A large sieve inequality of Elliott-Montgomery-Vaughan type for automorphic forms and two applications

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Abstract. In this paper, we establish a large sieve inequality of Elliott-Montgomery-Vaughan type for Fourier coefficients of newforms. As applications, we give a significant improvement on the principal result of Duke & Kowalski on Linnik's problem for modular forms and prove the upper part of the first conjecture of Montgomery-Vaughan in the context of automorphic L-functions.

§ 1. Introduction

The large sieve inequalities are fundamental tools in analytic number theory. The first idea was devised by Linnik to study a question of Vinogradov on the size of the smallest quadratic non-residue modulo a prime. Later, various large sieve inequalities were developed with vital applications, for instance, to problems on primes and Riemann zeta-function. Recently Kowalski [13] introduced a general abstract form of large sieve inequalities and gave many greatly interesting applications on algebraic problems. The present work is motivated by two important papers ([3], [8]). In [3], Duke & Kowalski generalized the large sieve inequality of Linnik to the automorphic form case and applied their estimate to study the analogue of Linnik's problem for automorphic forms/elliptic curves. In [8], Granville & Soundararajan proved the first conjecture of Montgomery-Vaughan [19] on the extreme values of the Dirichlet L-functions $L(1, \chi_d)$ associated to real characters. Very recently, we [17] found that in contrast with Linnik's inequality, the large sieve type inequalities adopted by Elliott [4], [5] or Montgomery and Vaughan [19] yields better almost-all results on the size of the smallest quadratic non-residue modulo a prime. We refer them as the large sieve inequalities of Elliott-Montgomery-Vaughan (E-M-V) type.

In this paper we derive a general large sieve inequality of E-M-V type for modular forms, which is of the same strength as the E-M-V inequality for real characters. As applications, we shall give a significant improvement on the principal result of Duke & Kowalski [3] on Linnik's problem for automorphic forms. Moreover, we use our large sieve inequality, differently than in the work of Granville & Soundararajan [8], to prove the upper part of the first conjecture of Montgomery-Vaughan in the context of automorphic L-functions (cf. Sections 2 and 3).

Let us fix our notation. For a positive even integer k and a positive squarefree integer N, we denote by $\mathrm{H}_k^*(N)$ the set of all normalized holomorphic primitive cusp forms of weight k for

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the congruence group $\Gamma_0(N)$. It is known that

$$|\mathrm{H}_k^*(N)| = \frac{k-1}{12}\varphi(N) + O\big((kN)^{2/3}\big), \tag{1.1}$$

where $\varphi(N)$ is the Euler function and the implied constant is absolute (cf. [10], (2.72)).

Let $\lambda_f(n)$ be the n-th normalized Fourier coefficient of $f \in \mathrm{H}_k^*(N)$ at the cusp ∞, i.e.

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \qquad (\Im m\, z > 0)$$

and $\lambda_f(1) = 1$. Following from the properties of Hecke operators, we have the Hecke relation

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} \lambda_f\Big(\frac{mn}{d^2}\Big) \tag{1.2}$$

for all positive integers m and n. Indeed $\lambda_f(n)$ is multiplicative, and for every prime p there are complex numbers $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-1}\beta_f(p) + \cdots + \beta_f(p)^\nu.$$

where $\alpha_f(p)$ and $\beta_f(p)$ are known, due to Deligne [2], to satisfy

$$\begin{cases} \alpha_f(p) = \varepsilon_f(p)p^{-1/2}, \quad \beta_f(p) = 0 & \text{if } p \mid N \\ |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1 & \text{if } p \nmid N \end{cases} \tag{1.3}$$

with $\varepsilon_f(p) = \pm 1$. Hence, $\lambda_f(n)$ is real and for all primes p and integers $\nu \geq 0$,

$$|\lambda_f(p^\nu)| \leq \nu + 1 \qquad \text{(Deligne's inequality)}. \tag{1.4}$$

Our first result is a large sieve inequality of Elliott-Montgomery-Vaughan type.

Theorem 1. *Let $\nu \geq 1$ be a fixed integer and let $\{b_p\}_p$ be a sequence of real numbers indexed by prime numbers such that $|b_p| \leq B$ for some constant B and for all primes p. Then we have*

$$\sum_{f \in \mathrm{H}_k^*(N)} \bigg| \sum_{\substack{P < p \leq Q \\ p \nmid N}} b_p \frac{\lambda_f(p^\nu)}{p} \bigg|^{2j} \ll_\nu k\varphi(N)\bigg(\frac{96B^2(\nu+1)^2 j}{P\log P}\bigg)^j + (kN)^{10/11}\bigg(\frac{10BQ^{\nu/10}}{\log P}\bigg)^{2j} \tag{1.6}$$

uniformly for

$$2 \mid k, \qquad B > 0, \qquad 2 \leq P < Q \leq 2P, \qquad N \geq 1 \ \text{(squarefree)}, \qquad j \geq 1.$$

The implied constant depends on ν only.

When $b_p \equiv 1$ and $N = 1$, a weaker form of (1.6) has been derived in ([16], Proposition). Essentially the estimate there contains an extra factor of j^j, which originates in the application of Cauchy-Schwarz's inequality. To save it, we approach Theorem 1 by another auxiliary tool which is simple but powerful. We shall use a trace formula without harmonic weights (see

Lemma 4.1 below and compare it with Corollary 2.10 of [10]). This trace formula is more effective for our purpose, though it is easily deduced from the work of Iwaniec, Luo & Sarnak [10]. Still, unlike quadratic characters, there are intrinsic technical difficulties in this problem. More specifically $\lambda_{\mathrm{sym}^\nu f}(n)$ is not completely multiplicative and satisfies no counterpart of the quadratic reciprocity law. The non-complete multiplicativity can be overcome with some preliminary calculation which will be done in Section 5. The exponent 10/11 in (1.6) can be done better by using more delicate technique (see Remark 4 below). Under the Grand Riemann Hypothesis (GRH), this exponent can be improved to 2/3. For our purpose, it is sufficient to have $1-\delta$ with some constant $\delta > 0$.

Sections 2 and 3 are devoted to the applications of Theorem 1, which give almost optimal results in Linnik's problem and the upper bound part of Montgomery-Vaughan's first two conjectures.

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§ 2. Linnik's problem for modular forms

This problem of Linnik was initially raised for Dirichlet characters and Dirichlet L-functions, stated as follows. Let d be a fundamental discriminant and $\chi_d(n) := \left(\frac{d}{n}\right)$ the Kronecker symbol. Then χ_d is a real primitive character with modulus $|d|$. Linnik's problem on χ_d is to investigate the least integer n such that

$$(|d|, n) = 1 \qquad \text{and} \qquad \chi_d(n) \neq 1.$$

Interested readers can find in [17] a historical account and some recent development on this problem.

The following generalization is formulated for automorphic forms. Given f and g two holomorphic primitive cusp forms of weights k_f and k_g, and of levels N_f and N_g, respectively. What is the smallest integer n for which

$$(n, N_f N_g) = 1 \qquad \text{and} \qquad \lambda_f(n) \neq \lambda_g(n)\,?$$

Denote this smallest integer by $n_{f,g}$ which must be a prime in view of (1.2). One way to evaluate $n_{f,g}$ is through the Rankin-Selberg L-function $L(s, f\otimes g)$ of f and g. Under GRH for $L(s, f\otimes g)$, it is known that

$$n_{f,g} \ll \{\log(k_f k_g N_f N_g)\}^2, \tag{2.1}$$

with an absolute implied constant. Using merely the convexity bound for $L(s, f\otimes g)$, for any $\varepsilon > 0$ one has

$$n_{f,g} \ll_\varepsilon (kN_f N_g)^{1+\varepsilon}, \tag{2.2}$$

where $k := \max\{k_f,\, k_g\}(|k_f - k_g| + 1)$ and the implied constant depends on ε only. As an application of their subconvexity bound for $L(\frac{1}{2} + i\tau, f \otimes g)$, Kowalski, Michel & Vanderkam [14] gave an improvement on the level aspect of (2.2),

$$n_{f,g} \ll_{k_f,g,\varepsilon} N_f^{1-1/40+\varepsilon},$$

where the implied constant depends on k_f, g and ε. Other relevant results in the weight aspect (with a fixed level) were obtained by Ram Murty [20], Sengupta [23] and Kohnen [11].

Of course, it is desirable to prove (2.1) unconditionally, which can at present be achieved in almost-all sense. Such a result was firstly obtained by Duke & Kowalski [3] with a generalization of Linnik's large sieve inequality to the automorphic form case. Their result ([3], Theorem 3) yields the following for squarefree conductors. See [10, Theorem 1.1] for more exposition.

Theorem A. *Let $Q \geq 2$ and k be an even integer. Let g be a given primitive form of weight k and conductor less than Q. For all $\alpha > 0$ and $\varepsilon > 0$, the number $\mathcal{N}$ of primitive non-CM modular forms f of weight k and squarefree conductor $q \leq Q$ such that*

$$\lambda_f(p) = \lambda_g(p) \quad for \quad p \leq (\log Q)^\alpha$$

satisfies

$$\mathcal{N} \ll Q^{10\alpha^{-1}+\varepsilon} \tag{2.3}$$

where the implied constant depends on α and ε only.

Theorem A can be viewed as a result concerning the number of the joint eigenfunctions of a Laplace operator and Hecke operators. The Strong Multiplicity One theorem asserts that a primitive form is uniquely determined by all of its Hecke eigenvalues. The interest in small α is also motivated from the multiplicity of Maass forms (see [22]). Let us return to the maass case in another occasion. The estimate (2.3) becomes trivial when $\alpha \leq 5$, for $\mathcal{N} \ll Q^2$. Our first application of Theorem 1 is to extend its range for non-trivial estimates when the level N is squarefree. (See Remark 1 (ii).) Here, we take generic sequences into account as in [12] where Kowalski was inspired by Sarnak [22].

Theorem 2. *Let N be squarefree. Let $\Lambda = \{\lambda(p)\}_p$ be a fixed real sequence indexed by prime numbers and $\nu \geq 1$ be a fixed integer. Let $\mathcal{P}$ be a set of prime numbers of positive density in the following sense*

$$\sum_{\substack{z<p\leq 2z\\ p\in\mathcal{P}}} \frac{1}{p} \geq \frac{\delta}{\log z} \qquad (z \geq z_0)$$

with some constants $\delta > 0$ and $z_0 > 0$. Then there are two positive constants C and c such that the number of forms $f \in \mathrm{H}_k^*(N)$ verifying

$$\lambda_f(p^\nu) = \lambda(p) \quad for \quad p \in \mathcal{P}, \quad p \nmid N \quad and \quad C\log(kN) < p \leq 2C\log(kN)$$

is bounded by

$$\ll kNe^{-c\log(kN)/\log_2(kN)},$$

where $\log_r$ is the r-fold iterated logarithm. Here C, c and the implied constant depend on Λ, ν and $\mathcal{P}$ only.

The following results follow straightforwardly from Theorem 2.

Corollary 3. *Let g be a primitive cusp form of weight k' and of level N' (not necessarily squarefree) and let $\mathcal{P}$ be stated as in Theorem 2. Then there are two positive constants $C = C(g,\mathcal{P})$ and $c = c(g,\mathcal{P})$ such that the number of forms $f \in \mathrm{H}_k^*(N)$ verifying*

$$\lambda_f(p) = \lambda_g(p) \quad \text{for} \quad p \in \mathcal{P}, \quad p \nmid N \quad \text{and} \quad C\log(kN) < p \leq 2C\log(kN)$$

is bounded by

$$\ll_{g,\mathcal{P}} kNe^{-c\log(kN)/\log_2(kN)}.$$

Corollary 4. *Let g be a primitive cusp form of weight k' and of level N' (not necessarily squarefree) and let $\mathcal{P}$ be as in Theorem 2. Then there are two positive constants $C = C(g,\mathcal{P})$ and $c = c(g,\mathcal{P})$ such that the number of primitive forms $f \in \mathrm{H}_k^*(N)$ verifying*

$$\lambda_f(p^2) = \lambda_g(p^2) \quad \text{for} \quad p \in \mathcal{P}, \quad p \nmid N \quad \text{and} \quad C\log(kN) < p \leq 2C\log(kN)$$

is bounded by

$$\ll_{g,\mathcal{P}} kNe^{-c\log(kN)/\log_2(kN)}.$$

Corollary 5. *For any quadratic field $\mathbb{K}/\mathbb{Q}$, there are two positive constants $C = C(\mathbb{K})$ and $c = c(\mathbb{K})$ such that the number of symmetric squares of forms $f \in \mathrm{H}_k^*(N)$ verifying*

$$\lambda_f(p) = 0 \quad \text{for} \quad C\log(kN) < p \leq 2C\log(kN) \quad \text{insert in} \quad \mathbb{K}$$

is bounded by

$$\ll_{\mathbb{K}} kNe^{-c\log(kN)/\log_2(kN)}.$$

Remark 1. (i) In the formulation of the Linnik problem, it is more appropriate to compare the normalized Fourier coefficient $\lambda_f(n)$ rather than the full coefficient $\lambda_f(n)n^{(k-1)/2}$, especially for the weight aspect. In the context of L-functions, the normalization process shifts the center of the critical strip to standardize the L-function. This removes the drastic amplifying effect of the factor $n^{(k-1)/2}$.

(ii) In contrast with Theorem A, Corollary 3 gives an upper bound uniformly for both level N and weight k, and above all, α can now assume the value 1. However, it is worthwhile to note that the main result of Duke & Kowalski ([3], Theorem 3) covers the situation of arbitrary conductors. Our work here supplements only the squarefree case.

(iii) Corollaries 4 and 5 relax the condition $\alpha > 5$ in Theorems 4.1 and 4.3 of [12] to $\alpha > 1$.

(iv) We have the query whether the estimate in Corollary 3 for the exceptional set is optimal. In the case of real primitive Dirichlet characters, we derive a similar result and the optimality is successfully shown (see [17], Theorem 2). The proof for optimality utilizes the weighted function

$$2^{-\pi(y)-1}(1+\chi_4(p)) \prod_{\substack{q \leq y \\ q \text{ primes}}} (1+\chi_p(q))$$

(where $\pi(y)$ denotes the number of primes $p \leq y$), and the most crucially, the quadratic reciprocity law and Graham-Ringrose's character sum estimates ([7], Theorem 5) for characters of friable/smooth moduli. We are unable to develop similar tools for modular forms. In fact, it might be too optimistic (or even too wild) to guess that the result in Corollary 3 is tight. Below is a probabilistic reasoning based on the referee's comment.

Since a real character assumes two values on generic primes, roughly there is a probability of $1/2^l$ that real characters coincide at l distinct primes. Now $l \approx \log N/\log_2 N$ which is the estimate of primes less than $\log N$. This explains the factor $e^{-c\log N/\log_2 N}$ in the case of real characters. However, $\lambda_f(p)$ may assume much more values, for example, as large as $p^{(k-1)/2}$ if f has integral Fourier coefficients. Repeating the probabilistic argument, one gets a much smaller value.

Finally, we outline the difference in the large sieve inequalities (and their generalizations) between Linnik's type and Elliott-Montgomery-Vaughan's type. The typical Linnik's large sieve inequality for primitive Dirichlet charaters is

$$\sum_{q\le Q} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\, q)} \Big|\sum_{n\le N} a_n\chi(n)\Big|^2 \le \big(N+Q^2\big)\sum_{n\le N}|a_n|^2 \tag{2.4}$$

for any sequence $\{a_n\}_{n\le N}$ of complex numbers. The upper estimate (2.4) is effective only when N is about Q^2, and relatively ineffective for N smaller than a power of $\log Q$. The same feature is carried over to its generalization in [3]. However the E-M-V inequality (1.6) is still good when Q is close to $\log(kN)$ with a suitable choice of j, which is the key to the proof of Theorem 2.

§ 3. Montgomery-Vaughan's conjectures for automorphic L-functions

The Montgomery-Vaughan's conjectures for Dirichlet L-functions $L(s,\chi_d)$ associated to primitive real character χ_d, proposed in [19] based on probabilistic models, are concerned with the proportion of the fundamental discriminant d for which the value $L(1,\chi_d)$ is exceptionally large or small.

Let $\mathcal{D}_x$ be the set of all fundamental discriminants d with $|d|\le x$. Suppose ξ is a positive function on $[100,\infty)$. As in [8], define the distribution functions

$$\Phi_x^+(\xi) := \frac{1}{|\mathcal{D}_x|}\sum_{\substack{d\in\mathcal{D}_x\\ L(1,\chi_d)>e^\gamma\xi(|d|)}} 1$$

and

$$\Phi_x^-(\xi) := \frac{1}{|\mathcal{D}_x|}\sum_{\substack{d\in\mathcal{D}_x\\ L(1,\chi_d)<(e^\gamma 6\pi^{-2}\xi(|d|))^{-1}}} 1,$$

where γ is the Euler constant. The (three) Montgomery-Vaughan conjectures in [19] can be expressed as follows: *There are positive constants x_0, $C>c>0$ and $0<\theta<\Theta<1$ such that*

$$e^{-C(\log x)/\log_2 x} \le \Phi_x^\pm(\log_2\cdot) \le e^{-c(\log x)/\log_2 x} \tag{C1}$$

and

$$x^{-\Theta} \le \Phi_x^\pm(\log_2\cdot+\log_3\cdot) \le x^{-\theta} \tag{C2}$$

for $x\ge x_0$. Further for any $\varepsilon>0$, one has

$$\Phi_x^\pm(\log_2\cdot+(1+\varepsilon)\log_3\cdot) \ll_\varepsilon x^{-1} \tag{C3}$$

for $x \geq 3$.

Recently Granville and Soundararajan [8] made great progress towards these conjectures. Their results depict $\Phi_x^{\pm}(\xi)$ in a precise and delicate way, and Conjecture (C1) follows unconditionally. Under GRH, their result ([8], Theorem 4) implies that for any large constant A there are positive constants x_0 and $0 < \theta < \Theta < 1$ such that

$$x^{-\Theta} \leq \Phi_x^{\pm}(\log_2 \cdot + \log_3 \cdot - A) \leq x^{-\theta}$$

for $x \geq x_0$ and hence the upper bound part of Conjecture (C2) holds (conditionally). Their proof uses the complex moment method and the saddle-point method. Two main ingredients are again quadratic reciprocity and Graham-Ringrose's estimate on character sums (see Remark 1(iii)).

Let us turn to automorphic L-functions. We consider the analogues of the above conjectures for symmetric ν-th power L-functions of $f \in \mathrm{H}_k^* := \mathrm{H}_k^*(1)$ in the k-aspect. In what follows we shall assume k to be any sufficiently large even integer. For $\nu \in \mathbb{N}$ and $f \in \mathrm{H}_k^*$, the associated symmetric ν-th power L-functions $L(s, \mathrm{sym}^\nu f)$ is given by

$$L(s, \mathrm{sym}^\nu f) := \prod_p \prod_{0 \leq j \leq \nu} \left(1 - \alpha_f(p)^{\nu - j} \beta_f(p)^j p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \lambda_{\mathrm{sym}^\nu f}(n) n^{-s}$$

for $\Re e\, s > 1$. Plainly by (1.3),

$$|\lambda_{\mathrm{sym}^\nu f}(n)| \leq \mathrm{d}_{\nu+1}(n) \qquad (n \geq 1),$$

where $\mathrm{d}_r(n)$ counts the number of ways of factorizing n into a product of r natural numbers. For $\nu = 1, 2, 3, 4$, the symmetric ν-th power functions $L(s, \mathrm{sym}^\nu f)$ can be analytically continued to the entire complex plane $\mathbb{C}$ and satisfies a functional equation. Here we are interested in the behavior of the extreme values of $L(1, \mathrm{sym}^\nu f)$ as the weight $k \to +\infty$.

In [15], we proved that under GRH for $L(s, \mathrm{sym}^\nu f)$ ($\nu \in \mathbb{N}$),

$$\{1 + o(1)\}(2B_\nu^- \log_2 k)^{-A_\nu^-} \leq L(1, \mathrm{sym}^\nu f) \leq \{1 + o(1)\}(2B_\nu^+ \log_2 k)^{A_\nu^+}. \tag{3.1}$$

Also, the constants $A_\nu^{\pm}$ and $B_\nu^{\pm}$ are explicitly evaluated:

$$\begin{cases} A_\nu^+ = \nu + 1, & B_\nu^+ = e^\gamma \qquad (\nu = 1, 2, 3, 4), \\ A_\nu^- = \nu + 1, & B_\nu^- = e^\gamma \zeta(2)^{-1} \qquad (\nu = 1, 3), \\ A_2^- = 1, & B_2^- = e^\gamma \zeta(2)^{-2}, \\ A_4^- = \frac{5}{4}, & B_4^- = e^\gamma B_4'^{-}, \end{cases}$$

where as usual $\zeta(s)$ is the Riemann zeta-function. The constant $B_4'^{-}$ is positive and given by a complicated Euler product (cf. [15], (1.16)). The inequality (3.1), if true unconditionally, is believed to be sharp up to the constant 2. Indeed, it is shown unconditionally that for $\nu = 1, 2, 3, 4$ there are $f_\nu^{\pm} \in \mathrm{H}_k^*$ such that for $k \to \infty$,

$$L(1, \mathrm{sym}^\nu f_\nu^+) \geq \{1 + o(1)\}(B_\nu^+ \log_2 k)^{A_\nu^+} \tag{3.2}$$

and

$$L(1, \mathrm{sym}^\nu f_\nu^-) \leq \{1 + o(1)\}(B_\nu^- \log_2 k)^{-A_\nu^-}. \tag{3.3}$$

In [16], we evaluated the size of the exceptional set in which (3.2) or (3.3) holds.

Theorem B. *Let $\varepsilon > 0$ be an arbitrarily small, $\nu = 1, 2, 3, 4$ and $2 \mid k$. Then there is a subset E_k^* of H_k^* such that*

$$|\mathrm{E}_k^*| \ll k e^{-(\log k)^{1/2-\varepsilon}}$$

and for each $f \in \mathrm{H}_k^* \smallsetminus \mathrm{E}_k^*$, we have

$$\big\{1 + O\big((\log k)^{-\varepsilon}\big)\big\}(B_\nu^- \log_2 k)^{-A_\nu^-} \le L(1, \mathrm{sym}^\nu f) \le \big\{1 + O\big((\log k)^{-\varepsilon}\big)\big\}(B_\nu^+ \log_2 k)^{A_\nu^+}.$$

The implied constants depend on ε only.

In order to describe more precisely the size of the exceptional set, one may consider the distribution functions

$$F_k^+(t, \mathrm{sym}^\nu) := \frac{1}{|\mathrm{H}_k^*|} \sum_{\substack{f \in \mathrm{H}_k^* \\ L(1, \mathrm{sym}^\nu f) > (B_\nu^+ t)^{A_\nu^+}}} 1$$

and

$$F_k^-(t, \mathrm{sym}^\nu) := \frac{1}{|\mathrm{H}_k^*|} \sum_{\substack{f \in \mathrm{H}_k^* \\ L(1, \mathrm{sym}^\nu f) < (B_\nu^- t)^{-A_\nu^-}}} 1.$$

Below are the analogues of Conjectures (C1) and (C2) of Montgomery and Vaughan: *For each $\nu \in \mathbb{N}$, there are positive constants $k_0 = k_0(\nu)$, $C_\nu > c_\nu > 0$ and $0 < \theta_\nu < \Theta_\nu < 1$ such that*

$$\text{(C1)}' \qquad e^{-C_\nu (\log k)/\log_2 k} \le F_k^\pm(\log_2 k, \mathrm{sym}^\nu) \le e^{-c_\nu (\log k)/\log_2 k}$$

and

$$\text{(C2)}' \qquad k^{-\Theta_\nu} \le F_k^\pm(\log_2 k + \log_3 k, \mathrm{sym}^\nu) \le k^{-\theta_\nu}$$

for $k \ge k_0$.

Towards the conjecture (C1)$'$, Liu, Royer & Wu [18] proved a weak form for $\nu = 1$: there are positive constants C, c_1 and c_2 such that for all large k,

$$\text{(3.4)} \qquad e^{-c_1 (\log k)/\{(\log_2 k)^{7/2} \log_3 k\}} \le F_k^\pm(T_k, \mathrm{sym}^1) \le e^{-c_2 (\log k)/\{(\log_2 k)^{7/2} \log_3 k\}}$$

where $T_k := \log_2 k - \frac{5}{2} \log_3 k - \log_4 k - 3C \sim \log_2 k$.

The next application of Theorem 1 is to derive some upper bounds for $F_k^\pm(\cdot, \mathrm{sym}^\nu)$ in (C1)$'$ and (C2)$'$ when $\nu = 1, 2, 3, 4$. Our result is unconditional and capable of establishing the upper estimate in the first Montgomery-Vaughan's conjecture (C1)$'$.

Theorem 6. *Let $\nu = 1, 2, 3, 4$. Then for any $\varepsilon > 0$, there are two positive constants $c = c(\varepsilon)$ and $k_0 = k_0(\varepsilon)$ such that*

$$F_k^\pm(\log_2 k + r, \mathrm{sym}^\nu) \le \exp\bigg(-c(|r| + 1)\frac{\log k}{\log_2 k}\bigg)$$

for all even integer $k \ge k_0$ and $\log \varepsilon \le r \le 9 \log_2 k$.

Remark 2. When $r = 0$, this gives the second inequality in (C1)$'$, i.e. the upper bound of the first Montgomery-Vaughan conjecture. The choice $r = \log_3 k$ gives (unconditionally) an upper bound for $F_k^\pm(\log_2 k + \log_3 k, \mathrm{sym}^\nu)$, which is however weaker than the conjectured value in (C2)$'$.

Concerning the lower bound of Conjecture (C1)$'$, we have the following result.

Theorem 7. *Let $\nu = 1, 2, 3, 4$. There are positive absolute constants k_0, c_1 and c_2 such that*

$$F_k^{\pm}(\log_2 k - c_1 \log_3 k, \mathrm{sym}^\nu) \geq e^{-c_2(\log k)/\{(\log_2 k)^2 \log_3 k\}} \tag{3.5}$$

for all even integer $k \geq k_0$.

Remark 3. (i) Theorem 7 can be regarded as a complement of the upper bound in (3.4), which applies to $\nu = 1$ only. Besides, the constant c_1 in (3.5) is indeed bigger than $\frac{5}{2}$ in (3.4). By virtue of the probabilistic model in [18], the distribution functions $F_k^{\pm}(t, \mathrm{sym}^\nu)$ behave like

$$\exp\left\{ - \frac{e^{t-\gamma_0}}{t}\left[1 + O\left(\frac{1}{t}\right)\right]\right\}$$

in a neighbourhood of the key value $\log_2 k$. (γ_0 is an explicit constant, see [18], Corollary 5.) Thus the expected size of $F_k^{\pm}(\log_2 k - c_1 \log_3 k, \mathrm{sym}^\nu)$ should be $e^{-c_2(\log k)/(\log_2 k)^{1+c_1}}$. The lower estimate in (3.5) is rather good.

(ii) As in [15] and [16], we do not want to impose further hypotheses, hence confine to the cases $1 \leq \nu \leq 4$.

(iii) Theorems 6 and 7 can be generalized to $\mathrm{H}_k^*(N)$. In this case the implied constants depend on the level N. Our method can also be applied to establish similar results in level aspects provided the level is squarefree and free of small prime factors. It seems possible to prove analogous results in both aspects k and N.

§ 4. An unweighted trace formula

As was indicated in §1, one of our new ingredients is the unweighted trace formula. This will simplify considerably the argument in [16] and is certainly of independent interest.

Lemma 4.1. *Let N be squarefree and $(mn, N^2) \mid N$. Then for any $\varepsilon > 0$ we have*

$$\begin{aligned}\sum_{f\in \mathrm{H}_k^*(N)} \lambda_f(m)\lambda_f(n) = \frac{k-1}{12}\varphi(N)\frac{\sigma((m,n))}{\sqrt{mn}}\delta_{mn=\square} \\ + O_\varepsilon\big(\{k^{29}N^{27}(mn)^3\}^{1/33}(kNmn)^\varepsilon\big),\end{aligned} \tag{4.1}$$

where $\delta_{\ell=\square} = 1$ for perfect square ℓ or 0 otherwise and $\sigma(\ell) := \sum_{d\mid\ell} d$.

Proof. First we establish the following asymptotic formula

$$L(1, \mathrm{sym}^2 f) = \zeta(2) \sum_{\substack{n\leq y\\(n,N)=1}} \frac{\lambda_f(n^2)}{n} + O_\varepsilon\big((kN/y)^{2/7}(kNy)^\varepsilon\big) \tag{4.2}$$

valable uniformly for $y \geq 1$.

According to (3.14)–(3.16) of [10], for $\Re e\, s > 1$ we have

$$L(s, \mathrm{sym}^2 f) = \zeta^{(N)}(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s} = \zeta^{(N)}(2s)\zeta_N(s+1) \sum_{\substack{n=1\\(n,N)=1}}^{\infty} \frac{\lambda_f(n^2)}{n^s}$$

where $\zeta_N(s) := \prod_{p|N}(1-p^{-s})^{-1}$ and $\zeta^{(N)}(s) := \zeta(s)/\zeta_N(s)$. Thus by using Perron's formula (Theorem II.2.2 of [24]) and Deligne's inequality, we can write

$$\sum_{\substack{n\le y\\ (n,N)=1}} \frac{\lambda_f(n^2)}{n} = \frac{1}{2\pi i}\int_{\varepsilon-iT}^{\varepsilon+iT} \zeta^{(N)}(2+2s)^{-1}\zeta_N(2+s)^{-1}L(1+s,\mathrm{sym}^2 f)\frac{y^s}{s}\,\mathrm{d}s$$
$$+ O_\varepsilon\bigg(y^\varepsilon \sum_{n\ge 1}\frac{\tau(n^2)}{n^{1+\varepsilon}(1+T|\log(y/n)|)}\bigg),$$

where T is a large parameter chosen up to our disposal. Separating, as usual, n into $|n-y|\ge \frac{1}{2}y$ or not, the O-term is bounded above by

$$\frac{y^\varepsilon}{T}\sum_{|n-y|\ge y/2}\frac{\tau(n^2)}{n^{1+\varepsilon}} + \frac{y^\varepsilon}{T}\sum_{1\le |n-y|\le y/2}\frac{\tau(n^2)}{n^\varepsilon|y-n|} + y^{\varepsilon-1} \ll \frac{y^\varepsilon}{T} + y^{-1+\varepsilon}.$$

Clearly both $\zeta^{(N)}(2+2s)$ and $\zeta_N(2+s)$ are holomorphic and of magnitude $\gg_\varepsilon 1$ on $\Re e\, s \ge -1/2+\varepsilon$. We shift the line of integration to $[-1/2+\varepsilon-iT, -1/2+\varepsilon+iT]$, the main term equals

$$L(1,\mathrm{sym}^2 f) + O\bigg(\bigg|\int_{-1/2+\varepsilon\pm iT}^{\varepsilon\pm iT}\bigg| + \bigg|\int_{-1/2+\varepsilon-iT}^{-1/2+\varepsilon+iT}\bigg|\bigg).$$

Invoking the convexity bound

$$\begin{aligned} L(\sigma+i\tau,\mathrm{sym}^2 f) &\ll \big\{N^2(|\tau|+1)(|\tau|+k)^2\big\}^{\max\{(1-\sigma)/2,\,0\}+\varepsilon} \\ &\ll \big\{(kN)^2(|\tau|+1)^3\big\}^{\max\{(1-\sigma)/2,\,0\}}(kN\tau)^\varepsilon \end{aligned}$$

with $\sigma = 1+\Re e\, s$, we see that

$$\bigg|\int_{-1/2+\varepsilon\pm iT}^{\varepsilon\pm iT}\bigg| + \bigg|\int_{-1/2+\varepsilon-iT}^{-1/2+\varepsilon+iT}\bigg| \ll T^{-1}(kNTy)^\varepsilon + (kN/y)^{1/2}T^{3/4}(kNTy)^\varepsilon.$$

Taking $T = \{y/(kN)\}^{2/7}$, we get the required asymptotic formula (4.2).

As in [10], for $\Re e\, s > 1$ define

$$Z(s,f) = \sum_{n=1}^\infty \lambda_f(n^2)n^{-s} = \zeta^{(N)}(2s)^{-1}L(s,\mathrm{sym}^2 f).$$

Then we write

$$\sum_{f\in \mathrm{H}_k^*(N)} \lambda_f(m)\lambda_f(n) = \zeta^{(N)}(2)^{-1}\sum_{f\in \mathrm{H}_k^*(N)}\frac{\lambda_f(m)\lambda_f(n)}{Z(1,f)}L(1,\mathrm{sym}^2 f), \tag{4.3}$$

in order to apply Corollary 2.10 of [10] (see [10], (2,54) as well).

Replacing $L(1,\mathrm{sym}^2 f)$ by the formula (4.2), we split the right-hand side of (4.3) into two parts accordingly. Since $1/\log(kN) \ll L(1,\mathrm{sym}^2 f) \ll \log(kN)$ (see [9] and [6]), the same bounds hold for $Z(1,f)$. The O-term in (4.2) contributes a term

$$\begin{aligned} &\ll (kN/y)^{2/7}(kNy)^\varepsilon \sum_{f\in \mathrm{H}_k^*(N)}\frac{|\lambda_f(m)\lambda_f(n)|}{Z(1,f)} \\ &\ll (kN)^{9/7}y^{-2/7}(kNmny)^\varepsilon. \end{aligned}$$

Inserting the main term in (4.2) into (4.3) and using the Hecke relation (1.2), the contribution of the main term in (4.2) to (4.3) is

$$
\begin{aligned}
(4.4) \qquad &= \zeta_N(2) \sum_{f\in \mathrm{H}_k^*(N)} \frac{\lambda_f(m)\lambda_f(n)}{Z(1,f)} \sum_{\substack{\ell\le y\\ (\ell,N)=1}} \frac{\lambda_f(\ell^2)}{\ell}\\
&= \sum_{\substack{d\mid(m,n)\\ (d,N)=1}} \sum_{\substack{\ell\le y\\ (\ell,N)=1}} \frac{\zeta_N(2)}{\ell} \sum_{f\in \mathrm{H}_k^*(N)} \frac{\lambda_f(\ell^2)\lambda_f(mn/d^2)}{Z(1,f)}.
\end{aligned}
$$

Note that $\zeta(2)\zeta^{(N)}(2)^{-1} = \zeta_N(2)$. As $(mn, N^2) \mid N$ implies that $(mn/d^2, N^2) \mid N$ for all $d \mid (m,n)$ and $(\ell, N) = 1$, we can apply Corollary 2.10 of [10] with $(m,n) = (\ell^2, mn/d^2)$ to the innermost sum,

$$
\zeta_N(2) \sum_{f\in \mathrm{H}_k^*(N)} \frac{\lambda_f(\ell^2)\lambda_f(mn/d^2)}{Z(1,f)} = \frac{k-1}{12}\varphi(N)\delta_{mn/d^2=\ell^2} + O\bigg(k^{1/6}\bigg(\frac{\ell^2 mn}{d^2}\bigg)^{1/4}(\ell N mn)^{\varepsilon}\bigg).
$$

Putting this into (4.4), we get that

$$
\frac{k-1}{12}\varphi(N) \sum_{\substack{\ell\le y\\ (\ell,N)=1}} \frac{1}{\ell} \sum_{\substack{d\mid(m,n)\\ (d,N)=1}} \delta_{mn/d^2=\ell^2} + O\big(\{k^2(mn)^3\}^{1/12} y^{1/2}(Nmny)^{\varepsilon}\big).
$$

If $y \ge (mn)^{1/2}$, the main term is apparently equal to

$$
\frac{k-1}{12}\varphi(N)\delta_{mn/(m,n)^2=\square}\frac{\sigma((m,n))}{\sqrt{mn}}.
$$

The condition $mn/(m,n)^2 = \square$ is equivalent to $mn = \square$. Combining the above estimates and optimizing y over $[(mn)^{1/2}, \infty)$, we obtain

$$
\begin{aligned}
\sum_{f\in \mathrm{H}_k^*(N)} \lambda_f(m)\lambda_f(n) = {} & \frac{k-1}{12}\varphi(N)\delta_{mn=\square}\frac{\sigma((m,n))}{\sqrt{mn}}\\
& + O\Big(\big\{(k^{29}N^{27})^{1/33}(mn)^{1/11} + k^{1/6}(mn)^{1/2}\big\}(kNmn)^{\varepsilon}\Big).
\end{aligned}
$$

This implies the asymptotic formula (4.1) since we can assume $mn \le (k^{47}N^{54})^{1/27}$, otherwise (4.1) is trivial. This completes the proof. $\square$

Remark 4. The error term in (4.1) can be improved by more delicate technique as in [16]. Under GRH, this error term is improved to

$$
O\bigg((kN)^{2/3}\frac{(mn)^{1/6}\tau((m,n))}{\sqrt{(mn,N)}}\bigg).
$$

In fact by using the Hecke relation (1.2), it follows that

$$
\sum_{f\in \mathrm{H}_k^*(N)} \lambda_f(m)\lambda_f(n) = \sum_{\substack{d\mid(m,n)\\ (d,N)=1}} \sum_{f\in \mathrm{H}_k^*(N)} \lambda_f\bigg(\frac{mn}{d^2}\bigg).
$$

Under GRH, we can apply Proposition 2.13 [(1)] of [10] to the inner sum over f for obtaining our required result.

§ 5. A preliminary lemma

In this section, we establish a technical lemma.

(1) It is worthy to indicate that this proposition assumes GRH.

Lemma 5.1. *Let $\Omega(n)$ be the number of all prime factors of n. For $2 \le P < Q \le 2P$, $j \ge 1$ and $n \ge 1$, we define*

$$a_j(n) = a_j(n; P, Q) := |\{(p_1, \dots, p_j) : p_1 \cdots p_j = n,\ P < p_1, \dots, p_j \le Q\}|.$$

Then the following three inequalities

$$\sum_n a_j(n^2)\frac{d^{\Omega(n)}}{n^2} \le \delta_{2|j}\left(\frac{3dj}{P\log P}\right)^{j/2}, \tag{5.1}$$

$$\sum_n{}^{\natural} a_j(n)\frac{d^{\Omega(n)}}{n} \le \left(\frac{12d^2j}{P\log P}\right)^{j/2}\left\{1+\left(\frac{j\log P}{54P}\right)^{j/6}\right\}, \tag{5.2}$$

$$\sum_m{}^{\flat} \sum_{(m,n)=1}{}^{\natural} a_j(mn)\frac{d^{\Omega(mn)}}{m^2n} \le \left(\frac{48d^2j}{P\log P}\right)^{j/2}\left\{1+\left(\frac{20j\log P}{P}\right)^{j/6}\right\} \tag{5.3}$$

hold for

$$d > 0, \qquad j \ge 1 \qquad \text{and} \qquad 2 \le P < Q \le 2P,$$

where

$$\delta_{2|j} := \begin{cases} 1 & \text{if } j \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The summations $\sum^{\natural}$ and $\sum^{\flat}$ run over squarefull and squarefree integers, respectively.

Proof. First we prove (5.1). The number of primes (counted with multiplicity) in a square is even, hence the case for odd j is trivial. Let $j = 2j'$. Then $a_j(n^2) = 0$ unless $\Omega(n) = j'$ and $n = p_1^{\nu_1} \cdots p_i^{\nu_i}$ where each prime factor lies in $(P, Q]$. In this case we have

$$\begin{aligned} a_{2j'}(n^2) &= \frac{(2j')!}{(2\nu_1)! \cdots (2\nu_i)!} \\ &= \frac{(2j')!}{j'!}\frac{\nu_1!}{(2\nu_1)!} \cdots \frac{\nu_i!}{(2\nu_i)!} a_{j'}(n) \\ &\le j'^{j'} a_{j'}(n). \end{aligned}$$

It follows that

$$\begin{aligned} \sum_n a_{2j'}(n^2)\frac{d^{\Omega(n)}}{n^2} &\le j'^{j'} \sum_n a_{j'}(n)\frac{d^{\Omega(n)}}{n^2} \\ &= \left(j' \sum_{P<p\le Q} \frac{d}{p^2}\right)^{j'} \\ &\le \left(\frac{6dj'}{P\log P}\right)^{j'}. \end{aligned}$$

Next, for (5.2) we note that every squarefull integer n is uniquely expressible in the form $n = \ell^2 m$, where m is squarefree and $m \mid \ell$. This can be seen by taking

$$\ell = p_1^{\nu_1} \cdots p_{i+i'}^{\nu_{i+i'}}, \qquad m = p_{i+1} \cdots p_{i+i'}$$

in the decomposition of

$$n = p_1^{2\nu_1} \cdots p_i^{2\nu_i} p_{i+1}^{2\nu_{i+1}+1} \cdots p_{i+i'}^{2\nu_{i+i'}+1}$$

into distinct prime powers. Observe that $a_j(\ell^2 m) = 0$ or $j = 2\nu_1 + \cdots + 2\nu_{i+i'} + i'$, and

$$\begin{aligned} a_j(\ell^2 m) &= \frac{j!}{(2\nu_1)!\cdots(2\nu_i)!(2\nu_{i+1}+1)!\cdots(2\nu_{i+i'}+1)!} \\ &\le \frac{j!}{(j-i')!3^{i'}}\frac{(j-i')!}{(2\nu_1)!\cdots(2\nu_{i+i'})!} \\ &= \frac{j!}{(j-i')!3^{i'}} a_{j-i'}(\ell^2) \\ &\le \left(\frac{j}{3}\right)^{i'} a_{j-i'}(\ell^2) \end{aligned}$$

whenever it is nonzero. We infer that

$$\begin{aligned} \sum_n{}^{\natural} a_j(n)\frac{d^{\Omega(n)}}{n} &= \sum_\ell \sum_{m|\ell}{}^{\flat} a_j(\ell^2 m)\frac{d^{\Omega(\ell^2 m)}}{\ell^2 m} \\ &\le \sum_{\substack{0\le i'\le j/3\\ i'\equiv j(\bmod 2)}} \left(\frac{j}{3}\right)^{i'} \sum_\ell a_{j-i'}(\ell^2)\frac{d^{2\Omega(\ell)}}{\ell^2} \sum_{\substack{m|\ell,\,\Omega(m)=i'\\ p|m\Rightarrow P<p\le Q}}{}^{\flat} \frac{d^{\Omega(m)}}{m}. \end{aligned}$$

Obviously

$$\sum_{\substack{m|\ell,\,\Omega(m)=i'\\ p|m\Rightarrow P<p\le Q}}{}^{\flat} \frac{d^{\Omega(m)}}{m} \le 2^{\Omega(\ell)}\left(\frac{d}{P}\right)^{i'}.$$

Together with (5.1), we have

$$\begin{aligned} \sum_n{}^{\natural} a_j(n)\frac{d^{\Omega(n)}}{n} &\le \sum_{\substack{0\le i'\le j/3\\ i'\equiv j(\bmod 2)}} \left(\frac{dj}{3P}\right)^{i'}\left(\frac{6d^2 j}{P\log P}\right)^{(j-i')/2} \\ &\le \left(\frac{6d^2 j}{P\log P}\right)^{j/2} \sum_{\substack{0\le i'\le j/3\\ i'\equiv j(\bmod 2)}} \left(\frac{j\log P}{54P}\right)^{i'/2} \\ &\le \left(\frac{6d^2 j}{P\log P}\right)^{j/2}\left(\frac{j/3+1}{2}+1\right)\left\{1+\left(\frac{j\log P}{54P}\right)^{j/6}\right\}, \end{aligned}$$

which implies (5.2) since $(j/3+1)/2+1 \le 2^{j/2}$ for $j \ge 2$.

Finally, we prove (5.3). Let us write $m = p_1\cdots p_i$ and $n = p_{i+1}^{\nu_{i+1}}\cdots p_{i+i'}^{\nu_{i+i'}}$ for squarefree m and squarefull n. We have $j = i + \nu_{i+1} + \cdots + \nu_{i+i'}$ when $a_j(mn) \ne 0$. Thus for $(m,n) = 1$,

$$\begin{aligned} a_j(mn) &= \frac{j!}{\nu_{i+1}!\cdots\nu_{i+i'}!} \\ &= \frac{j!}{(j-i)!i!} i! \frac{(j-i)!}{\nu_{i+1}!\cdots\nu_{i+i'}!} \\ &= \binom{j}{i} a_i(m)a_{j-i}(n). \end{aligned}$$

This shows that

$$\sum_m{}^{\flat} \sum_{(m,n)=1}{}^{\natural} a_j(mn)\frac{d^{\Omega(mn)}}{m^2 n} \le \sum_{0\le i\le j} \binom{j}{i} \sum_m{}^{\flat} a_i(m)\frac{d^{\Omega(m)}}{m^2} \sum_n{}^{\natural} a_{j-i}(n)\frac{d^{\Omega(n)}}{n}.$$

Clearly the sum over m is

$$\leq \biggl(\sum_{P<p\leq Q}\frac{d}{p^2}\biggr)^i \leq \biggl(\frac{6d}{P\log P}\biggr)^i.$$

Inserting it into the preceding inequality and applying (5.2) to the sum over n, we find that

$$\begin{aligned}\sum_m{}^{\flat}\sum_{(m,n)=1}{}^{\natural} a_j(mn)\frac{d^{\Omega(mn)}}{m^2n} &\leq \sum_{0\leq i\leq j}\binom{j}{i}\biggl(\frac{6d}{P\log P}\biggr)^i\biggl\{\biggl(\frac{12d^2j}{P\log P}\biggr)^{1/2}\biggl[1+\biggl(\frac{j\log P}{54P}\biggr)^{1/6}\biggr]\biggr\}^{j-i}\\ &\leq \biggl\{\frac{6d}{P\log P}+\biggl(\frac{12d^2j}{P\log P}\biggr)^{1/2}\biggl[1+\biggl(\frac{j\log P}{54P}\biggr)^{1/6}\biggr]\biggr\}^j\\ &\leq \biggl(\frac{12d^2j}{P\log P}\biggr)^{j/2}\biggl\{1+\biggl(\frac{20j\log P}{P}\biggr)^{1/6}\biggr\}^j.\end{aligned}$$

Consequently (5.3) follows since $(1+x)^j\leq 2^j(1+x^j)$ for any $x>0$. □

§ 6. Proof of Theorem 1

Without loss of generality, we suppose

$$j\leq P/(24\log P) \tag{6.1}$$

since, otherwise, the required result follows trivially. Indeed, if $j>P/(24\log P)$, we deduce by (1.1), (1.4), the hypothesis $|b_p|\leq B$ and the well-known estimate

$$\sum_{P<p\leq 2P}\frac{1}{p}\sim\frac{\log 2}{\log P}$$

that

$$\begin{aligned}\sum_{f\in \mathrm{H}_k^*(N)}\biggl|\sum_{\substack{P<p\leq Q\\ p\nmid N}} b_p\frac{\lambda_f(p^\nu)}{p}\biggr|^{2j} &\ll kN\biggl(\frac{2B(\nu+1)}{\log P}\biggr)^{2j}\\ &\ll_\nu kN\biggl(\frac{96B^2(\nu+1)^2j}{P\log P}\biggr)^j.\end{aligned}$$

Define $b_n:=\prod_{p^\mu\|n}b_p^\mu$. Multiplying out the product

$$\biggl|\sum_{\substack{P<p\leq Q\\ p\nmid N}} b_p\frac{\lambda_f(p^\nu)}{p}\biggr|^{2j},$$

we obtain a summation over integers in $(P^{2j},Q^{2j}]$ whose prime factors do not divide N. An integer decomposes uniquely into a product of coprime integers mn with m squarefree and n squarefull. It then follows that

$$\biggl|\sum_{\substack{P<p\leq Q\\ p\nmid N}} b_p\frac{\lambda_f(p^\nu)}{p}\biggr|^{2j} = \sum_{\substack{n\leq Q^{2j}\\ (n,N)=1}}^{\natural}\ \sum_{\substack{P^{2j}<mn\leq Q^{2j}\\ (m,nN)=1}}^{\flat} a_{2j}(mn)\frac{b_mb_n}{mn}\lambda_f(m^\nu)\prod_{p^\mu\|n}\lambda_f(p^\nu)^\mu, \tag{6.2}$$

where, as before, $\sum^\natural$ and $\sum^\flat$ run respectively over squarefull and squarefree integers.

Introducing the sets

$$\mathcal{E}_\mu^\nu(p) := \left\{ \mathbf{d}_p = (d_1, \dots, d_{\mu-1}) \in \mathbb{N}^{\mu-1} : d_j \Big| \left(\frac{p^{j\nu}}{(d_1 \cdots d_{j-1})^2}, \, p^\nu \right) \; (1 \le j \le \mu - 1) \right\}$$

with the convention $d_1 \cdots d_0 = 1$ and $\mathbb{N}^0 = \{1\}$, and

$$\mathcal{E}(n) := \left\{ (\mathbf{d}_p)_{p|n} : \; \mathbf{d}_p \in \mathcal{E}_\mu^\nu(p) \text{ where } p^\nu \| n \text{ for } p|n \right\}$$

where $(a_p)_{p|n}$ is an ordered tuple with $\omega(n)$ components.

We can write

$$\begin{aligned} \prod_{p^\mu \| n} \lambda_f(p^\nu)^\mu &= \prod_{p^\mu \| n} \left\{ \sum_{\mathbf{d}_p \in \mathcal{E}_\mu^\nu(p)} \lambda_f\left(\frac{p^{\nu\mu}}{|\mathbf{d}_p|^2} \right) \right\} \\ &= \sum_{(\mathbf{d}_p)_{p|n} \in \mathcal{E}(n)} \lambda_f\left(\prod_{p^\mu \| n} \frac{p^{\nu\mu}}{|\mathbf{d}_p|^2} \right), \end{aligned}$$

where $|\mathbf{d}_p| = d_1 \cdots d_{\mu-1}$. The right-hand side of (1.6) becomes

$$\sum_{\substack{n \le Q^{2j} \\ (n,N)=1}}^{\natural} \; \sum_{\substack{P^{2j} < mn \le Q^{2j} \\ (m,nN)=1}}^{\flat} a_{2j}(mn) \frac{b_m b_n}{mn} \sum_{(\mathbf{d}_p)_{p|n} \in \mathcal{E}(n)} \; \sum_{f \in \mathrm{H}_k^*(N)} \lambda_f\left(\prod_{p^\mu \| n} \frac{p^{\nu\mu}}{|\mathbf{d}_p|^2} \right) \lambda_f(m^\nu). \tag{6.3}$$

Since $(n, N) = 1$ and $(m, nN) = 1$, we have $(mn, N^2) = 1$. Thus the unweighted trace formula (4.1) is applicable, and leads (6.3) to two parts arising from the main term and error term in (4.1). Clearly we have

$$\begin{aligned} \sum_{(\mathbf{d}_p)_{p|n} \in \mathcal{E}(n)} \prod_{p^\mu \| n} |\mathbf{d}_p|^{-2/10} &= \prod_{p^\mu \| n} \left(\sum_{\mathbf{d}_p \in \mathcal{E}_\mu^\nu(p)} |\mathbf{d}_p|^{-1/5} \right) \\ &\le \prod_{p^\mu \| n} \left(\sum_{\nu' \ge 0} p^{-\nu'/5} \right)^{\mu-1} \\ &< 8^{\Omega(n)}. \end{aligned}$$

Noting $|b_p| \le B$, the contribution to (6.3) from the error term in (4.1) is, therefore,

$$\begin{aligned} &\ll (kN)^{10/11} \sum_{n \le Q^{2j}}^{\natural} \; \sum_{\substack{P^{2j}/n < m \le Q^{2j}/n \\ (m,n)=1}}^{\flat} a_{2j}(mn) \frac{B^{\Omega(mn)} 8^{\Omega(n)}}{(mn)^{1-\nu/10}} \\ &\ll (kN)^{10/11} Q^{j\nu/5} \sum_{P^{2j} < \ell \le Q^{2j}} a_{2j}(\ell) \frac{(8B)^{\Omega(\ell)}}{\ell} \\ &\ll (kN)^{10/11} Q^{j\nu/5} \left(\sum_{P < p \le Q} \frac{8B}{p} \right)^{2j} \\ &\ll (kN)^{10/11} \left(\frac{10BQ^{\nu/10}}{\log P} \right)^{2j}. \end{aligned} \tag{6.4}$$

Here we have used the uniqueness of decomposing an integer into the product of a squarefull and a squarefree number.

It remains to handle the contribution from the main term in (4.1), which is nonzero if and only if

$$\prod_{p^\mu \| n} \frac{p^{\nu\mu}}{|\mathbf{d}_p|^2} = \square \qquad \text{and} \qquad m^\nu = \square. \tag{6.5}$$

We separate into two cases.

(i) ν is odd. Then (6.5) is equivalent to

$$2 \mid \mu \quad (\text{i.e. } n = \square) \qquad \text{and} \qquad m = 1.$$

Thus by (5.1), the contribution is

$$\begin{aligned} &\leq k\varphi(N) \sum_n a_{2j}(n^2) \frac{[B(\nu+1)]^{2\Omega(n)}}{n^2} \\ &\leq k\varphi(N) \bigg(\frac{6B^2(\nu+1)^2 j}{P\log P}\bigg)^j. \end{aligned} \tag{6.6}$$

(ii) ν is even. Then (6.5) always holds. By (5.3) with $2j$ in place of j and (6.1), the contribution in this case is

$$\begin{aligned} &\ll k\varphi(N) \sum_m{}^{\flat} \sum_{(m,n)=1}{}^{\natural} a_{2j}(mn) \frac{(\nu+1)^{\Omega(n)} B^{\Omega(mn)}}{m^2 n} \\ &\ll k\varphi(N) \bigg(\frac{96B^2(\nu+1)^2 j}{P\log P}\bigg)^j. \end{aligned} \tag{6.7}$$

The proof is complete from (6.3), (6.4), (6.6) and (6.7). □

§ 7. Proof of Theorem 2

Define

$$\mathrm{E}_k^*(N, P; \mathcal{P}) := \big\{f \in \mathrm{H}_k^*(N) : \lambda_f(p^\nu) = \lambda(p) \ \text{ for } \ P < p \leq 2P, \ \ p \in \mathcal{P} \ \text{ and } \ p \nmid N\big\}.$$

It suffices to prove that there are two positive constants $C = C(\Lambda, \nu, \mathcal{P})$ and $c = c(\Lambda, \nu, \mathcal{P})$ such that

$$|\mathrm{E}_k^*(N, P; \mathcal{P})| \ll_{\Lambda,\nu,\mathcal{P}} kNe^{-c\log(kN)/\log_2(kN)} \tag{7.1}$$

uniformly for

$$2 \mid k, \qquad N \quad (\text{squarefree}), \qquad kN \geq X_0, \qquad C\log(kN) \leq P \leq (\log(kN))^{10}$$

for some sufficiently large number $X_0 = X_0(\Lambda, \nu, \mathcal{P})$. We may assume

$$|\lambda(p)| \leq \nu + 1 \tag{7.2}$$

for all $p \geq P \geq C\log(kN)$, otherwise the set $\mathrm{E}_k^*(N, P; \mathcal{P})$ is empty by Deligne's inequality.

Define for $1 \le \nu' \le \nu$,

$$\mathrm{E}_k^{\nu'}(N, P; \mathcal{P}) := \bigg\{ f \in \mathrm{H}_k^*(N) : \bigg| \sum_{\substack{P<p\le 2P\\ p\in\mathcal{P},\ p\nmid N}} \frac{\lambda_f(p^{2\nu'})}{p} \bigg| \ge \frac{\delta}{2\nu \log P} \bigg\}.$$

Take

$$\nu = 2\nu', \qquad Q = 2P \qquad \text{and} \qquad b_p = \begin{cases} 1 & \text{if } p\in\mathcal{P}\\ 0 & \text{otherwise}\end{cases}$$

in Theorem 1. Then we get

$$\begin{aligned}\left(\frac{\delta}{2\nu\log P}\right)^{2j} |\mathrm{E}_k^{\nu'}(N, P; \mathcal{P})| &\le \sum_{f\in \mathrm{H}_k^*(N)} \bigg| \sum_{\substack{P<p\le 2P\\ p\nmid N}} b_p \frac{\lambda_f(p^{2\nu'})}{p} \bigg|^{2j} \\ &\ll kN \left(\frac{96(2\nu'+1)^2 j}{P\log P}\right)^j + (kN)^{10/11} \left(\frac{10(2P)^{\nu'/5}}{\log P}\right)^{2j}.\end{aligned}$$

Hence,

$$|\mathrm{E}_k^{\nu'}(N, P; \mathcal{P})| \ll kN \left(\frac{3456\nu^4 j\log P}{\delta^2 P}\right)^j + (kN)^{10/11} P^{\nu j}, \tag{7.3}$$

provided $P \ge 2(20\nu/\delta)^{10/(3\nu)}$.

Let

$$b_p = \begin{cases} \lambda(p) & \text{if } p\in\mathcal{P}\\ 0 & \text{otherwise,}\end{cases}$$

¿From the definition of $\mathrm{E}_k^*(N, P; \mathcal{P})$, we deduce that

$$\begin{aligned}\sum_{f\in \mathrm{E}_k^*(N,P;\mathcal{P})} \bigg| \sum_{\substack{P<p\le 2P\\ p\in\mathcal{P},\ p\nmid N}} \frac{\lambda_f(p^\nu)^2}{p} \bigg|^{2j} &\le \sum_{f\in \mathrm{H}_k^*(N)} \bigg| \sum_{\substack{P<p\le 2P\\ p\nmid N}} b_p \frac{\lambda_f(p^\nu)}{p} \bigg|^{2j} \\ &\ll kN \left(\frac{96(\nu+1)^4 j}{P\log P}\right)^j + (kN)^{10/11} P^{\nu j/2},\end{aligned} \tag{7.4}$$

by Theorem 1 and (7.2). The Hecke relation (1.2) gives

$$\lambda_f(p^\nu)^2 = 1 + \lambda_f(p^2) + \cdots + \lambda_f(p^{2\nu}) \qquad (p\nmid N).$$

The left-hand side of (7.4) is

$$\begin{aligned}&\ge \sum_{f\in \mathrm{E}_k^*(N,P;\mathcal{P})\setminus(\cup_{\nu'=1}^{\nu} \mathrm{E}_k^{\nu'}(N,P;\mathcal{P}))} \bigg(\sum_{\substack{P<p\le 2P\\ p\in\mathcal{P},\ p\nmid N}} \frac{1}{p} - \sum_{1\le\nu'\le\nu} \bigg| \sum_{\substack{P<p\le 2P\\ p\in\mathcal{P},\ p\nmid N}} \frac{\lambda_f(p^{2\nu'})}{p} \bigg| \bigg)^{2j} \\ &\ge \sum_{f\in \mathrm{E}_k^*(N,P;\mathcal{P})\setminus(\cup_{\nu'=1}^{\nu} \mathrm{E}_k^{\nu'}(N,P;\mathcal{P}))} \bigg(\sum_{\substack{P<p\le 2P\\ p\in\mathcal{P},\ p\nmid N}} \frac{1}{p} - \frac{\delta}{2\log P}\bigg)^{2j}.\end{aligned}$$

Let $\omega(n)$ be the number of distinct prime factors of n. Using the hypothesis on $\mathcal{P}$ and the inequality

$$\omega(n) \le \{1+o(1)\}(\log n)/\log_2 n,$$

we infer that

$$\begin{aligned}\sum_{\substack{P<p\le 2P\\ p\in\mathcal{P},\, p\nmid N}} \frac{1}{p} - \frac{\delta}{2\log P} &\ge \sum_{\substack{P<p\le 2P\\ p\in\mathcal{P}}} \frac{1}{p} - \sum_{\substack{P<p\le 2P\\ p\,|N}} \frac{1}{p} - \frac{\delta}{2\log P}\\ &\ge \frac{\delta}{2\log P} - \frac{\omega(N)}{P}\\ &\ge \frac{\delta/2-2/C}{\log P}\\ &\ge \frac{\delta}{6\log P},\end{aligned}$$

provided $C\ge 6/\delta$.

Combining these estimates with (7.4), we conclude that

$$|\mathrm{E}_k^*(N,P;\mathcal{P})\setminus(\cup_{\nu'=1}^{\nu}\mathrm{E}_k^{\nu'}(N,P;\mathcal{P}))| \ll kN\bigg(\frac{3456(\nu+1)^4 j\log P}{\delta^2 P}\bigg)^j + (kN)^{10/11}P^{\nu j}.$$

Together with (7.3), it implies

$$|\mathrm{E}_k^*(N,P;\mathcal{P})| \ll kN\bigg(\frac{3456(\nu+1)^4 j\log P}{\delta^2 P}\bigg)^j + (kN)^{10/11}P^{\nu j} \tag{7.5}$$

uniformly for

$$2\mid k,\qquad N\quad(\text{squarefree}),\qquad C\log(kN)\le P\le(\log(kN))^{10},\qquad j\ge 1.$$

Take

$$j=\bigg[\delta^*\frac{\log(kN)}{\log P}\bigg]$$

where $\delta^*=\delta^2/(10(\nu+1))^4$. We can ensure $j>1$ once X_0 is chosen to be suitably large. A simple computation gives that

$$\bigg(\frac{3456(\nu+1)^4 j\log P}{\delta^2 P}\bigg)^j \ll e^{-c\log(kN)/\log_2(kN)}$$

for some positive constant $c=c(\Lambda,\nu,\mathcal{P})$ and $P^{\nu j}\ll (kN)^{1/1000}$, provided X_0 is large enough. Inserting them into (7.5), we get (7.1) and complete the proof. □

§ 8. Proof of Theorem 6

The proof is based on a more general result.

Proposition 8.1. *Let $\nu=1,2,3,4$. For any $0<\varepsilon<1$, there is a positive constant $c_0=c_0(\varepsilon)$ such that uniformly for*

$$2\mid k,\qquad k\ge 16,\qquad \varepsilon\log k\le z\le(\log k)^{10},$$

we have

$$L(1,\mathrm{sym}^\nu f)=\bigg\{1+O\bigg(\frac{1}{\log_2 k}\bigg)\bigg\}\prod_{p\le z}\prod_{0\le j\le\nu}\bigg(1-\frac{\alpha_f(p)^{\nu-2j}}{p}\bigg)^{-1} \tag{8.1}$$

for all but $O_\varepsilon\big(k^{1-c_0\{\log[2z/(\varepsilon\log k)]\}/\log_2 k}\big)$ primitive forms $f\in\mathrm{H}_k^*$. The implied constant in the O-term of (8.1) is absolute.

Before proving this Proposition 8.1, we need to establish a preliminary lemma.

Lemma 8.1. *Let $\nu \in \mathbb{N}$ be a fixed positive integer and let $0 < \varepsilon < 1$ be an arbitrary constant.*

(i) Define

$$\mathrm{E}_\nu^1(P,Q) := \bigg\{ f \in \mathrm{H}_k^* : \bigg| \sum_{P<p\le Q} \frac{\lambda_f(p^\nu)}{p} \bigg| > \frac{10(\nu+1)}{(\log k)(\log P)} \bigg\}. \tag{8.2}$$

We have

$$\big|\mathrm{E}_\nu^1(P,Q)\big| \ll_\nu k^{1-1/(250\nu)}$$

for

$$2 \mid k, \qquad k \ge 16, \qquad (\log k)^{10} \le P \le Q \le 2P \le \exp\big\{\sqrt{\log k}\big\}. \tag{8.3}$$

The implied constant depends on ν at most.

(ii) Let

$$\mathrm{E}_\nu^2(P,Q;z) := \Bigg\{ f \in \mathrm{H}_k^* : \bigg| \sum_{P<p\le Q} \frac{\lambda_f(p^\nu)}{p} \bigg| > \bigg(\frac{96(\nu+1)^2 z}{(\log_2 k)^2 P}\bigg)^{1/2} \Bigg\}. \tag{8.4}$$

There is a positive constant $c_0(\varepsilon,\nu)$ such that if

$$2 \mid k, \qquad k \ge 16, \qquad \varepsilon \log k \le z \le P \le Q \le 2P \le (\log k)^{10}, \tag{8.5}$$

then

$$\big|\mathrm{E}_\nu^2(P,Q;z)\big| \ll_{\varepsilon,\nu} k \exp\bigg\{ - c_0(\varepsilon,\nu)\frac{\log k}{\log_2 k} \log\bigg(\frac{2z}{\varepsilon \log k}\bigg)\bigg\},$$

where the implied constant depends on ε and ν at most.

Proof. Clearly we can assume $k \ge k_0$ (where $k_0 = e^{(200\nu)^2}$ for the assertion (i) and $k_0 = e^{(200\nu/\varepsilon)^2}$ for (ii)) and ignore the remaining cases by enlarging the $\ll$-constants. We shall apply Theorem 1 with the choice $N = 1$, $b_p = 1$, $B = 1$ and

$$j = \begin{cases} \bigg[\dfrac{\log k}{100\nu \log P}\bigg] & \text{if (8.3) holds} \\[2ex] \bigg[\dfrac{\varepsilon \log k}{100\nu \log_2 k}\bigg] & \text{if (8.5) holds} \end{cases}$$

to count $|\mathrm{E}_\nu^1(P,Q)|$ and $|\mathrm{E}_\nu^2(P,Q;z)|$. The right-hand side of (1.6) is plainly

$$\ll k\bigg\{\bigg(\frac{96(\nu+1)^2 j}{P\log P}\bigg)^j + \frac{Q^{j\nu}}{k^{1/11}}\bigg\}. \tag{8.6}$$

By (8.3) and (8.6), we get that

$$\begin{aligned}
|\mathrm{E}_\nu^1(P,Q)| &\ll k\bigg\{\bigg(\frac{96(\nu+1)^2 j}{P\log P}\bigg)^j + \frac{Q^{j\nu}}{k^{1/11}}\bigg\}\bigg(\frac{(\log k)^2(\log P)^2}{100(\nu+1)^2}\bigg)^j \\
&\ll k\bigg\{\bigg(\frac{j(\log P)(\log k)^2}{P}\bigg)^j + \frac{Q^{2\nu j}}{k^{1/11}}\bigg\} \\
&\ll k\bigg\{\bigg(\frac{(\log k)^3}{P}\bigg)^j + \frac{e^{2\nu j \log Q}}{k^{1/11}}\bigg\} \\
&\ll k^{1-1/(250\nu)},
\end{aligned}$$

for $j\log Q\le \frac{1}{100\nu}\log k$.

Next for the case (8.5), we have $\log P\ge \frac12\log_2 k$ and $z\ge(\log_2 k)^2$, whence it follows that

$$\begin{aligned}|\mathrm{E}_\nu^2(P,Q;z)| &\ll k\bigg\{\bigg(\frac{96(\nu+1)^2j}{P\log P}\bigg)^j+\frac{Q^{j\nu}}{k^{1/11}}\bigg\}\bigg(\frac{P(\log_2 k)^2}{96(\nu+1)^2z}\bigg)^j\\ &\ll k\bigg\{\bigg(\frac{j\log_2 k}{z}\bigg)^j+\frac{Q^{2\nu j}}{k^{1/11}}\bigg\}\\ &\ll k\bigg\{\bigg(\frac{\varepsilon\log k}{2z}\bigg)^j+\frac{e^{2\nu j\log Q}}{k^{1/11}}\bigg\}\\ &\ll k\exp\bigg\{-\frac{\varepsilon}{101\nu}\frac{\log k}{\log_2 k}\log\bigg(\frac{2z}{\varepsilon\log k}\bigg)\bigg\}.\end{aligned}$$

This completes the proof of Lemma 8.1. □

Now we prove the proposition 8.1.

Let $\eta\in(0,\frac{1}{100}]$ be fixed and $\nu=1,2,3,4$. We let

$$\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta):=\big\{f\in\mathrm{H}^*_k: L(s,\mathrm{sym}^\nu f)\ne0,\ s\in\mathcal{S}\big\} \tag{8.7}$$

where $\mathcal{S}:=\{s:\sigma\ge1-\eta,\ |\tau|\le100k^\eta\}\cup\{s:\sigma\ge1\}$, and

$$\mathrm{H}^-_{k,\mathrm{sym}^\nu}(1;\eta):=\mathrm{H}^*_k\smallsetminus\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta). \tag{8.8}$$

¿From (1.11) of [15], we have

$$\big|\mathrm{H}^-_{k,\mathrm{sym}^\nu}(1;\eta)\big|\ll_\eta k^{31\eta}. \tag{8.9}$$

Define

$$y_0=\exp\big\{\sqrt{(\log k)/\{7(\nu+4)\}}\big\},\qquad y_1:=(\log k)^{10},\qquad y_2:=\varepsilon(\log k).$$

By Lemma 3.1 of [16] with the choice of $\delta_0=\frac12$, we have

$$\log L(1,\mathrm{sym}^\nu f)=\sum_{p\le y_0}\sum_{0\le j\le\nu}\log\bigg(1-\frac{\alpha_f(p)^{\nu-2j}}{p}\bigg)^{-1}+O\bigg(\frac{1}{\sqrt{\log k}}\bigg)$$

for any $f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)$. The implied constant is absolute.

For each $f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)$, we further write

$$\log L(1,\mathrm{sym}^\nu f)=\sum_{p\le y_1}\sum_{0\le j\le\nu}\log\bigg(1-\frac{\alpha_f(p)^{\nu-2j}}{p}\bigg)^{-1}+O\bigg(\frac{1}{\sqrt{\log k}}\bigg)+R_1(\mathrm{sym}^\nu f)$$

where

$$\begin{aligned}R_1(\mathrm{sym}^\nu f)&:=\sum_{y_1<p\le y_0}\sum_{0\le j\le\nu}\log\bigg(1-\frac{\alpha_f(p)^{\nu-2j}}{p}\bigg)^{-1}\\ &=\sum_{y_1<p\le y_0}\frac{\lambda_f(p^\nu)}{p}+O\bigg(\frac{1}{y_1}\bigg).\end{aligned}$$

To treat the last sum, we divide it dyadically and apply Lemma 8.1(i). Define

$$P_\ell(y_1):=2^{\ell-1}y_1,\qquad Q_\ell(y_1,y_0):=\min\{2^\ell y_1,y_0\} \tag{8.10}$$

and

$$\mathrm{E}_\nu^1 := \mathrm{H}_{k,\mathrm{sym}^\nu}^-(1;\eta) \cup \bigcup_\ell \mathrm{E}_\nu^1\big(P_\ell(y_1),\, Q_\ell(y_1,y_0)\big), \tag{8.11}$$

where $\mathrm{E}_\nu^1(P,\, Q)$ is defined as in (8.2). There are at most $(\log y_0)/(\log 2)+1$ values of ℓ which occur in the union. By Lemma 8.1(i), we see that

$$\begin{aligned} |\mathrm{E}_\nu^1| &\ll k^{31\eta} + \sum_\ell \big|\mathrm{E}_\nu^1\big(P_\ell(y_1),\, Q_\ell(y_1,y_0)\big)\big| \\ &\ll k^{31\eta} + k^{1-1/(250\nu)}\sqrt{\log k} \\ &\ll k^{1-1/(251\nu)}. \end{aligned} \tag{8.12}$$

For all $f\in \mathrm{H}_k^*\smallsetminus\mathrm{E}_\nu^1$, we have

$$\begin{aligned} R_1(\mathrm{sym}^\nu f) &\ll \sum_\ell \bigg|\sum_{P_\ell(y_1)<p\le Q_\ell(y_1,y_0)} \frac{\lambda_f(p^\nu)}{p}\bigg| + \frac{1}{y_1} \\ &\ll \sum_\ell \frac{10(\nu+1)}{(\log k)\log P_\ell(y_1)} + \frac{1}{y_1} \\ &\ll \frac{\log_2 k}{\log k}. \end{aligned}$$

Therefore, for all $2\mid k$, $k\ge 16$ and $f\in \mathrm{H}_k^*\smallsetminus\mathrm{E}_\nu^1$ we have

$$\log L(1,\mathrm{sym}^\nu f) = \sum_{p\le y_1}\sum_{0\le j\le\nu} \log\bigg(1-\frac{\alpha_f(p)^{\nu-2j}}{p}\bigg)^{-1} + O\bigg(\frac{1}{\log_2 k}\bigg),$$

where the implied constant is absolute.

Finally we consider $y_2\le z\le y_1$. In the same fashion, it remains to evaluate

$$R_2(\mathrm{sym}^\nu f) := \sum_{z<p\le y_1}\sum_{0\le j\le\nu} \log\bigg(1-\frac{\alpha_f(p)^{\nu-2j}}{p}\bigg)^{-1}.$$

Set

$$\mathrm{E}_\nu^2(z) := \mathrm{E}_\nu^1 \cup \bigcup_\ell \mathrm{E}_\nu^2\big(P_\ell(z),\, Q_\ell(z,y_1); z\big),$$

where E_ν^1, $\mathrm{E}_\nu^2(P,Q;z)$, $P_\ell(z)$ and $Q_\ell(z,y_1)$ are defined as in (8.11), (8.4) and (8.10), respectively. Here the number of sets in the union over ℓ is at most $(\log y_1)/(\log 2)+1\ll \log_2 k$. By (8.12) and Lemma 8.1(ii), we have

$$\begin{aligned} |\mathrm{E}_\nu^2(z)| &\ll_\varepsilon k^{1-1/(251\nu)} + k^{1-2c_0\{\log(2z/y_2)\}/\log_2 k}\log_2 k \\ &\ll_\varepsilon k^{1-c_0\{\log(2z/y_2)\}/\log_2 k} \end{aligned}$$

for all even integer $k\ge 16$, where $c_0=c_0(\varepsilon)$ is a positive constant depending on ε.

For all even integer $k \geq 16$ and $f \in \mathrm{H}_k^* \smallsetminus \mathrm{E}_\nu^2(z)$, we have

$$\begin{aligned}
R_2(\mathrm{sym}^\nu f) &\ll \sum_\ell \bigg| \sum_{P_\ell(z) < p \leq Q_\ell(z, y_1)} \frac{\lambda_f(p^\nu)}{p} \bigg| + \frac{1}{z} \\
&\ll \sum_\ell \left(\frac{z}{P_\ell(z)(\log_2 k)^2} \right)^{1/2} + \frac{1}{z} \\
&\ll \sum_\ell \frac{1}{2^{\ell/2} \log_2 k} + \frac{1}{z} \\
&\ll \frac{1}{\log_2 k}.
\end{aligned}$$

Hence

$$\log L(1, \mathrm{sym}^\nu f) = \sum_{p \leq z} \sum_{0 \leq j \leq \nu} \log \left(1 - \frac{\alpha_f(p)^{\nu - 2j}}{p} \right)^{-1} + O\bigg(\frac{1}{\log_2 k} \bigg)$$

for all even integer $k \geq 16$ and $f \in \mathrm{H}_k^* \smallsetminus \mathrm{E}_\nu^2(z)$, where the implied constant is absolute. This proves Proposition 8.1.

Finally we are ready to complete the proof of Theorem 6.

According to Proposition 8.1, for any $\varepsilon > 0$ there are two suitable positive constants $c_0 = c_0(\varepsilon)$ and $k_0 = k_0(\varepsilon)$ such that for even integer $k \geq k_0$ and $\varepsilon \log k \leq z \leq (\log k)^{10}$ we can find a subset $\mathrm{E}_k^*(z) \subset \mathrm{H}_k^*$ with cardinality

$$|\mathrm{E}_k^*(z)| < k \exp\left\{ - c_0 \log \left(\frac{2z}{\varepsilon \log k} \right) \frac{\log k}{\log_2 k} \right\} \tag{8.13}$$

such that for all $f \in \mathrm{H}_k^* \setminus \mathrm{E}_k^*(z)$, the formula (8.1) holds. Thus for these f, we can deduce

$$\begin{aligned}
L(1, \mathrm{sym}^\nu f) &\leq \left\{ 1 + O\bigg(\frac{1}{\log_2 k} \bigg) \right\} \prod_{p \leq z} \left(1 - \frac{1}{p} \right)^{-(\nu+1)} \\
&\leq \left\{ 1 + O\bigg(\frac{1}{\log_2 k} \bigg) \right\} \big(e^\gamma \log z \big)^{\nu+1} \\
&\leq \big\{ e^\gamma (\log z + C_0) \big\}^{\nu+1}
\end{aligned} \tag{8.14}$$

and (similarly)

$$L(1, \mathrm{sym}^\nu f) \geq \big\{ B_\nu^- (\log z + C_0) \big\}^{-A_\nu^-}, \tag{8.15}$$

where C_0 is an absolute positive constant. Now Theorem 6 follows from (8.13), (8.14) and (8.15) with ε replaced by εe^{-C_0} and the choice of $z = e^{\log_2 k + r - C_0}$. $\square$

§ 9. Proof of Theorem 7

The proof of Theorem 7 is similar to that of théorème A of [21]. The essential difference is that our asymptotic formula for moments (see [15], Proposition 6.1) holds in a larger domain. As a result we can obtain a better estimate. Of course, some modifications are necessary since only the symmetric square case was considered there.

Let $\eta \in (0, \frac{1}{100}]$, $\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)$ be defined as in (8.7) and $\Upsilon_{k,\nu} > 0$ be a parameter to be specified later on. By the Cauchy–Schwarz inequality, we infer that for any integer $n \geq 1$,

$$\Bigg(\sum_{\substack{f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)\\ L(1,\mathrm{sym}^\nu f)>\Upsilon_{k,\nu}}} L(1,\mathrm{sym}^\nu f)^n\Bigg)^2 \leq |\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta,\Upsilon_{k,\nu})| \sum_{f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)} L(1,\mathrm{sym}^\nu f)^{2n},$$

where

$$\mathrm{H}^+_{k,\mathrm{sym}^\nu}\big(1;\eta,\Upsilon_{k,\nu}\big) := \big\{f \in \mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta) : L(1,\mathrm{sym}^\nu f) > \Upsilon_{k,\nu}\big\}.$$

On the other hand, we can write

$$\begin{aligned}\sum_{\substack{f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)\\ L(1,\mathrm{sym}^\nu f)>\Upsilon_{k,\nu}}} L(1,\mathrm{sym}^\nu f)^n &= \sum_{f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)} L(1,\mathrm{sym}^\nu f)^n - \sum_{\substack{f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)\\ L(1,\mathrm{sym}^\nu f)\leq\Upsilon_{k,\nu}}} L(1,\mathrm{sym}^\nu f)^n\\ &\geq \sum_{f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)} L(1,\mathrm{sym}^\nu f)^n - \Upsilon_{k,\nu}^n\, |\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)|.\end{aligned}$$

¿From these two estimates, we deduce that

$$\big|\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta,\Upsilon_{k,\nu})\big| \geq \frac{\big(\mathcal{M}_k^n - \Upsilon_{k,\nu}^n\big)^2}{\mathcal{M}_k^{2n}} |\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)| \tag{9.1}$$

provided

$$\mathcal{M}_k^n \geq \Upsilon_{k,\nu}^n, \tag{9.2}$$

where

$$\mathcal{M}_k^n := |\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)|^{-1} \sum_{f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)} L(1,\mathrm{sym}^\nu f)^n. \tag{9.3}$$

To evaluate $\mathcal{M}_k^n$, we recall at first Proposition 6.1 of [15]: There are two positive constants $\delta = \delta(\eta)$ and $C_1 = C_1(\eta)$ such that

$$W^z_{\mathrm{sym}^\nu} := \sum_{f\in\mathrm{H}^+_{k,\mathrm{sym}^\nu}(1;\eta)} \omega_f L(1,\mathrm{sym}^\nu f)^z = M^z_{\mathrm{sym}^\nu} + O_\eta\big(e^{-\delta\log k/\log_2 k}\big)$$

uniformly for

$$2 \mid k \qquad \text{and} \qquad |z| \leq C_1 \log k/(\log_2(8k)\log_3(8k)),$$

where

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}\|f\|} = \frac{12\zeta(2)}{(k-1)L(1,\mathrm{sym}^2 f)} \tag{9.4}$$

is the harmonic weight. The main term $M^z_{\mathrm{sym}^\nu}$ is given by

$$M^z_{\mathrm{sym}^\nu} := \prod_p \frac{2}{\pi}\int_0^\pi \prod_{0\leq j\leq\nu} \big(1 - e^{i(\nu-2j)\theta}p^{-1}\big)^{-z} \sin^2\theta\,\mathrm{d}\theta,$$

from which one deduces $M_{\mathrm{sym}^2}^{-1} = \zeta(3)^{-1}$, $M_{\mathrm{sym}^2}^{2} = \zeta(2)^3$ and the following asymptotic formula (see [1], Theorem 1.12)

$$\log M_{\mathrm{sym}^\nu}^{\pm n} = A_\nu^\pm n \log\big(B_\nu^\pm \log n\big) + O_\nu\bigg(\frac{n}{\log n}\bigg).$$

As the sum in (9.3) is not weighted, we need to use (9.4) and the Cauchy-Schwarz inequality to resolve it. Applying the Cauchy-Schwarz inequality to (9.3) and using (9.4), we infer that

$$\begin{aligned}\big(\mathcal{M}_k^{2n}\big)^2 &\le |\mathrm{H}_{k,\mathrm{sym}^\nu}^+(1;\eta)|^{-2} W_{\mathrm{sym}^\nu}^{4n} \sum_{f\in \mathrm{H}_{k,\mathrm{sym}^\nu}^+(1;\eta)} \omega_f^{-1}\\ &= |\mathrm{H}_{k,\mathrm{sym}^\nu}^+(1;\eta)|^{-2}[(k-1)/(12\zeta(2))]^2 W_{\mathrm{sym}^\nu}^{4n} W_{\mathrm{sym}^2}^{2}.\end{aligned}$$

In view of (1.1), (8.7) and (8.9), it follows that $|\mathrm{H}_{k,\mathrm{sym}^\nu}^+(1;\eta)| \sim k/12$. Thus for any integer $n\ge 1$, we have

$$\mathcal{M}_k^{2n} \le C_2\big(M_{\mathrm{sym}^\nu}^{4n}\big)^{1/2}. \tag{9.5}$$

Like the constant C_1, we use here (and in the sequel) C_i $(i=2,3,\dots)$ to denote suitable positive constants depending on η only.

Similarly we have

$$\begin{aligned}\big(W_{\mathrm{sym}^\nu}^{n/2}\big)^2 &\le |\mathrm{H}_{k,\mathrm{sym}^\nu}^+(1;\eta)| \mathcal{M}_k^n \sum_{f\in \mathrm{H}_{k,\mathrm{sym}^\nu}^+(1;\eta)} \omega_f^{2}\\ &\le [12\zeta(2)/(k-1)] |\mathrm{H}_{k,\mathrm{sym}^\nu}^+(1;\eta)| \mathcal{M}_k^n W_{\mathrm{sym}^2}^{-1},\end{aligned}$$

which implies

$$\mathcal{M}_k^n \ge \frac{\big(W_{\mathrm{sym}^\nu}^{n/2}\big)^2}{2W_{\mathrm{sym}^2}^{-1}} \ge C_3\big(M_{\mathrm{sym}^\nu}^{n/2}\big)^2.$$

Taking

$$\Upsilon_{k,\nu} = \Upsilon_{k,\nu}(n) := \bigg(\frac{C_3}{2}\big(M_{\mathrm{sym}^\nu}^{n/2}\big)^2\bigg)^{1/n}, \tag{9.6}$$

we see that (9.2) is satisfied and thus, in view of (9.5),

$$\frac{\big(\mathcal{M}_k^n - \Upsilon_{k,\nu}^n\big)^2}{\mathcal{M}_k^{2n}} \ge C_5 \frac{\big(M_{\mathrm{sym}^\nu}^{n/2}\big)^4}{\big(M_{\mathrm{sym}^\nu}^{4n}\big)^{1/2}} \ge C_6 \exp\bigg\{-\frac{C_7 n}{\log n}\bigg\}$$

holds uniformly for $1\le n\le C_1 \log k/(\log_2(8k)\log_3(8k))$. By (9.1), we conclude that

$$\big|\mathrm{H}_{k,\mathrm{sym}^\nu}^+(1;\eta,\Upsilon_{k,\nu}(n))\big| \ge C_6 \exp\bigg\{-\frac{C_7 n}{\log n}\bigg\} \tag{9.7}$$

for $1\le n\le C_1 \log k/(\log_2(8k)\log_3(8k))$.

Finally, with the choice of $n = \big[C_1\log k/(4\log_2(8k)\log_3(8k))\big]$, it is plain that (9.6) gives

$$\begin{aligned}\Upsilon_k &\ge (1 - C_8 \log_3 k/\log_2 k)(B_\nu^+ \log_2 k)^{A_\nu^+}\\ &\ge \{B_\nu^+(\log_2 k - C\log_3 k)\}^{A_\nu^+}\end{aligned}$$

and (9.7) yields the desired lower bound (3.5).

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