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Permutable entire functions and their Julia sets

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Abstract

In 1922–23, Julia and Fatou proved that any 2 rational functions f and g of degree at least 2 such that $f(g(z)) = g(f(z))$, have the same Julia set. Baker then asked whether the result remains true for nonlinear entire functions. In this paper, we shall show that the answer to Baker's question is true for almost all nonlinear entire functions. The method we use is useful for solving functional equations. It actually allows us to find out all the entire functions g which permute with a given f which belongs to a very large class of entire functions.

1. *Introduction and main results*

The Fatou set $F(f)$ of an entire function f of one complex variable is the subset of the complex plane where the family $\{f^n\}$ of iterates of f is normal. Its Julia set $J(f)$ is equal to $\mathbb{C}\backslash F(f)$. The Fatou set and Julia set of a rational function can be defined as subsets of the Riemann sphere in a similar way. A well known property of the Julia set of an entire or rational function f is that $J(f) = J(f^n)$. Other basic knowledge of iterations of rational or transcendental functions can be found in [**5**, **6**, **13**].

In 1922–23, Julia [**16**] and Fatou [**12**] proved that for any 2 rational functions f and g of degree at least 2 such that f and g are permutable, i.e. $f \circ g = g \circ f$, then their Julia sets will be the same. It is natural to consider the following open problem which was first mentioned in [**4**] by Baker.

Problem A. Let f and g be nonlinear entire functions. If f and g are permutable, is $J(f) = J(g)$?

Julia and Fatou's works were motivated by the problem of characterizing all permutable rational functions f and g. Julia [**16**] and Fatou [**12**] solved this problem when both f and g are polynomials. They proved that for permutable nonlinear polynomials f and g, there exist natural numbers m, n such that (up to a conjugacy of linear maps) either (i) $f^m(z) = g^n(z)$; (ii) $f(z) = z^m$ and $g(z) = z^n$ or (iii) $f(z) = T_m(z)$ and $g(z) = T_n(z)$, where T_k is the Tchebycheff polynomial determined by the equation $\cos kw = T_k(\cos w)$.

The rational case was first solved completely by Ritt [**24**] in 1923. However, Ritt did not use methods from complex dynamics. A proof of Ritt's result in the spirit of the ideas of Julia and Fatou was provided by Eremenko [**10**] in 1989. It is also natural to ask the following open problem.

Problem B. Is there a complete classification of all pairs of nonlinear permutable entire functions?

In 1958–59, Baker [**1**] and Iyer [**15**] started the investigations of permutable entire functions. They both proved that if a nonconstant polynomial f is permutable with a transcendental entire function g, then $f(z) = e^{2m\pi i/k}z + b$ for some $m, k \in \mathbb{N}$ and complex number b. It follows from this result, as well as Julia and Fatou's results, that in order to answer Problems A and B, we only need to consider permutable transcendental entire functions. Let $m, n \in \mathbb{N}$ and h be a transcendental entire function. Suppose that $az+b$ and $cz+d$ permute with h^m and h^n respectively. If $az+b$ also permutes with $cz+d$, then $f = ah^m+b$ permutes with $g = ch^n+d$. Up to a conjugacy of linear maps, almost all known examples of permutable transcendental entire functions are of this form. Note that a and c must be a pth root and qth root of unity for some $p, q \in \mathbb{N}$. If both $a, b \neq 1$, then it is easy to check that $f^p = h^{mp}$ and $g^q = h^{nq}$ so that $f^{npq} = h^{mnpq} = g^{mpq}$. This is case (i) above. Recently, the following interesting example of permutable transcendental entire functions is mentioned in [**14**].

Example 1. Let $a, c \in \mathbb{C}$ such that $e^{4a} = -1$ and $c \neq 0$.

Define $f(z) = ci[\exp((ai/2c^2)z^2) + \exp((-ai/2c^2)z^2)]$ and $g(z) = c[\exp((ai/2c^2)z^2) - \exp((-ai/2c^2)z^2)]$. Then f permutes with g.

In [**1**], Baker characterized all nonlinear entire functions which permute with the exponential function and proved the following result.

THEOREM A. *Let g be a nonlinear entire function which is permutable with $f(z) = ae^{bz} + c$ $(ab \neq 0, a, b, c \in \mathbb{C})$, then $g = f^n$. Hence $J(f) = J(f^n) = J(g)$.*

This result shows that there are only countably infinitely many nonlinear entire functions which permute with $f(z) = e^z$. This is in fact true for general f (see [**3**]). Theorem A also answers Problems A and B for the special case that $f(z) = e^z$. Concerning Problem A, Baker proved the following result in [**4**].

THEOREM B. *Suppose that f and g are transcendental entire functions such that $g(z) = af(z) + b$, where a and b are complex numbers. If g permutes with f, then $J(f) = J(g)$.*

In fact Baker only proved the case $a = 1$, but the general case can be proved similarly (see [**22**]). In the same paper, after a careful analysis of Julia and Fatou's original arguments, Baker also proved the following result.

THEOREM C. *If f and g are permutable transcendental entire functions and if ∞ is neither a limit function of any subsequence of $\{f^n\}$ in a component of $F(f)$, nor of any subsequence of $\{g^n\}$ in a component of $F(g)$, then $J(f) = J(g)$.*

From the classification of components of Fatou sets (see [**6**]), there are only two ways in which f^n can tend to infinity locally uniformly on a component U of $F(f)$. One possibility is that U is a wandering domain of f, i.e. $f^m(U) \neq f^n(U)$ for all $n \neq m$. The other is $f^r(U) \subset V$ for some $r \geqslant 0$ and Baker domain V. V is a Baker domain if $\infty \in V$ and $f^n(V) \subset V$ for some $n \geqslant 0$. The following result of Bergweiler and

Hinkkanen [**7**] shows that the presence of Baker domain is not a problem provided that both f and g do not have wandering domain.

THEOREM D. *Let f and g be permutable transcendental entire functions. If both f and g have no wandering domains, then $J(f) = J(g)$.*

A slightly weaker version of Theorem D was first proved by Langley [**18**]. Besides Theorem D, there are other partial results concerning Problem A (see [**13**, chapter 7], [**22**, **23**]). There are quite a lot of results about permutable entire functions which are related to Problem B. They can be found in [**14**, **17**, **25**–**28**]. The following typical example of these results can be found in [**25**].

THEOREM E. *Let $f(z) = p(z)e^{q(z)}$, where $p(z)$ and $q(z)$ are polynomials. If g is a finite order entire function which permutes with f. Then $g(z) = af(z)$ for some $a \in \mathbb{C}$.*

All these results require both f and g to satisfy certain conditions. This is rather restrictive because in general given an entire function f, it is difficult to check whether the entire functions which permute with f satisfy the required condition or not. The situation will be much clearer if we rephrase Problem A into an equivalent problem as follows: let f be a nonlinear entire function. If g is a nonlinear entire function which permutes with f, is $J(f) = J(g)$? We immediately see that Problem A has only been solved for polynomials and transcendental entire functions $ae^{bz} + c$.

In this paper, we shall answer Problems A and B for a class of entire functions including $e^z + p(z)$, $\sin z + p(z)$, where p is a nonconstant polynomial. In fact, we shall prove that for any nonlinear entire function g which permutes with f in this class, $g(z) = af^n(z) + b$ for some $a, b \in \mathbb{C}$. Note that $f^n \circ g = g \circ f^n$ and hence by Theorem B, $J(f) = J(f^n) = J(g)$. By factoring out f^n, we also have $f^n(az + b) = af^n(z) + b$. The result of Baker and Iyer mentioned before tells us that a is a kth root of unity.

Before stating our main result, we recall that an entire function F is prime (left-prime) in the entire sense if whenever $F(z) = f(g(z))$ for some entire functions f, g, then either f or g is linear (f is linear whenever g is transcendental). For example, $e^z + z$, ze^z are prime functions (see [**8**] for more examples).

THEOREM 1. *Let f be a transcendental entire function which satisfies the following conditions.*

(A1) *f is not of the form $H \circ Q$, where H is periodic entire and Q is a polynomial.*
(A2) *f is left-prime in the entire sense.*
(A3) *f' has at least two distinct zeros.*
(A4) *There exists a natural number N such that for any complex number c, the simultaneous equations $f(z) = c$, $f'(z) = 0$ have at most N solutions.*
(A5) *The orders of zeros of f' are bounded by M for some $M \in \mathbb{N}$.*

Let g be a nonlinear entire function which permutes with f. Then $g(z) = af^n(z) + b$, where a is a kth root of unity and $b \in \mathbb{C}$. Hence $J(f) = J(g)$.

The conditions (A4) and (A2) are related. For example, Ozawa [**21**] proved that if f is of finite order and for any $c \in \mathbb{C}$, the simultaneous equations $f(z) = c$, $f'(z) = 0$ have only a finite number of solutions, then f is left-prime in the entire sense. Other similar results can also be found in [**20**] and [**21**]. It is easy to check that for any nonconstant polynomial p, $e^z + p(z)$ and $\sin z + p(z)$ satisfy conditions (A4) and (A5). By Ozawa's result, they are left-prime. Using Borel's Lemma ([**8**, theorem 1·7]), it is not difficult to show that the dervatives of $e^z + p(z)$ and $\sin z + p(z)$ have infinitely many zeros. Suppose that $e^z + p(z)$ or $\sin z + p(z)$ is of the form $H \circ Q$ in (A1). Then Q cannot be linear because $e^z + p(z)$ and $\sin z + p(z)$ are nonperiodic. Now Q is of degree greater than 1, then it can be shown that the order of $H \circ Q$ will be greater than 1 which is also impossible as both $e^z + p(z)$ and $\sin z + p(z)$ are of order 1. Therefore, $e^z + p(z)$ and $\sin z + p(z)$ satisfy conditions (A1)–(A5).

The conditions (A1)–(A5) are not restrictive. In certain sense, almost all entire functions satisfy these conditions as can be seen from the following result of Noda [**20**].

THEOREM F. *Let f be a transcendental entire function. There exists a countable set $E_f \subset \mathbb{C}$ such that for all $a \notin E_f$, $f_a(z) = f(z) + az$ satisfies the following conditions.*

(B1) *f_a is nonperiodic.*
(B2) *f_a is prime in the entire sense.*
(B3) *f'_a has infinitely many zeros.*
(B4) *For any complex number c, the simultaneous equations $f_a(z) = c$, $f'_a(z) = 0$ has at most one solution.*
(B5) *The orders of zeros of $f'_a(z)$ are equal to* 1.

Noda's original proof only shows that we can find an exceptional set $D_f \subset \mathbb{C}$ such that f_a satisfies (B1)–(B4) for all $a \notin D_f$. (B5) will also be satisfied if we replace D_f by $E_f = D_f \cup \{-f(c) | f''(c) = 0\}$. Clearly, Bi implies Ai for $i = 2, 3, 4, 5$. It is not difficult to check that conditions (B1) and (B2) together imply condition (A1). Now, combining Theorem 1 and Theorem F, we obtain the following result which says that in certain sense, the answer to Baker's question is yes for almost all entire functions.

THEOREM 2. *Let f be a transcendental entire function and define $f_a(z) = f(z) + az$. Then there exists a countable set $E_f \subset \mathbb{C}$ such that for each $a \notin E_f$, any nonlinear entire function g permutes with f_a is of the form $g(z) = cf_a^n(z) + d$, where c is a* kth *root of unity and $d \in \mathbb{C}$. Hence $J(f_a) = J(g)$.*

The method developed in this paper will be useful for solving functional equations. Actually we can also use it to prove the following result.

THEOREM 3. *Let q be a nonconstant entire function and p be a polynomial with at least two distinct zeros. Suppose that $f(z) = p(z)e^{q(z)}$ is prime in the entire sense. Then any nonlinear entire function g which permutes with f is of the form $g(z) = af^n(z) + b$, where a is a* kth *root of unity and $b \in \mathbb{C}$. Hence $J(f) = J(g)$.*

It is known that if q is a polynomial and p and q do not have a nonlinear common right factor, then $f(z) = p(z)e^{q(z)}$ is prime in the entire sense.

2. *The Common Right Factor Theorem*

To prove Theorem 1, we first prove the Common Right Factor Theorem which gives a sufficient condition for the existence of a nonlinear generalized common right factor of two entire functions.

Definition 1. Let $F(z)$ be a nonconstant entire function. An entire function $g(z)$ is a generalized right factor of F (denoted by $g \leqslant F$) if there exists a function f, which is analytic on the range of g, such that $F = f \circ g$. If $h \leqslant f$ and $h \leqslant g$, we say that h is a generalized common right factor of f and g.

The following lemma is crucial. It is extracted from the proof of theorem 1·1 in a paper of Eremenko and Rubel [**11**]. A quite detailed proof of it can also be found in [**19**].

LEMMA 1. *Let f, g be two entire functions. For $i = 1, \ldots, k$, $k \geqslant 2$, let $S_i = \{z_{in}\}_{n\in\mathbb{N}}$ be a sequence of distinct complex numbers with limit point z_i. Suppose that all the limit points z_i are distinct and for all $n \in \mathbb{N}$,*

$$\begin{cases} f(z_{1n}) = f(z_{2n}) = \cdots = f(z_{kn}) \\ g(z_{1n}) = g(z_{2n}) = \cdots = g(z_{kn}). \end{cases}$$

Then there exists an entire function h (which depends on f and g only and is independent of k and the sequences S_i) satisfying $h \leqslant f$, $h \leqslant g$ and $h(z_1) = h(z_i)$ for all $2 \leqslant i \leqslant k$.

Example 2. Let $f(z) = \cos z$ and $g(z) = \sin z$. Let $z_{1n} = 1/n$, $z_{2n} = 2\pi + 1/n$ and $z_{3n} = -2\pi + 1/n$. Then $\lim_{n\to\infty} z_{1n} = 0$, $\lim_{n\to\infty} z_{2n} = 2\pi$, $\lim_{n\to\infty} z_{3n} = -2\pi$ and for all $n \in \mathbb{N}$,

$$\begin{cases} f(z_{1n}) = f(z_{2n}) = f(z_{3n}) \\ g(z_{1n}) = g(z_{2n}) = g(z_{3n}). \end{cases}$$

Note that there exists an entire function $h(z) = e^{iz}$ satisfying $h \leqslant f$, $h \leqslant g$ and $h(0) = h(-2\pi) = h(2\pi)$.

In many situations, Lemma 1 is not so easy to use because of the difficulties in finding the sequences required in the lemma. We shall prove the Common Right Factor Theorem below which is quite powerful and easy to use.

THEOREM 4 (Common Right Factor Theorem). *Let f and g be two entire functions and $z_1, \ldots, z_k$ be $k \geqslant 2$ distinct complex numbers such that*

$$\begin{cases} f(z_1) = f(z_2) = \cdots = f(z_k) = A \\ g(z_1) = g(z_2) = \cdots = g(z_k) = B. \end{cases}$$

Suppose that there exist nonconstant functions f_1 and g_1 such that $f_1 \circ f \equiv g_1 \circ g$ on $\bigcup_{i=1}^{k} U_i$, where U_i is some open neighbourhood containing z_i. If f_1 is analytic in a neighbourhood of A and the order of f_1 at A is $K < k$, then there exists an entire function h (which depends on f and g only and is independent of k and z_i) with $h \leqslant f$, $h \leqslant g$. Moreover, among the z_i, there exist at least $m = [k - 1/K] + 1$ distint points $z_{n_1}, \ldots, z_{n_m}$ such that $h(z_{n_1}) = \cdots = h(z_{n_m})$.

We immediately have the following

COROLLARY 1. *Let f and g be two entire functions and $\{z_n\}_{n\in\mathbb{N}}$ be an infinite sequence of distinct complex numbers such that for all $n \in \mathbb{N}$, $f(z_n) = A$ and $g(z_n) = B$. Suppose that there exist nonconstant functions f_1 and g_1 such that $f_1 \circ f \equiv g_1 \circ g$ on $\bigcup_{i=1}^{\infty} U_i$, where U_i is some open neighbourhood containing z_i. If f_1 is analytic in a neighbourhood of A, then there exists a transcendental entire function h with $h \leqslant f$, $h \leqslant g$.*

Remark. In Theorem 4, the condition that $k > K$ is essential. Let $f(z) = z^2$, $g(z) = e^{iz}$, $f_1(z) = \cos\sqrt{z}$ and $g_1(z) = \frac{1}{2}(z + z^{-1})$. Then $\cos z = f_1 \circ f(z) = g_1 \circ g(z)$. Although $f(-\pi) = f(\pi) = \pi^2$ and $g(-\pi) = g(\pi) = -1$, f and g do not have a nonlinear generalized common right factor. Note that in this case, the order K of f_1 at π^2 is exactly two.

Proof of Theorem 4. Replacing $f_1(z)$ by $f_1(z+A)$ and $f(z)$ by $f(z)-A$ if necessary, we may assume that $A=0$. Recall that f_1 is analytic at A with order K. So if V is a sufficiently small neighbourhood of $A=0$ and $a,b\in V$ such that $f_1(a)=f_1(b)$, then $b\approx e^{l\frac{2\pi i}{K}}a$ for some $0\leqslant l\leqslant K-1$. Hence if we take any ray L starting from $f_1(A)$, then $f^{-1}_{1|V}(L)$ consists of K curves starting at $A=0$ which divide V into K open sector shaped regions V_j such that f_1 is injective on each region V_j.

For each $n\in\mathbb{N}$ and $1\leqslant i\leqslant k$, let $D_n(z_i)=\{z\colon |z-z_i|<1/n\}$. There is no harm in assuming that all $D_n(z_i)\subset U_i$. Since $g(z_1)=g(z_i)=B$ and g is entire, $\bigcap_{i=1}^n g(D_n(z_i))$ is an open set containing B. Choose a $z_{1n}\in D_n(z_1)$ such that $g(z_{1n})\in\bigcap_{i=1}^n g(D_n(z_i))$, $g(z_{1n})\neq g(z_1)$ and $\arg\{f_1(f(z_{1n}))-f_1(A)\}=\pi$. The last condition means that $f(z_{1n})$ is on a curve through A which approximately bisects V_1. Now for each $2\leqslant i\leqslant k$, there exists $z_{in}\in D_n(z_i)$ such that $g(z_{1n})=\cdots=g(z_{kn})$. Since $g(z_{1n})\neq g(z_1)=g(z_i)=B$, we have $z_{in}\neq z_i$, for all $n\in\mathbb{N}$. Clearly, $z_{in}\to z_i$ as $n\to\infty$ and hence $f(z_{in})\to f(z_i)=A$ as $n\to\infty$. By passing to a subsequence if necessary, we may assume that for each i, $\{z_{in}\}_{n\in\mathbb{N}}$ is a sequence of distinct complex numbers with limit point z_i.

From the condition that $f_1\circ f\equiv g_1\circ g$ on $\bigcup_{i=1}^k U_i$, we have $f_1(f(z_{1n}))=\cdots=f_1(f(z_{kn}))$. Hence for each fixed n, $f(z_{jn})\approx e^{l_j\frac{2\pi i}{K}}f(z_{1n})$ for some $0\leqslant l_j\leqslant K-1$. Since $f(z_{1n})$ is on a curve through A which approximately bisects V_1, each $f(z_{jn})$ is also on a curve through A which approximately bisects some V_i. Therefore there exists one sector shaped region V_i which contains at least $m=[k-1/K]+1$ of the $f(z_{jn})'s$, say $f(z_{j_1n}),\ldots,f(z_{j_mn})$. Since $f_1(f(z_{j_1n}))=\cdots=f_1(f(z_{j_mn}))$ and f_1 is injective on V_i, we must have $f(z_{j_1n})=\cdots=f(z_{j_mn})$. Therefore for each $n\in\mathbb{N}$, we obtain a subset (depending on n) $\{j_1,\ldots,j_m\}$ of $\{1,\ldots,k\}$. As there are only finitely many subsets of $\{1,\ldots,k\}$ containing exactly m elememts, we can find a $\{j_1,\ldots,j_m\}$ which corresponds to infinitely many n. For these n, we have

$$\begin{cases} f(z_{j_1n})=f(z_{j_2n})=\cdots=f(z_{j_mn}) \\ g(z_{j_1n})=g(z_{j_2n})=\cdots=g(z_{j_mn}). \end{cases}$$

Clearly, $S_{j_i}=\{z_{j_in}\}$ are the required sequence in Lemma 1 and we are done.

3. *Proof of Theorems* 1 *and* 3

Let $\bar{n}(r,1/f)$ and $\bar{n}(r_o\leqslant|z|\leqslant r,1/f)$ be the number of distinct zeros of f in $|z|\leqslant r$ and $r_o\leqslant|z|\leqslant r$ respectively. The following lemma is due to Clunie [**9**].

LEMMA 2. *Let k be entire and transcendental. Given $K>0$ there is a number $r_0>0$ and an increasing sequence $\{r_n\}_{n\in\mathbb{N}}$ with $r_1>r_0$ and $r_n\to\infty$ (as $n\to\infty$) such that for $n\geqslant 1$ and all r in $r_n\leqslant r\leqslant r_n^2$ and all a satisfying $r_0\leqslant|a|\leqslant r$ we have*

$$\bar{n}\left(r,\frac{1}{k-a}\right)>K.$$

LEMMA 3. *Let h,k be entire and transcendental. Suppose that h has infinitely many zeros. Then for each $N\in\mathbb{N}$, there exists a zero a_N of h such that $k(z)=a_N$ has at least N distinct roots which are not the zeros of h.*

Proof of Lemma 3. Assume the contrary, then for each zero a_i of h, all (except at most $N-1$) distinct roots of $k(z)=a_i$ are zeros of h. This implies that

$$\sum_{\substack{h(a_i)=0\\|a_i|\leqslant r}}\left[\bar{n}\left(r,\frac{1}{k-a_i}\right)-(N-1)\right]\leqslant \bar{n}\left(r,\frac{1}{h}\right).$$

Hence,

$$\sum_{\substack{h(a_i)=0\\|a_i|\leqslant r}}\bar{n}\left(r,\frac{1}{k-a_i}\right)\leqslant \bar{n}\left(r,\frac{1}{h}\right)+(N-1)\bar{n}\left(r,\frac{1}{h}\right)=N\bar{n}\left(r,\frac{1}{h}\right).$$

We therefore have for all $0<r_0<r$,

$$\sum_{\substack{h(a_i)=0\\r_0\leqslant|a_i|\leqslant r}}\bar{n}\left(r,\frac{1}{k-a_i}\right)\leqslant N\bar{n}\left(r,\frac{1}{h}\right). \tag{1}$$

Applying Lemma 2 to k and $K=2N$, we get the r_0 and required sequence $\{r_n\}_{n\in\mathbb{N}}$ such that

$$\sum_{\substack{h(a_i)=0\\r_0\leqslant|a_i|\leqslant r_n}}\bar{n}\left(r_n,\frac{1}{k-a_i}\right)>\sum_{\substack{h(a_i)=0\\r_0\leqslant|a_i|\leqslant r_n}}2N=2N\bar{n}\left(r_o\leqslant|z|\leqslant r_n,\frac{1}{h}\right).$$

Since h has infinitely many zeros, for all sufficiently large r_n, $\bar{n}(r_o\leqslant|z|\leqslant r_n,1/h)>\bar{n}(r_o,1/h)$. Hence for all sufficiently large r_n,

$$\sum_{\substack{h(a_i)=0\\r_0\leqslant|a_i|\leqslant r_n}}\bar{n}\left(r_n,\frac{1}{k-a_i}\right)>N\bar{n}\left(r_n,\frac{1}{h}\right).$$

This contradicts (1).

Lemma 4. *Let f be a transcendental entire function such that f' has at least two distinct zeros. Let g be a nonlinear entire function permutes with f. Then for each $K\in\mathbb{N}$, there exists $a_K\in\mathbb{C}, g'(a_K)=0$ such that $f-a_K$ and $f'\circ g$ have at least K common distinct zeros.*

Proof of Lemma 4. As g is an nonlinear entire function which permutes with the transcendental entire function f, the result of Baker and Iyer mentioned earlier guarantees that g is transcendental. Now $f\circ g=g\circ f$ implies that $f'(g(z))g'(z)=g'(f(z))f'(z)$. Suppose that $g'\circ f$ has finitely many (say M) zeros. Then all (except M) zeros of $f'\circ g$ are zeros of f'. Since f' has at least 2 zeros and g is transcendental entire, by the Little Picard Theorem $f'\circ g$ has infinitely many zeros. It follows that f' also has infinitely many zeros. Now by Lemma 3, there exists $a\in\mathbb{C}$, $f'(a)=0$ such that $g(z)=a$ has at least $M+1$ roots which are not the zeros of f'. This is a contradiction. Therefore $g'\circ f$ has an infinite number of zeros and hence g' has at least one zero. Clearly at least one zero of g' is not a Picard exceptional value of f, otherwise $g'\circ f$ will have finitely many zeros only. So there exists some $b\in\mathbb{C}, g'(b)=0$ such that $f(z)=b$ has an infinite number of roots. Suppose that g' has only finitely many zeros. It follows from $f'(g(z))g'(z)=g'(f(z))f'(z)$ that $f-b$ and $f'\circ g$ have infinitely many common zeros and we are done in this case. Now suppose that g' has infinitely many zeros. By Lemma 3, for each $K\in\mathbb{N}$, there exists $a_K\in\mathbb{C}, g'(a_K)=0$ such that $f-a_K$ has at least K distinct zeros which are not the zeros of g'. Since

$f'(g(z))g'(z) = g'(f(z))f'(z)$, $f - a_K$ and $f'(g)$ must have at least K distinct common zeros.

We also need the following result of Baker ([**1**, p. 145]).

LEMMA 5. *If f and g are permutable entire transcendental functions, then there exists a positive integer n and $R > 0$, such that $M(r, g) < M(r, f^n)$ holds for all $r > R$, where $M(r, g), M(r, f^n)$ denote the maxmium modulus function of g and f^n respectively.*

Proof of Theorem 1. For each natural number K which is a multiple of $2N(M + 1)$, by Lemma 4 we can now find some $a_K \in \mathbb{C}, g'(a_K) = 0$ such that $f - a_K$ and $f'(g)$ has at least K distinct common zeros, say $z_1, \ldots, z_K$. Now $f(z_i) = a_K$ implies that $f(g(z_i)) = g(f(z_i)) = g(a_K)$. Moreover, $f'(g(z_i)) = 0$. By condition (A4), the system of equations $f(z) = g(a_K)$, $f'(z) = 0$ has at most N solutions. Therefore, at least K/N of $g(z_i)$ are equal (say $g(z_1), \ldots, g(z_{K/N})$). Hence we have

$$\begin{cases} f(z_1) = f(z_2) = \cdots = f(z_{K/N}) = a_K \\ g(z_1) = g(z_2) = \cdots = g(z_{K/N}) = B. \end{cases}$$

According to condition (A5), the order of f at B is at most $M+1$ and $K/N > M+1$. By the Common Right Factor Theorem, there exists an entire function h (which depends on f and g only) with $h \leqslant f$, $h \leqslant g$. Moreover, among the $z_1, \ldots, z_{K/N}$, there exist at least $m = K/N(M + 1)$ distinct points at which h takes the same value. Since K as well as m can be arbitrarily large and h is independent of K, h is transcendental.

As $h \leqslant f$ and $h \leqslant g$, $f = f_1 \circ h$ and $g = g_1 \circ h$ for some f_1, g_1 which are analytic on the range of h. h is transcendental entire, by the Little Picard Theorem, h can omit at most one complex number. If the range of h is $\mathbb{C}\backslash\{a\}$ for some $a \in \mathbb{C}$, then $h = a + e^q$ for some entire function q and $f(z) = f_1(a + e^w) \circ q(z)$. Note that q cannot be transcendental because by condition (A2), f is left-prime. Therefore q must be a polynomial which is also impossible by condition (A1). So the range of h must be the whole plane. This implies that both f_1, g_1 are entire. Since f is prime and h is transcendental, f_1 must be linear. Hence $h = f_1^{-1} \circ f$ and $g = g_1 \circ f_1^{-1} \circ f = g_2 \circ f$ where $g_2 = g_1 \circ f_1^{-1}$. From $f \circ g = g \circ f$, we have $f \circ g_2 \circ f = g_2 \circ f \circ f$. Note that the range of f equals that of h. Therefore, $f \circ g_2 \equiv g_2 \circ f$ on $\mathbb{C}$. If g_2 is nonlinear, by repeating the same arguments, we can find an entire g_3 which permutes with f such that $g_2 = g_3 \circ f$ and $g = g_3 \circ f^2$. Inductively, we have $g = g_{m+1} \circ f^m$ provided that g_m is nonlinear. If there exists some m such that g_m is linear, then we are done. So assume that all g_m are nonlinear. Since each g_m permutes with f, g_m must be transcendental. By Lemma 5, there exists a positive integer n and $R > 0$, such that $\log M(r, g) < \log M(r, f^n)$ holds for all $r > R$.

On the other hand, a result of Clunie ([**9**, theorem 1]) implies that

$$\limsup_{r\to\infty} \frac{\log M(r, g)}{\log M(r, f^n)} = \limsup_{r\to\infty} \frac{\log M(r, g_{n+1} \circ f^n)}{\log M(r, f^n))} = \infty.$$

This is a contradiction and we are done.

Proof of Theorem 3. Note that g is transcendental as it is nonlinear. Since p has at least two distinct zeros, $p(g(z))$ has infinitely many zeros by the Little Picard Theorem. It follows from $p(g(z))e^{q(g(z))} = g(p(z)e^{q(z)})$ that $g(p(z)e^{q(z)})$ also has infinite number of zeros. Therefore there exists a zero b of g such that $p(z)e^{q(z)} - b$ has infinitely many zeros $\{z_i\}_{i\in\mathbb{N}}$. Note that the z_i's are zeros of $p(g(z))e^{q(g(z))} = g(p(z)e^{q(z)})$. Hence, $p(g(z_i)) = 0$ for all $i \in \mathbb{N}$. Therefore we can find an infinite subsequence $\{z_{n_i}\}_{i\in\mathbb{N}}$ such that $f(z_{n_i}) = b$ and $g(z_{n_i}) = a$ for some zero a of p. By Corollary 1, there exists a transcendental entire function h with $h \leqslant f$ and $h \leqslant g$. Since f is nonperiodic and prime, we can repeat the arguments used in the proof of Theorem 1 to obtain the required conclusions.

4. *Final remarks*

Provided that the below conjecture is true, we can replace condition (A4) in Theorem 1 by a much weaker condition: for any $c \in \mathbb{C}$, the simultaneous equations $f(z) = c, f'(z) = 0$ have only a finite number of solutions.

Conjecture. Let f be a transcendental entire function such that f' has at least two distinct zeros. Let g be a nonlinear entire function which permutes with f. Then there exists $a \in \mathbb{C}, g'(a) = 0$ such that $f - a$ and $f' \circ g$ have infinitely many distinct common zeros.

It is expected that the Common Right Factor Theorem and its corollary will also be useful to solve other functional equations (e.g. $f \circ f = g \circ g$). These results reduce the problem of solving one functional equation to a problem of solving systems of simultaneous equations. Therefore it would be nice to know whether it is true that if $f \circ g = g \circ f$ or $f \circ f = g \circ g$, then we can always find two distinct points z_1, z_2 such that $f(z_1) = f(z_2)$ and $g(z_1) = g(z_2)$, where f, g are transcendental entire functions.

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