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Uniruled projective manifolds with irreducible reductive G-structures

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We will call a G-structure modeled after a compact irreducible Hermitian symmetric space S of rank $\geqq 2$, an S-structure. (See section 3 or [KO] for a precise definition. Note that our S-structure is called $G(S)$-structure in [KO].) Such structures were studied by many authors in the 60's (see [Oc] and the references there). From the 80's, they were studied by people working on twistor theory (see [Ba], [Ma] and the references there). When one studies these works, what is rather amazing, at least to the authors, is the lack of a nonflat example among compact manifolds. One may even expect that S-structures are always flat under mild conditions. One result along this line is

Theorem (Kobayashi-Ochiai [KO]). *An S-structure on a compact Kähler-Einstein manifold X is always flat, and the universal cover of X is a Hermitian symmetric space.*

Since the existence of a Kähler-Einstein metric is a very strong condition when X is Fano ($c_1(X) > 0$), it would be much nicer, if the flatness is true for general Fano manifolds. Our main result says that this is indeed the case. We get a slightly more general statement. Note that Fano manifolds are uniruled by [Mo].

Main Theorem. *Let $G \subset GL(V)$ be an irreducible faithful representation of a connected reductive complex Lie group G. Let M be a uniruled projective manifold with a G-structure. Then the G-structure is flat. Furthermore, if $G \neq GL(V)$, then M is biholomorphic to a compact irreducible Hermitian symmetric space S of rank $\geqq 2$.*

For the four classical types, we can describe S-structures more explicitly:

Corollary. *Let M be a uniruled projective manifold.*

(I) *If there exist two vector bundles U, W of rank $\geqq 2$, such that $T(M) \cong U \otimes W$, then M is biholomorphic to a Grassmannian.*

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(II) *If there exists a vector bundle U with $T(M) \cong \Lambda^2 U$, then M is biholomorphic to a quadric Grassmannian of m-dimensional linear subspaces in a $2m$-dimensional hyperquadric.*

(III) *If there exists a vector bundle U with $T(M) \cong S^2 U$, where S^2 denotes the symmetric square, then M is biholomorphic to a Lagrangian Grassmannian, i.e. the variety of Lagrangian subspaces of a symplectic vector space.*

(IV) *If there exists a line bundle L and a section of $S^2 T^*(M) \otimes L$ defining a non-degenerate symmetric bilinear form on $T(M)$, then M is biholomorphic to the hyperquadric.*

One can describe the exceptional S-structures as a certain spinor structure and an octanionic structure on the tangent bundle. But we will skip it.

(IV) was proved earlier by [Ye]. His proof uses the Kobayashi-Ochiai criterion for hyperquadrics and needs a rather detailed study of the full space of minimal rational curves. Our approach in case (IV) is simply showing the vanishing of the Weyl conformal curvature tensor directly from the information about a generic minimal rational curve. So even for case (IV), we have a different and simple proof.

Flatness of G-structures gives rise to uniformizing coordinates which define a corresponding pseudogroup structure on the underlying manifold. As such our characterization of Fano manifolds with G-structures comes close to the perspective on the uniformization of complex manifolds as expounded by Gunning [Gn].

As is well-known the projective space and the hyperquadric of dimension $\geqq 3$ can be characterized in terms of ample lines bundles. Our Main Theorem gives the first algebro-geometric characterizations of other irreducible Hermitian symmetric manifolds of the compact type, without the assumption of homogeneity. As such, it might be useful in the study of Fano manifolds X with numerically effective tangent bundles, which are conjectured in Campana-Peternell [CP] to be always rational homogeneous. The present article suggests the approach of constructing not necessarily reductive G-structures on X from cones of minimal rational curves and the possibility of recovering the structure of rational homogeneous manifolds from such G-structures by some generalization of our Main Theorem.

1. Flatness conditions for G-structures

We will briefly recall the basic notions of the theory of G-structures. We will follow Guillemin's presentation ([Gu]).

Throughout this paper, V is a fixed n-dimensional complex vector space. V can be canonically identified with the vector space of constant vector fields on V. For notational convenience, let us write $\mathbf{f} = \mathbf{gl}(V)$. We define $\mathbf{f}^{(k)} = V \otimes S^{k+1} V^*$, where S^k means the k-th symmetric power. In particular, $\mathbf{f}^{(0)} = \mathbf{f}$. Note that $\mathbf{f}^{(k)}$ can be naturally identified with the vector space of homogeneous vector fields of degree $k+1$ on V. The infinite sum $V + \mathbf{f} + \mathbf{f}^{(1)} + \cdots$ can be identified with the vector space of formal vector fields on V, and has a natural Lie algebra structure arising from this identification. Note that the Lie bracket satisfies $[\,,\,] : \mathbf{f}^{(k)} \otimes \mathbf{f}^{(l)} \to \mathbf{f}^{(k+l)}$. We define $\mathbf{f}^k = \mathbf{f} + \mathbf{f}^{(1)} + \cdots + \mathbf{f}^{(k)}$. It is a Lie algebra under $[\,,\,]$ modulo $\mathbf{f}^{(k+1)} + \mathbf{f}^{(k+2)} + \cdots$.

Let $C^{k,l} = \mathbf{f}^{(k-1)} \otimes \wedge^l V^*$ be the space of (l-form)-valued homogeneous vector fields of degree k. Taking differentials of the coefficients, we get a derivative of degree $(-1,1)$, $\delta : C^{k,l} \to C^{k-1,l+1}$. Then $\delta^2 = 0$ and the complex $(C^{k,l}, \delta)$ is exact, which is just a formal version of the Poincaré lemma.

Given a complex manifold X, we define the frame bundle $\mathscr{F}(X)$ as the principal $GL(V)$-bundle with the fiber at $x \in X$, $\mathscr{F}_x(X) = \mathrm{Isom}(V, T_x(X))$. We can also view an element of $\mathscr{F}_x(X)$ as the 1-jet of a local biholomorphism of pointed spaces $(V,0) \to (X,x)$. If we consider the $(k+1)$-jets, we get a fiber bundle $\mathscr{F}^k(X)$ whose structure group corresponds to the Lie algebra $\mathbf{f}^k$. On $\mathscr{F}^k(X)$, there is a canonical $(V + \mathbf{f} + \mathbf{f}^{(1)} + \cdots + \mathbf{f}^{(k-1)})$-valued 1-form $\Lambda^k = \mu + \nu^0 + \cdots + \nu^{k-1}$, where μ is a V-valued 1-form and ν^i is an $\mathbf{f}^{(i)}$-valued 1-form. The V-valued 1-form μ at $\phi \in \mathscr{F}^k_x(X)$ is given by simply projecting the tangent vector to a tangent vector at $x \in X$, and then use the identification of V with $T_x(X)$ given by the 1-jet of ϕ. ν^i is defined similarly using the $(i+2)$-jet of ϕ, $i < k$.

Let $G \subset GL(V)$ be a connected complex closed subgroup. A G-structure on X means a G-subbundle $\mathscr{G} \subset \mathscr{F}(X)$. Given a G-structure $\mathscr{G}$ on X, and a biholomorphic map $f: X \to Y$ to another complex manifold Y, we get an induced G-structure $f_*\mathscr{G}$ on Y. V has a canonical G-structure $\mathscr{G}(V) = G \times V \subset \mathscr{F}(V)$. A G-structure on a complex manifold X, $\mathscr{G} \subset \mathscr{F}(X)$ is flat, if for each point $x \in X$, there exists a local biholomorphism from a neighborhood of $0 \in V$ to a neighborhood of $x \in X$, which induces an isomorphism of $\mathscr{G}(V)$ and $\mathscr{G}$.

Given two manifolds with G-structures $(X, \mathscr{G})$ and $(X', \mathscr{G}')$, a local biholomorphism $f: (X,x) \to (X',x')$ is k-th order structure preserving, if $f_*\mathscr{G} \subset \mathscr{F}(X')$ contains $\mathscr{G}'_{x'}$ and the two submanifolds $\mathscr{G}'$ and $f_*\mathscr{G}$ of $\mathscr{F}(X')$ are tangent to order $\geqq k$ along $\mathscr{G}'_{x'}$, namely the ideal defining $f_*\mathscr{G}$ restricted to $\mathscr{G}'$ has multiplicity $> k$ along $\mathscr{G}'_{x'}$. This notion depends only on the $(k+1)$-jet of f.

Let $\mathbf{g} \subset \mathbf{f}$ be the Lie algebra of G. It can be regarded as the vector space of linear vector fields on V, which generate local biholomorphisms of $(V,0)$, which are 0-th order structure preserving with respect to the trivial G-structure $\mathscr{G}(V)$ at $0 \in V$. Similarly, we consider $\mathbf{g}^{(k)} \subset \mathbf{f}^{(k)}$ which generate k-th order structure preserving automorphisms.

Define $C^{k,l}(\mathbf{g}) \subset C^{k,l}$ as the subspace $\mathbf{g}^{(k-1)} \otimes \wedge^l V^* \subset \mathbf{f}^{(k-1)} \otimes \wedge^l V^*$. Then we get a complex $(C^{k,l}(\mathbf{g}), \delta)$, whose cohomology groups will be denoted by $H^{k,l}(\mathbf{g})$. They are called the Spencer cohomology groups associated to $\mathbf{g} \subset \mathbf{f}$. We will be mostly interested in $H^{k,2}(\mathbf{g})$, namely, in the sequence

$$\mathbf{g}^{(k)} \otimes V^* \to \mathbf{g}^{(k-1)} \otimes \wedge^2 V^* \to \mathbf{g}^{(k-2)} \otimes \wedge^3 V^* .$$

A G-structure $\mathscr{G}$ on X is uniformly k-flat, if at each point $x \in X$, there exists a $(k+1)$-jet of a local biholomorphism $(V,0) \to (X,x)$, which is k-th order structure preserving. Any G-structure is uniformly 0-flat. By Cartan-Kähler theorem, a G-structure is flat, if and only if it is uniformly k-flat for all positive integers k (see [SS]).

If $\mathscr{G} \subset \mathscr{F}(X)$ is uniformly k-flat, we can define a subbundle $\mathscr{G}^k \subset \mathscr{F}^k(X)$, whose fiber at $x \in X$ is the set of $(k+1)$-jets of local biholomorphisms $(V,0) \to (X,x)$ which are k-th order structure preserving. $\mathscr{G}^k$ is a principal bundle with the Lie algebra

$\mathbf{g}^k = \mathbf{g} + \mathbf{g}^{(1)} + \cdots + \mathbf{g}^{(k)}$, a subalgebra of $\mathbf{f}^k$. In the same way, as we defined Λ^k on $\mathscr{F}^k(X)$, we can define $(V + \mathbf{g} + \cdots + \mathbf{g}^{(k-1)})$-valued 1-form $\Gamma^k = \omega + \Omega^0 + \Omega^1 + \cdots + \Omega^{k-1}$ on $\mathscr{G}^k$.

We define a function $c^k : \mathscr{G}^k \to H^{k,2}(\mathbf{g})$ as follows. Given a point $p \in \mathscr{G}^k$, choose a horizontal subspace $H \subset T_p(\mathscr{G}^k)$, which is in the kernel of $\Omega^0, \Omega^1, \ldots, \Omega^{k-1}$. Such a subspace always exists. Define $\gamma_H : \Lambda^2 V^* \to \mathbf{g}^{(k-1)}$ by $\gamma_H(v_1, v_2) = d\Omega^{k-1}(V_1 \wedge V_2)$, where V_i is the element of H whose projection to X coincides with the push-forward of v_i by p. The element $\gamma_H \in \mathbf{g}^{(k-1)} \otimes \Lambda^2 V^*$ depends on the choice of H. But its cohomology class in $H^{k,2}(\mathbf{g})$ does not ([Gu], Prop. 4.3). We will denote this cohomology class by c^k. c^k is constant along the fibers of $\mathscr{G}^k \to \mathscr{G}$. Hence we get an $H^{k,2}(\mathbf{g})$-valued function c^k on $\mathscr{G}$, when $\mathscr{G}$ is uniformly k-flat. This function is precisely the obstruction to the flatness:

Proposition 1 (Guillemin [Gu], Corollary to Theorem 4.1). *Suppose $\mathscr{G}$ is uniformly k-flat, where k is any nonnegative integer. Then it is uniformly $(k+1)$-flat, if and only if $c^k \equiv 0$.*

When G is reductive, we can interpret c^k in terms of more classical tensors:

Proposition 2. *Suppose $G \subset GL(V)$ is a connected reductive subgroup. Let $\mathscr{G} \subset \mathscr{F}(X)$ be a G-structure on a complex manifold X. If $H^0(X, \mathcal{O}(T(X) \otimes S^k T^*(X) \otimes \Lambda^2 T^*(X))) = 0$ for all nonnegative integers k, then $\mathscr{G}$ is flat.*

Proof. Since G is reductive, we can regard $H^{k,2}(\mathbf{g})$ as a G-submodule of $\mathbf{g}^{(k-1)} \otimes \Lambda^2 V^* \subset V \otimes S^k V^* \otimes \Lambda^2 V^*$ in a canonical way. Along the fibers of $\mathscr{G} \to X$, the function c^k varies according to the natural G-action on $V \otimes S^k V^* \otimes \Lambda^2 V^*$. Hence c^k induces a holomorphic section of $T(X) \otimes S^k T^*(X) \otimes \Lambda^2 T^*(X)$. By the assumption, $c^k = 0$.

By the above result of Guillemin, $\mathscr{G}$ is uniformly k-flat for all nonnegative integers k. By Cartan-Kähler theorem ([SS]), $\mathscr{G}$ is flat. □

2. Flatness of G-structures on uniruled projective manifolds

Let M be a uniruled projective manifold. Mori's bend-and-break shows that through a generic point of M, we can find a rational curve C with $T(M)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$, where $T(M)|_C$ denotes the pull-back of the tangent bundle of M to the normalization of C (e.g. (2.4.3) in [M1]). Let $\mathscr{K}$ be the irreducible component of the Chow space containing $[C]$, the point corresponding to C, and $\mathscr{R} \subset \mathscr{K}$ be the Zariski-dense subset corresponding to rational curves where $T(M)$ splits as $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$. Define

$$\mathscr{U} = \{([C_1], x) \in \mathscr{R} \times M, x \in C_1\}$$

and let $\psi : \mathscr{U} \to \mathscr{R}$ and $\phi : \mathscr{U} \to M$ be the projections. They are the universal family maps when we regard $\mathscr{R}$ as a subset of the Hilbert scheme of M. Let $x \in M$ be a generic point and define $\mathscr{K}_x$ as the reduced Zariski closure of $\mathscr{R}_x := \psi(\phi^{-1}(x)) \subset \mathscr{R}$ in $\mathscr{K}$. Then each component of $\mathscr{K}_x$ has dimension p. For each $y \in \mathscr{R}_x$, $\psi^{-1}(y)$ corresponds to an immersed rational curve in M passing through x. Hence we get a holomorphic immersion $\Theta_x : \mathscr{R}_x \to \mathbb{P} T_x(M)$ assigning the tangent vectors at x tangential to the curves. The closure of the image $\Theta_x(\mathscr{R}_x)$ will be denoted by $\mathscr{C}_x \subset \mathbb{P}T_x(M)$ and will be called the cone of minimal rational tangents at x. This is defined for $x \in M$ outside a proper subvariety. Note that $\mathscr{C}_x$ is an equidimensional subvariety, but needs not be irreducible.

Let $G \subset GL(V)$ be a faithful irreducible representation of a connected reductive group G. Let $\mathscr{F}(M)$ be the frame bundle with $\mathscr{F}(M)_x = \mathrm{Isom}(V, T_x(M))$ and $\mathscr{G} \subset \mathscr{F}(M)$ be a G-structure on M. Then the vector bundle associated to $\mathscr{G}$ via $G \subset GL(V)$ is the tangent bundle $T(M)$. Let $\lambda \in V$ be a highest weight vector of G and $\mathscr{W} \subset \mathbb{P}V$ be the orbit of λ. By the irreducibility of the representation, $\mathscr{W} \subset \mathbb{P}V$ is linearly nondegenerate, namely, it is not contained in a hyperplane. For each $x \in M$, we define the cone of highest weight tangents at x, denoted by $\mathscr{W}_x \subset \mathbb{P}T_x(M)$ as the image of $\mathscr{W}$ under any element of $\mathscr{G}_x \subset \mathrm{Isom}(V, T_x(M))$. This image does not depend on the choice of elements of $\mathscr{G}_x$.

We recall Grothendieck's classification of G-principal bundles over $\mathbb{P}^1$. Let G be a connected complex reductive Lie group and $H \subset G$ be a maximal algebraic torus. Let $\mathscr{O}(1)^*$ be the $\mathbb{C}^*$-principal bundle on $\mathbb{P}^1$, which is just the complement of the zero section of $\mathscr{O}(1)$.

Proposition 3 (Grothendieck [Gt]). *Let $\mathscr{G}$ be a principal G-bundle on $\mathbb{P}^1$. Then there exists an algebraic one-parameter subgroup $\varrho : \mathbb{C}^* \to H$ such that $\mathscr{G}$ is equivalent to the G-bundle associated to $\mathscr{O}(1)^*$ via the action ϱ. Furthermore, let $\mathscr{V}$ be a vector bundle associated to $\mathscr{G}$ via a representation $\mu : G \to GL(V)$. Then $\mathscr{V}$ splits as the direct sum of line bundles $\mathscr{O}(\langle \mu_i, \varrho \rangle)$, where $\mu_i : H \to \mathbb{C}^*$ are the weights of μ and $\langle \mu_i, \varrho \rangle$ denotes the integral exponent of the homomorphism $\mu_i \circ \varrho : \mathbb{C}^* \to \mathbb{C}^*$.*

Let $\mathbf{g} \subset \mathbf{gl}(V)$ be the Lie algebra of G. Since it is reductive, we can write $\mathbf{g} = \mathbf{z} + \mathbf{l}$, where $\mathbf{z}$ is the center and $\mathbf{l} = [\mathbf{g}, \mathbf{g}]$ is semisimple. It follows from Lie's theorem (e.g. [Hu], 19.1) that $\mathbf{z} = 0$ or 1-dimensional, and acts with a single weight $\sigma \in \mathbf{z}^*$, namely, $z \cdot v = \sigma(z)v$ for any $v \in V$. Let $\mathbf{h} \subset \mathbf{g}$ be a Cartan subalgebra, and Φ^+ be a fixed choice of positive roots with respect to $\mathbf{h}$. Let $\Lambda \subset \mathbf{h}^*$ be the weight lattice. Then a one-parameter subgroup $\varrho : \mathbb{C}^* \to H$ gives rise to an element $\tilde{\varrho} \in \mathbf{h}$ and the weights μ_i have corresponding elements $\tilde{\mu}_i \in \Lambda$ so that $\tilde{\mu}_i(\tilde{\varrho}) = \langle \mu_i, \varrho \rangle$. In fact, $\tilde{\mu}_i = \sigma + v_i$ where v_i's are weights of the corresponding representation of the semisimple $\mathbf{l}$ and σ is the single weight of the center $\mathbf{z}$. Let v_1 be the highest weight. Then $v_1 - v_i$ is a sum of elements of Φ^+. It follows that $\tilde{\mu}_1 - \tilde{\mu}_i$ is a sum of elements of Φ^+.

Proposition 4. *In the above situation, $\mathscr{C}_x \subset \mathscr{W}_x$, for any generic point $x \in M$.*

Proof. Given a generic element α of any irreducible component of $\mathscr{C}_x$, we can find a rational curve C through x which is tangent to α and

$$T(M)|_C = \mathscr{O}(2) \oplus [O(1)]^p \oplus \mathscr{O}^q,$$

where $\mathscr{O}(2) = T(C)$. Let ϱ be the one-parameter subgroup defining the G-principal bundle $\mathscr{G}|_C$. Modulo Weyl group action, we can assume that $a(\tilde{\varrho}) \geqq 0$ for any positive root $a \in \Phi^+$. Hence $\tilde{\mu}_1(\tilde{\varrho}) - \tilde{\mu}_i(\tilde{\varrho}) \geqq 0$. It follows that $\tilde{\mu}_1(\tilde{\varrho}) = 2$ and the $\mathscr{O}(2)$-factor of $T(M)|_C$ corresponds to the highest weight vector. Hence $T_x(C)$ lies in the G-orbit of highest weight vector in $T_x(M)$, implying $\alpha \in \mathscr{W}_x$. □

Proposition 5. *In the above situation, suppose $\mathscr{W} \neq \mathbb{P}V$. Then for any generic point $x \in M$, $\mathscr{W}_x \subset \mathscr{C}_x$.*

Proof. Since $\mathbf{g}$ is reductive, the nondegenerate symmetric bilinear form $\mathrm{Tr}(a, b) = \mathrm{Tr}(a \circ b)$ on $\mathbf{gl}(V)$ is nondegenerate on $\mathbf{g}$ (e.g. [OV], Theorem 2, p. 138). Let $\mathbf{g}^{\perp}$ be the orthogonal complement so that $\mathbf{gl}(V) = \mathbf{g} \oplus \mathbf{g}^{\perp}$. A G-action on V induces a G-action on $\mathbf{gl}(V)$ and this action preserves $\mathbf{g}$ and $\mathbf{g}^{\perp}$. The associated vector bundle $\mathrm{End}(T(M))$ splits as $U \oplus U^{\perp}$ where U is the bundle associated to $\mathbf{g}$ and $U^{\perp}$ is the bundle associated to $\mathbf{g}^{\perp}$.

Choose a generic element α in an irreducible component of $\mathscr{C}_x$. From the previous proposition, we know that $\alpha \in \mathscr{W}_x$. Let C be a rational curve tangent to α with $T(M)|_C = \mathscr{O}(2) \oplus [\mathscr{O}(1)]^p \oplus \mathscr{O}^q$. Then q is the codimension of $\mathscr{C}_x \subset \mathbb{P}T_x(M)$. Hence it is enough to show that $q \leqq \mathrm{codim}(\mathscr{W}_x \subset \mathbb{P}T_x(M))$.

We have

$$\mathrm{End}(T(M))|_C = [\mathscr{O}(2)]^q \oplus [\mathscr{O}(1)]^{p(q+1)} \oplus \mathscr{O}^{p^2+q^2+1} \oplus [\mathscr{O}(-1)]^{p(q+1)} \oplus [\mathscr{O}(-2)]^q .$$

Note that $[\mathscr{O}(2)]^q$-factor is uniquely defined as a subbundle of $\mathrm{End}(T(M))|_C$. From the above, we have another decomposition $\mathrm{End}(T(M))|_C = U|_C \oplus U^{\perp}|_C$.

Claim 1. $[\mathscr{O}(2)]^q_x \subset U^{\perp}_x$.

Suppose not. By the uniqueness of Grothendieck decomposition, $U|_C$ must have an $\mathscr{O}(2)$-factor. Elements of $[\mathscr{O}(2)]^q_x \subset T_x(M) \otimes T^*_x(M)$ are of the form $T_x(C) \otimes \beta$, $\beta \in T^*_x(M)$. We fix an element of $\mathscr{G}_x$ and identify $T_x(M)$ with V by that element. By the previous proposition, $T_x(C)$ can be regarded as the span of a highest weight vector α in V. The assumption that $U|_C$ has an $\mathscr{O}(2)$-factor implies that $\alpha \otimes \beta^* \in \mathbf{g}$ for some $\beta^* \in V^*$, after we identify $\mathbf{gl}(V) = V \otimes V^*$. Let $\gamma \in V$ be any element in the orbit of the highest weight vector. Then $(\alpha \otimes \beta^*)(\gamma) = \beta^*(\gamma)\alpha$. Hence $\mathbf{g} \cdot \gamma$ contains a multiple of α for any γ. But $\mathbf{g} \cdot \gamma$ corresponds to the tangent space to the orbit of highest weight at γ. Thus the cone $\mathscr{W}_x \subset \mathbb{P}T_x(M)$ has a point α which lies in the tangent space to any point in $\mathscr{W}_x$. Since $\mathscr{W}_x$ is a homogeneous projective submanifold, α can be chosen to be any point of $\mathscr{W}_x$, which implies $\mathscr{W}_x$ is linear. This is impossible, because $\mathscr{W} \subset \mathbb{P}V$ is a linearly nondegenerate proper subvariety, by the assumption. □

Claim 2. Suppose $\alpha \otimes W \subset \mathbf{g}^{\perp}$ for some subspace $W \subset V^*$. Then W annihilates $\mathbf{g} \cdot \alpha$.

Suppose not, then we have $\alpha \otimes w \in \mathbf{g}^{\perp}$ such that $w(g \cdot \alpha) = c \neq 0$ for some $w \in W$, $g \in \mathbf{g}$. From the definition of $\mathbf{g}^{\perp}$, $\mathrm{Tr}(k \circ (\alpha \otimes w)) = 0$ for any $k \in \mathbf{g}$. Put $k = g$, we get $\mathrm{Tr}(g \circ (\alpha \otimes w)) = 0$. But $(g \circ (\alpha \otimes w))(g \cdot \alpha) = c(g \cdot \alpha)$, while for any element $v \in V$, $(g \circ (\alpha \otimes w))(v) = w(v)(g \cdot \alpha)$. Hence $\mathrm{Tr}(g \circ (\alpha \otimes w)) = c \neq 0$, a contradiction. □

Note that $[\mathscr{O}(2)]^q$ is of the form $\alpha \otimes W$, which lies in $\mathbf{g}^{\perp}$ by Claim 1. By Claim 2, $q \leqq \dim(\text{annihilator of } \mathbf{g} \cdot \alpha \text{ in } V^*)$. Since $\mathbf{g} \cdot \alpha$ is the tangent to $\mathscr{W}_x$ at α, we get $q \leqq$ codim$(\mathscr{W}_x \subset \mathbb{P}T_x(M))$. □

Proposition 4 and Proposition 5 give

Theorem 1. *Let $G \subset GL(V)$ be a faithful irreducible representation of a connected reductive complex Lie group G. Let $\mathscr{G} \subset \mathscr{F}(M)$ be a G-structure on a uniruled projective manifold M. Assume that $G \neq GL(V)$. Then at a generic point $x \in M$, the cone of minimal rational tangents $\mathscr{C}_x$ coincides with the cone of highest weight tangents $\mathscr{W}_x$.*

Proof. Suppose $\mathscr{W} = \mathbb{P}V$. There are four possibilities for G, namely, $GL(V)$, $SL(V)$, $Sp(V)$, $CSp(V)$. Here $Sp(V)$ means the symplectic group and $CSp(V)$ means the conformal symplectic group with respect to a symplectic form on V. They are possible only when $n = \dim(M)$ is even.

First we claim that $G \neq SL(V), Sp(V)$. Consider a rational curve C with $[C] \in \mathscr{R}$ as before and get $\tilde{\varrho}, \tilde{\mu}_i$. For the standard representation of $\mathbf{sl}(V)$ or $\mathbf{sp}(V)$, $\sum \tilde{\mu}_i = 0$. On the other hand, $\sum \tilde{\mu}_i(\tilde{\varrho}) = 2 + p$, a contradiction.

Suppose M admits a conformal symplectic structure. Then we have a line bundle L such that $T(M) \cong T^*(M) \otimes L$ induced by a symplectic structure. Restricting the bundle isomorphism to C chosen as before we see that $L|_C \cong \mathscr{O}(2)$ and

$$T(M)|_C \cong \mathscr{O}(2) + [\mathscr{O}(1)]^{n-2} \oplus \mathscr{O},$$

and $\mathscr{C}_x$ is a hypersurface in $\mathbb{P}T_x(M)$. The symplectic form induces a map $\omega : \Lambda^2 T \to L$. From $L|_C = \mathscr{O}(2)$, we can see that $\omega(\mathscr{O}(2), \mathscr{O}(1)) = 0$. This means that the homogeneization of the tangent space to a generic point of $\mathscr{C}_x$ is isotropic with respect to the symplectic form. Hence $\dim(\mathscr{C}_x) \leq \frac{1}{2}\dim(M)$. The only possibility is $\dim(M) = 2$, but then $CSp(V) = GL(V)$. □

Note that a line in $\mathbb{P}T_x(M)$ can be viewed as a point on $\mathbb{P}\Lambda^2 T_x(M)$. Let $\mathscr{T}_x \subset \mathbb{P}\Lambda^2 T_x(X)$ be the variety of tangential lines to $\mathscr{W}_x \subset \mathbb{P}T_x(M)$. The following proposition was proved in [HM]. We give the proof for the readers' convenience.

Proposition 6. *For any $x \in M$, $\mathscr{T}_x$ is linearly nondegenerate in $\mathbb{P}\Lambda^2 T_x(M)$.*

Proof. We may prove it for a generic point $x \in M$. If $\mathscr{W} = \mathbb{P}V$, this is obvious. So from Theorem 1, we can assume that $\mathscr{C}_x = \mathscr{W}_x$.

Let $\alpha \in \mathscr{C}_x$ be generic so that we have C tangential to α with

$$T(M)|_C = \mathscr{O}(2) \oplus [\mathscr{O}(1)]^p \oplus \mathscr{O}^q.$$

As before, we have a one-parameter subgroup $\varrho : \mathbb{C}^* \to G$ defining $\mathscr{G}|_C$. From the splitting type of $T(M)|_C$, $\varrho(\mathbb{C}^*)$ acts on V with three exponents $2, 1, 0$ and gives rise to a decomposition $V = \mathbb{C}\alpha \oplus \mathscr{H}_\alpha \oplus \mathscr{N}_\alpha$. This action preserves the cone $\mathscr{C}_x \subset \mathbb{P}V$. Taking the inverse and tensoring with a scalar representation, we get a $\mathbb{C}^*$-action on $\mathbb{P}V$ preserving $\mathscr{C}_x$ which fixes $\mathbb{C}\alpha$, acts as t on $\mathscr{H}_\alpha$ and acts as t^2 on $\mathscr{N}_\alpha$, $t \in \mathbb{C}^*$. Choose a generic point $\alpha + \xi + \zeta$ on $\mathscr{C}_x$. The orbit of the $\mathbb{C}^*$-action is $\alpha + t\xi + t^2\zeta$. At $t = t_0$, we further consider the curve $\alpha + e^s t_0 \xi + e^{2s} t_0^2 \zeta$. Taking derivative with respect to s, we get the tangent vector $t_0\xi + 2t_0^2\zeta$ to $\mathscr{C}_x$ at the point $\alpha + t_0\xi + t_0^2\zeta$. The corresponding element of $\mathscr{T}_x$ is

$$(\alpha + t_0\xi + t_0^2\zeta) \wedge (t_0\xi + 2t_0^2\zeta) = t_0 \alpha \wedge \xi + 2t_0^2 \alpha \wedge \zeta + t_0^3 \xi \wedge \zeta.$$

Thus the linear span of $\mathcal{T}_x$ contains $\alpha \wedge \xi, \alpha \wedge \zeta$, and $\xi \wedge \zeta$ for any generic $\alpha \in \mathscr{C}_x$ and $\alpha + \xi + \zeta \in \mathscr{C}_x$. As we vary ξ on $\mathscr{H}_\alpha$, the corresponding $\zeta \in \mathscr{N}_\alpha$ spans $\mathscr{N}_\alpha$. Otherwise, $\mathscr{C}_x$ will be contained in the linear subspace $\mathbb{C}\alpha \oplus \mathscr{H}_\alpha \oplus \mathscr{N}'_\alpha$ for some proper subspace $\mathscr{N}'_\alpha \subset \mathscr{N}_\alpha$, contradiction to the nondegeneracy of $\mathscr{C}_x$. It follows that $\mathcal{T}_x$ is linearly nondegenerate. □

We are ready to prove the first half of Main Theorem.

Theorem 2. *Let $G \subset GL(V)$ be a faithful irreducible representation of a connected reductive complex Lie group and $\mathscr{G} \subset \mathscr{F}(M)$ be a G-structure on a uniruled projective manifold M. Then $\mathscr{G}$ is flat.*

Proof. If $G = GL(V)$, then the flatness of $\mathscr{G} = \mathscr{F}(M)$ is obvious. Thus assume $G \neq GL(V)$, so that Theorem 1 applies. By Proposition 2, it is enough to show that $H^0(M, \mathcal{O}(T(M) \otimes S^k T^*(M) \otimes \wedge^2 T^*(M))) = 0$ for all nonnegative integers k.

Let $\theta: \wedge^2 T(M) \to T(M) \otimes S^k T^*(M)$ be a holomorphic section of

$$T(M) \otimes S^k T^*(M) \otimes \wedge^2 T^*(M).$$

It is enough to show that θ vanishes at any generic point $x \in M$. From Proposition 6, it suffices to show $\theta(u, v) = 0$, where $u \in \mathscr{C}_x$ is a generic point of the cone of minimal rational tangent and $v \in \mathbb{P}T_x(M)$ lies on the tangent space of $\mathscr{C}_x \subset \mathbb{P}T_x(M)$ at u.

Let C be a minimal rational curve tangent to u, with $T(M)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$. The subspace of $T_x(M)$ corresponding to $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p$ is precisely the tangent space to $\mathscr{C}_x \subset \mathbb{P}T_x(M)$ at u (e.g. (3.1) in [M1]). Thus u is a vector in $\mathcal{O}(2)$ and v is a vector in $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p$ at x.

Now we can find a section $\tilde{u}$ of $T(M)|_C$ with two zeroes such that $\tilde{u}(x) = u$ and a section $\tilde{v}$ of $T(M)|_C$ with at least one zero such that $\tilde{v}(x) = v$. Then $\theta|_C(\tilde{u}, \tilde{v})$ is a section of $T(M)|_C \otimes S^k T^*(M)$ which has three zeroes. But from the splitting type, a nonzero section of $T(M)|_C \otimes S^k T^*(M)$ cannot have three zeroes. It follows that $\theta|_C(\tilde{u}, \tilde{v}) = 0$ and $\theta(u, v) = 0$. □

3. Reduction to irreducible Hermitian symmetric spaces

Let S be a compact irreducible Hermitian symmetric space of rank $\geqq 2$. Choose a base point $o \in S$ and consider the representation of the connected isotropy group at o on the tangent space $V = T_o(S)$. The image in $GL(V)$ is $K^{\mathbb{C}}$, the complexification of the isometric isotropy group K with respect to a Kähler-Einstein metric on S. $K^{\mathbb{C}}$ is reductive and the irreducibility of the symmetric space implies that the isotropy representation is irreducible. Thus $K^{\mathbb{C}} \subset GL(V)$ is a faithful irreducible representation of a connected reductive complex Lie group. Let $\mathscr{C}_o \subset \mathbb{P}V$ be the cone of the orbit of a highest weight vector. The following properties of $\mathscr{C}_o$ are well-known:

(*) $\mathscr{C}_o$ is a compact Hermitian symmetric space and $K^{\mathbb{C}}$ is the identity component of the subgroup of $GL(V)$ consisting of linear transformations preserving $\mathscr{C}_o$.

A G-structure modeled after this irreducible representation is called an S-structure. We have the following result of Ochiai on flat S-structures.

Proposition 7 (Ochiai [Oc]). *Let M be a simply connected compact complex manifold with a flat S-structure. Then M is biholomorphic to the irreducible Hermitian symmetric space S.*

An S-structure is also an irreducible reductive G-structure. From Theorem 2 and Proposition 7, we get

Proposition 8. *Let M be a uniruled projective manifold with an S-structure. Then M is biholomorphic to the irreducible Hermitian symmetric space S.*

Proof. It suffices to show that M is simply connected. We claim that M is rationally connected ([KMM]), which implies that it is simply connected ([KMM], (2.5.3), [Ca]) Suppose not. Then all rational curves through a generic point $x \in M$ are contained in the fibers of the MRC-fibration ([KMM], Theorem 2.7). In particular, the cone $\mathscr{C}_x$ is contained in the tangent space to the MRC-fiber, a contradiction to the linear nondegeneracy. □

Given an S-structure on a manifold M, we have the cone of highest weight tangents $\mathscr{C} \subset \mathbb{P}T(M)$. Conversely, given a subbundle $\mathscr{C} \subset \mathbb{P}T(M)$, whose fiber is projectively equivalent to $\mathscr{C}_o \subset \mathbb{P}V$, we get an S-structure, because of the property $(*)$ of $\mathscr{C}_o$. Hence to finish the proof of Main Theorem, it suffices to prove

Proposition 9. *Let $G \subset GL(V)$ be a faithful irreducible representation of a connected reductive complex Lie group with $G \neq GL(V)$. Given a G-structure on a uniruled projective manifold M, the cone $\mathscr{W}_x \subset \mathbb{P}T_x(M)$ is projectively equivalent to the cone $\mathscr{C}_o$ associated to one of the irreducible Hermitian symmetric spaces.*

Proof. Recall that a submanifold $Z \subset \mathbb{P}V$ is the cone $\mathscr{C}_o$ associated to an irreducible Hermitian symmetric space, if and only if it is one of the following (Table 1 in [HM], or Appendix 3 in [M2]).

(i) The complex projective space $\mathbb{P}_l$ embedded by the second Veronese embedding.

(ii) A rank-2 irreducible Hermitian symmetric space embedded by the ample generator of the Picard group.

(iii) The Segre embedding of the product of two projective spaces.

From the proof of Proposition 6, for any $[\alpha] \in \mathscr{W}_x$, we have a $\mathbb{C}^*$-action on $T_x(M)$, which fixes α, acts by t on the tangent space $T_{[\alpha]}(\mathscr{W}_x)$, and t^2 modulo $T_{[\alpha]}(\mathscr{W}_x)$. The action of $-1 \in \mathbb{C}^*$ on the tangent space $T_{[\alpha]}(\mathscr{W}_x)$ is an involution with the isolated fixed point α. Hence $\mathscr{W}_x \subset \mathbb{P}T_x(M)$ is an equivariant embedding of a Hermitian symmetric space Z. Note that a generic orbit $\alpha + t\xi + t^2\zeta$ of the $\mathbb{C}^*$-action can be compactified to a curve of degree 2 in $\mathbb{P}T_x(M)$, because it is contained in the plane spanned by α, ξ and ζ. Thus $Z \subset \mathbb{P}V$ has the property that given any tangent vector to Z at a point, there exists a curve through that point with the given tangent direction, which has degree 2.

The polydisc theorem (Ch. 5, (1.1) in [M2]) tells us that a curve through a given point of a compact Hermitian symmetric space of rank r in a generic tangential direction has degree $\geq r$ with respect to any ample line bundle. Moreover, if the degree of such a curve is r, then the ample line bundle must be the minimal one. It follows that our Z has rank ≤ 2. If it is of rank 1, then the existence of $\mathbb{C}^*$-actions of the above type implies that Z is one of (i). If Z is of rank 2, the existence of a curve of degree 2 in a generic tangential direction implies that Z is one of (ii) or (iii). □

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References

[Ba] *R. Baston*, Almost Hermitian symmetric manifolds, Duke Math. J. **63** (1991), 81–138.

[Ca] *F. Campana*, On twistor spaces of the class $\mathscr{C}$, J. Diff. Geom. **33** (1991), 541–549.

[CP] *F. Campana, T. Peternell*, Projective manifolds whose tangent bundles are numerically effective, Math. Ann. **289** (1991), 169–187.

[Gt] *A. Grothendieck*, Sur la classification des fibrés holomorphes sur la sphère de Riemann, Amer. J. Math. **79** (1957), 121–138.

[Gu] *V. Guillemin*, The integrability problem for G-structures, Trans. A.M.S. **116** (1965), 544–560.

[Gn] *R. Gunning*, On the uniformizaton of complex manifolds: the role of connections, Mathematical Notes **22**, Princeton University Press, Princeton 1978.

[Hu] *J. Humphreys*, Introduction to Lie algebras and representation theory, Grad. Texts Math. **9**, Springer, New York–Heidelberg–Berlin 1972.

[HM] *J.-M. Hwang, N. Mok*, Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation, Invent. Math., to appear.

[KO] *S. Kobayashi, T. Ochiai*, Holomorphic structures modeled after compact hermitian symmetric spaces, in: Hano et al. (eds.), Manifolds and Lie groups, Papers in honor of Yozo Matsushima, Progr. Math. **14** (1981), 207–221.

[KMM] *J. Kollár, Y. Miyaoka, S. Mori*, Rationally connected varieties, J. Alg. Geom. **1** (1992), 429–448.

[Ma] *Y. Manin*, Gauge field theory and complex geometry, Grundl. Math. Wiss. **289**, Springer, Berlin–Heidelberg 1988.

[M1] *N. Mok*, The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature, J. Diff. Geom. **27** (1988), 179–214.

[M2] *N. Mok*, Metric rigidity theorems on Hermitian locally symmetric manifolds, Ser. Pure Math. **6**, World Scientific, Singapore–New Jersey–London–Hong Kong 1989.

[Mo] *S. Mori*, Projective manifolds with ample tangent bundles, Ann. Math. **124** (1979), 593–606.

[Oc] *T. Ochiai*, Geometry associated with semisimple flat homogeneous spaces, Trans. A.M.S. **152** (1970), 159–193.

[OV] *A. Onishchik, E. Vinberg*, Lie Groups and Algebraic Groups, Springer, Berlin–Heidelberg–New York 1990.

[SS] *I. Singer, S. Sternberg*, The infinite groups of Lie and Cartan, J. anal. Math. **15** (1965), 1–114.

[Ye] *Y. Ye*, Extremal rays and null geodesics on a complex conformal manifold, Intern. J. Math. **5** (1994), 141–168.

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