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Large values of error terms of a class of arithmetical functions

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Abstract. We consider the error terms of a class of arithmetical functions whose Dirichlet series satisfy a functional equation with multiple gamma factors. Our aim is to establish $\Omega_{\pm}$ results to a subclass of these arithmetical functions with a good localization of the occurrence of the extreme values. As applications, we improve the $\Omega_{\pm}$ results of some special 3-dimensional ellipsoids of other writers and extend our result to other ellipsoids.

1. Introduction

Our objective in this paper is to investigate the occurrence of large values of the error term in the summatory formula $\sum_{n \leqq x} a(n)$ of an arithmetical function $a(n)$. We shall consider a class of arithmetical functions $a(n)$ whose associated Dirichlet series satisfy a type of functional equations with multiple gamma factors. This class is very wide and contains a lot of well-known and classical examples, such as the Ramanujan function $\tau(n)$, the divisor function $d(n)$ in Dirichlet's divisor problem, the counting function $r(n)$ of representations of n as a sum of two squares in the circle problem, some of other divisor functions and the enumerating function of representations of an integer by a quadratic form.

The formulation and research in the general context was enhanced by Chandrasekharan and Narasimhan (see [6], [7]) although the characteristic (i.e. the relation of satisfying a functional equation) had been known earlier. Their work was later continued by other authors, such as Berndt [2]–[5], Hafner [10], [11], Ivić [12], and Redmond [14], [15]. Up-to-date, the studied area covers the Voronoi-type series expansion, mean square formulas, $\Omega_{\pm}$-results, localization of large values and sign-changes. Concerning the large values, one should note the articles of Hafner [11] and Ivić [12]. The former gave the best Ω-results (or $\Omega_{\pm}$-results) to date but was unable to localize the occurrence of the large values. The latter one [12] can do this but with the extreme values not as sharp as those obtained in [11].

In this paper, we focus on a subclass of these arithmetical functions for which we can give extreme values sharper than those obtained in Theorem 1 of [12] and at the same time, provide good localization on the occurrence of such values. (See our main result Theorem 1 in Section 3.) Particularly interesting examples in this subclass include the generalized divisor function $\sigma_{-1/2}(n)$ and the counting function $r(Q,n)$ of representations of an integer n by a positive definite ternary quadratic form Q (refer to Theorem 2 in Section 5). Furthermore, due to the different conditions required, we can deduce some consequences which are not covered in Hafner [11]. More specifically, let us consider the problem of counting lattice points in three-dimensional ellipsoids. (The corresponding arithmetical function is $r(Q,n)$.) Hafner's approach cannot give $\Omega_{\pm}$-results in this case (see [11], Section 5.2, p. 72–73). In fact, a recent paper of Adhikari and Pétermann [1] proved that the error terms in the lattice points problem of six different ellipsoids, including the sphere, are $\Omega_{\pm}(X^{1/2}\log\log X)$. With our approach, we can replace the $\log\log X$ by $\sqrt{\log X}$ and extend the (improved) results to other three-dimensional ellipsoids (those determined by integral positive definite quadratic forms). It should be remarked that the Ω_{-}-result for the case of a sphere was obtained long ago by Szegö [16] and the Ω_{+}-result was proved recently by the second author [17].

2. Definitions and some properties

Throughout this paper, we use $Y \gg Z$ (or $Z \ll Y$) to mean that $|Z| \leqq CY$ for some constant $C > 0$, and $Y \asymp Z$ to mean both $Y \ll Z$ and $Z \ll Y$ hold.

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers, not identically zero. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two strictly increasing sequences of positive numbers, both of which tend to ∞. Suppose that the series $\phi(s) = \sum_{n=1}^{\infty} a_n\lambda_n^{-s}$ and $\psi(s) = \sum_{n=1}^{\infty} b_n\mu_n^{-s}$ both converge absolutely in some half-planes $\Re e\, s > \sigma_a^*$ and $\Re e\, s > \sigma_b^*$ respectively (σ_a^* and σ_b^* are their abscissas of absolute convergence). For each $\nu = 1,2,\dots,N$, we let $\alpha_\nu > 0$, $\beta_\nu \in \mathbb{C}$ and define

$$\Delta(s) = \prod_{\nu=1}^{N} \Gamma(\alpha_\nu s + \beta_\nu), \qquad \alpha = \sum_{\nu=1}^{N} \alpha_\nu.$$

Let $\delta \in \mathbb{R}$ and suppose ϕ and ψ satisfy the functional equation

$$\Delta(s)\phi(s) = \Delta(\delta - s)\psi(\delta - s)$$

in the following sense: there exists a compact set S, which contains all the singularities of $\Delta(s)\phi(s)$, and there exists a meromorphic function $\chi(s)$, which is holomorphic in the complement of S, such that

(i) $\lim_{|t|\to\infty} \chi(\sigma + it) = 0$ uniformly in every interval $\eta_1 \leqq \sigma \leqq \eta_2$, and

(ii)

$$\chi(s) = \begin{cases} \Delta(s)\phi(s) & \text{for } \sigma > \sigma_a^*, \\ \Delta(\delta - s)\psi(\delta - s) & \text{for } \sigma < \delta - \sigma_b^*. \end{cases}$$

Let

$$s_0 = \sup\{|s| \colon s \in S\} \quad \text{and} \quad t_0 = \max\{|\beta_\nu|\alpha_\nu^{-1} \colon \nu = 1,2,\dots,N\}.$$

Choose two constants $c > \max(\sigma_a^*, \sigma_b^*, s_0, t_0)$, $R > \max(s_0, t_0)$. Let $\gamma > c + \delta$ be any sufficiently large but fixed number such that both $\delta - \gamma$ and $\delta - (\gamma + 1/\alpha)$ are not integers. Let

$\mathscr{C}_y$ be the boundary of the rectangle with vertices at $c \pm iR$ and $\delta - \gamma \pm iR$, taken in the anti-clockwise direction (so it encircles S). Define for $x > 0$,

$$E_\rho(x) = A_\rho(x) - M_\rho(x) \tag{2.1}$$

where

$$A_\rho(x) = \frac{1}{\Gamma(\rho+1)} \sum_{\lambda_n \leqq x}{}' a_n (x - \lambda_n)^\rho \quad \text{and} \quad M_\rho(x) = \frac{1}{2\pi i} \int_{\mathscr{C}_y} \frac{\Gamma(s)}{\Gamma(s+\rho+1)} \phi(s) x^{s+\rho}\, ds.$$

The prime in $\sum'$ means the last term in the sum is equal to $\frac{1}{2} a_n$ if $\rho = 0$ and $x = \lambda_n$.

Lemma 2.1. *Suppose that for each $u > \sigma_b^*$,*

$$\sup_{0 \leqq t \leqq 1} \left| \sum_{X^{2\alpha} \leqq \lambda_n \leqq (X+t)^{2\alpha}} \frac{b_n}{\mu_n^{u-1/(2\alpha)}} \right| \to 0 \tag{2.2}$$

as $X \to \infty$. Then for any $y > 0$ and any $\rho > 2\alpha\sigma_b^* - \alpha\delta - 3/2$, we have

$$E_\rho(y) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^{\delta+\rho}} f_\rho(y\mu_n) \tag{2.3}$$

where

$$f_\rho(y) = \frac{1}{2\pi i} \int_{\mathscr{C}_R(\lambda,\gamma)} \frac{\Gamma(\delta - s)\Delta(s)}{\Gamma(\delta+\rho+1-s)\Delta(\delta - s)} y^{\delta+\rho-s}\, ds.$$

Here $\lambda = \min(\sigma_b^* - 2/\alpha, \delta/2 - 1/(2\alpha))$ and $\mathscr{C}_R(\lambda,\gamma)$ denotes the contour which joins the points $\lambda - i\infty$, $\lambda - iR$, $\gamma - iR$, $\gamma + iR$, $\lambda + iR$ and $\lambda + i\infty$ in such order. The series in (2.3) converges uniformly on any finite closed interval in $(0,\infty)$ where $A_\rho(x)$ is continuous.

Proof. This is Hafner [10], Theorem B.

In [10], Lemma 2.1, Hafner proved an asymptotic formula for $f_\rho(y)$ in which the constant implicit in the O-symbol is dependent upon ρ. In the following Lemma 2.2, we show that, when ρ lies in a fixed finite interval, the constant implicit in the O-symbol in (2.4) below can be made independent of ρ.

Lemma 2.2. *Let ρ_0 be fixed such that $2\alpha\gamma - \delta\alpha - 7/2 > \rho_0 > \min(2\alpha\sigma_b^* - \alpha\delta - 4, -1)$. Then for any $y > 0$ and any $\rho \in [\rho_0, \rho_0 + 1]$, we have*

$$f_\rho(y) = \sum_{\nu=0,1} e_\nu(\rho) y^{\theta_\rho - \nu/(2\alpha)} \cos(hy^{1/(2\alpha)} + k_\nu(\rho)\pi) + O(y^{\theta_\rho - 1/\alpha}), \tag{2.4}$$

where the O-constant is independent of ρ, and

$$\theta_\rho = \frac{\delta}{2} - \frac{1}{4\alpha} + \rho\left(1 - \frac{1}{2\alpha}\right),$$

$$h = 2\alpha \exp\left(-\alpha^{-1} \sum_{\nu=1}^{N} \alpha_\nu \log \alpha_\nu\right),$$

$$e_0(\rho) = (2\alpha/h)^\rho (h\pi)^{-1/2},$$

$$e_1(\rho) = (2\alpha/h)^\rho \times (\textit{a quadratic polynomial in } \rho),$$

$$k_\nu(\rho) = -\frac{1}{2}\alpha\delta - \frac{1}{4} - \frac{\rho}{2} - \sum_{j=1}^{N}\left(\beta_j - \frac{1}{2}\right) + \frac{\nu - 1}{2}.$$

Proof. By Stirling's formula,

$$\log \Gamma(z) = (z - 1/2)\log z - z + \frac{1}{2}\log 2\pi + \frac{1}{12z} + O\left(\frac{1}{|z|^3}\right)$$

for $|\arg z| \leqq \pi - \varepsilon$ and $|z| \to \infty$. Assume $w \in \mathbb{C}$ with $|w| \leqq W$ where $W > 0$ is a fixed constant. Then for $|z|$ sufficiently large, we have

$$\log \Gamma(z + w) = (z + w - 1/2)\log z - z + \frac{1}{2}\log 2\pi + \frac{c_1(w)}{z} + \frac{c_2(w)}{z^2} + O\left(\frac{1}{|z|^3}\right),$$

where $c_i(w)$ $(i = 1, 2)$ are polynomials in w and the O-constant depends only on W. Let

$$r_\rho = k_\nu(\rho) - \nu/2,$$

$$a_\rho = -(\delta/2 + 1/(4\alpha) + \rho/(2\alpha)),$$

$$G_\rho(s) = \frac{\Gamma(\delta - s)\Delta(s)}{\Gamma(\delta + \rho + 1 - s)\Delta(\delta - s)},$$

and

$$F_\nu^\rho(s) = e_\nu(\rho) 2\alpha h^{-(2\alpha s + 2\alpha a_\rho - \nu)} \Gamma(2\alpha s + 2\alpha a_\rho - \nu)\cos\left(\pi(\alpha s + \alpha a_\rho + r_\rho)\right)$$

where $e_0(\rho) = (2\alpha/h)^\rho (h\pi)^{-1/2}$ and $e_\nu(\rho) = (2\alpha/h)^\rho \times$ (a polynomial in ρ). Following through the computation in [8], (9)–(11), we get

$$G_\rho(s) = F_0^\rho(s) + F_1^\rho(s) + F_2^\rho(s) + F_0^\rho(s)\mathscr{R}(s) \tag{2.5}$$

with

$$\mathscr{R}(s) = O(|s|^{-3}) \tag{2.6}$$

where the O-constant is independent of ρ (but depends on ρ_0). Moreover, for any fixed real numbers σ' and σ'', we have

$$|F_\nu^\rho(s)| \asymp |\Gamma(2\alpha s + 2\alpha a_\rho - \nu)\cos\left(\pi(\alpha s + \alpha a_\rho + r_\rho)\right)| \asymp |t|^{2\alpha\sigma + 2\alpha a_\rho - \nu - 1/2} \tag{2.7}$$

and

$$|G_\rho(s)| \sim |F_0^\rho(s)| \asymp |t|^{\alpha(2\sigma-\delta)-(\rho+1)}$$

uniformly in $\sigma' \leqq \sigma \leqq \sigma''$ and $|t| \geqq R$. Hence, we can shift the path of integration to deduce

$$\text{(2.8)} \qquad f_\rho(y) = \sum_{\nu=0}^{2} \frac{1}{2\pi i} \int_{\mathscr{C}_R(-a_\rho,\gamma)} F_\nu^\rho(s) y^{\delta+\rho-s}\, ds + \frac{1}{2\pi i} \int_{\mathscr{C}_R(-a_\rho,\gamma)} F_0^\rho(s)\mathscr{R}(s) y^{\delta+\rho-s}\, ds.$$

As $2\alpha\gamma - \delta\alpha - 5/2 > \rho_0 + 1 \geqq \rho$, we have $2\alpha(\gamma + a_\rho) > 2$, so $F_\nu^\rho(s)$ $(\nu = 0, 1, 2)$ has no poles on the right side of $\mathscr{C}_R(-a_\rho, \gamma)$. Together with (2.7) and applying Cauchy's Theorem,

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathscr{C}_R(-a_\rho,\gamma)} F_\nu^\rho(s) y^{\delta+\rho-s}\, ds \\ &= e_\nu(\rho) y^{\delta+\rho+a_\rho-\nu/(2\alpha)} \frac{2\alpha}{2\pi i} \int_{\mathscr{C}_R(-a_\rho,\gamma)} \Gamma(2\alpha s + 2\alpha a_\rho - \nu) \\ &\quad \times \cos\big(\pi(\alpha s + \alpha a_\rho + r_\rho)\big) \big(h y^{1/(2\alpha)}\big)^{-(2\alpha s + 2\alpha a_\rho - \nu)}\, ds \\ &= e_\nu(\rho) y^{\theta_\rho - \nu/(2\alpha)} \frac{2\alpha}{2\pi i} \int_{\sigma_\nu - i\infty}^{\sigma_\nu + i\infty} \Gamma(2\alpha s + 2\alpha a_\rho - \nu) \\ &\quad \times \cos\big(\pi(\alpha s + \alpha a_\rho + r_\rho)\big) \big(h y^{1/(2\alpha)}\big)^{-(2\alpha s + 2\alpha a_\rho - \nu)}\, ds \end{aligned}$$

where $\sigma_\nu = -a_\rho + \nu/(2\alpha) + 1/(8\alpha)$. Using the fact that

$$1/(2\pi i) \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s) \cos(\beta + \pi s/2) y^{-s}\, ds = \cos(y + \beta) \quad \text{for } 0 < \sigma < 1,$$

we see that the last integral equals

$$\int_{1/4-i\infty}^{1/4+i\infty} \Gamma(s) \cos\big(k_\nu(\rho)\pi + \pi s/2\big) \big(h y^{1/(2\alpha)}\big)^{-s}\, ds = \frac{\pi i}{\alpha} \cos\big(h y^{1/(2\alpha)} + k_\nu(\rho)\pi\big)$$

and hence

$$\text{(2.9)} \qquad \frac{1}{2\pi i} \int_{\mathscr{C}_R(-a_\rho,\gamma)} F_\nu^\rho(s) y^{\delta+\rho-s}\, ds = e_\nu(\rho) y^{\theta_\rho - \nu/(2\alpha)} \cos\big(h y^{1/(2\alpha)} + k_\nu(\rho)\pi\big).$$

From (2.5), we observe that $F_0^\rho(s)\mathscr{R}(s) = G_\rho(s) - \sum_{\nu=0}^{2} F_\nu^\rho(s)$ represents a meromorphic function, and it has at most $O(\alpha^{-1})$ simple poles, contributed by the factor $\Gamma(\delta - s)$ of $G_\rho(s)$, in the region between $\mathscr{C}_R(-a_\rho, \gamma)$ and $\mathscr{C}_R(-a_\rho + 1/\alpha, \gamma + 1/\alpha)$. No pole of $F_0^\rho(s)\mathscr{R}(s)$ lies on $\mathscr{C}_R(-a_\rho + 1/\alpha, \gamma + 1/\alpha)$ due to the condition that $\delta - (\gamma + 1/\alpha)$ is not an integer. Using (2.7), we now shift the path of integration of the last integral in (2.8) and it becomes

$$(2.10)\qquad \frac{1}{2\pi i}\int_{\mathscr{C}_R(-a_\rho+1/\alpha,\gamma+1/\alpha)} F_0^\rho(s)\mathscr{R}(s)y^{\delta+\rho-s}\,ds + O(y^{\delta+\rho-\gamma}) \ll y^{\theta_\rho-1/\alpha}$$

by (2.6), where the implied constants are independent of ρ. Again, we have used

$$2\alpha\gamma-\delta\alpha-5/2>\rho$$

and (2.7) to derive the last bound in (2.10). In view of (2.8), our assertion then follows from (2.9) and (2.10).

3. Assumptions and the main result

From now on we consider the subclass of $\{a_n\}$ for which the following assumptions are valid:

(∗) *All the b_n's are real, $\alpha=1$, $\delta\geqq 0$, $\sigma_a^*\leqq\sigma_b^*$, $|\mu_n-\mu_m|\gg 1$ for $m\neq n$, the condition (2.2) holds, and for some absolute constants η and κ,*

$$(3.1)\qquad \sum_{\mu_n\leqq x} b_n^2=\eta x^{\delta+1/2}\log^{2\kappa}x+O(x^{\delta+1/2}\log^{2\kappa-1}x).$$

Under these assumptions, we see that $\sum_{\mu_n\leqq x}|b_n|\ll x^{\delta/2+3/4}\log^\kappa x$ and hence $\sigma_b^*\leqq\delta/2+3/4$. Moreover, we have the following.

Lemma 3.1. *For any H, $r>0$ and any small $\varepsilon>0$, we have*

$$\text{(a)}\quad \sum_{\mu_n\leqq H}\frac{|b_n|}{\mu_n^r}\ll 1+H^{\delta/2+3/4-r+\varepsilon},\qquad \text{(b)}\quad \sum_{\mu_n\leqq H}\frac{b_n^2}{\mu_n^\delta}\ll H^{1/2+\varepsilon},$$

$$\text{(c)}\quad \sum_{\mu_n\leqq H}\frac{b_n^2}{\mu_n^{\delta+1/2}}\asymp\log^{2\kappa+1}H,\qquad \text{(d)}\quad \sum_{\mu_n\geqq H}\frac{b_n^2}{\mu_n^{\delta+1/2+r}}\ll H^{-r+\varepsilon},$$

where the implied constants depend at most on ε.

Proof. This is proved by using (3.1) in conjunction with partial summation and Cauchy-Schwarz's inequality.

By Lemma 2.1 and Lemma 2.2 with $\rho_0=0$, we have for $0<\rho\leqq 1$ ($\rho>0$ is required in Lemma 2.1),

$$(3.2)\qquad \begin{aligned} E_\rho(y) &= e_0(\rho)y^{\theta_\rho}\sum_{n=1}^{\infty}\frac{b_n}{\mu_n^{\delta/2+1/4+\rho/2}}\cos(h\sqrt{\mu_n y}+k_0(\rho)\pi)\\ &\quad + e_1(\rho)y^{\theta_\rho-1/2}\sum_{n=1}^{\infty}\frac{b_n}{\mu_n^{\delta/2+3/4+\rho/2}}\cos(h\sqrt{\mu_n y}+k_1(\rho)\pi)\\ &\quad + O(y^{\theta_\rho-1}) \end{aligned}$$

where the implied constant is independent of ρ. This point is important as we shall later consider $\rho\to 0+$. According to Lemma 2.1, the first sum in (3.2) converges uniformly on any finite closed interval in $(0, \infty)$, while by Lemma 3.1 (a), the second sum converges absolutely for any fixed $\rho > 0$. From the definition (2.1), we see that the function

$$E(y) = \lim_{\rho \to 0+} E_\rho(y)$$

exists and $E(y) = E_0(y)$ for all $y \neq \lambda_n$. We can now state our main result.

Theorem 1. *Suppose that the conditions in* $(*)$ *hold and that for some constant* $D > 0$,

$$\sum_{\substack{n,m,l=1 \\ |\sqrt{\mu_m}+\sqrt{\mu_n}-\sqrt{\mu_l}| \ll \mu_l^{-D}}}^{\infty} \frac{|b_m b_n b_l|}{(\mu_m \mu_n \mu_l)^{\delta/2+1/4}} \ll 1. \tag{3.3}$$

Then for any sufficiently large $L \leqq \sqrt{X}$, *we have*

$$\sup_{v \in [X, X+L\sqrt{X}]} \pm E_0(v) \gg X^{\theta_0} \log^{\kappa+1/2} L.$$

(*Here* $\sup \pm E_0(v)$ *denotes both* $\sup E_0(v)$ *and* $\sup(-E_0(v))$.)

An application of our result to 3-dimensional ellipsoids will be given in the last section.

To prove Theorem 1, we need one more lemma.

Lemma 3.2. *Let h be a real-valued integrable function defined on an interval I. If*

$$|I|^{-1} \left| \int_I h^3 \right| \leqq \theta \left(|I|^{-1} \int_I h^2 \right)^{3/2}$$

for some $\theta < 1$, *then*

$$\sup_I (\pm h) \geqq \left(\frac{1-\theta}{2} \right)^{1/3} \left(|I|^{-1} \int_I h^2 \right)^{1/2}.$$

Proof. This is [17], Lemma 1.

4. Proof of Theorem 1

Following the method in [17], we shall derive our $\Omega_\pm$ result by computing the second and third power moments of a convolution. The reason for taking convolution is to truncate the infinite series expansion (first sum on the right hand side of (3.2)) into a manageable finite sum.

Let L be sufficiently large and $L \leqq \sqrt{X}$. We shall use the kernel

$$K(u) = B \left(\frac{\sin(\pi B u)}{\pi B u} \right)^2$$

where $B = [L^{1+(4+2D)^{-1}}] L^{-1}$ and D is as in (3.3). Note that $B \asymp L^{(4+2D)^{-1}}$ and BL is an integer so that $K(L) = 0$. This helps to simplify our argument.

Since we do not have a series expansion for $E_0(v)$ in hand (note that the validity of (3.2) does not include the case $\rho = 0$) so instead of treating $E_0(v)$ directly, we first consider

$$F_\rho(t) = \int_{-L}^{L} \frac{E_\rho((t+u)^2)}{(t+u)^{2\theta_\rho}} K(u)\, du, \quad \text{for } 0 < \rho \leqq 1 \text{ and } t \geqq 2L.$$

After evaluating the second and third power moments of $F_\rho(t)$ by means of (3.2), we then let $\rho \to 0+$ to deduce our result. So in the estimations below, we have to keep all implied constants in the $\ll$ and O-symbols independent of ρ.

First of all, we find that

$$\int_{-\infty}^{\infty} K(u)e^{iuy}\, du = \max\left(0, 1 - \left|\frac{y}{2\pi B}\right|\right),$$

$K(u) \ll \min(B^{-1}u^{-2}, B)$, $K'(u) \ll \min(B^3|u|, u^{-2})$ and $K''(u) \ll Bu^{-2}$. Hence by partial integration,

$$\begin{aligned}\int_{-L}^{L} K(u)e^{iuy}\, du &= \left(\int_{-\infty}^{\infty} - \int_{|u|>L}\right) K(u)e^{iuy}\, du \\ &= \max\left(0, 1 - \left|\frac{y}{2\pi B}\right|\right) + 2K(L)\frac{\sin(yL)}{y} + O(BL^{-1}y^{-2}) \\ &= \max\left(0, 1 - \left|\frac{y}{2\pi B}\right|\right) + O(BL^{-1}y^{-2}),\end{aligned}$$

since $K(L) = 0$. Furthermore for $|t| \geqq 2L$, by partial integration,

$$\int_{-L}^{L} (t+u)^{-1} K(u)e^{iuy}\, du \ll B|t|^{-1}y^{-1}.$$

Using also the estimate $\int_{-L}^{L} K(u)\, du \ll 1$ and those in Lemma 3.1, we deduce from (3.2) that

$$F_\rho(t) = \Sigma_\rho(t) + O(BL^{-1}) \tag{4.1}$$

where

$$\Sigma_\rho(t) = e_0(\rho) \sum_{\mu_n \leqq (2\pi B/h)^2} \frac{b_n}{\mu_n^{\delta/2+1/4+\rho/2}} w_n \cos\left(h\sqrt{\mu_n}t + k_0(\rho)\pi\right) \tag{4.2}$$

and $w_n = 1 - h\sqrt{\mu_n}/(2\pi B)$.

By squaring out and then integrating term by term, we get

$$L^{-1}\int_T^{T+L}\Sigma_\rho(t)^2\,dt$$

$$= e_0(\rho)^2 \sum_{\mu_m,\mu_n\leqq(2\pi B/h)^2}\frac{b_m b_n}{(\mu_m\mu_n)^{\delta/2+1/4+\rho/2}}w_m w_n$$

$$\times\frac{1}{2L}\int_T^{T+L}\left\{\cos\left(h(\sqrt{\mu_m}-\sqrt{\mu_n})t\right)+\cos\left(h(\sqrt{\mu_m}+\sqrt{\mu_n})t+2k_0(\rho)\pi\right)\right\}dt$$

$$=\frac{e_0(\rho)^2}{2}\sum_{\mu_n\leqq(2\pi B/h)^2}\frac{b_n^2}{\mu_n^{\delta+1/2+\rho}}w_n^2$$

$$+O\left(L^{-1}\sum_{\mu_m\neq\mu_n\leqq(2\pi B/h)^2}\frac{|b_m b_n|}{(\mu_m\mu_n)^{\delta/2+1/4+\rho/2}}\left|\sqrt{\mu_m}-\sqrt{\mu_n}\right|^{-1}\right)$$

$$+O\left(L^{-1}\sum_{\mu_m,\mu_n\leqq(2\pi B/h)^2}\frac{|b_m b_n|}{(\mu_m\mu_n)^{\delta/2+1/4+\rho/2}}\left(\sqrt{\mu_m}+\sqrt{\mu_n}\right)^{-1}\right).$$

Here we have used the simple bound

$$\int_T^{T+L}\cos(ut+\tau)\,dt \ll \min(L,|u|^{-1}). \tag{4.3}$$

The first O-term above is

$$\ll L^{-1}\left(\sum_{\mu_n\leqq\mu_m/2\ll B^2}+\sum_{\mu_m/2<\mu_n<\mu_m\ll B^2}\right).$$

Clearly, by Lemma 3.1 (a),

$$\sum_{\mu_n\leqq\mu_m/2\ll B^2}\ll\sum_{\mu_m\ll B^2}\left(\sum_{\mu_n\leqq\mu_m}|b_n|\mu_n^{-(\delta/2+1/4)}\right)|b_m|\mu_m^{-(\delta/2+3/4)}\ll B^{1+\varepsilon},$$

and

$$\sum_{\mu_m/2<\mu_n<\mu_m\ll B^2}\frac{|b_m b_n|}{(\mu_m\mu_n)^{\delta/2+1/4}}\frac{\sqrt{\mu_m}}{(\mu_m-\mu_n)}$$

$$\ll\sum_{\mu_m\ll B^2}\frac{b_m^2}{\mu_m^\delta}\sum_{\mu_m/2<\mu_n<\mu_m}(\mu_m-\mu_n)^{-1}+\sum_{\mu_n\ll B^2}\frac{b_n^2}{\mu_n^\delta}\sum_{\mu_n<\mu_m<2\mu_n}(\mu_m-\mu_n)^{-1}$$

$$\ll\sum_{\mu_m\ll B^2}\frac{b_m^2}{\mu_m^\delta}\log\mu_m \quad (\text{as } |\mu_m-\mu_n|\gg 1 \text{ for } n\neq m)$$

$$\ll B^{1+\varepsilon},$$

by Lemma 3.1 (b). The second O-term is easier and is treated by similar argument. Their overall contribution is $\ll B^{1+\varepsilon}L^{-1}$. Thus,

$$(4.4)\qquad L^{-1}\int_T^{T+L}\Sigma_\rho(t)^2\,dt=\frac{e_0(\rho)^2}{2}\sum_{\mu_n\leqq(2\pi B/h)^2}\frac{b_n^2}{\mu_n^{\delta+1/2+\rho}}w_n^2+O(B^{1+\varepsilon}L^{-1}).$$

Since $w_n\ll 1$, by Lemma 3.1 (c),

$$(4.5)\qquad L^{-1}\int_T^{T+L}\Sigma_\rho(t)^2\,dt\ll\log^{2\kappa+1}B.$$

Thus, in view of (4.4) and (4.1), we have

$$(4.6)\qquad L^{-1}\int_T^{T+L}F_\rho(t)^2\,dt=\frac{e_0(\rho)^2}{2}\sum_{\mu_n\leqq(2\pi B/h)^2}\frac{b_n^2}{\mu_n^{\delta+1/2+\rho}}w_n^2+O(B^{1+\varepsilon}L^{-1}).$$

We come now to show that

$$L^{-1}\int_T^{T+L}F_\rho(t)^3\,dt\ll 1.$$

From (4.1), by applying Cauchy-Schwarz's inequality and using the bound (4.5), we find that

$$(4.7)\qquad L^{-1}\int_T^{T+L}F_\rho(t)^3\,dt=L^{-1}\int_T^{T+L}\Sigma_\rho(t)^3\,dt+O(B^{1+\varepsilon}L^{-1}).$$

Multiply out the finite series for $\Sigma_\rho(t)$ given in (4.2) and then integrate term by term, we find that

$$L^{-1}\int_T^{T+L}\Sigma_\rho(t)^3\,dt$$

$$=e_0(\rho)^3\sum_{\mu_m,\mu_n,\mu_l\leqq(2\pi B/h)^2}\prod_{j=m,n,l}\frac{b_jw_j}{\mu_j^{\delta/2+1/4+\rho/2}}L^{-1}\int_T^{T+L}\prod_{j=m,n,l}\cos\big(h\sqrt{\mu_j}t+k_0(\rho)\pi\big)\,dt.$$

Since

$$\cos A\cos B\cos C$$

$$=\frac{1}{4}\big(\cos(A+B+C)+\cos(A+B-C)+\cos(A-B+C)+\cos(A-B-C)\big),$$

it follows, after using (4.3) and renaming m, n, l, that

$$L^{-1}\int_T^{T+L}\Sigma_\rho(t)^3\,dt \ll L^{-1}\sum_{\mu_m,\mu_n,\mu_l\ll B^2}\frac{|b_m b_n b_l|}{(\mu_m\mu_n\mu_l)^{\delta/2+1/4}}\min(L,|\sqrt{\mu_m}+\sqrt{\mu_n}-\sqrt{\mu_l}|^{-1})$$
$$+L^{-1}\sum_{\mu_m,\mu_n,\mu_l\ll B^2}\frac{|b_m b_n b_l|}{(\mu_m\mu_n\mu_l)^{\delta/2+1/4}}(\sqrt{\mu_m}+\sqrt{\mu_n}+\sqrt{\mu_l})^{-1}$$
$$=T_1+T_2,$$

say. By Lemma 3.1 (a), we have $T_2 \ll L^{-1}B^{2+\varepsilon}$. To evaluate T_1, we use our additional assumption (3.3) stated in Theorem 1. We split the sum in T_1 into two parts according as $|\sqrt{\mu_m}+\sqrt{\mu_n}-\sqrt{\mu_l}| \ll \mu_l^{-D}$ or $\gg \mu_l^{-D}$. By (3.3), the first part is $\ll 1$. Applying Lemma 3.1 (a) again, the second part is $\ll L^{-1}B^{3+2D+\varepsilon}$. Hence, $T_1 \ll L^{-1}B^{3+2D+\varepsilon} \ll 1$, since $B \asymp L^{1/(4+2D)}$. Putting this into (4.7), we get

$$L^{-1}\int_T^{T+L} F_\rho(t)^3\,dt \ll 1. \tag{4.8}$$

Since, from (2.1), $E_\rho(x)$ remains bounded for $0<\rho\leqq 1$ and x lying in any finite interval, we can pass the limit $\rho\to 0+$ inside the integrand sign to obtain

$$\lim_{\rho\to 0+}F_\rho(t)=\int_{-L}^{L}\lim_{\rho\to 0+}\frac{E_\rho((t+u)^2)}{(t+u)^{2\theta_0}}K(u)\,du$$
$$=\int_{-L}^{L}\frac{E_0((t+u)^2)}{(t+u)^{2\theta_0}}K(u)\,du, \tag{4.9}$$

since $\lim_{\rho\to 0+}E_\rho(y)=E_0(y)$ except for $y=\lambda_n$.

On the other hand, by (4.6), Lemma 3.1 (c) and the fact that $B\asymp L^{(4+2D)^{-1}}$, we have

$$L^{-1}\int_T^{T+L}\lim_{\rho\to 0+}F_\rho^2(t)\,dt=\frac{e_0(0)^2}{2}\sum_{\mu_n\leqq(2\pi B/h)^2}\frac{b_n^2}{\mu_n^{\delta+1/2}}w_n^2+O(B^{1+\varepsilon}L^{-1})$$
$$\geqq\frac{e_0(0)^2}{8}\sum_{\mu_n\leqq(\pi B/h)^2}\frac{b_n^2}{\mu_n^{\delta+1/2}}+O(B^{1+\varepsilon}L^{-1})$$
$$\gg \log^{2\kappa+1}L.$$

Note that $w_n\geqq 1/2$ for $\mu_n\leqq(\pi B/h)^2$. Also, by (4.8),

$$L^{-1}\int_T^{T+L}\lim_{\rho\to 0+}F_\rho(t)^3\,dt \ll 1.$$

Applying Lemma 3.2, we deduce that

$$\sup_{t\in[T,T+L]}\left(\pm\lim_{\rho\to 0+}F_\rho(t)\right)\gg \log^{\kappa+1/2}L.$$

Finally, since $\int_{-L}^{L} K(u)\,du \ll 1$, we find from (4.9) that

$$\sup_{t\in[T,T+L]}\ \sup_{u\in[-L,L]} \pm\frac{E_0((t+u)^2)}{(t+u)^{2\theta_0}} \gg \log^{\kappa+1/2} L.$$

Choosing $T=\sqrt{X}+L$, our Theorem 1 follows.

5. Lattice points in ellipsoids

Let Q be a 3×3 positive definite symmetric integral matrix with even diagonal elements, $q(x)=\frac{1}{2}x^TQx$ be the associated quadratic form in 3 variables and denote the Epstein zeta-function of Q by

$$\zeta_Q(s)=\sum_{n=1}^{\infty}\frac{r(Q,n)}{n^s}=\sum_{x\in\mathbb{Z}^3-\{0\}} q(x)^{-s}\quad(\Re e\, s>3/2),$$

where $r(Q,n)$ counts the number of integral solutions of $q(x)=n$. Suppose that Q is primitive (i.e. $Q=(a_{ij})$ with g.c.d.$\big((a_{ii}/2,a_{ij})_{1\leqq i+j\leqq m}\big)=1$). Then, $\zeta_Q(s)$ can be meromorphically continued to the whole complex plane with a simple pole at $s=3/2$ of residue $\mathrm{res}_{s=3/2}\,\zeta_Q(s)=|\det Q|^{-1/2}\Gamma(3/2)^{-1}(2\pi)^{3/2}$, which gives rise to the main term in the summatory formula of $r(Q,n)$. Besides, the following functional equation is satisfied by $\zeta_Q(s)$:

$$(2\pi)^{-s}\Gamma(s)\zeta_Q(s)=|\det Q|^{-1/2}\left(\frac{2\pi}{q}\right)^{s-3/2}\Gamma\left(\frac{3}{2}-s\right)\zeta_{qQ^{-1}}\left(\frac{3}{2}-s\right)$$

where q is the smallest positive integer such that qQ^{-1} is an integral matrix with even diagonal elements, called the level of Q. Hence, in the notation of Section 2, $\delta=3/2$, $\alpha=1$, $\theta_0=1/2$, $\phi(s)=(2\pi)^{-s}\zeta_Q(s)$ and $\psi(s)=|\det Q|^{-1/2}\left(\frac{2\pi}{q}\right)^{-s}\zeta_{qQ^{-1}}(s)$. Define

$$P_Q(x)=\sum_{n<x} r(Q,n)-|\det Q|^{-1/2}\frac{(2\pi)^{3/2}}{\Gamma(5/2)}x^{3/2}. \tag{5.1}$$

Landau proved that

$$P_Q(x)\ll x^{3/4+\varepsilon} \tag{5.2}$$

(reproved in Müller [13], p. 150, as well). Now direct application of our Theorem 1 yields the following result in the opposite direction.

Theorem 2. $P_Q(x)=\Omega_\pm(x^{1/2}\sqrt{\log x})$.

Remark. This improves the $\Omega_\pm$-results of the 3-dimensional ellipsoids discussed in [1].

To prove it, we first quote Theorem 6.1 of Müller [13] which gives

$$\sum_{n \leqq x} r(Q,n)^2 = B_Q x^2 + O(x^{14/9}),$$

for some positive constant B_Q. (So $\kappa = 0$ in (3.1).) From (5.1) and (5.2),

$$\sum_{X^2 \leqq n \leqq (X+t)^2} r(Q,n) n^{1/2-u} \ll X^{1-2u} \sum_{X^2 \leqq n \leqq (X+t)^2} r(Q,n)$$
$$\ll X^{1-2u}\left(X^2 + |P_Q(X^2)| + |P_Q((X+1)^2)|\right) \ll X^{3-2u}.$$

As $\sigma_a^* = 3/2$, we see that (2.2) is valid and thus condition $(*)$ holds. To see the condition (3.3), we note that $|\sqrt{m} + \sqrt{n} - \sqrt{l}|$ is either equal to 0 or $\gg l^{-3/2}$. As $r(Q,n) \ll n^{1/2+\varepsilon}$, we see that

$$\sum_{\sqrt{m}+\sqrt{n}=\sqrt{l}} \frac{r(Q,m)r(Q,n)r(Q,l)}{mnl} \ll \sum_{\substack{a,b \\ s \text{ squarefree}}} s^{-3/2+\varepsilon}\left(ab(a+b)\right)^{-1+\varepsilon} \ll 1,$$

since m, n and l satisfy $\sqrt{m} + \sqrt{n} = \sqrt{l}$ if and only if they are of the form $m = a^2 s$, $n = b^2 s$ and $l = (a+b)^2 s$ where a, b, s are positive integers and s is squarefree. Theorem 2 is thus proved.

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References

[1] *S. D. Adhikari* and *Y.-F. S. Pétermann*, Lattice points in ellipsoids, Acta Arith. **59** (1991), 329–338.
[2] *B. C. Berndt*, Identities involving the coefficients of a class of Dirichlet series I, Trans. Amer. Math. Soc. **137** (1969), 345–359.
[3] *B. C. Berndt*, On the average order of some arithmetical functions, Bull. Amer. Math. Soc. **76** (1970), 856–859.
[4] *B. C. Berndt*, On the average order of some arithmetical functions I, J. Number Th. **3** (1971), 184–203.
[5] *B. C. Berndt*, On the average order of some arithmetical functions II, J. Number Th. **3** (1971), 288–305.
[6] *K. Chandrasekharan* and *R. Narasimhan*, Hecke's functional equation and arithmetical identities, Ann. Math. **74** (1961), 1–23.
[7] *K. Chandrasekharan* and *R. Narasimhan*, Hecke's functional equation and the average order of arithmetical functions, Acta Arith. **6** (1961), 487–503.
[8] *K. Chandrasekharan* and *R. Narasimhan*, Approximate functional equations for a class of zeta-functions, Math. Ann. **152** (1963), 30–64.
[9] *K. Chandrasekharan* and *R. Narasimhan*, On the mean value of the error term for a class of arithmetical functions, Acta Math. **112** (1964), 41–67.
[10] *J. L. Hafner*, On the representation of the summatory function of a class of arithmetical functions, Analytic Number Theory, M. I. Knopp, ed., Springer Lect. Notes Math. **899** (1981).
[11] *J. L. Hafner*, On the average order of a class of arithmetical functions, J. Number Th. **15** (1982), 36–76.
[12] *A. Ivić*, Large values of certain number-theoretic error terms, Acta Arith. **56** (1990), 135–159.

[13] *W. Müller*, The mean square of Dirichlet series associated with automorphic forms, Mh. Math. **113** (1992), 121–159.
[14] *D. Redmond*, Mean value theorems for a class of Dirichlet series, Pac. J. Math. **78** (1978), 191–231.
[15] *D. Redmond*, Omega theorems for a class of Dirichlet series, Rocky Mountain J. Math. **9** No. 4 (1979), 733–748.
[16] *G. Szegö*, Beiträge zur Theorie der Laguerreschen Polynome II: Zahlentheoretische Anwendungen, Math. Z. **25** (1926), 388–404.
[17] *K. Tsang*, Counting lattice points in the sphere, Bull. Lond. Math. Soc. **32** (2000), 679–688.

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