On Exact Exponential Stability Assignment of Retarded Delay Systems via Static Output-Feedback Controllers

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Abstract: This paper addresses the static output-feedback (SOF) exponential stabilization problem of retarded delay systems with a specification on the decay rate and decay coefficient. A novel Lyapunov-Krasovskii functional equipped with appropriate exponential terms is provided to establish a new condition for exponential stability. Based on this and an augmentation technique, a necessary and sufficient condition for quadratic SOF exact exponential stabilizability is proposed. An efficient iteration algorithm is then given to solve the proposed condition. Using the proposed approach, no structural constraints are imposed on the slack matrices when the controller matrix is parameterized, and the obtained controller not only stabilizes the original system but also guarantees that the closed-loop system possess desirable transient properties. Numerical examples are used to show the effectiveness of the theoretical results.

Key–Words: Augmentation, exponential stability, linear matrix inequality (LMI), Lyapunov-Krasovskii functional, retarded delay systems, static output-feedback.

1 Introduction

Over the past decades, stability and stabilization of time-delay systems have received much attention, since time delay is encountered in various practical systems such as chemical processes, long transmission lines in pneumatic systems [1], and is considered as a major cause for instability and poor performance of dynamic systems. Typical stability analysis is concerned with asymptotic stability, whereas from the standpoint of practical application, exponential stability is more useful since the transient process of a system can be characterized more clearly once the decay rate is determined. Unlike linear systems without time delay, the eigenvalues of delay systems cannot be computed analytically owing to the existence of transcendental characteristic equations. This motivates researchers to study the exponential estimating problem (the α-stability) of time-delay systems. By using the Lyapunov-Razumikhin approach, some results on retarded delay systems have been obtained in [2, 3]. Based on the generalized Gronwall-Bellman Lemma and the matrix measure concept, exponential estimates of retarded delay systems have been obtained in [4] and [5], respectively. Some improvements have been presented in [6, 7]. Recently, a new estimating approach for retarded delay systems, which bases on a new Lyapunov-Krasovskii functional and the LMI technique, has been presented in [8]. Very recently, an improved estimating approach for retarded delay systems has been obtained in [9]. As for static output-feedback control of delay systems, we refer readers to [10, 11] and references therein.

Despite the available literature on exponential stability and its decay rate estimation, there are only a few results concerning with the synthesis problem with α-stability constraint. In [12], a delay-independent approach has been provided to design a controller for linear systems with constant delay such that the closed-loop system satisfies a prescribed bound of the decay rate. However, the derived condition based on exponential scaling is not monotonic with respect to the decay rate. This may bring difficulties into optimizing the decay rate by feedback controllers. In addition, the delay-independence of the condition may cause some conservatism when the time delay is not large. As for the optimization of the decay coefficient, which has significant application in minimization of transient energy, few results are available to the authors' knowledge.

In this paper, we investigate the exact exponential stability assignment problem for retarded delay systems via static output-feedback controllers. A new characterization on stability of the closed-loop system with the decay rate and decay coefficient constraint is established in terms of a novel Lyapunov-Krasovskii functional. Slack matrices are introduced to reduce the conservatism as well. Based on this and an augmentation technique, a necessary and sufficient condition for quadratic SOF exact exponential stabilizability is proposed. Although the derived condition is nonconvex, it can be solved efficiently by an iterative algorithm. Numerical examples are provided to illustrate the effectiveness of the theoretical results.

Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). The superscript "T" represents the transpose. $|\cdot|$ denotes the Euclidean norm for vectors and $\|\cdot\|$ denotes the spectral norm for matrices. $C_\perp$ denotes an orthogonal complement matrix of matrix C. The symbol $\#$ is used to denote a matrix which can be inferred by symmetry. $\mathcal{C}([-d,0],\mathbb{R}^n)$ denotes the family of continuous functions ϕ from $[-d,0]$ to $\mathbb{R}^n$ with norm $|\phi|_d = \sup_{-d\leq s\leq 0}|\phi(s)|$.

2 Preliminaries and Problem Formulation

Consider the following class of time-delay systems:

$$\begin{cases} \dot{x}(t) &= Ax(t) + A_d x(t-d) + Bu(t), \\ x(s) &= \phi(s), \;\; s \in [-d,0], \\ y(t) &= Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, and $y(t) \in \mathbb{R}^l$ are the system state, the external input, and the measured output, respectively, and A, A_d, B, C are the system matrices. The initial function $\phi(t)$ is assumed to belong to $\mathcal{C}([-d,0],\mathbb{R}^n)$. The function d represents the unknown time delay, and is assumed to satisfy $0 < d \leq \bar{d}$.

Definition 1

1. *The system in (1) is said to be exponentially stable if, when $u(t) \equiv 0$, for all initial function $\phi \in \mathcal{C}([-d,0],\mathbb{R}^n)$, there exist $\lambda > 0$ and $\sigma \geq 1$ such that*

$$|x(t)| \leq \sigma \mathrm{e}^{-\lambda t} |\phi|_d ,$$

where λ and σ are called the decay rate and decay coefficient, respectively.

2. *The system in (1) is said to be (λ,σ)-exponentially stable if it is exponentially stable with a decay rate no less than λ and a decay coefficient no larger than σ.*

The goal of this paper is to design a static output-feedback controller,

$$u(t) = Ky(t), \tag{2}$$

such that the resulting closed-loop system is (λ,σ)-exponentially stable for prescribed $\lambda > 0$ and $\sigma \geq 1$. This specification is very useful in practical applications since the transient process of a dynamic system can be controlled more accurately once the decay rate and decay coefficient are determined.

Lemma 2 ([13]) *Let G, U, and V be real matrices with G being symmetric. There exists matrix X such that $G + UXV^T + VX^TU^T > 0$, if and only if $U_\perp^T G U_\perp > 0$ and $V_\perp^T G V_\perp > 0$, which, by Finsler's Lemma, are equivalent to, for some scalar ε, $G - \varepsilon VV^T > 0$ and $G - \varepsilon UU^T > 0$.*

3 Main Results

3.1 (λ,σ)-Exponential Stability Analysis

Theorem 3 *For prescribed $\lambda > 0$ and $\sigma \geq 1$, if there exist matrices P, Q, R, Z, Y_i, $(i = 1,2,3,4)$, and scalars δ_p, δ_P, δ_Q, δ_R, δ_Z such that*

$$0 < \delta_p I < P < \delta_P I, \tag{3}$$

$$0 < Q < \delta_Q I, \tag{4}$$

$$0 < R < \delta_R I, \tag{5}$$

$$0 < Z < \delta_Z I, \tag{6}$$

$$0 < R - 2\lambda Z, \tag{7}$$

$$\delta_P + \rho_q(\lambda,\bar{d})\,\delta_Q + \rho_r(\lambda,\bar{d})\,\delta_R + \rho_z(\bar{d})\,\delta_Z - \sigma^2\delta_p < 0, \tag{8}$$

$$\Phi(\lambda,\bar{d}) = \mathcal{G}(\lambda,\bar{d}) + \mathcal{I}\mathcal{Y} + (\mathcal{I}\mathcal{Y})^T < 0, \tag{9}$$

where

$$\rho_q(\lambda,\bar{d}) = \frac{\mathrm{e}^{2\lambda\bar{d}} - 1}{2\lambda},$$

$$\rho_r(\lambda,\bar{d}) = \frac{\mathrm{e}^{2\lambda\bar{d}} - 2\lambda\bar{d} - 1}{4\lambda^2},$$

$$\rho_z(\bar{d}) = 4\bar{d},$$

$$\mathcal{I} = \begin{bmatrix} I & -I & 0 & -I \end{bmatrix}^T,$$

$$\mathcal{Y} = \begin{bmatrix} Y_1 & Y_2 & Y_3 & Y_4 \end{bmatrix},$$

$$\mathcal{G}(\lambda,\bar{d}) = \left[\begin{matrix} \mathcal{G}_{11}(\lambda,\bar{d}) & \# \\ A_d^T P & -Q \\ -2\lambda Z - Z(A+BKC) & -ZA_d \\ 0 & 0 \end{matrix}\right.$$

$$\begin{bmatrix} \# & \# \\ \# & \# \\ -\bar{d}^{-1}R+2\bar{d}^{-1}\lambda Z & \# \\ 0 & -Z \end{bmatrix},$$

$$\mathcal{G}_{11}(\lambda,\bar{d}) = (A+BKC)^T P + P(A+BKC) + 2\lambda P + \mathrm{e}^{2\lambda\bar{d}}Q + \frac{\mathrm{e}^{2\lambda\bar{d}}-1}{2\lambda}R,$$

then the closed-loop system is (λ,σ)-exponentially stable for any $0 < d \le \bar{d}$.

Proof: To show exponential stability, a novel Lyapunov-Krasovskii functional with appropriately constructed exponential terms is constructed as follows:

$$V(x_t,t) = \sum_{i=1}^{4} V_i(x_t,t),$$

where $x_t = x(t+\theta)$, $-d \le \theta \le 0$, and

$$V_1(x_t,t) = \mathrm{e}^{2\lambda t}x^T(t)(P-dZ)x(t),$$
$$V_2(x_t,t) = \int_{t-d}^{t} \mathrm{e}^{2\lambda(\alpha+d)}x^T(\alpha)Qx(\alpha)\,\mathrm{d}\alpha,$$
$$V_3(x_t,t) = \int_{-d}^{0}\int_{t+\beta}^{t} \mathrm{e}^{2\lambda(\alpha-\beta)}x^T(\alpha)^T Rx(\alpha)\mathrm{d}\alpha\mathrm{d}\beta,$$
$$V_4(x_t,t) = \int_{t-d}^{t} \mathrm{e}^{2\lambda t}m^T(t,s)Zm(t,s)\,\mathrm{d}s,$$

and

$$m(\alpha,\beta) = x(\alpha) - x(\beta).$$

Then, the derivative of $V_i(x_t,t)$, $i = 1,2,3,4$, along the trajectories of the closed-loop system for $t > 0$ can be evaluated as

$$\begin{aligned}\dot{V}_1(x_t,t) &= 2\lambda\mathrm{e}^{2\lambda t}x^T(t)(P-dZ)x(t) \\ &\quad + 2\mathrm{e}^{2\lambda t}x^T(t)(P-dZ) \\ &\quad \times [(A+BKC)x(t) + A_d x(t-d)],\\ \dot{V}_2(x_t,t) &= \mathrm{e}^{2\lambda(t+d)}x^T(t)Qx(t) \\ &\quad - \mathrm{e}^{2\lambda t}x^T(t-d)Qx(t-d),\\ \dot{V}_3(x_t,t) &= \mathrm{e}^{2\lambda t}\frac{e^{2\lambda d}-1}{2\lambda}x^T(t)Rx(t) \\ &\quad - \mathrm{e}^{2\lambda t}\int_{t-d}^{t} x^T(s)^T Rx(s)\mathrm{d}s,\\ \dot{V}_4(x_t,t) &= -\mathrm{e}^{2\lambda t}m^T(t,t-d)Zm(t,t-d) \\ &\quad + 2\frac{\mathrm{e}^{2\lambda t}}{d}\int_{t-d}^{t} m^T(t,s)(\lambda dZ)m(t,s)\,\mathrm{d}s \\ &\quad + 2\frac{\mathrm{e}^{2\lambda t}}{d}\int_{t-d}^{t} m^T(t,s)(dZ)\,\mathrm{d}s \\ &\quad \times [(A+BKC)x(t) + A_d x(t-d)]^T.\end{aligned}$$

It is noted that the following relationships hold for any matrices Y_i, $(i = 1,2,3,4)$, with compatible dimensions,

$$\begin{aligned}\operatorname{slack}(t,s) &\triangleq 2\zeta^T(t,s)\mathcal{Y}^T \times \\ &\qquad [x(t) - x(t-d) - m(t,t-d)] \\ &= 0, \end{aligned} \tag{10}$$

where

$$\zeta(t,s) = \left[x^T(t), x^T(t-d), dx^T(s), m^T(t,t-d)\right]^T.$$

Hence,

$$\begin{aligned}\dot{V}(x_t,t) &= \sum_{i=1}^{4}\dot{V}_i(x_t,t) + \frac{\mathrm{e}^{2\lambda t}}{d}\int_{t-d}^{t}\operatorname{slack}(t,s)\,\mathrm{d}s \\ &\le \frac{\mathrm{e}^{2\lambda t}}{d}\int_{t-d}^{t}\zeta^T(t,s)^T\Phi(\lambda,d)\zeta(t,s)\mathrm{d}s,\end{aligned}$$

which together with (9) implies that

$$\dot{V}(x_t,t) < 0.$$

With this, one obtains that, when $t > 0$,

$$\begin{aligned} & V(x_t,t) \\ &\le V(x_0,0) \\ &\le \|P-dZ\|\,|\phi|_d^2 + \frac{\mathrm{e}^{2\lambda d}-1}{2\lambda}\|Q\|\,|\phi|_d^2 \\ &\quad + \frac{\mathrm{e}^{2\lambda d}-2\lambda d-1}{4\lambda^2}\|R\|\,|\phi|_d^2 + 4d\|Z\|\,|\phi|_d^2 \\ &\le [\delta_P + \rho_q(\lambda,\bar{d})\delta_Q + \rho_r(\lambda,\bar{d})\delta_R \\ &\quad + \rho_z(\bar{d})\delta_Z]\,|\phi|_d^2. \end{aligned} \tag{11}$$

On the other hand,

$$V(x_t,t) \ge \mathrm{e}^{2\lambda t}\delta_p|x(t)|^2. \tag{12}$$

Therefore, it follows from (8), (11), and (12) that

$$|x(t)| \le \sigma\mathrm{e}^{-\lambda t}|\phi|_d.$$

This completes the proof. □

Remark 4 *In [9], to reduce the conservatism, some functionals like $\int_{-d}^{0}\int_{t+\beta}^{t}\dot{x}^T(\alpha)^T R\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta$ are used. These functionals, however, will cause problems in the decay coefficient estimation and related synthesis since, as shown in [9], the envelop of $x(t)$ over $(0,d]$ needs to be estimated. To overcome the drawback of such functionals, a new functional $V_4(x_t,t)$, which is not explicitly dependent on any information on the derivative of $x(s)$, is introduced to construct slack matrix equations. By virtue of this functional, it is not necessary to estimate the envelop of $x(t)$ over $(0,d]$, and the synthesis procedure becomes easier.*

3.2 Quadratic SOF (λ,σ)-Exponential Stabilizability and Controller Design

Definition 5 *The system in (1) is said to be quadratic SOF (λ,σ)-exponential stabilizable if there exists a SOF controller in (2) such that the closed-loop system has corresponding solution to (3)–(9).*

It can be seen readily that if the system in (1) is quadratic SOF (λ,σ)-exponential stabilizable, then there exists an SOF controller in (2) such that the closed-loop system is (λ,σ)-exponential stable. The following theorem gives a necessary and sufficient condition to quadratic SOF (λ,σ)-exponential stabilizable.

Theorem 6 *For prescribed $\lambda > 0$ and $\sigma \geq 1$, the system in (1) is quadratic SOF (λ,σ)-exponential stabilizable if and only if there exist matrices P, Q, R, Z, $S > 0$, L, and scalars δ_p, δ_P, δ_Q, δ_R, δ_Z, μ such that (3)–(8) hold, and*

$$\hat{\mathcal{G}}\left(\lambda,\bar{d}\right) - \mu\hat{\mathcal{I}}\hat{\mathcal{I}}^T < 0, \tag{13}$$

where

$$\hat{\mathcal{G}}\left(\lambda,\bar{d}\right) = \begin{bmatrix} \hat{\mathcal{G}}_{11}\left(\lambda,\bar{d}\right) & \# & \# & \# & \# \\ LC+B^TP & -S & \# & \# & \# \\ A_d^TP & 0 & -Q & \# & \# \\ -2\lambda Z - ZA & -ZB & -ZA_d & -\bar{d}^{-1}R + 2\bar{d}^{-1}\lambda Z & \# \\ 0 & 0 & 0 & 0 & -Z \end{bmatrix},$$

$$\begin{aligned} \hat{\mathcal{G}}_{11}\left(\lambda,\bar{d}\right) &= 2\lambda P + PA + A^TP + \mathrm{e}^{2\lambda\bar{d}}Q \\ &\quad + \frac{\mathrm{e}^{2\lambda\bar{d}}-1}{2\lambda}R - C^TL^TM - M^TLC \\ &\quad + M^TSM, \\ \hat{\mathcal{I}} &= \begin{bmatrix} I & 0 & -I & 0 & -I \end{bmatrix}^T. \end{aligned}$$

Under the condition, a controller gain can be obtained as

$$K = S^{-1}L. \tag{14}$$

Proof: It suffices to prove (9)⇔(13).

(Sufficiency) Define

$$T = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ KC & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}.$$

By pre- and post-multiplying (13) with T^T and its transpose, and noting (14) and

$$\begin{aligned} & -C^TL^TS^{-1}LC \\ \leq\ & -C^TL^TM - M^TLC + M^TSM \end{aligned} \tag{15}$$

one obtains that

$$\begin{bmatrix} \mathcal{G}\left(\lambda,\bar{d}\right) - \mu\mathcal{I}\mathcal{I}^T & \# \\ \Sigma & -S \end{bmatrix} < 0, \tag{16}$$

where $\Sigma = \begin{bmatrix} B^TP & 0 & -B^TZ & 0 \end{bmatrix}$. By Lemma 2, one further obtains that (9) holds.

(Necessity) Assume that (9) holds. By Lemma 2, we obtain that $\mathcal{G}\left(\lambda,\bar{d}\right) - \mu\mathcal{I}\mathcal{I}^T < 0$. Set $S > 0$ to be a sufficient large positive definite matrix such that $-S + \Sigma\left[\mathcal{G}\left(\lambda,\bar{d}\right) - \mu\mathcal{I}\mathcal{I}^T\right]^{-1}\Sigma^T < 0$. By this, Schur complement equivalence, and noting that T is nonsingular, we obtain that

$$\begin{aligned} & \hat{\mathcal{G}}\left(\lambda,\bar{d}\right) - \mu\hat{\mathcal{I}}\hat{\mathcal{I}}^T \\ &= T^{-T}\begin{bmatrix} \mathcal{G}\left(\lambda,\bar{d}\right) - \mu\mathcal{I}\mathcal{I}^T & \# \\ \Sigma & -S \end{bmatrix}T^{-1} \\ &< 0. \end{aligned}$$

This completes the proof. □

Remark 7 *Whether in previous works such as the descriptor transformation based approach ([14]), the free weight matrices based approach ([15]), or in recent work on the projection approach ([16]), structural constraints on the slack matrix variables are inevitable when the controller matrix is parameterized, and tuning parameters are needed to be specified, whereas, in Theorem 6, neither structural constraints on the slack matrix variables nor tuning parameters are required. Thus those slack matrix variables can be eliminated to reduce the computational burden without increasing conservatism.*

When M is fixed, (13) becomes a strict LMI, which could be verified easily by conventional LMI solver. It can be seen from (15) that the scalar ϵ satisfying $\hat{\mathcal{G}}\left(\lambda,\bar{d}\right) - \mu\hat{\mathcal{I}}\hat{\mathcal{I}}^T < \epsilon I$ achieves its minimum when $M = S^{-1}L$, which can be used to construct an iteration rule.

Algorithm 8

1. *Set $\nu = 1$, and let a scalar $c > 0$. Select an initial value M_ν such that $A + A_d + BM_\nu$ is Hurwitz.*

2. *For fixed M_ν, solve the following sequential convex optimization problem:*

   Reduce ϵ_ν subject to

$$\mathcal{G}\left(\lambda, \bar{d}\right) - \mu \mathcal{I}\mathcal{I}^T \quad < \quad \epsilon_\nu I, \tag{17}$$

$$\epsilon_\nu \quad > \quad -c, \tag{18}$$

   until $\epsilon_\nu \leq 0$ or $\epsilon_\nu > 0$ is minimized.

   If $\epsilon_\nu \leq 0$, then a desired controller matrix can be obtained as (14). ***STOP****. Otherwise, denote ϵ_ν^* as the minimized value of ϵ_ν satisfying (17) and (18).*

3. *If $\left|\epsilon_\nu^* - \epsilon_{\nu-1}^*\right| \leq \delta$, a prescribed tolerance, then go to next step, else update $M_{\nu+1}$ as*

$$M_{\nu+1} = \left(S_\nu\right)^{-1} L_\nu,$$

   and set $\nu = \nu + 1$, then go to Step 2.

4. *There may not exist a solution.* ***STOP*** *(or choose other initial value M_1, then run the algorithm again).*

Remark 9 *It can be shown easily from (15) that the sequence ϵ_ν^* is monotonic decreasing with respect to ν, that is, $\epsilon_\nu^* \leq \epsilon_{\nu-1}^*$. On the other hand, (18) implies that ϵ_ν^* is bounded from below by $-c$. Therefore, the convergence of the algorithm is guaranteed.*

Remark 10 *The initial value M_1 is a state-feedback stabilizing controller matrix for $A + A_d$, which can be found by existing approaches. If no such matrices can be found, it can be concluded immediately that the system cannot be stabilized. Like many other iterative algorithms, the sequence of iteration depends on the selection of M_1, and an appropriate selection of M_1 will improve solvability. The optimization of initial values may consist of an interesting problem for future study.*

4 Numerical Example

Consider a delay system in (1) with the following system matrices:

$$A = \begin{bmatrix} 0.3 & -0.1 & 0.2 \\ 0 & 0.2 & 0.3 \\ -0.1 & 0 & -3 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0 & 0.2 & -0.1 \\ -0.13 & 0.1 & 0 \\ -0.1 & 0.2 & -0.2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0.5 & 1 & 0.5 \\ 0.15 & 0.7 & 0.5 \end{bmatrix}.$$

It is assumed that $0 < d \leq 0.7$. The open-loop system is unstable (see Figure 1).

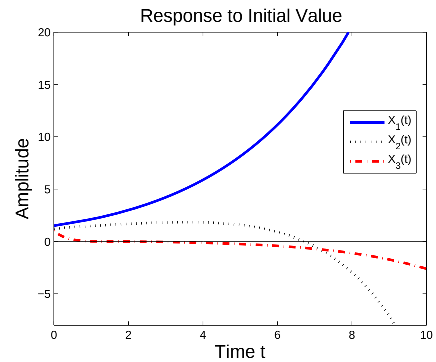

Figure 1: Open-Loop Response to Initial Value.

For different σ's and λ's, different stabilizing controllers, by Theorem 6 and corresponding algorithms, are obtained as follows:

$$K_{\sigma=10,\ \lambda=0.05} = \begin{bmatrix} -5.6179 & 8.1718 \\ 3.0995 & -7.4062 \end{bmatrix},$$

$$K_{\sigma=1.5,\ \lambda=0.85} = \begin{bmatrix} -18.3940 & 24.3318 \\ 2.1322 & -8.3843 \end{bmatrix}.$$

Figures 2 and 3 give the response $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T$ to the initial condition $\phi(s) = \begin{bmatrix} 1.5 & 1.2 & 1 \end{bmatrix}^T$.

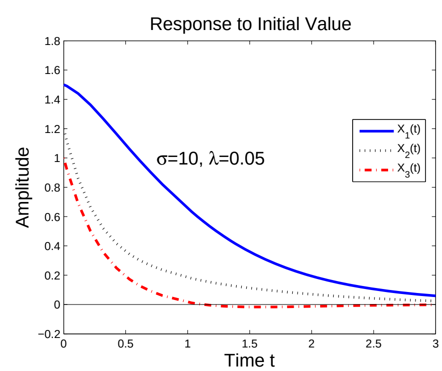

Figure 2: Closed-Loop Response with $\sigma = 10$, $\lambda = 0.05$.

5 Conclusion

A stability characterization with decay rate and decay coefficient constraints is established in terms of a novel Lyapunov-Krasovskii functional with appropriately constructed exponential terms. Based on this and an augmentation technique, a necessary and sufficient

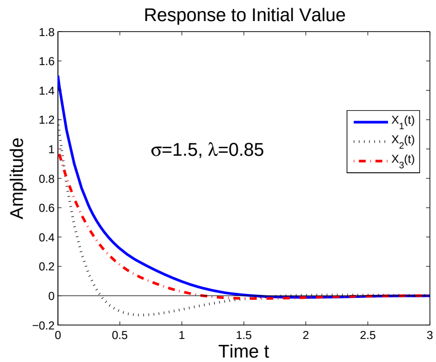

Figure 3: Closed-Loop Response with $\sigma = 1.5$, $\lambda = 0.85$.

condition for quadratic SOF exact exponential stabilizability is proposed. An efficient iteration algorithm is then given to solve the proposed condition. Using the proposed approach, no structural constraints are imposed on the slack matrices when the controller matrix is parameterized, and the obtained controller not only stabilizes the original system but also guarantees the closed-loop system to possess desirable transient properties. Numerical examples are employed to show the effectiveness of the theoretical results.

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