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TECHNICAL NOTE

Pore pressure coefficient for soil and rock and its relation to compressional wave velocity

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KEYWORDS: compressibility; dynamics; laboratory tests; partial saturation; pore pressures

INTRODUCTION

In laboratory triaxial tests it has been common to check the specimen's degree of saturation by determining the pore pressure coefficient, B. The method is known as the B-value test and basically originated from Skempton's work on evaluation of the pore pressures in clays in the design of earth dams, in which he derived the well-known expression for B (Skempton, 1954):

$$B = \frac{\Delta u}{\Delta \sigma} = \frac{1}{1 + n(K_b/K_f)} \tag{1}$$

Here $\Delta\sigma$ is a small increase in all-round or confining pressure applied to an element of soil in an earth dam or to the sample in the tests, and Δu is the resulting change in pore pressure measured under undrained conditions. The porosity of soil is denoted in equation (1) by n, K_b denotes the bulk modulus of the soil skeleton, and K_f the bulk modulus of the mixture of pore water and air.

Since the compressibility of water is negligible compared with that of the soil skeleton, equation (1) indicates that $B \approx 1$ for saturated soils, whereas for partially saturated soils $0 < B < 1$, with a typical range of 0·1 to 0·5 at the optimum water content.

In deriving equation (1) the solid grains were assumed as incompressible, which has been a widely accepted assumption in soil mechanics, but may not be applicable in rock mechanics. For soils and rocks with compressible particles, Bishop (1973) showed later on that the pore pressure coefficient was expressed by

$$B = \frac{\Delta u}{\Delta \sigma} = \frac{1}{1 + n(1/K_w - 1/K_s)/(1/K_b - 1/K_s)} \tag{2}$$

in which K_s is the bulk modulus of solid particles.

Recently, Kokusho (2000) presented the following expression for B using Biot's formulation given by Zienkiewicz & Bettess (1982):

$$B = \frac{1/K_b - 1/K_s}{1/K_b + n/K_f - n/K_s} \tag{3}$$

It is not difficult to note that equation (3) differs from Bishop's expression when the latter is written in an alternative form as

$$B = \frac{1/K_b - 1/K_s}{1/K_b + n/K_f - (1+n)/K_s} \tag{4}$$

Since Biot's theory describes the behaviour of porous media such as soils and rocks more rigorously, a concern naturally arises over the correctness of either Bishop's or Kokusho's expression, as well as the cause leading to the difference between them. It is the purpose of the present study to clarify this issue, both analytically and numerically.

Besides the pore pressure parameter B, a new indicator of saturation for soils—the velocity of compressional waves (i.e. P-waves), V_p—has been advocated, and has received increasing attention (Yang & Sato, 1998, 2000a; Kokusho, 2000; Ishihara *et al.*, 2001). Compared with the pore pressure parameter B, the P-wave velocity can be measured conveniently in the field and thus has an advantage in practical applications. The effectiveness of the use of P-wave velocity in identifying in-situ partially saturated zones has been well demonstrated by a borehole array site (Yang & Sato, 2000a). More recently, an application of P-wave velocity in interpreting laboratory test data for the cyclic strength of partially saturated sands has been proposed by Yang (2002), who showed a useful relation between B and V_p:

$$V_p = \left[\frac{4\mu/3 + K_b/(1-B)}{\rho}\right]^{1/2} \tag{5}$$

in which μ (or G) is the shear modulus of the soil skeleton and ρ is total mass density.

In deriving equation (5) the compressibility of solid grains was not of primary concern and therefore not included in the expressions for either B or V_p. In this study a general relation between B and V_p will be established within the framework of Biot's theory, and the effect of particle compressibility will be identified numerically. This is a second purpose of the present study.

DERIVATION OF PORE PRESSURE PARAMETER

The analysis is based on classical Biot's theory (Biot, 1956, 1962), which models the complex interactions between the solid and fluid parts using the macroscopic laws of mechanics. In its simplest form the analysis requires the following assumptions:

(*a*) The porous medium is statistically isotropic in such a way that for any cross-section the same ratio of the fluid area to the solid area applies.
(*b*) The void space of the porous medium is interconnecting, and the sealed pore space is part of the solid.
(*c*) Both the solid skeleton and the solid material forming it are elastic and isotropic.
(*d*) The solid grains, solid skeleton and pore fluid are compressible.
(*e*) The pore fluid is viscous, and Darcy's law governs its flow.

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It is customary to denote the displacement of the solid part by u_i and the displacement of fluid relative to the solid by w_i. Thus the field equations accounting for both inertial and viscous interactions between the two parts in the porous medium can be given as

$$\mu u_{i,jj} + (\lambda + \alpha^2 M + \mu) e_{,i} - \alpha M \zeta_{,i} = \rho \ddot{u}_i + \rho_f \ddot{w}_i \tag{6}$$

$$\alpha M e_{,i} - M \zeta_{,i} = \rho_f \ddot{u}_i + \frac{\rho_f}{n} \ddot{w}_i + \frac{\eta}{k'} \dot{w}_i \tag{7}$$

where $e = u_{j,j}$ and $\zeta = -w_{j,j}$; ρ_s is the mass density of solid grains, ρ_f is the mass density of fluid, and $\rho = (1 - n)\rho_s + n\rho_f$; λ is the Lamé constant of the solid frame; and α and M are parameters accounting for the compressibilities of solid and fluid constituents, and given by

$$\alpha = 1 - \frac{K_b}{K_s} \tag{8}$$

$$M = \frac{K_s^2}{K_s\left[1 + n\left(\frac{K_s}{K_f} - 1\right)\right] - K_b} \tag{9}$$

Obviously, for incompressible solid grains $\alpha = 1$ and $1/M = n/K_f$.

Note that k' in the field equations differs from the permeability coefficient k (m/s) that is used in soil mechanics. They are related by

$$k' = k \frac{\eta}{\rho_f g} \tag{10}$$

in which η is fluid viscosity and g is the gravitation acceleration at which the permeability is measured.

The field equations (6) and (7) have been obtained by entering into the equilibrium conditions the following relationships between stress, pore pressure, and strain:

$$\sigma_{ij} = \lambda e \delta_{ij} + 2\mu \varepsilon_{ij} - \alpha \delta_{ij} p_f \tag{11}$$

$$p_f = M\zeta - \alpha M e \tag{12}$$

where σ_{ij} is the stress tensor, $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$ is the strain tensor, p_f is pore pressure, and δ_{ij} is the Kronecker delta.

Here a concept of homogenisation has been introduced that assumes that the mixture of pore water and air can approximately be treated as an equivalent homogeneous pore fluid completely filling the voids with a single pore pressure. Note that equation (11) implies the effective stress law in the following form:

$$\sigma_{ij} = \sigma'_{ij} - \delta_{ij} \alpha p_f \tag{13}$$

in which σ'_{ij} is the effective stress tensor. Clearly, equation (13) can be reduced to the well-known effective stress equation introduced by Terzaghi if the compressibility of solid grains is ignored.

Equation (12) can readily be rewritten as

$$w_{i,i} = \alpha u_{i,i} + \frac{p_f}{M} \tag{14}$$

By specifying the undrained conditions that there is no flow of pore fluid, one arrives at

$$\alpha u_{i,i} + \frac{p_f}{M} = 0 \tag{15}$$

or

$$\left(1 - \frac{K_b}{K_s}\right)\left(\frac{\sigma'_{ii}}{3K_b}\right) + \left(\frac{\alpha}{K_s} + \frac{n}{K_f} - \frac{n}{K_s}\right) p_f = 0 \tag{16}$$

Introducing the effective stress law expressed in equation (13) into the above equation leads to

$$\left(\frac{1}{K_b} - \frac{1}{K_s}\right)\left(\frac{\sigma_{ii}}{3}\right) + \left(\frac{n}{K_f} - \frac{n}{K_s} + \frac{\alpha}{K_b}\right) p_f = 0 \tag{17}$$

from which the pore pressure coefficient can readily be derived as

$$B = -\frac{3p_f}{\sigma_{ii}} = \frac{1/K_b - 1/K_s}{(1/K_b - 1/K_s) + n(1/K_f - 1/K_s)} \tag{18}$$

Note that the negative sign is required by the sign convention embedded in equation (11).

Now, it is of interest to observe that the expression derived above is the same as that by Bishop (1973), implying that the expression due to Kokusho (2000) is in error. It is found that a major cause of the error was incorrect use of the effective stress law: the effective stress equation by Terzaghi rather than the general form in equation (13) was used, and a consistency was lost.

Furthermore, by introducing the commonly used expression for the compressibility of pore fluid (Yang & Sato, 1998, 2000a),

$$\frac{1}{K_f} = \frac{1}{K_w} + \frac{1 - S_r}{p_a} \tag{19}$$

where S_r is the degree of saturation, K_w is the bulk modulus of pore water, and p_a is the absolute fluid pressure, equation (18) can be written as a function of saturation:

$$B = \frac{1/K_b - 1/K_s}{1/K_b - (1 + n)/K_s + n/K_w + n(1 - S_r)/p_a} \tag{20}$$

A GENERAL EXPRESSION FOR COMPRESSIONAL WAVE VELOCITY

In the same context of Biot's theory, the field equations (6) and (7) can be solved with the aid of potential functions, yielding two compressional waves and one shear wave. In the low-frequency range to which the frequencies involved in most soil dynamics problems belong, only the fast compressional wave exists, with its velocity being independent of frequency, while the slow one is a diffusion process owing to the viscous coupling between the solid and fluid phases. The interested reader may refer to Yang & Sato (2000b), where more details about the two compressional waves are presented, with a special reference to the vibration of a soil column.

The velocity of the fast compressional wave at low frequencies is given by (Yang & Sato, 2000a)

$$V_p = \left(\frac{\lambda + 2\mu + \alpha^2 M}{\rho}\right)^{1/2} \tag{21}$$

It can be verified that for a limiting case of incompressible grains the above expression takes the form as given in Yang (2002):

$$V_p = \left(\frac{\lambda + 2\mu + K_f/n}{\rho}\right)^{1/2} \tag{22}$$

Similarly, the general expression in equation (21) can be written as a function of the degree of saturation as

$$V_p = \left[\frac{K_b + 4\mu/3}{\rho} + \frac{\alpha^2}{(\alpha - n)K_s + n/K_w + n(1 - S_r)/p_a}\frac{1}{\rho}\right]^{1/2} \tag{23}$$

For a fully saturated porous medium with incompressible

particles, equation (23) reduces to the following simple form:

$$V_p = \left(\frac{K_b + 4\mu/3 + K_w/n}{\rho} \right)^{1/2} \tag{24}$$

With equations (18) and (21) in hand a general relationship between B and V_p can be obtained:

$$V_p = \left[\frac{4\mu/3 + K_b/(1 - \alpha B)}{\rho} \right]^{1/2} \tag{25}$$

To the author's knowledge, equation (25) is the first proposal of the explicit relationship between B and V_p in which the particle compressibility is consistently accounted for in the framework of Biot's theory. It can be shown that in a limiting case of incompressible grains this general relation becomes the simple one given by Yang (2002).

An alternative form for the above expression can be given by introducing the shear wave velocity, V_s, an engineering property directly linked with the shear modulus of soil and rock:

$$B = \frac{1}{\alpha} \left[1 - \frac{3K_b}{\rho \left(3V_p^2 - 4V_s^2 \right)} \right] \tag{26}$$

EFFECT OF PARTICLE COMPRESSIBILITY

The influence of particle compressibility, skeleton compressibility and porosity on the value of B at full saturation is illustrated in Fig. 1(a), where comparison of the values given by Skempton's expression (equation (1)) and by equation (18) is included. Here the bulk modulus of solid particles is assumed as a typical value (i.e. $K_s = 36$ GPa), and the bulk modulus of the skeleton, K_b, is normalised by K_s. For purposes of comparison, the values calculated by equation (3) (Kokusho's expression) are presented together with those given by equation (18) in Fig. 1(b).

It is interesting to note from Fig. 1(a) that for porous media of high skeleton compressibility, say $K_b/K_s < 0{\cdot}1$, the difference between the values of B given by equations (1) and (18) is not significant. In other words, whether or not the particle compressibility is taken account of will not impact on the prediction of B. However, as the skeleton compressibility approaches the compressibility of solid grains, major errors will arise from the neglect of the particle compressibility. Indeed, a wide range of the skeleton compressibility has been reported for porous or fissured rocks, for which it is not uncommon that K_b/K_s may be of the order of 0·5.

Figure 1(a) also indicates that, at a specific large value of K_b/K_s, the difference caused by neglect of the particle compressibility is sensitive to the value of porosity. The difference will become much greater as the porosity decreases, with Skempton's expression giving a higher value of B.

Based on the general expression given in equation (20), the relation between the pore pressure ratio B and the degree of saturation S_r is plotted in Fig. 2 for various values of K_b/K_s, where Fig. 2(a) is for a porosity of 0·35 and Fig. 2(b) for a porosity of 0·05. In the calculation the absolute fluid pressure is taken as the atmospheric pressure. It is clear from Fig. 2 that at a specific degree of saturation the value of B is strongly dependent on the ratio of K_b/K_s. For example, for the porous medium with a porosity of 0·05, B at full saturation is about 0·999 for $K_b/K_s = 0{\cdot}001$, but only 0·565 for $K_b/K_s = 0{\cdot}5$. On the other hand, Fig. 2 implies that, for a specific value of B, the porous medium with a stiffer solid skeleton can achieve a higher degree of saturation.

Figure 3(a) illustrates the effects of the particle compressibility and of the porosity on the P-wave velocity in fully saturated porous media, where the solid lines are generated using equation (21) (i.e. the particle compressibility is taken into account) and the broken lines are generated using equation (22) (i.e. the particle compressibility is neglected). In computation the shear modulus, μ, is related to K_b using a

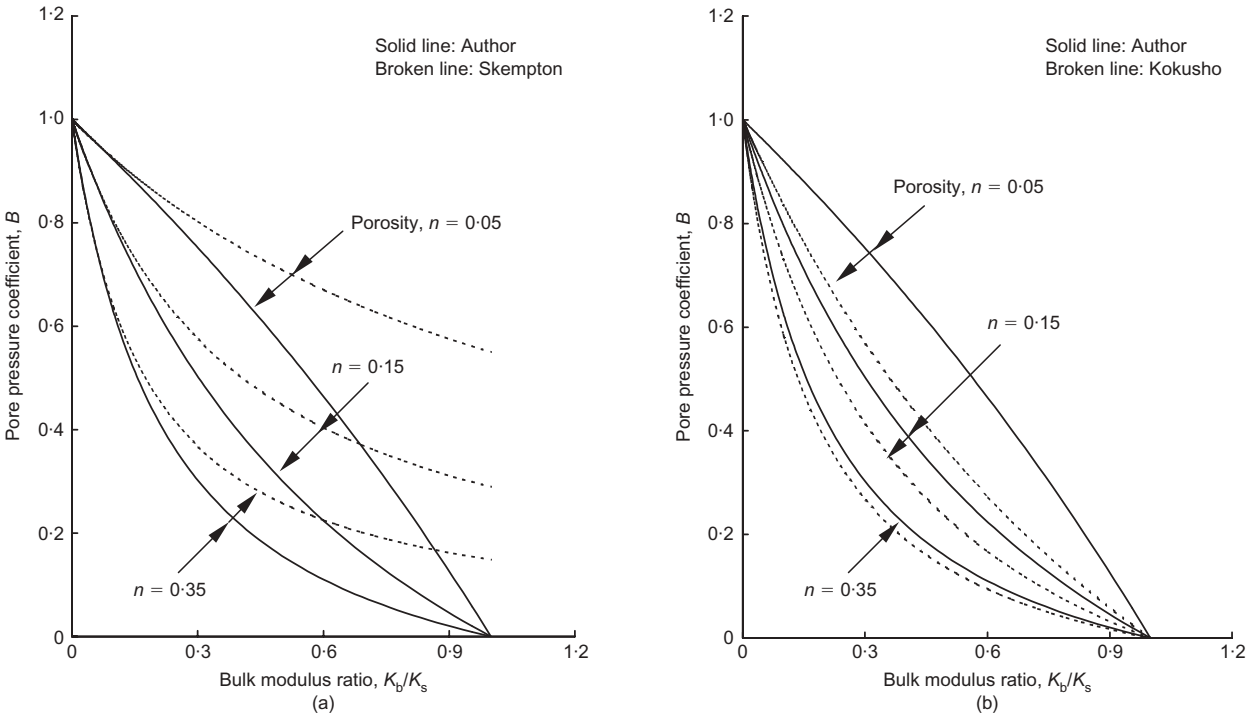

Fig. 1. Influence of particle compressibility on pore pressure coefficient

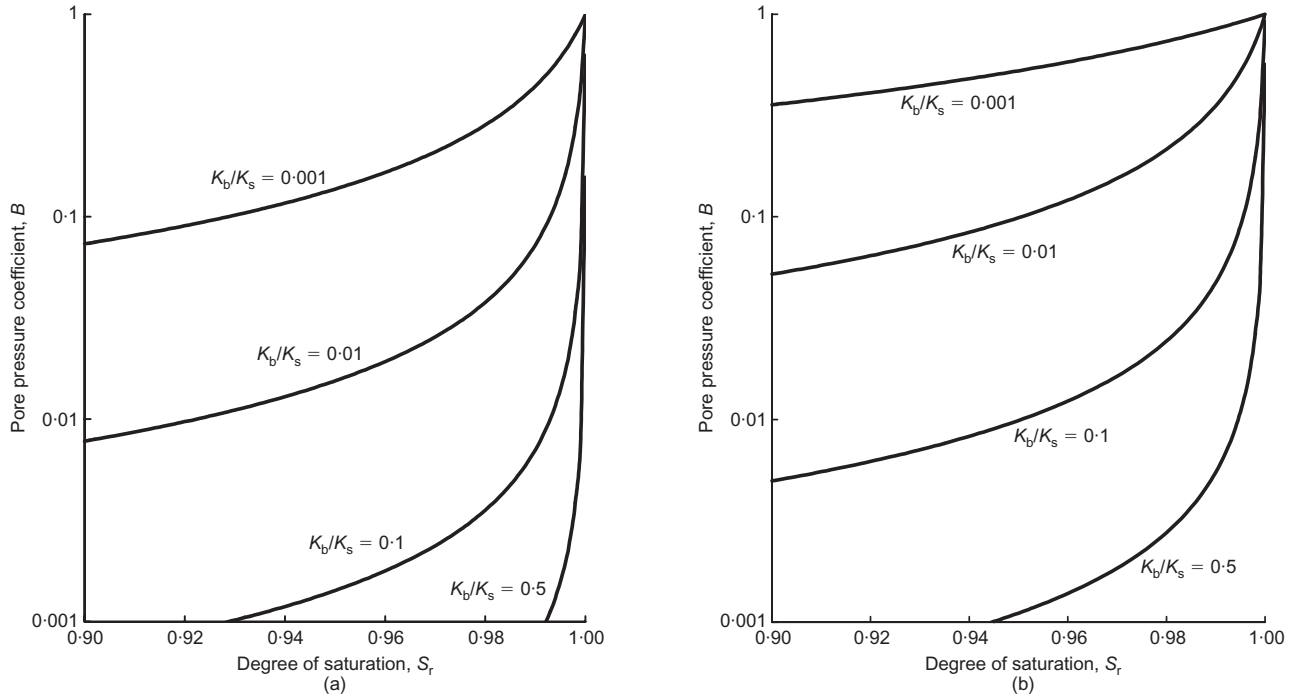

Fig. 2. Pore pressure coefficient as a function of degree of saturation: (a) porosity, $n = 0{\cdot}35$; (b) porosity, $n = 0{\cdot}05$

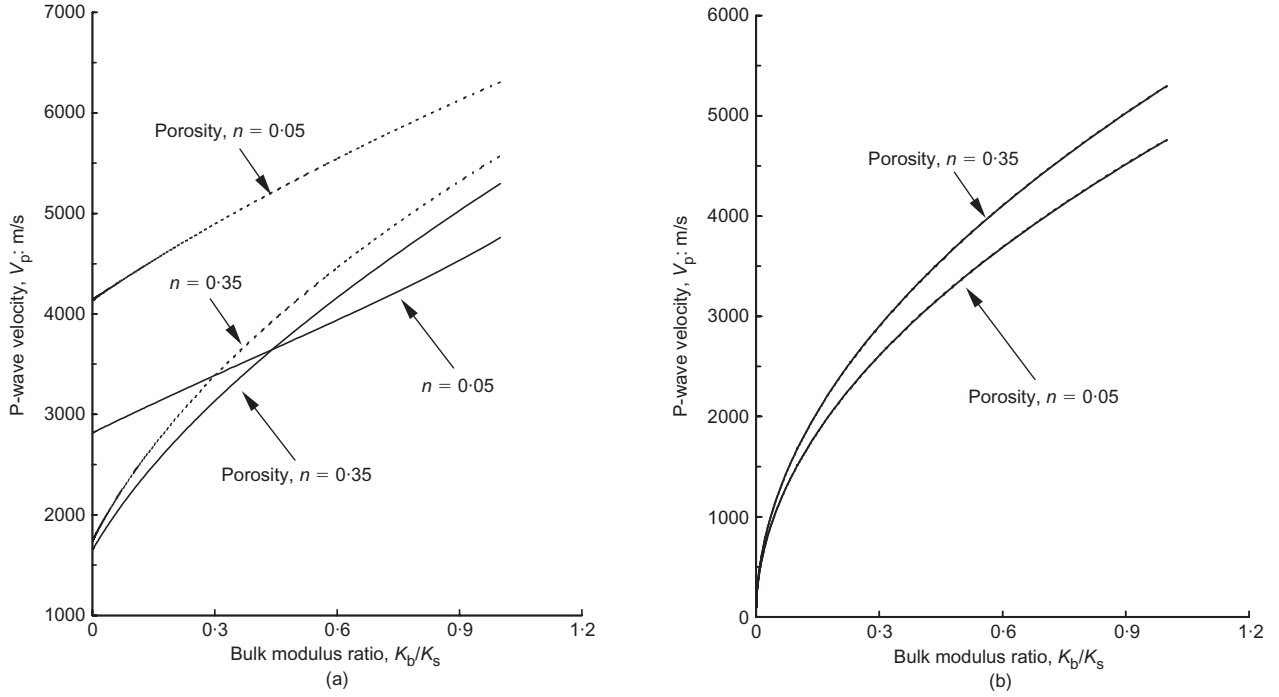

Fig. 3. Influence of particle compressibility on V_p (solid line, particle compressibility included; broken line, particle compressibility excluded): (a) $S_r = 100\%$; (b) $S_r = 95\%$

common value of Poisson's ratio, 0·3, and the specific gravity of solid grains is assumed as 2·65.

It is evident from Fig. 3(a) that for porous media with small porosity the difference between the values of V_p given by equations (21) and (22) is very significant and exists over an entire range of the ratio K_b/K_s. For porous media of high porosity, however, the difference between the predictions given by equations (21) and (22) becomes less significant, especially at low values of K_b/K_s.

In the case of partially saturated porous media with either low or high porosity, the deviation of the value of V_p due to neglect of the particle compressibility is too slight to distinguish, as shown in Fig. 3(b).

The derived explicit expression in equation (25) enables a quantitative identification of the effect of particle compressibility on the relationship between B and V_p, as shown in Fig. 4, where the thick lines are obtained with equation (25) and the thin lines are generated by equation (5), a limiting case

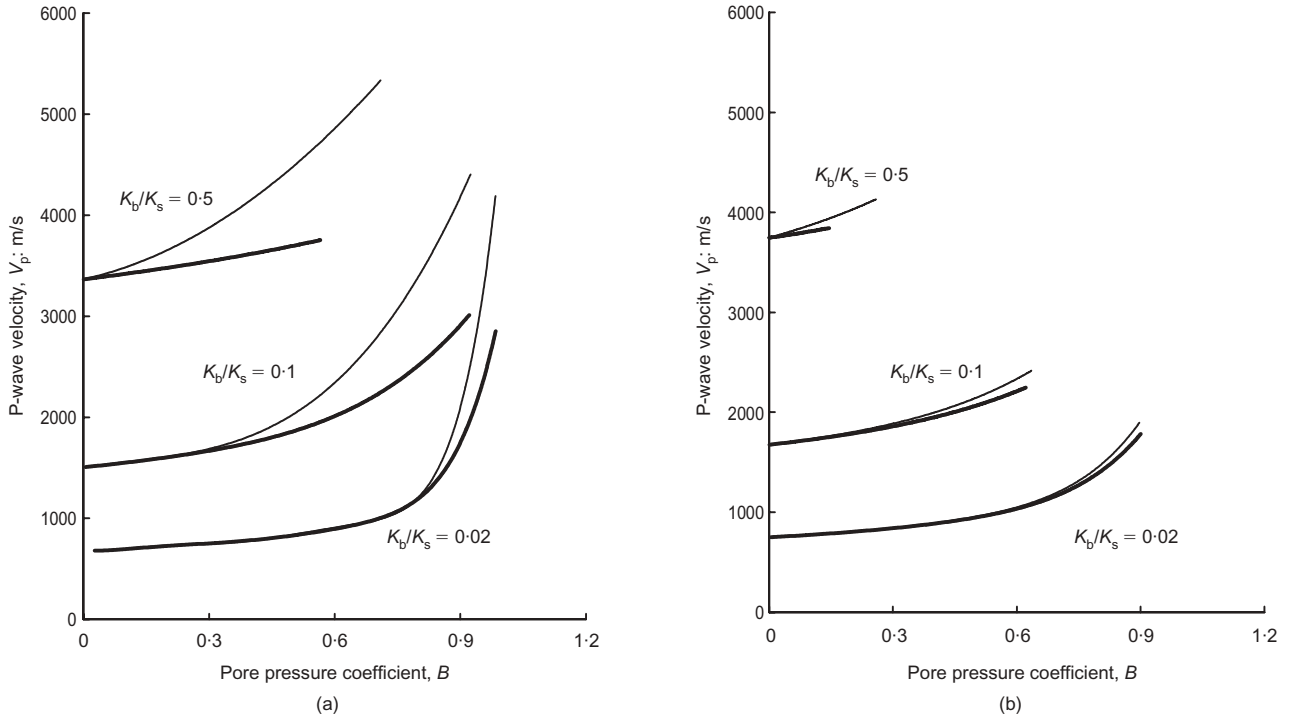

Fig. 4. Relation between B and V_p (thick line, particle compressibility included; thin line, particle compressibility excluded): (a) porosity, $n = 0{\cdot}05$; (b) porosity, $n = 0{\cdot}35$

Table 1. Values of B and V_p at full saturation for various ratios of K_b/K_s

Bulk modulus ratio, K_b/K_s	Porosity = 0·05				Porosity = 0·35			
	Particle compressibility excluded		Particle compressibility included		Particle compressibility excluded		Particle compressibility included	
	B	V_p: m/s	B	V_p: m/s	B	V_p: m/s	B	V_p: m/s
0·002	0·998	4145	0·998	2820	0·989	1758	0·989	1664
0·005	0·996	4153	0·996	2826	0·974	1781	0·974	1685
0·1	0·924	4405	0·921	3012	0·636	2416	0·622	2247
0·5	0·710	5335	0·566	3753	0·259	4131	0·145	3843
0·9	0·576	6126	0·126	4533	0·162	5318	0·017	5028

Note: The two shaded areas indicate respectively the representative cases for rock and soil.

of equation (25). In the same graph the influence of porosity and the skeleton compressibility is also illustrated.

Figure 4(a) indicates that, for porous media of low porosity, the larger the value of K_b/K_s, the more significant the deviation due to neglect of the particle compressibility. For instance, for a rock with $K_b/K_s = 0{\cdot}5$, the predicted value of B at full saturation is 0·566 by using the general expression in equation (18), and the corresponding V_p is around 3753 m/s. If the compressibility of solid particles is ignored, the predicted value of B becomes 0·710 and the corresponding V_p is as large as 5335 m/s (Table 1).

For porous media of high porosity, the difference caused by ignoring the particle compressibility will become less significant, as shown in Fig. 4(b). But caution should still be exercised, since the difference may still be appreciable at high values of K_b/K_s (Table 1).

CONCLUDING REMARKS

With the purpose of clarifying the difference between the expressions given by Bishop (1973) and by Kokusho (2000), the pore pressure parameter B was revisited in this study. It has been shown that Bishop's expression can also be derived in a rigorous way based on classical Biot's theory, and the error in the expression by Kokusho (2000) was caused mainly by the incorrect use of the effective stress equation. In the same context of Biot's theory, a first proposal of the explicit relationship between the pore pressure coefficient, B, and the compressional wave velocity, V_p, has been established. The relationship accounts for the compressibility of solid particles in a consistent way and is applicable to both soils and rocks.

The effects of skeleton compressibility, particle compressibility and porosity have been identified numerically with respect to B and its relation to V_p. The results indicate that major errors in the value of B will arise from neglect of the particle compressibility as the skeleton compressibility approaches the compressibility of solid grains, and meanwhile the amount of errors is sensitive to the value of porosity. For porous media of low porosity and low skeleton compressibility, such as rocks, the effect of particle compressibility may be very significant on the values of

both B and V_p at full saturation, whereas for porous media of high porosity and high skeleton compressibility, such as soils, neglect of the particle compressibility leads only to a slightly greater value of V_p but almost the same prediction for B.

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NOTATION

B pore pressure coefficient
g acceleration of gravity
K_b bulk modulus of solid skeleton
K_f bulk modulus of pore fluid
K_s bulk modulus of solid grains
K_w bulk modulus of pore water
k' permeability (m^2)
k permeability coefficient (m/s)
n porosity
p_f pore pressure
S_r degree of saturation
u_i displacement of solid phase
V_p compressional wave velocity
V_s shear wave velocity
w_i relative displacement of fluid phase
α, M parameters accounting for the compressibility of constituents
ε_{ij} strain tensor
η fluid viscosity
λ Lamé constant of solid skeleton
μ shear modulus of solid skeleton
ρ total density
ρ_f density of fluid
ρ_s density of solid grains
σ_{ij} total stress
σ'_{ij} effective stress

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