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Delay-dependent $\mathcal{L}_2$-$\mathcal{L}_\infty$ Model Reduction for Polytopic Systems with Time-varying Delay

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***Abstract*— This paper considers the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problems for polytopic system with time-varying delay. In terms of the solution of linear matrix inequalities (LMIs) and inverse constraints, sufficient conditions are presented to construct the reduced order models such that the $\mathcal{L}_2$-$\mathcal{L}_\infty$ gain of the error system between the full order model and the reduced order one is less than a given scalar.**

***Index Terms*— model reduction problems, polytopic systems with time-varying delay, $\mathcal{L}_2$-$\mathcal{L}_\infty$ norm**

I. Introduction

The $\mathcal{L}_2$-$\mathcal{L}_\infty$ gain of a system Σ is defined as

$$\|\Sigma\|_{\mathcal{L}_2\text{-}\mathcal{L}_\infty} = \sup_{\|u\|_{\mathcal{L}_2}\le 1} \|y\|_{\mathcal{L}_\infty}$$

with zero initial state, where $y(t)$ is the output of system Σ and $u(t)$ is the input of system Σ with $\|u\|_{\mathcal{L}_2} = \left(\int_0^\infty \|u(t)\|^2 dt\right)^{1/2}$ and $\|u\|_{\mathcal{L}_\infty} = \sup_{t\ge 0}\|u(t)\|$, $\|u(t)\| = \sqrt{u^T(t)\,u(t)}$ [1] and [16]. The $\mathcal{L}_2$-$\mathcal{L}_\infty$ gain of error system, which is also referred as the energy-to-peak gain in [14] is a natural criterion for model reduction. The $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem for continuous-time and discrete-time systems without delay is studied in [11]. Necessary and sufficient conditions are given in terms of LMIs and a rank constraint, which can be solved by the alternating projection algorithm [4, Chap.13]. The model reduction problem has been extensively studied in the literature, as can be seen from [13] and the references listed therein.

In many physical, industrial and engineering systems, control systems cannot be described accurately without the introduction of delay element. Delays can lead to poor performance and instability of control systems. Considerable research has been carried out on systems with time-varying delay in recent years. The criteria for asymptotic stability of such systems can be classified as delay-independent [18] or delay-dependent [3], [6], [9], [10], which is less conservative than delay-independent stability criteria in general. For polytopic systems with constant delay, the robust stability and stabilization are studied in [17]. The extension to time-varying delay is investigated in [12] with stability criteria less conservative through the introduction of slack variables in the LMIs. The polytopic system has been extensively studied in the literature, as can be seen from [8], [19] and the references listed therein. To the authors knowledge, the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problems for polytopic systems with time-varying delay are a challenging topic worthwhile to tackle.

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II. $\mathcal{L}_2$-$\mathcal{L}_\infty$ Model Reduction

A polytopic system Σ with time-varying delay is given by

$$\begin{aligned} \Sigma : \dot{x}(t) &= A(t)x(t) + A_h(t)x(t-h(t)) + B(t)u(t) \\ y(t) &= C(t)x(t) + C_h(t)x(t-h(t)) \\ x(t) &= 0, \quad \forall t \in [-\bar{h}, 0] \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input which belongs to $\mathcal{L}_2[0,\infty)$ and $y(t) \in \mathbb{R}^p$ is the controlled output. $A(t)$, $A_h(t)$, $B(t)$, $C(t)$ and $C_h(t)$ are appropriately dimensioned continuous functions of time t, and satisfy the real convex polytopic model

$$(A(t)\, A_h(t)\, B(t)) = \sum_{i=1}^{q} \alpha_i(t)\,(A_i\ A_{hi}\ B_i),$$

$$(C(t)\ \ C_h(t)) = \sum_{i=1}^{q} \alpha_i(t)\,(C_i\ C_{hi})$$

$\forall \alpha_i(t) \ge 0$, $\sum_{i=1}^q \alpha_i(t) = 1$, where $\alpha_i(t)$, $i = 1, 2, \ldots, q$, are time-varying functions of t, C_i, C_{hi}, B_i, A_i and A_{hi}, $i = 1, 2, \ldots, q$, are constant matrices with appropriate dimensions. $h(t)$ is the time-varying delay satisfying $0 < h(t) \le \bar{h} < \infty$, $\dot{h}(t) \le \mu < 1$ with $\bar{h} > 0$ and $\mu > 0$. System Σ, which is to be reduced, is assumed to be quadratically stable.

Definition 2.1: If there exists an $\hat{n}$th-order quadratically stable system $\hat{\Sigma}$

$$\begin{aligned} \hat{\Sigma} : \dot{\hat{x}}(t) &= \hat{A}(t)\hat{x}(t) + \hat{A}_h(t)\hat{x}(t-h(t)) + \hat{B}(t)u(t) \\ \hat{y}(t) &= \hat{C}(t)\hat{x}(t) + \hat{C}_h(t)x(t-h(t)) \\ \hat{x}(t) &= 0, \quad \forall t \in [-\bar{h}, 0] \end{aligned}$$

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where $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$, $\hat{y}(t) \in \mathbb{R}^{p}$, and $\hat{n} < n$, such that

$$\left\| \Sigma - \hat{\Sigma} \right\|_{\mathcal{L}_2\text{-}\mathcal{L}_\infty} = \sup_{\|u\|_{\mathcal{L}_2} \le 1} \|y - \hat{y}\|_{\mathcal{L}_\infty} < \gamma$$

then we say the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem for system Σ is solvable.

The $\mathcal{L}_2$-$\mathcal{L}_\infty$ gain of retarded system Σ_R

$$\begin{aligned} \Sigma_R: \quad \dot{x}(t) &= Ax(t) + A_h x(t - h(t)) + Bu(t) \\ y(t) &= Cx(t) + C_h x(t - h(t)) \end{aligned}$$

can be characterized by the following algebraic conditions, where A, A_h, B, C and C_h are contant matrices.

Lemma 2.2: [15] If there exist matrices $P > 0$, $Q > 0$, $X > 0$, $Z > 0$ and Y, and a scalar $0 < \alpha < 1$ such that

$$\begin{bmatrix} \begin{array}{c} A^T P + PA + \\ \bar{h}X + Y \\ +Y^T + Q \end{array} & PA_h - Y & PB & \tau A^T Z \\ A_h^T P - Y^T & -(1-\mu)Q & 0 & \tau A_h^T Z \\ B^T P & 0 & -I_m & \tau B^T Z \\ \tau Z A & \tau Z A_h & \tau Z B & -\tau Z \end{bmatrix} < 0$$

$$\begin{bmatrix} -\alpha P & 0 & C^T \\ 0 & -(1-\alpha)P & C_h^T \\ C & C_h & -\gamma^2 I_p \end{bmatrix} < 0$$

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \ge 0$$

then system Σ_R is asymptotically stable and $\|\Sigma_R\|_{\mathcal{L}_2\text{-}\mathcal{L}_\infty} < \gamma$, where $\tau = \frac{\bar{h}}{1-u}$. In the case of $h(t) = h$, if there exists matrices $P > 0$, $Q > 0$, and a scalar $0 < \alpha < 1$ such that

$$\begin{bmatrix} \begin{array}{c} A^T P + PA + \\ \bar{h}X + Y + \\ Y^T + Q \end{array} & PA_h - Y & PB \\ A_h^T P - Y^T & -(1-\mu)Q & 0 \\ B^T P & 0 & -I_m \end{bmatrix} < 0$$

$$\begin{bmatrix} -\alpha P & 0 & C^T \\ 0 & -(1-\alpha)P & C_h^T \\ C & C_h & -\gamma^2 I_p \end{bmatrix} < 0$$

The following theorem gives the solution of $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem for system Σ.

Theorem 2.3: With $\tilde{n} = n + \hat{n}$, if there exist matrices $P > 0$, $\tilde{P} > 0$, $X > 0$, $\tilde{X} > 0$, $Q > 0$, $\tilde{Q} > 0$, $M > 0$, $\tilde{M} > 0$, $N > 0$, $\tilde{N} > 0$, $Z > 0$, $\tilde{Z} > 0$ and Y, and a scalar $0 < \alpha < 1$ such that for $i = 1, 2, \ldots, q$,

$$\begin{bmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{12}^T & \mathcal{X}_{22} \end{bmatrix} < 0 \qquad (1)$$

$$\begin{bmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \mathcal{Y}_{12}^T & \mathcal{Y}_{22} \end{bmatrix} < 0 \qquad (2)$$

$$\begin{bmatrix} -\alpha \mathcal{I}_1 P \mathcal{I}_1^T & 0 & C_i^T \\ * & -(1-\alpha)\mathcal{I}_1 P \mathcal{I}_1^T & C_{hi}^T \\ * & * & -\gamma^2 I_p \end{bmatrix} < 0 \qquad (3)$$

$$\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} \ge 0 \qquad (4)$$

$$\begin{bmatrix} -N & Y \\ * & -M \end{bmatrix} \le 0 \qquad (5)$$

$$P\tilde{P} = I_{\tilde{n}}, \; Q\tilde{Q} = I_{\tilde{n}}, \; X\tilde{X} = I \qquad (6)$$

$$Z\tilde{Z} = I_{\tilde{n}}, \; \tilde{M}M = I_{\tilde{n}}, \; \tilde{N}N = I_{\tilde{n}} \qquad (7)$$

where $*$ denotes the symmetric terms in a symmetric matrix and

$$\mathcal{X}_{11} = \begin{bmatrix} \Lambda_{i11} & \Lambda_{i12} & B_i & \Lambda_{i14} & \Lambda_{i15} \\ * & \Lambda_{22} & 0 & \Lambda_{i12}^T & 0 \\ * & * & -I_m & B_i^T & 0 \\ * & * & * & \Lambda_{i44} & \Lambda_{i45} \\ * & * & * & * & \Lambda_{i55} \end{bmatrix}$$

$$\mathcal{X}_{12} = \begin{bmatrix} \mathcal{I}_1\tilde{P} & \mathcal{I}_1\tilde{P} & \mathcal{I}_1\tilde{P} & \mathcal{I}_1\tilde{P} \\ 0 & 0 & -I_{\tilde{n}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathcal{I}_2\tilde{P} & -\mathcal{I}_2\tilde{P} & -\mathcal{I}_2\tilde{P} & -\mathcal{I}_2\tilde{P} \end{bmatrix}$$

$$\mathcal{X}_{22} = \begin{bmatrix} -\bar{h}^{-1}\tilde{X} & 0 & 0 & 0 \\ * & -\tilde{Q} & 0 & 0 \\ * & * & -\tilde{M} & 0 \\ * & * & * & -\tilde{N} \end{bmatrix}$$

$$\mathcal{Y}_{11} = \begin{bmatrix} A_i^T \mathcal{I}_1 P \mathcal{I}_1^T + \mathcal{I}_1 P \mathcal{I}_1^T A_i \\ +\mathcal{I}_1 \begin{pmatrix} \bar{h}X + Y \\ +Y^T + Q \end{pmatrix} \mathcal{I}_1^T \end{bmatrix}$$

$$\mathcal{Y}_{12} = \begin{bmatrix} \mathcal{I}_1 P \mathcal{I}_1^T A_{hi} - \mathcal{I}_1 Y \mathcal{I}_1^T & \tau A_i^T \mathcal{I}_1 Z \end{bmatrix}$$

$$\mathcal{Y}_{22} = \begin{bmatrix} -(1-\mu)\mathcal{I}_1 Q \mathcal{I}_1^T & \tau A_{hi}^T \mathcal{I}_1 Z \\ * & -\tau Z \end{bmatrix}$$

and

$$\Lambda_{i11} = \mathcal{I}_1 \tilde{P} \mathcal{I}_1^T A_i^T + A_i \mathcal{I}_1 \tilde{P} \mathcal{I}_1^T, \quad \Lambda_{i12} = A_{hi}\mathcal{I}_1 \qquad (8)$$

$$\Lambda_{i14} = \mathcal{I}_1 \tilde{P} \mathcal{I}_1^T A_i^T, \quad \Lambda_{i15} = -A_i \mathcal{I}_1 \tilde{P} \mathcal{I}_2^T \qquad (9)$$

$$\Lambda_{22} = -(1-\mu)Q, \quad \Lambda_{i44} = -\tau^{-1}\mathcal{I}_1 \tilde{Z} \mathcal{I}_1^T \qquad (10)$$

$$\Lambda_{i55} = -\tau^{-1}\mathcal{I}_2 \tilde{Z} \mathcal{I}_2^T \qquad (11)$$

$$\Lambda_{i45} = -\tau^{-1}\mathcal{I}_1 \tilde{Z} \mathcal{I}_2^T - A_i \mathcal{I}_1 \tilde{P} \mathcal{I}_2^T \qquad (12)$$

then there exists a quadratically stable system $\hat{\Sigma}$ such that the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem is sovable with $\left\| \Sigma - \hat{\Sigma} \right\|_{\mathcal{L}_2\text{-}\mathcal{L}_\infty} < \gamma$. In this case, a desired reduced system corresponding to a feasible solution (P, $\tilde{P}$, X, $\tilde{X}$, Q, $\tilde{Q}$, M, $\tilde{M}$, N, $\tilde{N}$, Z, $\tilde{Z}$, Y, α) to (1)–(7) is given by

$$\begin{pmatrix} \hat{B}(t) & \hat{A}(t) & \hat{A}_h(t) \end{pmatrix} = \sum_{i=1}^{q} \alpha_i(t) \bar{G}_{i1} \qquad (13)$$

$$\begin{pmatrix} \hat{C}(t) & \hat{C}_h(t) \end{pmatrix} = \sum_{i=1}^{q} \alpha_i(t) \bar{G}_{i2} \qquad (14)$$

where

$$\begin{aligned}\bar{G}_{i1} &= \left[\begin{array}{ccc}\hat{B}_i & \hat{A}_i & \hat{A}_{hi}\end{array}\right] \\ &= -U_{i1}^{-1}\bar{\Omega}_1^T V_{i1}\bar{\Lambda}_{i1}^T\left(\bar{\Lambda}_1 V_{i1}\bar{\Lambda}_1^T\right)^{-1} \\ &\quad +U_{i1}^{-1}W_{i1}^{\frac{1}{2}}L_{i1}\left(\bar{\Lambda}_1 V_{i1}\bar{\Lambda}_1^T\right)^{-\frac{1}{2}} \qquad (15)\\ \bar{G}_{i2} &= \left[\begin{array}{cc}\hat{C}_i & \hat{C}_{hi}\end{array}\right] \\ &= -U_{i2}^{-1}\bar{\Omega}_2^T V_{i2}\bar{\Lambda}_2^T\left(\bar{\Lambda}_2 V_{i2}\bar{\Lambda}_2^T\right)^{-1} \\ &\quad +U_{i2}^{-1}W_{i2}^{\frac{1}{2}}L_{i2}\left(\bar{\Lambda}_2 V_{i2}\bar{\Lambda}_2^T\right)^{-\frac{1}{2}} \qquad (16)\end{aligned}$$

and for $i = 1, 2, \ldots, q$, L_{i1} and L_{i2} are any matrices satisfying $\|L_{i1}\| < 1$ and $\|L_{i2}\| < 1$, and $U_{i1} > 0$ and $U_{i2} > 0$ such that

$$\begin{aligned}V_{i1} &= \left(\bar{\Omega}_1 U_{i1}^{-1}\bar{\Omega}_1^T - \bar{\Phi}_{i1}\right)^{-1} > 0 \\ V_{i2} &= \left(\bar{\Omega}_2 U_{i2}^{-1}\bar{\Omega}_2^T - \bar{\Phi}_2\right)^{-1} > 0 \\ W_{i1} &= U_{i1} - \bar{\Omega}_1^T\left(\begin{array}{c}V_{i1} - V_{i1}\bar{\Lambda}_1^T \times \\ \left(\bar{\Lambda}_1 V_{i1}\bar{\Lambda}_1^T\right)^{-1}\bar{\Lambda}_1 V_{i1}\end{array}\right)\bar{\Omega}_1 \\ W_{i2} &= U_{i2} - \bar{\Omega}_2^T\left(\begin{array}{c}V_2 - V_{i2}\bar{\Lambda}_2^T \times \\ \left(\bar{\Lambda}_2 V_{i2}\bar{\Lambda}_2^T\right)^{-1}\bar{\Lambda}_2 V_{i2}\end{array}\right)\bar{\Omega}_2\end{aligned}$$

$$\bar{\Phi}_{i1} = \left[\begin{array}{cccc}\begin{array}{c}\bar{A}_i^T P + \\ P\bar{A}_i + \\ \bar{h}X + Y + \\ Y^T + Q\end{array} & P\bar{A}_{hi} - Y & P\bar{B}_i & \tau\bar{A}_i^T Z \\ * & -(1-\mu)Q & 0 & \tau\bar{A}_{hi}^T Z \\ * & * & -I_m & \tau\bar{B}_i^T Z \\ * & * & * & -\tau Z\end{array}\right] \qquad (17)$$

$$\bar{\Phi}_{i2} = \left[\begin{array}{ccc}-\alpha P & 0 & \bar{C}_i^T \\ * & -(1-\alpha)P & \bar{C}_{hi}^T \\ * & * & -\gamma^2 I_p\end{array}\right] \qquad (18)$$

$$\bar{\Omega}_1 = \left[\begin{array}{c}P\left[\begin{array}{c}0 \\ I_{\hat{n}}\end{array}\right] \\ 0 \\ 0 \\ \tau Z\left[\begin{array}{c}0 \\ I_{\hat{n}}\end{array}\right]\end{array}\right], \quad \bar{\Omega}_2 = \left[\begin{array}{c}0 \\ 0 \\ I_p\end{array}\right] \qquad (19)$$

$$\bar{\Lambda}_1 = \left[\begin{array}{cc|cc|c|c}0 & 0 & 0 & 0 & I_m & 0 \\ 0 & I_{\hat{n}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\hat{n}} & 0 & 0\end{array}\right] \qquad (20)$$

$$\bar{\Lambda}_2 = \left[\begin{array}{cc|cc|c}0 & -I_{\hat{n}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{\hat{n}} & 0\end{array}\right] \qquad (21)$$

$$\bar{A}_i = \left[\begin{array}{cc}A_i & 0 \\ 0 & 0\end{array}\right], \quad \bar{A}_{hi} = \left[\begin{array}{cc}A_{hi} & 0 \\ 0 & 0\end{array}\right] \qquad (22)$$

$$\bar{B}_i = \left[\begin{array}{c}B_i \\ 0\end{array}\right], \quad \bar{C}_i = \left[\begin{array}{cc}C_i & 0\end{array}\right] \qquad (23)$$

$$\bar{C}_{hi} = \left[\begin{array}{cc}C_{hi} & 0\end{array}\right] \qquad (24)$$

$$\mathcal{I}_1 = \left[\begin{array}{cc}I_n & 0\end{array}\right], \quad \mathcal{I}_2 = \left[\begin{array}{cc}0 & I_{\hat{n}}\end{array}\right] \qquad (25)$$

Proof. We first consider the error system $\Sigma - \hat{\Sigma}$ given by

$$\begin{aligned}\dot{\tilde{x}}(t) &= \tilde{A}(t)\tilde{x}(t) + \tilde{A}_h(t)\tilde{x}(t-h(t)) + \tilde{B}(t)u(t) \\ \tilde{y}(t) &= \tilde{C}(t)\tilde{x}(t) + \tilde{C}_h(t)\tilde{x}(t-h(t))\end{aligned}$$

where $\tilde{x}(t) = \left[\begin{array}{cc}x^T(t) & \hat{x}^T(t)\end{array}\right]^T$, $\tilde{y}(t) = y(t) - \hat{y}(t)$,

$$\begin{aligned}&\left[\begin{array}{ccc}\tilde{B}(t) & \tilde{A}(t) & \tilde{A}_h(t)\end{array}\right] = \sum_{i=1}^{q}\alpha_i(t) \times \\ &\left[\bar{B}_i + \bar{F}\bar{G}_{i1}\bar{N} \quad \bar{A}_i + \bar{F}\bar{G}_{i1}\bar{H} \quad \bar{A}_{hi} + \bar{F}\bar{G}_{i1}\bar{M}\right] \qquad (26)\\ &\left[\begin{array}{cc}\tilde{C}(t) & \tilde{C}_h(t)\end{array}\right] \\ &= \sum_{i=1}^{q}\alpha_i(t)\left[\begin{array}{cc}\bar{C}_i + \bar{G}_{i2}\bar{J} & \bar{C}_{hi} + \bar{G}_{i2}\bar{K}\end{array}\right] \qquad (27)\end{aligned}$$

with $\bar{G}_{i1}$ and $\bar{G}_{i2}$, $i = 1, 2, \ldots, q$, defined in (15) and (16), $\bar{B}_i$, $\bar{A}_i$, $\bar{A}_{hi}$, $\bar{C}_i$ and $\bar{C}_{hi}$ given in (22) and (23), and

$$\bar{F} = \left[\begin{array}{c}0 \\ I_{\hat{n}}\end{array}\right], \quad \bar{H} = \left[\begin{array}{cc}0 & 0 \\ 0 & I_{\hat{n}} \\ 0 & 0\end{array}\right] \qquad (28)$$

$$\bar{M} = \left[\begin{array}{cc}0 & 0 \\ 0 & 0 \\ 0 & I_{\hat{n}}\end{array}\right], \quad \bar{N} = \left[\begin{array}{c}I_m \\ 0 \\ 0\end{array}\right] \qquad (29)$$

$$\bar{J} = \left[\begin{array}{cc}0 & -I_{\hat{n}} \\ 0 & 0\end{array}\right], \quad \bar{K} = \left[\begin{array}{cc}0 & 0 \\ 0 & -I_{\hat{n}}\end{array}\right] \qquad (30)$$

From Lemma 2.2, if there exist matrices $\bar{G}_{i1}$ and $\bar{G}_{i2}$ satisfying (4) and

$$\left[\begin{array}{cccc}\begin{array}{c}\left(\bar{A}_i + \bar{F}\bar{G}_{i1}\bar{H}\right)^T P \\ +P\left(\bar{A}_i + \bar{F}\bar{G}_{i1}\bar{H}\right) \\ +\bar{h}X + Y + \\ Y^T + Q\end{array} & \begin{array}{c}P\left(\bar{A}_{hi} + \bar{F}\bar{G}_{i1}\bar{M}\right) \\ -Y\end{array} & P\left(\bar{B}_i + \bar{F}\bar{G}_{i1}\bar{N}\right) & \tau\left(\bar{A}_i + \bar{F}\bar{G}_{i1}\bar{H}\right)^T Z \\ * & -(1-\mu)Q & 0 & \tau\left(\bar{A}_{hi} + \bar{F}\bar{G}_{i1}\bar{M}\right)^T Z \\ * & * & -I_m & \tau\left(\bar{B}_i + \bar{F}\bar{G}_{i1}\bar{N}\right)^T Z \\ * & * & * & -\tau Z\end{array}\right] < 0 \qquad (31)$$

$$\left[\begin{array}{ccc}-\alpha P & 0 & \left(\bar{C}_i + \bar{G}_{i2}\bar{J}\right)^T \\ * & -(1-\alpha)P & \left(\bar{C}_{hi} + \bar{G}_{i2}\bar{K}\right)^T \\ * & * & -\gamma^2 I_p\end{array}\right] < 0 \qquad (32)$$

It follows from (31) and (32) that

$$\begin{bmatrix} \tilde{A}(t)^T P + P\tilde{A}(t) + \bar{h}X + Y + Y^T + Q & P\tilde{A}_h(t) - Y & P\left(\bar{B}_i + \bar{F}\bar{G}_{i1}\bar{N}\right) & \tau\tilde{A}(t)^T Z \\ * & -(1-\mu)Q & 0 & \tau\tilde{A}_h(t)^T Z \\ * & * & -I_m & \tau\tilde{B}(t)^T Z \\ * & * & * & -\tau Z \end{bmatrix} < 0$$

$$\begin{bmatrix} -\alpha P & 0 & \tilde{C}(t)^T \\ * & -(1-\alpha)P & \tilde{C}_h(t)^T \\ * & * & -\gamma^2 I_p \end{bmatrix} < 0$$

Then we have $\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{L}_2\text{-}\mathcal{L}_\infty} < \gamma$. It is easy to show that matrix inequalities (31) and (32) can be rewritten as

$$\bar{\Phi}_{i1} + \bar{\Omega}_1\bar{G}_{i1}\bar{\Lambda}_1 + \left(\bar{\Omega}_1\bar{G}_{i1}\bar{\Lambda}_1\right)^T < 0 \qquad (33)$$

$$\bar{\Phi}_{i2} + \bar{\Omega}_2\bar{G}_{i2}\bar{\Lambda}_2 + \left(\bar{\Omega}_2\bar{G}_{i2}\bar{\Lambda}_2\right)^T < 0 \qquad (34)$$

where $\bar{\Phi}_{ij}$, $\bar{\Omega}_j$, and $\bar{\Lambda}_j$, $i = 1, 2, \ldots, q$, $j = 1, 2$, are defined in (17)–(21). LMIs (33) and (34) have solutions $\bar{G}_{i1}$ and $\bar{G}_{i2}$, if and only if

$$\bar{\Omega}_1^{\perp}\bar{\Phi}_{i1}\bar{\Omega}_1^{\perp T} < 0, \quad \bar{\Lambda}_1^{T\perp}\bar{\Phi}_{i1}\bar{\Lambda}_1^{T\perp T} < 0 \qquad (35)$$

$$\bar{\Omega}_2^{\perp}\bar{\Phi}_{i2}\bar{\Omega}_2^{\perp T} < 0, \quad \bar{\Lambda}_2^{T\perp}\bar{\Phi}_{i2}\bar{\Lambda}_2^{T\perp T} < 0 \qquad (36)$$

$\bar{\Omega}_j^{\perp}$ and $\bar{\Lambda}_j^{T\perp}$, $j = 1, 2$, can be chosen as follows:

$$\bar{\Omega}_1^{\perp} = \begin{bmatrix} \mathcal{I}_1 P^{-1} & 0 & 0 & 0 \\ 0 & I_{\tilde{n}} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & \tau^{-1}\mathcal{I}_1 Z^{-1} \\ -\mathcal{I}_2 P^{-1} & 0 & 0 & \tau^{-1}\mathcal{I}_2 Z^{-1} \end{bmatrix}$$

$$\bar{\Lambda}_1^{T\perp} = \begin{bmatrix} \mathcal{I}_1 & 0 & 0 & 0 \\ 0 & \mathcal{I}_1 & 0 & 0 \\ 0 & 0 & 0 & I_{\tilde{n}} \end{bmatrix}$$

$$\bar{\Omega}_2^{\perp} = \begin{bmatrix} I_{\tilde{n}} & 0 & 0 \\ 0 & I_{\tilde{n}} & 0 \end{bmatrix} \quad \bar{\Lambda}_2^{T\perp} = \begin{bmatrix} \mathcal{I}_1 & 0 & 0 \\ 0 & \mathcal{I}_1 & 0 \\ 0 & 0 & I_p \end{bmatrix}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined in (25). Then, by Schur complement and the conditions in (6) and (7), we obtain that $\bar{\Omega}_1^{\perp}\bar{\Phi}_{i1}\bar{\Omega}_1^{\perp T} < 0$, equals to

$$\begin{bmatrix} \Lambda_{i11} & \Lambda_{i12} & B_i & \Lambda_{i14} & \Lambda_{i15} & \mathcal{I}_1\tilde{P} & \mathcal{I}_1\tilde{P} \\ * & -(1-\mu)Q & 0 & \Lambda_{i12}^T & 0 & 0 & 0 \\ * & * & -I_m & B_i^T & 0 & 0 & 0 \\ * & * & * & \Lambda_{i44} & \Lambda_{i45} & 0 & 0 \\ * & * & * & * & \Lambda_{i55} & -\mathcal{I}_2\tilde{P} & -\mathcal{I}_2\tilde{P} \\ * & * & * & * & * & -\bar{h}^{-1}\tilde{X} & 0 \\ * & * & * & * & * & * & -\tilde{Q} \end{bmatrix} + \mathcal{M}_1 Y^T \mathcal{N}_1 + \left(\mathcal{M}_1 Y^T \mathcal{N}_1\right)^T < 0 \qquad (37)$$

where

$$\mathcal{M}_1 = \begin{bmatrix} \tilde{P}\mathcal{I}_1^T & -I_{\tilde{n}} & 0 & 0 & -\tilde{P}\mathcal{I}_2^T & 0 & 0 \end{bmatrix}^T$$
$$\mathcal{N}_1 = \begin{bmatrix} \tilde{P}\mathcal{I}_1^T & 0 & 0 & 0 & -\tilde{P}\mathcal{I}_2^T & 0 & 0 \end{bmatrix}$$

and Λ_{i11}, Λ_{i12}, Λ_{i14}, Λ_{i15}, Λ_{i44} and Λ_{i45}, $i = 1, 2, \ldots, q$, are defined in (8)–(12). Since for any matrices $\mathcal{M}$, $\mathcal{N}$, Y and $M > 0$ and $N > 0$, $\mathcal{M}Y^T\mathcal{N} + (\mathcal{M}Y^T\mathcal{N})^T \leq \mathcal{M}M\mathcal{M}^T + \mathcal{N}^T N\mathcal{N}$, holds if $YM^{-1}Y^T \leq N$. LMI (37) is true if (1) and (5) hold. $\bar{\Lambda}_1^{T\perp}\bar{\Phi}_{i1}\bar{\Lambda}_1^{T\perp T} < 0$, is equivalent to (2). For the inequalities in (36),

$$\bar{\Omega}_2^{\perp}\bar{\Phi}_{i2}\bar{\Omega}_2^{\perp T} = \begin{bmatrix} -\alpha P & 0 \\ 0 & -(1-\alpha)P \end{bmatrix} < 0$$

and $\bar{\Lambda}_2^{T\perp}\bar{\Phi}_{i2}\bar{\Lambda}_2^{T\perp T} < 0$ is equivalent to (3). From LMIs (1)–(5), if there exist matrices P, $\tilde{P}$, X, $\tilde{X}$, Q, $\tilde{Q}$, M, $\tilde{M}$, N, $\tilde{N}$, Z, $\tilde{Z}$ and Y satisfying (35) and (36), there exist matrices $\bar{G}_{i1}$ and $\bar{G}_{i2}$ such that LMIs (33) and (34) hold. Therefore, we have $\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{L}_2\text{-}\mathcal{L}_\infty} < \gamma$. All the parameters of the reduced order models satisfying (33) and (34) can be constructed by the parametrization method of Gahinet and Apkarian [7]. This completes the proof. □

In the case of $h(t) = h$, a delay-independent sufficient condition is given for the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem from Lemma 2.2 and the proof of Theorem 2.3.

Corollary 2.4: If there exist matrices $P > 0$, $\tilde{P} > 0$, $Q > 0$ and $\tilde{Q} > 0$, and a scalar $0 < \alpha < 1$ such that for $i =$

$1, 2, \ldots, q,$

$$\begin{bmatrix} \mathcal{I}_1\tilde{P}\mathcal{I}_1^T A_i^T + & & & \\ A_i\mathcal{I}_1\tilde{P}\mathcal{I}_1^T & A_{hi}\mathcal{I}_1 & B_i & \mathcal{I}_1\tilde{P} \\ * & -Q & 0 & 0 \\ * & * & -I_m & 0 \\ * & * & * & -\tilde{Q} \end{bmatrix} < 0$$

$$\begin{bmatrix} A_i^T\mathcal{I}_1 P\mathcal{I}_1^T + & \\ \mathcal{I}_1 P\mathcal{I}_1^T A_i + \mathcal{I}_1 Q\mathcal{I}_1^T & \mathcal{I}_1 P\mathcal{I}_1^T A_{hi} \\ * & -\mathcal{I}_1 Q\mathcal{I}_1^T \end{bmatrix} < 0$$

$$\begin{bmatrix} -\alpha\mathcal{I}_1 P\mathcal{I}_1^T & 0 & C_i^T \\ * & -(1-\alpha)\mathcal{I}_1 P\mathcal{I}_1^T & C_{hi}^T \\ * & * & -\gamma^2 I_p \end{bmatrix} < 0$$

$$P\tilde{P} = I_{\tilde{n}}, \quad Q\tilde{Q} = I_{\tilde{n}}$$

there exists a quadratically stable system $\hat{\Sigma}$ with $h(t) = h$ solving the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem with $\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{L}_2\text{-}\mathcal{L}_\infty} < \gamma$.

Remark 2.5: If P, $\tilde{P} = P^{-1}$ and Q have the following special form

$$\begin{aligned} P &= \begin{bmatrix} \hat{X} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \hat{Y}^{-1} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \\ Q &= \begin{bmatrix} \hat{Z} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \end{aligned} \quad (38)$$

Corollary 2.4 can be expressed by LMIs with rank constraint: if there exist matrices $\hat{X} > 0$, $\hat{Y} > 0$ and $\hat{Z} > 0$, and a scalar $0 < \alpha < 1$ such that for $i = 1, 2, \ldots, q$,

$$\begin{bmatrix} A_i^T\hat{Y} + \hat{Y}A_i + & & \\ \hat{Z} + \hat{Y} - \hat{X} & \hat{Y}A_{hi} & \hat{Y}B_i \\ * & -\hat{Z} + \hat{Y} - \hat{X} & 0 \\ * & * & -I_m \end{bmatrix} < 0$$

$$\begin{bmatrix} A_i^T\hat{X} + \hat{X}A_i + \hat{Z} & \hat{X}A_{hi} \\ * & -\hat{Z} \end{bmatrix} < 0$$

$$\begin{bmatrix} -\alpha\hat{X} & 0 & C_i^T \\ * & -(1-\alpha)\hat{X} & C_{hi}^T \\ * & * & -\gamma^2 I_p \end{bmatrix} < 0$$

$$\text{rank}(\hat{Y} - \hat{X}) \leq \hat{n}$$

there exists a quadratically stable system $\hat{\Sigma}$ with $h(t) = h$ solving the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem with $\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{L}_2\text{-}\mathcal{L}_\infty} < \gamma$.

If $h(t) = 0$, $C_h = 0$ and $A_h = 0$, $q = 1$, system Σ reduces to a continuous time-invariant system without delay

$$\Sigma_C: \quad \begin{aligned} \dot{x}(t) &= A_1 x(t) + B_1 u(t) \\ y(t) &= C_1 x(t) \end{aligned}$$

Corollary 2.6: If there exist matrices $P > 0$ and $\tilde{P} > 0$ such that

$$\mathcal{I}_1\tilde{P}\mathcal{I}_1^T A_1^T + A_1\mathcal{I}_1\tilde{P}\mathcal{I}_1^T + B_1 B_1^T < 0 \quad (39)$$

$$A_1^T\mathcal{I}_1 P\mathcal{I}_1^T + \mathcal{I}_1 P\mathcal{I}_1^T A_1 < 0 \quad (40)$$

$$-\mathcal{I}_1 P\mathcal{I}_1^T + \gamma^{-2} C_1^T C_1 < 0 \quad (41)$$

$$P\tilde{P} = I_{n+\hat{n}} \quad (42)$$

there exists an asymptotically stable continuous system $\hat{\Sigma}_C$

$$\hat{\Sigma}_C: \quad \begin{aligned} \dot{\hat{x}}(t) &= \hat{A}_1\hat{x}(t) + \hat{B}_1 u(t) \\ \hat{y}(t) &= \hat{C}_1\hat{x}(t) \end{aligned}$$

that solves the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem with $\left\|\Sigma_C - \hat{\Sigma}_C\right\|_{\mathcal{L}_2\text{-}\mathcal{L}_\infty} < \gamma$. If $P > 0$ has special form (38), conditions (39)–(42) are equivalent to the existence of matrices $\hat{X} > 0$ and $\hat{Y} > 0$ such that $A_1^T\hat{Y} + \hat{Y}A_1 + B_1 B_1^T < 0, A_1^T\hat{X} + \hat{X}A_1 < 0, -\hat{X} + \gamma^{-2}C_1^T C_1 < 0$ and rank$(\hat{Y} - \hat{X}) \leq \hat{n}$, which recovers the result in [11]. In this particular case, the model reduction condition is both necessary and sufficient.

III. Model Reduction Algorithm

In order to use the cone complementarity linearization (CCL) algorithm [5] to solve the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem with fixed $0 < \alpha < 1$, we define a convex set $\mathcal{C} := \{\mathcal{X} \mid \mathcal{X}$ satisfies LMIs (1)–(5)$\}$ and a nonconvex set $\mathcal{T} = \{\mathcal{X} \mid \mathcal{X}$ satisfies the inverse constraints in (6-7)$\}$ where

$$\begin{aligned} \mathcal{X} &= (P > 0,\ \tilde{P} > 0,\ X > 0,\ \tilde{X} > 0,\ Q > 0,\ \tilde{Q} > 0, \\ Z &> 0,\ \tilde{Z} > 0,\ M > 0,\ \tilde{M} > 0,\ N > 0,\ \tilde{N} > 0,\ Y) \end{aligned}$$

The solvability of the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem can be translated into the following nonconvex feasibility problem:

$$\text{Find } \mathcal{X} \in \mathcal{C} \quad \text{subject to } \mathcal{X} \in \mathcal{T} \quad (43)$$

through the sufficient condition in Theorem 2.3. Then, by the CCL approach, the above nonconvex problem (43) has a solution if and only if the minimization problem

$$\min_{\mathcal{X}\in\mathcal{C}\cap\mathcal{C}_{in}} \left\{ \begin{array}{c} \text{trace}(P\tilde{P} + X\tilde{X} + Q\tilde{Q} + \\ Z\tilde{Z} + M\tilde{M} + N\tilde{N}) \end{array} \right\} \quad (44)$$

where

$$\mathcal{C}_{in} = \left\{ \begin{array}{c} \begin{bmatrix} P & I \\ I & \tilde{P} \end{bmatrix} \geq 0, \begin{bmatrix} X & I \\ I & \tilde{X} \end{bmatrix} \geq 0, \\ \begin{bmatrix} Q & I \\ I & \tilde{Q} \end{bmatrix} \geq 0, \begin{bmatrix} Z & I \\ I & \tilde{Z} \end{bmatrix} \geq 0, \\ \begin{bmatrix} M & I \\ I & \tilde{M} \end{bmatrix} \geq 0, \begin{bmatrix} N & I \\ I & \tilde{N} \end{bmatrix} \geq 0 \end{array} \right\}$$

achieves a minimum, $12\tilde{n}$, that is, an optimal solution of problem (44) satisfying

$$\begin{aligned}\text{trace}(P\tilde{P}) &= \text{trace}(X\tilde{X}) = \text{trace}(Q\tilde{Q}) = \text{trace}(Z\tilde{Z}) \\ &= \text{trace}(M\tilde{M}) = \text{trace}(N\tilde{N}) = \tilde{n}\end{aligned}$$

Otherwise (43) is infeasible. Therefore, in order to solve the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem for systems with polytopic uncertainties and time-varying delay, we transform it to a global solution of the minimization problem (44). The CCL algorithm solves problem like (44) effectively although it is still a nonconvex optimization problem [2]. Interested readers may refer to [5] for the details of the CCL algorithm.

An algorithm is presented to solve the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem based on the above analysis.

$\mathcal{L}_2$-$\mathcal{L}_\infty$ CCL Model Reduction Algorithm:

Step 1 Given system Σ, the order of the reduced order model $\hat{n}$, prescribed model reduction error $\gamma > 0$ and $\delta > 0$ is a sufficiently small prescribed a scalar to control the convergence accuracy.

Step 2 Set $k = 0$ and for a fixed α, choose any initial guess

$$\begin{aligned}\mathcal{X}_0 &= (P_0, \tilde{P}_0, X_0, \tilde{X}_0, Q_0, \tilde{Q}_0, Z_0, \\ &\quad \tilde{Z}_0,\ M_0, \tilde{M}_0, N_0,\ \tilde{N}_0,\ Y_0) \in \mathcal{C}\end{aligned}$$

Step 3 Define

$$\begin{aligned}& f_k(P, \tilde{P}, X, \tilde{X}, Q, \tilde{Q}, Z, \tilde{Z}, M, \tilde{M}, N, \tilde{N}) \\ = \ & \text{trace}(P_k\tilde{P} + \tilde{P}_kP + X_k\tilde{X} + \\ & \tilde{X}_kX + Q_k\tilde{Q} + \tilde{Q}_kQ + Z_k\tilde{Z} + \tilde{Z}_kZ \\ & + M_k\tilde{M} + \tilde{M}_kM + N_k\tilde{N} + \tilde{N}_kN)\end{aligned}$$

Solve the following convex minimization problem:

$$\min_{\mathcal{X}\in\mathcal{C}\cap\mathcal{C}_{in}} \left\{ \begin{array}{c} f_k(P, \tilde{P}, X, \tilde{X}, Q, \tilde{Q}, \\ Z, \tilde{Z}, M, \tilde{M}, N, \tilde{N}) \end{array} \right\}$$

and denote the minimizer $\mathcal{X}_k^*$ and compute the minimum value $f_k^* = f_k(\mathcal{X}_k^*)$.

Step 4 If $|f_k^* - 12\tilde{n}| < \delta$, then construct a reduced order model based on (13) and (14); otherwise set $k = k + 1$, and assign

$$\begin{aligned}\mathcal{X}_{k-1}^* &= (P_k,\ \tilde{P}_k, X_k,\ \tilde{X}_k, Q_k, \tilde{Q}_k, Z_k, \\ &\quad \tilde{Z}_k, M_k, \tilde{M}_k, N_k, \tilde{N}_k, Y_k)\end{aligned}$$

and go to step 3.

It can be seen that Step 2 is a simple LMI feasibility problem, and Step 3 is a convex programming with LMI constraints. From the explanation in [2], [5], $\{f_k^*\}$ is decreasing and bounded below by $12\tilde{n}$. Once it converges, then (43) is feasible, which implies that the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problem is solvable for given $\gamma > 0$.

IV. CONCLUSION

This paper presents a delay-dependent sufficient condition for the $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction problems of polytopic systems with time-varying delay. An explicit formula for the construction of reduced order models has been given in terms of a set of the solution of LMIs and inverse constraints. An effective algorithm has been exploited to solve the LMI problems with inverse constraints.

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