Affine varieties with equivalent cylinders

Vladimir Shpilrain

and

Jie-Tai Yu*

Abstract. A well-known *cancellation problem* asks when, for two algebraic varieties $V_1, V_2 \subseteq \mathbf{C}^n$, the isomorphism of the cylinders $V_1 \times \mathbf{C}$ and $V_2 \times \mathbf{C}$ implies the isomorphism of V_1 and V_2.

In this paper, we address a related problem: when the *equivalence* (under an automorphism of $\mathbf{C}^{n+1}$) of two cylinders $V_1 \times \mathbf{C}$ and $V_2 \times \mathbf{C}$ implies the equivalence of their bases V_1 and V_2 under an automorphism of $\mathbf{C}^n$? We concentrate here on hypersurfaces and show that this problem establishes a strong connection between the Cancellation conjecture of Zariski and the Embedding conjecture of Abhyankar and Sathaye. We settle the problem for a large class of polynomials. On the other hand, we give examples of equivalent cylinders with inequivalent bases (those cylinders, however, are not hypersurfaces).

Another result of interest is that, for an arbitrary field K, the equivalence of two polynomials in m variables under an automorphism of $K[x_1, ..., x_n]$, $n \geq m$, implies their equivalence under a *tame* automorphism of $K[x_1, ..., x_{2n}]$.

1. Introduction

Let $\mathbf{C}[x_1, ..., x_n]$ be the polynomial algebra in n variables over the field $\mathbf{C}$. Any collection of polynomials $p_1, ..., p_m$ from $\mathbf{C}[x_1, ..., x_n]$ determines an algebraic variety $\{p_i = 0,\ i = 1, ..., m\}$ in the affine space $\mathbf{C}^n$. We shall denote this algebraic variety by $V(p_1, ..., p_m)$.

We say that two algebraic varieties $V(p_1, ..., p_m)$ and $V(q_1, ..., q_k)$ are *isomorphic* if the algebras of residue classes $\mathbf{C}[x_1, ..., x_n]/\langle p_1, ..., p_m\rangle$ and $\mathbf{C}[x_1, ..., x_n]/\langle q_1, ..., q_k\rangle$ are isomorphic. Here $\langle p_1, ..., p_m\rangle$ denotes the ideal of $\mathbf{C}[x_1, ..., x_n]$ generated by $p_1, ..., p_m$. Thus, isomorphism that we consider here is algebraic, not geometric, i.e., we actually consider isomorphism of what is called *affine schemes*.

On the other hand, we say that two algebraic varieties $V(p_1, ..., p_m)$ and $V(q_1, ..., q_k)$ are *equivalent* if there is an automorphism of $\mathbf{C}^n$ that takes one of them onto the other. Algebraically, this means there is an automorphism of $\mathbf{C}[x_1, ..., x_n]$ that takes the ideal $\langle p_1, ..., p_m\rangle$ to the ideal $\langle q_1, ..., q_k\rangle$.

Furthermore, a variety equivalent to $V \times \mathbf{C}$ is called a *cylinder*; a variety of the form $\{p = 0\}$ is called a *hypersurface*, and a hypersurface equivalent to $\{x_1 = 0\}$ is called

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a *hyperplane*. In particular, a *cylindrical hypersurface* in $\mathbf{C}^n$ is a variety of the form $\{p(x_1, ..., x_m) = 0\}$, where $m < n$.

Below we list 3 conjectures relevant to the subject of the present paper. The first two of them are very well known.

Conjecture 1. (The Cancellation conjecture of Zariski). Let $V \subseteq \mathbf{C}^n$ be a hypersurface. If $V \times \mathbf{C}$ is isomorphic to a hyperplane in $\mathbf{C}^{n+1}$, then V is isomorphic to a hyperplane in $\mathbf{C}^n$.

Conjecture 2. (The Embedding conjecture of Abhyankar and Sathaye). If a hypersurface $V(p)$ in $\mathbf{C}^n$ is isomorphic to the hyperplane $V(x_1)$, then it is equivalent to it.

Conjecture 3. Let $V \subseteq \mathbf{C}^n$ be a hypersurface. If $V \times \mathbf{C}$ is equivalent to a hyperplane in $\mathbf{C}^{n+1}$, then V is equivalent to a hyperplane in $\mathbf{C}^n$. Or, in purely algebraic language: if $p = p(x_1, ..., x_n)$ and $\varphi(p) = x_1$ for some automorphism φ of $\mathbf{C}[x_1, ..., x_{n+1}]$, then also $\alpha(p) = x_1$ for some automorphism α of $\mathbf{C}[x_1, ..., x_n]$.

These conjectures are all unsettled in general, but there are some partial results. Conjecture 2 is settled for $n = 2$ in [2]; Conjecture 1 is settled for $n = 2$ in [1] and [14], and for $n = 3$ in [8]. In [8], Fujita, based on the work of Miyanishi and Sugie [15], proved that $V \times \mathbf{C}^n$ isomorphic to $\mathbf{C}^{n+2}$ implies that V is isomorphic to $\mathbf{C}^2$. In [3], Asanuma gives an idea for constructing a (possible) counterexample to Conjecture 1 over the field of reals.

For more information, see the survey [13].

The focus of the present paper is on Conjecture 3. It looks similar to Conjecture 1; the difference is that isomorphism here is replaced by equivalence.

All the three conjectures seem to be related; however, it is not so easy to pinpoint precise relations between them. Here we prove the following

Proposition 1.1.

(a) Conjectures 1 and 2 together imply Conjecture 3.

(b) Conjectures 3 and 2 together imply Conjecture 1.

(c) Conjecture 3 is true for $n = 2$ and 3.

Thus, Conjecture 2 is, in a way, the strongest of the three; in fact, since Conjecture 3 is most likely true for any n, it is probably the case that Conjecture 2 just implies Conjecture 1.

Our proof of part (c) in the case $n = 3$ is based on deep results cited above, whereas the case $n = 2$ can also be handled by somewhat more elementary methods, based on a result of [5].

In the course of our work on Conjecture 3, we were able to prove the following result of independent interest. Before we give the statement, we need a few more

Definitions. Let K be an arbitrary field. An automorphism of $K[x_1, ..., x_n]$ is called *elementary* if it fixes all variables but one, say, x_i, and maps x_i to $x_i + f(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. An automorphism of $K[x_1, ..., x_n]$ is called *tame* if it is a product of elementary and linear automorphisms. A polynomial p is called a *coordinate polynomial* (or simply a *coordinate*) if there is an automorphism of $K[x_1, ..., x_n]$ that takes p to x_1. A polynomial $p \in K[x_1, ..., x_n]$ is called a *tame coordinate* if it is a component of a tame automorphism, i.e., if there is a tame automorphism of $K[x_1, ..., x_n]$ that takes p to x_1. Finally, a polynomial $p \in K[x_1, ..., x_n]$ is called a *stably tame coordinate* if it is a component of a tame automorphism of $K[x_1, ..., x_N]$ for some $N \geq n$. More generally, two polynomials $p, q \in K[x_1, ..., x_n]$ are called *stably (tame) equivalent* if $\alpha(p) = q$ for some (tame) automorphism α of $K[x_1, ..., x_N]$, $N \geq n$.

We were able to prove:

Theorem 1.2. If two polynomials $p, q \in K[x_1, ..., x_n]$ are equivalent, then they are stably tame equivalent.

In particular:

Corollary 1.3. Every coordinate is stably tame.

Thus, to settle Conjecture 3, it is sufficient to consider cylinders equivalent under a tame automorphism.

Conjecture 3 also motivates the following generalization:

Problem 1. Are any two stably equivalent polynomials equivalent? Or, in geometric language: is it true that equivalence of two cylindrical hypersurfaces $V(p) \times \mathbf{C}$ and $V(q) \times \mathbf{C}$ in $\mathbf{C}^{n+1}$ implies equivalence of their bases $V(p)$ and $V(q)$ in $\mathbf{C}^n$?

A weaker version of this problem is still quite hard:

Problem 2. Is it true that equivalence of two cylindrical hypersurfaces $V(p) \times \mathbf{C}$ and $V(q) \times \mathbf{C}$ in $\mathbf{C}^{n+1}$ implies isomorphism of their bases $V(p)$ and $V(q)$ in $\mathbf{C}^n$?

A generalization of Conjecture 3 in another direction is

Problem 3. Can a cylinder in $\mathbf{C}^n$ be equivalent to a non-cylinder in $\mathbf{C}^n$ under an automorphism of $\mathbf{C}^m$, $m > n$?

Problem 1 is, in our opinion, the most interesting one, and it probably holds the key to many mysteries of affine algebraic geometry.

We were able to prove that the answer to Problem 1 is affirmative for two rather large classes of polynomials (Proposition 1.4 and Theorem 1.5 below). The first result is just a simple observation:

Proposition 1.4. Let $p = p(x_1, ..., x_n)$ be of the form $x_1^{M_1} \cdot ... \cdot x_n^{M_n} + \sum_j c(j) \cdot x_1^{i_1(j)} ... x_n^{i_n(j)}$, where $c(j)$ are some coefficients, all $M_i > 0$, all monomials under the sum are different from $x_1^{M_1} \cdot ... \cdot x_n^{M_n}$, and, for every k, j, one has $i_k(j) \leq M_k$. Then any polynomial $q = q(x_1, ..., x_n)$ stably equivalent to p, is equivalent to it. Or, in geometric language: for any hypersurface $V(q)$ in $\mathbf{C}^n$, equivalence of $V(p) \times \mathbf{C}$ and $V(q) \times \mathbf{C}^{\mathbf{k}}$ in $\mathbf{C}^{n+k}$ implies equivalence of $V(p)$ and $V(q)$ in $\mathbf{C}^n$.

We note that if p is a coordinate in $\mathbf{C}[x_1, ..., x_n]$, then p cannot possibly satisfy the condition of Proposition 1.4 since, by a result of Hadas [9], all vertices of the Newton polytope of a coordinate polynomial must be on coordinate hyperplanes. Therefore, Proposition 1.4 does not settle Conjecture 3.

On the other hand, it should be pointed out that, even if a given polynomial p does not satisfy the condition of Proposition 1.4, there might be an automorphic image of p that does, and then the result will hold for p as well. For example, $p(x, y) = (x + y)^2 y^2$ does not have the form required in Proposition 1.4, but it has an automorphic image $q(x, y) = x^2 y^2$ that does. We discuss the two-variable case in more detail in Section 2, after the proof of Proposition 1.4.

Another, more interesting, class of polynomials for which Problem 1 has the affirmative answer, is given by the following

Theorem 1.5. Suppose a polynomial $p = p(x_1, ..., x_n)$ has the following property: if $\psi(p) = p$ for some injective polynomial mapping ψ of $\mathbf{C}[x_1, ..., x_n]$, then ψ must be an automorphism of $\mathbf{C}[x_1, ..., x_n]$. Then any polynomial $q = q(x_1, ..., x_n)$ stably equivalent to p, is equivalent to it.

Polynomials that satisfy the condition of Theorem 1.5 are called *test polynomials* for injective polynomial mappings – see [7]. Jelonek [10] showed that a generic polynomial of degree $> n$ in $\mathbf{C}[x_1, ..., x_n]$ is a test polynomial for injective mappings. (The statement "a generic polynomial of degree d has a property W" means that there exists a Zariski open dense subset $U \subseteq \mathcal{H}_d$ of the set $\mathcal{H}_d$ of all polynomials of degree d, such that every element of U has the property W.) Thus, the class of polynomials covered by Theorem 1.5 is really large. Still, it does not include any coordinate polynomials, as was observed in [7].

Then, we were able to answer Problem 3 in the affirmative:

Theorem 1.6. For any $k \geq 1$ and $n \geq k + 2$, there is a variety $V \times \mathbf{C}^k$ in $\mathbf{C}^n$ and a non-cylinder U in $\mathbf{C}^n$, such that U is equivalent to $V \times \mathbf{C}^k$ in $\mathbf{C}^{2n}$.

The simplest example (that we know) illustrating Theorem 1.6 would be $U = \{x(1 + xy + z^2) = 1\}$ in $\mathbf{C}^3$ and $V = \{xy = 1\}$ in $\mathbf{C}^2$. The varieties U and $V \times \mathbf{C}$ are inequivalent in $\mathbf{C}^3$, but are equivalent in $\mathbf{C}^6$ (in fact, they are equivalent even in $\mathbf{C}^4$).

We also mention here an example (due to Danielewski [6], unpublished) of *isomorphic* cylindrical hypersurfaces $V(p) \times \mathbf{C}$ and $V(q) \times \mathbf{C}$ with non-isomorphic bases $V(p)$ and $V(q)$ in $\mathbf{C}^3$. In his example, $p = p(x, y, z) = xy - z^2 + 1$; $q = q(x, y, z) = x^2 y - z^2 + 1$.

Recently, Bandman and Makar-Limanov [4] uncovered a geometric reason why two cylinders with non-isomorphic bases can possibly be isomorphic. In our concluding Section 3, we give an explicit algebraic isomorphism (due to P. Russell) for the cylinders in Danielewski's example, to show how complicated it is. This also establishes the fact that Danielewski's cylinders are isomorphic over any ground field of characteristic 0, not just over $\mathbf{C}$.

Furthermore, a combination of Danielewski's example with our characterization of isomorphic varieties [16, Corollary 1.2] yields an example of *equivalent* cylinders with inequivalent (even non-isomorphic !) bases; those cylinders, however, are not hypersurfaces:

Proposition 1.7. As in Danielewski's example, let $p(x,y,z) = xy - z^2 + 1$; $q(x,y,z) = x^2y - z^2 + 1$. Then the varieties $V_1 = V(p(x,y,z),t,u,v,w)$ and $V_2 = V(q(x,y,z),t,u,v,w)$ in $\mathbf{C}^7$ are not isomorphic, whereas the cylinders $V_1 \times \mathbf{C}$ and $V_2 \times \mathbf{C}$ are equivalent in $\mathbf{C}^8$.

2. Proofs

Proof of Proposition 1.1. We start with part **(a)**. We are going to prove this statement in the following form: if both Conjectures 1 and 2 are true for any n, then for any $k \geq 1$, whenever $V \times \mathbf{C}^k$ is equivalent to a hyperplane in $\mathbf{C}^{n+k}$, one has V equivalent to a hyperplane in $\mathbf{C}^n$.

Let $V = V(p)$, $p = p(x_1, ..., x_n)$, and suppose that p is a coordinate in $\mathbf{C}[x_1, ..., x_N]$, $N = n + k$. Then, in particular, the hypersurface $\{p = 0\}$ in $\mathbf{C}^N$ is isomorphic to $\mathbf{C}^{N-1}$. If Conjecture 1 is true, this implies that $\{p = 0\}$ in $\mathbf{C}^n$ is isomorphic to $\mathbf{C}^{n-1}$. Now if Conjecture 2 is true, this implies that p is a coordinate in $\mathbf{C}[x_1, ..., x_n]$, hence V is equivalent to a hyperplane in $\mathbf{C}^n$. □

For part **(b)**, let $V = V(p)$, $p = p(x_1, ..., x_n)$, and suppose that the hypersurface $\{p = 0\}$ in $\mathbf{C}^N$, $N > n$, is isomorphic to $\mathbf{C}^{N-1}$. If Conjecture 2 is true, this implies that p is a coordinate in $\mathbf{C}[x_1, ..., x_N]$. Then Conjecture 3 implies that p is a coordinate in $\mathbf{C}[x_1, ..., x_n]$ as well. In particular, the hypersurface $\{p = 0\}$ in $\mathbf{C}^n$ is isomorphic to $\mathbf{C}^{n-1}$.

For part **(c)**, we start with $n = 2$. Let $V = V(p)$, $p = p(x_1, x_2)$, and suppose that for some $N \geq 3$, p is a coordinate in $\mathbf{C}[x_1, ..., x_N]$. Then, in particular, the hypersurface $\{p = 0\}$ in $\mathbf{C}^N$ is isomorphic to $\mathbf{C}^{N-1}$. Then, by the result of Abhyankar-Eakin-Heinzer [1] and Miyanishi [14] cited in the Introduction, the hypersurface $\{p = 0\}$ in $\mathbf{C}^2$ is isomorphic to $\mathbf{C}$. This implies, by the Abhyankar-Moh theorem [2], that p is a coordinate in $\mathbf{C}[x_1, x_2]$.

Now we get to $n = 3$. The proof here is similar. Let $V = V(p)$, $p = p(x_1, x_2, x_3)$. If p is a coordinate in $\mathbf{C}[x_1, ..., x_N]$ for some $N > 3$, then also for any $c \in \mathbf{C}$, $p - c$ is a coordinate in $\mathbf{C}[x_1, ..., x_N]$. Therefore, for any $c \in \mathbf{C}$, the hypersurface $\{p = c\}$ in $\mathbf{C}^N$

is isomorphic to $\mathbf{C}^{N-1}$. Then, again for any $c \in \mathbf{C}$, the hypersurface $\{p = c\}$ in $\mathbf{C}^3$ is isomorphic to $\mathbf{C}^2$ by the result of Fujita [8] cited in the Introduction. Now this implies, by a recent result of Kaliman [11] (see also [12]), that p is a coordinate in $\mathbf{C}[x_1, x_2, x_3]$, hence V is equivalent to a hyperplane in $\mathbf{C}^3$. $\square$

Proof of Theorem 1.2. Let φ be an automorphism of $K[x_1, ..., x_n]$. By just replacing all x_i with y_i, we make a "copy" of φ acting on $K[y_1, ..., y_n]$. We are going to show that the following automorphism of $K[x_1, ..., x_n, y_1, ..., y_n]$ is tame: $\psi : x_i \to \varphi(x_i), 1 \le i \le n;\ y_i \to -\varphi^{-1}(y_i), 1 \le i \le n$.

Let $\alpha : x_i \to x_i + \varphi(y_i),\ y_i \to y_i,\ 1 \le i \le n,$ and $\beta : x_i \to x_i,\ y_i \to y_i - \varphi^{-1}(x_i),\ 1 \le i \le n$, be two automorphisms of $K[x_1, ..., x_n, y_1, ..., y_n]$. They both are obviously tame.

Now let

$$\psi_1 = \alpha \circ \beta = \beta(\alpha) : x_i \to x_i + \varphi(y_i - \varphi^{-1}(x_i)) = \varphi(y_i),\ y_i \to y_i - \varphi^{-1}(x_i).$$

Compose ψ_1 with a linear automorphism $\pi : x_i \to y_i;\ y_i \to x_i$. We get: $\psi_2 = \psi_1 \circ \pi = \pi(\psi_1) : x_i \to \varphi(x_i),\ y_i \to x_i - \varphi^{-1}(y_i)$.

Finally, compose ψ_2 with $\tau : x_i \to x_i,\ y_i \to y_i + \varphi(x_i)$ to get

$$\psi = \tau(\psi_2) : x_i \to \varphi(x_i),\ y_i \to x_i - \varphi^{-1}(y_i + \varphi(x_i)) = -\varphi^{-1}(y_i).$$

This completes the proof. $\square$

Remark 2.1. The same argument establishes a somewhat more general result: if $p_i, q_i \in K[x_1, ..., x_n],\ 1 \le i \le m$, and $\varphi(p_i) = q_i,\ 1 \le i \le m$, for some automorphism φ of $K[x_1, ..., x_n]$, then $\psi(p_i) = q_i,\ 1 \le i \le m$, for some tame automorphism ψ of $K[x_1, ..., x_{2n}]$. In particular, every *coordinate tuple* of polynomials in $K[x_1, ..., x_n]$ is stably tame in the sense that it is part of a tame automorphism of $K[x_1, ..., x_{2n}]$.

Remark 2.2. The same argument can be used, in fact, for many type of free algebras, thus establishing the same result, in particular, for a free associative algebra $K\langle x_1, ..., x_n\rangle$. This also implies that every coordinate of $K[x_1, ..., x_n]$ can be *lifted* to a primitive element (this is what coordinates are called in non-commutative setting) of $K\langle x_1, ..., x_{2n}\rangle$.

Proof of Proposition 1.4. Let $\varphi(p) = q$ for some automorphism φ of $\mathbf{C}[x_1, ..., x_N],\ N > n$. Then all x_i with $i > n$ have to cancel out in $\varphi(p)$. However, because of the presence of the "dominating" monomial $x_1^{M_1} \cdot ... \cdot x_n^{M_n}$ in p, if there is some x_i with $i > n$ in some of the $\varphi(x_j),\ 1 \le j \le n$, then this x_i will not cancel out in $\varphi(p)$. Therefore, none of the $\varphi(x_j),\ 1 \le j \le n$, depends on any $x_i,\ i > n$, in which case the restriction of φ to $\mathbf{C}[x_1, ..., x_n]$ must be an automorphism of $\mathbf{C}[x_1, ..., x_n]$. $\square$

Remark 2.3. As we have mentioned in the Introduction, even if a given polynomial p does not satisfy the condition of Proposition 1.4, there might be an automorphic image of p that does, and then the result will hold for p as well. Here we note that, in the case where p is a two-variable polynomial over $\mathbf{C}$, it is algorithmically possible to find out whether some automorphic image of p has the form required by Proposition 1.4. Indeed, Wightwick [17] showed that, if some automorphic image of p has lower degree than p does, then there is a single elementary automorphism that reduces the degree of p. At the same time, it is easy to see that no elementary automorphism can reduce the degree of a polynomial that satisfies the condition of Proposition 1.4. Thus, given a polynomial p, we keep applying elementary automorphisms that reduce the degree, until the degree becomes irreducible. Then we check if there is a linear or elementary automorphism preserving the degree of the obtained polynomial, that produces a polynomial in the form required by Proposition 1.4. For more details, we refer to [17].

Proof of Theorem 1.5. Let $\varphi(p) = q$ for some automorphism φ of $\mathbf{C}[x_1, ..., x_N]$, $N > n$. It will be sufficient to prove that there is an injective mapping ψ_φ of $\mathbf{C}[x_1, ..., x_n]$ such that $\psi_\varphi(p) = q$. Indeed, if this is the case, then, similarly, there is an injective mapping $\psi_{\varphi^{-1}}$ of $\mathbf{C}[x_1, ..., x_n]$ such that $\psi_{\varphi^{-1}}(q) = p$. Then $\psi_{\varphi^{-1}}\psi_\varphi(p) = p$, and $\psi_{\varphi^{-1}}\psi_\varphi$ is injective. The result follows.

We also may assume, by our Theorem 1.2, that the automorphism φ is tame. Thus, the proof of the theorem will be complete if we establish the following

Proposition 2.4. Let φ be a tame automorphism of $\mathbf{C}[x_1, ..., x_{n+k}]$, $n, k \geq 1$, and let $\varphi_{q_1,...,q_k}$ be the restriction to $\mathbf{C}[x_{i_1}, ..., x_{i_n}]$ of the homomorphism obtained by replacing some k variables $x_{j_1}, ..., x_{j_k}$ with polynomials q_j, that depend on other variables $x_{i_1}, ..., x_{i_n}$ only, in every $\varphi(x_{i_j})$, $1 \leq j \leq n$. Then, for some polynomials q_j, the mapping $\varphi_{q_1,...,q_k}$ is injective on $\mathbf{C}[x_{i_1}, ..., x_{i_n}]$.

We are going to give a proof here for $k = 1$ since in the general case, the proof goes along exactly the same lines, but the notation becomes intractable. Also for notational convenience, we will assume that φ_q is the restriction to $\mathbf{C}[x_1, ..., x_n]$ of the homomorphism φ. Thus, we are going to give a proof of the following

Proposition 2.4.1. Let φ be a tame automorphism of $\mathbf{C}[x_1, ..., x_{n+1}]$, $n \geq 1$, and let φ_q be the restriction of φ to $\mathbf{C}[x_1, ..., x_n]$ obtained by replacing x_{n+1} with a polynomial $q = q(x_1, ..., x_n)$ in every $\varphi(x_i)$, $1 \leq i \leq n$. Then, for some polynomial q, the mapping φ_q is injective on $\mathbf{C}[x_1, ..., x_n]$.

It will be convenient to single out one (obvious) preliminary statement:

Lemma 2.5. In the notation of Proposition 2.4.1, let the mapping φ_q be injective on $\mathbf{C}[x_1, ..., x_n]$. Let $\varphi_q : x_i \to h_i$, $1 \leq i \leq n$. Then, for any $m \geq 0$ and j, $1 \leq i \leq n$, the mapping $\varphi_{q'}$ is injective on $\mathbf{C}[x_1, ..., x_n]$, too, where $q' = q \cdot h_j^m$.

Proof of Lemma 2.5. This statement says that if polynomials $h_1, h_2, ..., h_n, q$ are algebraically independent, then polynomials $h_1, h_2, ..., h_n, q \cdot h_j^m$ are algebraically independent, too. This is fairly obvious from considering the quotient fields $\mathbf{C}(h_1, ..., h_n, q)$ and $\mathbf{C}(h_1, ..., h_n, q \cdot h_j^m)$. $\square$

The reason why we need this flexibility in choosing the polynomial q, will be clear below.

Proof of Proposition 2.4.1. We use induction on the number of elementary and linear automorphisms in a decomposition of φ. Thus, we assume that for some particular φ, we have found a polynomial q such that φ_q is injective on $\mathbf{C}[x_1, ..., x_n]$, and now we want to find such a q for a composition of φ with an elementary or linear automorphism. A linear automorphism, in its turn, is a product of elementary automorphisms, permutations of variables, and multiplications (of some of the variables) by non-zero constants. Composing φ with an automorphism of one of the latter two kinds does not present any difficulty. Thus, we may assume that we are composing φ with an elementary automorphism, call it ϵ. Here we have to consider two cases:

(i) ϵ fixes all variables except some x_i with $1 \le i \le n$. We may as well assume that $i = 1$. Let $\varphi : x_i \to g_i,\ 1 \le i \le n+1$, and let $\epsilon : x_1 \to x_1 + f(x_2, ..., x_{n+1}); x_i \to x_i,\ 2 \le i \le n+1$. We claim that the mapping $(\epsilon(\varphi))_{q'}$ is injective on $\mathbf{C}[x_1, ..., x_n]$ for some q' of the form $q \cdot g_j^m$.

By way of contradiction, assume that for some polynomial p, one has

$$p(g_1(x_1 + f(x_2, ..., x_n, q'),\ x_2, ..., x_n, q'), ..., g_n(x_1 + f(x_2, ..., x_n, q'),\ x_2, ..., x_n, q')) = 0.$$

On the other hand, by the inductive assumption, we must have

$$p(g_1(x_1, x_2, ..., x_n, q'), ..., g_n(x_1, x_2, ..., x_n, q')) \neq 0.$$

The former equality is obtained from the latter inequality by applying the following mapping of the algebra $\mathbf{C}[x_1, ..., x_n]$: $x_1 \to x_1 + f(x_2, ..., x_n, q'),\ x_i \to x_i,\ 2 \le i \le n$. This mapping is injective since x_1 cannot cancel out in $x_1 + f(x_2, ..., x_n, q')$ – this is where we use the flexibility in choosing the polynomial q' provided by Lemma 2.5. Upon multiplying q by an appropriate g_j^m, we make sure that either $f(x_2, ..., x_n, q')$ has no x_1 whatsoever, or it has x_1 in a monomial of degree at least 2.

Thus, we have got a contradiction, which completes the proof in this case.

(ii) ϵ fixes all variables except x_{n+1}. Let $\epsilon : x_{n+1} \to x_{n+1} + f(x_1, ..., x_n); x_i \to x_i,\ 1 \le i \le n$. In this case, from the inductive assumption, it is obvious that, if we take $q' = q - f$, then the mapping $(\epsilon(\varphi))_{q'}$ is going to be injective on $\mathbf{C}[x_1, ..., x_n]$. $\Box$

We conclude this section with the

Proof of Theorem 1.6. First we give a proof for $k = 1,\ n = 3$ to simplify the notation; then we explain how this proof generalizes easily to arbitrary $k \ge 1$ and $n \ge k + 2$.

We are going to show that $U = \{x(1+xy+z^2) = 1\}$ is isomorphic to $W = \{xy = 1\}$ in $\mathbf{C}^3$. The latter variety obviously is of the form $V \times \mathbf{C}$, whereas the former is not.

As in [16], it will be technically more convenient to write algebras of residue classes as "algebras with relations", i.e., for example, instead of $\mathbf{C}[x_1, ..., x_n]/\langle p(x_1, ..., x_n)\rangle$ we shall write $\langle x_1, ..., x_n \mid p(x_1, ..., x_n) = 0\rangle$.

Now we get the following chain of "elementary" isomorphisms:

$\langle x, y, z \mid x(1 + xy + z^2) = 1\rangle = \langle x, y, z \mid x(1 + xy + z^2) = 1,\ xy(1 + xy + z^2) = y\rangle =$
$\langle x, y, z \mid x + x^2y + xz^2 = 1,\ xy + x^2y^2 + xyz^2 = y\rangle\ \cong$
$\langle x, y, z, u \mid x + x^2y + xz^2 = 1,\ xy + x^2y^2 + xyz^2 = y,\ u = xy\rangle\ \cong$
$\langle x, y, z, u \mid x + xu + xz^2 = 1,\ y = u + u^2 + uz^2,\ u = xy\rangle\ \cong$
$\langle x, z, u \mid x + xu + xz^2 = 1,\ u = x(u + u^2 + uz^2)\rangle\ \cong$
$\langle x, y, z \mid x + xy + xz^2 = 1,\ y = x(y + y^2 + yz^2)\rangle = \langle x, y, z \mid x + xy + xz^2 = 1\rangle =$
$\langle x, y, z \mid x(1 + y + z^2) = 1\rangle\ \cong \langle x, y, z \mid xy = 1\rangle.$

Thus, $U = \{x(1 + xy + z^2) = 1\}$ is isomorphic to $W = \{xy = 1\}$ in $\mathbf{C}^3$. Upon replacing U with $U_m = \{x(1 + xy + z_1^2 + ... + z_m^2) = 1\}$, where $z_1, ..., z_m$ are variables, and using the same chain of "elementary" isomorphisms, we get examples, for any $m \geq 1$, of non-cylinders U_m isomorphic to $W = \{xy = 1\}$ in $\mathbf{C}^{m+2}$. Then, upon replacing U_m with $U_{m,r} = \{x \cdot t_1 \cdot ... \cdot t_r \cdot (1 + xy + z_1^2 + ... + z_m^2) = 1\}$ and W with $W_r = \{xy \cdot t_1 \cdot ... \cdot t_r = 1\}$, where $t_1, ..., t_r$ are variables, and using the same chain of isomorphisms, we get isomorphism of $U_{m,r}$ to W_r in $\mathbf{C}^{m+2+r}$ for any $m \geq 1,\ r \geq 0$.

To show that, for instance, the polynomial $x(1+xy+z_1^2+...+z_m^2)$ is not equivalent to any polynomial in less than $(m+2)$ variables, it is clearly sufficient to show the same for the polynomial $q = 1 + xy + z_1^2 + ... + z_m^2$. The gradient of the latter polynomial is $(y,\ x,\ 2z_1, ...,\ 2z_m)$. Now the "chain rule" for partial derivatives shows that, if we apply an automorphism ϕ to q, then the gradient of $\phi(q)$ must contain all the variables $x, y, z_1, ..., z_m$. Then the same is true for the polynomial $\phi(q)$ itself.

Finally, to get the equivalence claimed in the statement of Theorem 1.6 out of the above isomorphism, we just recall (see e.g. [16, Corollary 1.2]) that if two algebraic varieties $V(p_1, ..., p_m)$ and $V(q_1, ..., q_k)$ in $\mathbf{C}^n$ are isomorphic, then the varieties $V(p_1, ..., p_m, x_{n+1}, ..., x_{2n})$ and $V(q_1, ..., q_k, x_{n+1}, ..., x_{2n})$ in $\mathbf{C}^{2n}$ are equivalent under a (tame) automorphism of $\mathbf{C}^{2n}$. $\Box$

3. Around Danielewski's example

In Danielewski's example [6] of two isomorphic cylindrical hypersurfaces $V(p) \times \mathbf{C}$ and $V(q) \times \mathbf{C}$ with non-isomorphic bases $V(p)$ and $V(q)$ in $\mathbf{C}^3$, we have $p = p(x, y, z) = xy - z^2 + 1;\ q = q(x, y, z) = x^2y - z^2 + 1$. We start this section by giving an explicit algebraic isomorphism (due to P.Russell) for cylinders in Danielewski's example. We are grateful to P.Russell for kindly permitting us to use his observation here.

Remark 3.1. (Russell, unpublished). Let $p = p(x_1, y_1, z_1) = x_1y_1 - z_1^2 + 1;\ q = q(x_2, y_2, z_2) = x_2^2y_2 - z_2^2 + 1$, and let K be an arbitrary field of characteristic 0. The following mapping φ from $K[x_1, y_1, z_1, u]$ to $K[x_2, y_2, z_2, v]$ induces an isomorphism between algebras of residue classes $K[x_1, y_1, z_1, u]/\langle p\rangle$ and $K[x_2, y_2, z_2, v]/\langle q\rangle$:

$\varphi : x_1 \to x_2;\ y_1 \to x_2v^2 + 2z_2v + x_2y_2;\ z_1 \to x_2v + z_2;$
$u \to x_2v^3 + 3z_2v^2 + 3x_2y_2v + y_2z_2.$

It is easy to check that φ induces a homomorphism between $K[x_1, y_1, z_1, u]/\langle p\rangle$ and $K[x_2, y_2, z_2, v]/\langle q\rangle$.

We are going to show that φ is *onto*. We already have $x_2 \pmod{\langle q\rangle}$ in the image; a straightforward computation shows that $\varphi(y_1 z_1) - x_2 \cdot \varphi(u) = -2x_2^2 y_2 v + 2z_2^2 v = 2v(z_2^2 - x_2^2 y_2) = 2v \pmod{\langle q\rangle}$. Thus, we have $v \pmod{\langle q\rangle}$, and therefore also $z_2 \pmod{\langle q\rangle}$ in the image. Now inspection of $\varphi(y_1)$ shows that we have $x_2 y_2 \pmod{\langle q\rangle}$, hence also $x_2^2 y_2^2 \pmod{\langle q\rangle}$ in the image. From $\varphi(u)$ we now see that we have $y_2 z_2 \pmod{\langle q\rangle}$, hence also $y_2 z_2^2 \pmod{\langle q\rangle}$ in the image. This finally gives $y_2 z_2^2 - x_2^2 y_2^2 = y_2 \pmod{\langle q\rangle}$ in the image.

Thus, φ is *onto*. If φ were not one-to-one, then the algebra $K[x_2, y_2, z_2, v]/\langle q\rangle$ would be isomorphic to an algebra $K[x_1, y_1, z_1, u]/J$, where J is an ideal that properly contains $\langle p\rangle$. This isomorphism then implies that J can be generated (as an ideal) by a single polynomial. Since p is irreducible, this generating polynomial has to be p.

Therefore, φ is one-to-one, hence an isomorphism.

Now we get to

Proof of Proposition 1.7. First of all, the varieties $V_1 = V(p(x, y, z), t, u, v, w)$ and $V_2 = V(q(x, y, z), t, u, v, w)$ in $\mathbf{C}^7$ are not isomorphic since if they were, then $V(p(x, y, z))$ and $V(q(x, y, z))$ would be isomorphic in $\mathbf{C}^3$ which is known not to be the case.

On the other hand, we know that the algebras of residue classes $\mathbf{C}[x, y, z, t]/\langle p\rangle$ and $\mathbf{C}[x, y, z, t]/\langle q\rangle$ are isomorphic. This implies, by [16, Corollary 1.2], that the varieties $V(p(x_1, y_1, z_1), u, x_2, y_2, z_2, v)$ and $V(q(x_1, y_1, z_1), u, x_2, y_2, z_2, v)$ are equivalent under an automorphism of $\mathbf{C}^8$. We note that the latter equivalence also follows from a result of Asanuma [3]. $\square$

Finally, we make one more observation inspired by Danielewski's example:

Proposition 3.2. Let $y, x_1, ..., x_m,\ m \geq 1$, be variables, and $p = p(x_1, ..., x_m)$ an arbitrary polynomial. Then hypersurfaces $y \cdot p = 1$ and $y \cdot p^k = 1$ are isomorphic in $\mathbf{C}^{m+1}$ for any $k \geq 1$.

Speaking somewhat informally, adding a polynomial $q(z_1, ..., z_k)$ in new variables z_i, to both polynomials in the statement of Proposition 3.2, can "spoil" this simple but delicate isomorphism (in the course of the proof, we shall see why), but, apparently, sometimes this isomorphism "survives" in higher dimensions, hence the isomorphism of cylinders. Understanding when exactly this happens can be the key to constructing a counterexample to (some of) the conjectures mentioned in this paper.

Proof of Proposition 3.2. Here we get the following chain of "elementary" isomorphisms:

$\langle y, x_1, ..., x_m \mid y \cdot p = 1\rangle = \langle y, x_1, ..., x_m \mid y \cdot p = 1, y^2 \cdot p = y\rangle\ \cong$
$\langle y, x_1, ..., x_m, u \mid y \cdot p = 1, y^2 \cdot p = y, u = y^2\rangle\ \cong$
$\langle y, x_1, ..., x_m, u \mid u \cdot p^2 = 1, u \cdot p = y, u = u^2 p^2\rangle\ \cong$

$\langle x_1, ..., x_m, u \mid u \cdot p^2 = 1, u = u^2p^2\rangle = \langle x_1, ..., x_m, u \mid u \cdot p^2 = 1\rangle \cong$
$\langle y, x_1, ..., x_m \mid y \cdot p^2 = 1\rangle$.

Upon applying the same trick $(k-1)$ times, we will have $y \cdot p = 1$ isomorphic to $y \cdot p^k = 1$. $\square$

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References

[1] S. S. Abhyankar, P. Eakin and W. J. Heinzer, *On the uniqueness of the coefficient ring in a polynomial ring*, J. Algebra **23** (1972), 310–342.

[2] S. S. Abhyankar, T.-T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1975), 148–166.

[3] T. Asanuma, *Non-linearizable algebraic k^*-actions on affine spaces*, Invent. Math. **138** (1999), 281–306.

[4] T. Bandman and L. Makar-Limanov, *Affine surfaces with isomorphic cylinders*, preprint.

[5] A. Campbell, J.-T. Yu, *Two dimensional coordinate polynomials and dominant maps*, Comm. Algebra **28** (2000), 2297–2301.

[6] W. Danielewski, *On the cancellation problem and automorphism groups of affine algebraic varieties*, preprint, Warsaw 1989.

[7] A. van den Essen, V. Shpilrain, *Some combinatorial questions about polynomial mappings*, J. Pure Appl. Algebra **119** (1997), 47–52.

[8] T. Fujita, *On Zariski problem*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), 106–110.

[9] O. Hadas, *On the vertices of Newton polytopes associated with an automorphism of the ring of polynomials.* J. Pure Appl. Algebra **76** (1991), 81–86.

[10] Z. Jelonek, *Test polynomials*, J. Pure Appl. Algebra **147** (2000), 125–132.

[11] S. Kaliman, *Polynomials with general $\mathbf{C}^2$-fibers are variables*, preprint.

[12] S. Kaliman, M. Zaidenberg, *Families of affine planes: the existence of a cylinder*, preprint.

[13] H. Kraft, *Challenging problems on affine n-space*, Astérisque **237** (1996), 295–317.

[14] M. Miyanishi, *Some remarks on polynomial rings*, Osaka J. Math. **10** (1973), 617–624.

[15] M. Miyanishi, T. Sugie, *Affine surfaces containing cylinderlike open sets*, J. Math. Kyoto Univ. **20** (1980), 11–42.

[16] V. Shpilrain and J.-T. Yu, *Embeddings of hypersurfaces in affine spaces*, J. Algebra **239** (2001), 161–173.

[17] P. Wightwick, *Equivalence of polynomials under automorphisms of* $\mathbb{C}^2$, J. Pure Appl. Algebra **157** (2001), 341–367.

Department of Mathematics, The City College of New York, New York, NY 10031
e-mail address: shpil@groups.sci.ccny.cuny.edu
http://zebra.sci.ccny.cuny.edu/web/shpil

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
e-mail address: yujt@hkusua.hku.hk
http://hkumath.hku.hk/˜ jtyu