-degrees of these highest-degree monomials. We see that these $x$-degrees cannot be equal unless $f(x)$ is a constant, contradicting the assumption. This completes the proof in Case I.

In Case II, we are going to prove the following somewhat stronger statement:

**Proposition.** Let $\varphi$ be an injective endomorphism of $\mathbb{C}[x,y]$, and suppose that there are no coordinate polynomials in $\varphi(\mathbb{C}[x,y])$. Then $\varphi^{\infty}(\mathbb{C}[x,y]) = \mathbb{C}$.

**Proof.** Let $\varphi(x) = u = u(x,y)$, $\varphi(y) = v = v(x,y)$, and let $D(u,v)$ denote the determinant of the Jacobian matrix of $\varphi$. Since $\varphi$ is injective, $D(u,v) \neq 0$. Now there are two cases:

1. $\deg(D(u,v)) = 0$, i.e., $D(u,v)$ is a non-zero constant. Then, by a result of Kraft [5], we have $\varphi^{\infty}(\mathbb{C}[x,y]) = \mathbb{C}$.
2. $\deg(D(u,v)) > 0$. Note that for any $k \geq 1$, there are no coordinate polynomials in $\varphi^k(\mathbb{C}[x,y])$. Indeed, if there were a coordinate polynomial in $\varphi^k(\mathbb{C}[x,y])$, then, by the result of [4], there would have to be a coordinate polynomial in $\varphi^{k-1}(\mathbb{C}[x,y])$. This would lead to a contradiction with the assumption that there are no coordinate polynomials in $\varphi(\mathbb{C}[x,y])$.

Let $\varphi^k(x) = u^{(k)}$, $\varphi^k(y) = v^{(k)}$. Then from $\deg(D(u,v)) > 0$ and from the "chain rule" we get $\deg(D(u^{(k)},v^{(k)})) \geq k$. Now the Proposition will follow from the lemma below. Before we get to it, we need one more definition.

We call a pair $(p,q)$ of polynomials from $K[x,y]$ **elementary reduced** if the sum of their degrees cannot be reduced by a (non-degenerate) linear transformation or a transformation of one of the following two types:

1. $(p,q) \longrightarrow (p + \mu \cdot q^k, q)$ for some $\mu \in K^*; \ k \geq 2$;
2. $(p,q) \longrightarrow (p, q + \mu \cdot p^k)$.

Now we are ready for our

**Lemma.** Let $p = p(x,y)$ and $q = q(x,y)$ be two algebraically independent polynomials such that the pair $(p,q)$ is elementary reduced. Let $n = \deg(p) < m = \deg(q); \ m, n \geq 2, \ \deg(D(p,q)) \geq k$. Let $w = w(x,y) \in \mathbb{C}[p,q])$. Then, unless $w$ is a linear combination of $p$ and $q$, one has $\deg(w) > \min(n,k)$.

**Proof.** The proof here is based on a result of Shestakov and Umirbaev [7] Theorem 3]. Let $N = N(p,q) = mn \left(\frac{mn}{g.c.d.(n,m)} - mn - n + \deg(D(p,q)) + 2 \right)$. Following [7], we may
assume that the highest homogeneous parts of \( p \) and \( q \) are algebraically dependent; otherwise, \( \deg(w) > n \) is immediate (unless \( w \) is a linear combination of \( p \) and \( q \)). Then \( \frac{mn}{\gcd(n,m)} - m - n \geq 0 \). Indeed, if \( \gcd(n,m) = n \), then the pair \((p,q)\) would not be elementary reduced, contradicting the assumption. If \( \gcd(n,m) < n \), then \( \frac{n}{\gcd(n,m)} \geq 2 \), therefore \( \frac{mn}{\gcd(n,m)} \geq 2m \), hence \( \frac{mn}{\gcd(n,m)} - m - n \geq 0 \).

Thus, from now on we assume \( N = N(p,q) = \deg(D(p,q)) + 2 \).

Suppose now that the \( y \)-degree of \( w = w(x,y) \) is of the form \( \frac{n}{\gcd(n,m)} \cdot b + r \neq 0 \), where \( 0 \leq r < \frac{n}{\gcd(n,m)} \). Then, by [9, Theorem 3], we have

\[
\deg(w(p,q)) \geq b \cdot N + mr.
\]

If \( b \neq 0 \), this implies \( \deg(w(p,q)) \geq N \geq k + 2 > k \). If \( b = 0 \), then \( r \neq 0 \), implying \( \deg(w(p,q)) \geq m > n \).

It remains to consider the case where the \( y \)-degree of \( w = w(x,y) \) is 0. Then the \( x \)-degree of \( w \) must be nonzero; suppose it is of the form \( \frac{m}{\gcd(n,m)} \cdot b_1 + r_1 \neq 0 \), where \( 0 \leq r_1 < \frac{m}{\gcd(n,m)} \). Then, again by [9, Theorem 3], we have

\[
\deg(w(p,q)) \geq b_1 \cdot N + nr_1.
\]

As before, \( b_1 \neq 0 \) implies \( \deg(w(p,q)) > k \). If \( b_1 = 0 \), then \( r_1 \geq 2 \) because we assume that \( w(x,y) \) is not linear. Then we have \( \deg(w(p,q)) \geq 2n > n \), which completes the proof of the lemma. \( \square \)

Continuing with the proof of the Proposition, we aim at showing that for any integer \( M \), there is an integer \( k \) such that the degree of any polynomial in \( \varphi^k(\mathbb{C}[x,y]) \) is \( > M \). The above lemma “almost” does it if we use it with \( p = \varphi^k(x) = u^{(k)} \), \( q = \varphi^k(y) = v^{(k)} \), but it has one extra condition on the pair \((p,q)\) to be elementary reduced, whereas a pair \((u^{(k)},v^{(k)})\) may not be elementary reduced. However, if we denote by \( (\overline{u}^{(k)},\overline{v}^{(k)}) \) an elementary reduced pair obtained from \((u^{(k)},v^{(k)})\) by elementary transformations, we shall have all conditions of the lemma satisfied for this pair while obviously \( \mathbb{C}[\overline{u}^{(k)},\overline{v}^{(k)}] = \mathbb{C}[u^{(k)},v^{(k)}] \). In particular, the inequality \( \deg(D(\overline{u}^{(k)},\overline{v}^{(k)})) \geq k \) follows from the fact that the mapping \( x \to \overline{u}^{(k)}, y \to \overline{v}^{(k)} \) is a composition of \( \varphi^k \) with an automorphism \( \alpha \) of \( \mathbb{C}[x,y] \) in such a way that \( \alpha \) is applied first. Therefore, the “chain rule” applied to this composition yields \( \deg(D(\overline{u}^{(k)},\overline{v}^{(k)})) = \deg(D(u^{(k)},v^{(k)})) \). Thus, our lemma is applicable to the pair \((\overline{u}^{(k)},\overline{v}^{(k)})\), which completes the proof of the Proposition and therefore of Theorem 2. \( \square \)

### 3 Proof of Theorem 1 and Corollary

The “only if” part of Theorem 1 follows from a result of [9] rather easily. If \( p = p(x,y) \) belongs to a proper retract \( \mathbb{C}[q] \) of \( \mathbb{C}[x,y] \), then, by [9], for some automorphism \( \alpha \), \( \alpha(p) \) belongs to \( \mathbb{C}[x+y \cdot u] \) for some polynomial \( u = u(x,y) \). Then the mapping
$x \to x + y \cdot u, \ y \to 0$ fixes the polynomial $x + y \cdot u$, and therefore also fixes $\alpha(p)$. Thus, $\alpha(p)$ is not a test polynomial, and neither is $p$.

For the “if” part of Theorem 1, suppose that $p$ does not belong to any proper retract of $\mathbb{C}[x, y]$, and let $\varphi(p) = p$ for some mapping $\varphi$ of $\mathbb{C}[x, y]$. Then, by the result of [3], $\varphi$ must be injective. Then, by our Theorem 2, $\varphi$ must be an automorphism, hence $p$ is a test polynomial. □

**Proof of Corollary 1.** By way of contradiction, assume that $\varphi$ is not an automorphism. Then, by our Theorem 2, $p \in \mathbb{C}[q]$, where $q$ is a coordinate polynomial of $\mathbb{C}[x, y]$. Therefore, the composite mapping $\psi \varphi$ fixes a polynomial $f(q)$ in $q$. Then it is easy to see (by looking at the highest degree monomial in $f(q)$) that $\psi(\varphi(q)) = c \cdot q$ for some $c \in \mathbb{C}^*$, which implies, by the result of [1], that $\varphi(q)$ is coordinate. A mapping of $\mathbb{C}[x, y]$ with invertible Jacobian matrix that takes a coordinate polynomial to a coordinate polynomial is obviously an automorphism, a contradiction. □

In conclusion, we recall a result of [3] Theorem 1.3] saying that if, for a mapping $\varphi$ of $\mathbb{C}[x, y]$ with invertible Jacobian matrix, $\varphi(x)$ generates a proper retract of $\mathbb{C}[x, y]$, then $\varphi$ is an automorphism of $\mathbb{C}[x, y]$. Then, the case where $\varphi(x)$ belongs to a proper retract but does not generate it, can be ruled out since in that case, $\varphi(x) = f(p(x, y))$, where $p(x, y)$ generates a proper retract of $\mathbb{C}[x, y]$, and $f$ is some one-variable polynomial of degree $> 1$. The gradient of such a polynomial cannot form a row of any invertible Jacobian matrix, which can be easily seen from the “chain rule” applied to $f(p(x, y))$.

Therefore, by Theorem 1 of the present paper, if $\varphi$ is a counterexample to the Jacobian conjecture for $\mathbb{C}[x, y]$, then $\varphi(x)$ must be a test polynomial. Perhaps a way to prove the Jacobian conjecture for $\mathbb{C}[x, y]$ could be through showing that the gradient of a test polynomial cannot form a row of any invertible Jacobian matrix. This is known to be the case with (non-commutative) partial derivatives of a test element of a free group of rank 2, see [3] Corollary 2.2.8.

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