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EDGE GUARDS ON A FORTRESS

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Edge Guards on A Fortress

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Abstract

The *fortress* problem was originally posed to determine the number of vertex guards sufficient to cover the exterior of a polygon with n vertices And O'Rourke and Wood [J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, 1987] showed that $\lceil n/2 \rceil$ vertex guards are sometimes necessary and always sufficient In this article, we consider a variation of the problem which allows a guard to patrol an edge of the polygon instead of standing stationarily. Tight bounds of $\lceil n/3 \rceil$ and $\lfloor n/4 \rfloor + 1$ are obtained for general polygons and rectilinear polygons, respectively.

1 Introduction

The fortress problem is a variation of the *art gallery problem*. Refer to [4, 5] for an excellent description of the latter. This paper deals with the following problem: Given a simple polygon with n vertices, select a set of edges such that every point in the exterior of the polygon is *covered* by at least one of the selected edges. An exterior point x is said to be covered by an edge AB if there exists a point y on AB such that xy does not intersect the interior of the polygon. In the real-world analogy of the problem, each selected edge of a polygonal fortress can be thought of as being patrolled by a guard (called an *edge guard*) to protect the fortress against invasion, and thus the problem's name. In section 2, we will show that there exist polygons requiring $\lceil n/3 \rceil$ edge guards and the bound will be proved to be tight. *Rectilinear* polygons (ones whose edges alternate between horizontal and vertical) are considered in section 3. $\lfloor n/4 \rfloor + 1$ edge guards are sometimes necessary and always sufficient in this case.

2 General Polygons

Given a polygon, each connected region inside its convex hull but exterior to the polygon is called a *pocket*. For example in figure 1, p_1, p_2, and p_3 are pockets where p_1 has 2 edges, p_2 has 3 edges, while p_3 has 4 edges. A *triangulation graph* of a pocket is a graph whose embedding is a triangulation of the pocket. Its vertices are the vertices of the pocket and its edges are the edges of the pocket plus the diagonals of the triangulation. A triangulation graph is *dominated* by a set of edges (guards) if every triangular face of the graph shares a vertex with at least one of the guards. Note that domination of a triangulation graph is a stricter requirement than the corresponding geometric problem.

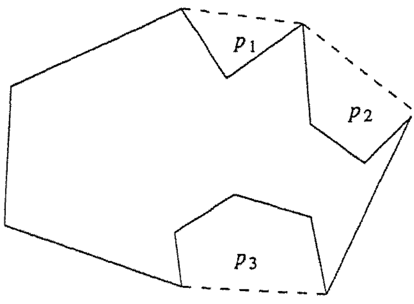

Figure 1

Lemma 1: If the polygon is convex or all of its pockets have less than four edges, then $\lceil n/3 \rceil$ edge guards suffice to cover the exterior of the polygon. Also the guards dominate any triangulation graphs of the pockets.

Proof: Color the vertices of the polygon with colors 0, 1 and 2 in a clockwise direction starting at an arbitrary vertex. The guards are then placed on the edges which start with color 0. There are at most $\lceil n/3 \rceil$ guards. Divide the exterior of the polygon into two types of regions (figure 2), R_1 and R_2. Each R_1 region consists of one of the pockets and the region enclosed between the extensions of its two side edges. R_2 regions are those lying between two R_1 regions. In the case where there is only one R_1 region, the remainder of the exterior will form an R_2 region. An R_2 region may be empty and two adjacent R_1 regions may overlap.

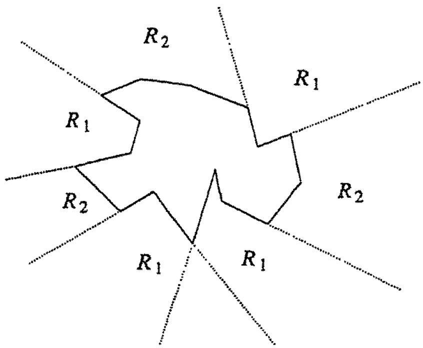

Figure 2

For a pocket with two edges, an edge guard will be able to reach either vertex a or b and certainly R_1 is covered and the guard will also dominate the triangulation graph of that pocket (figure 3a). For a pocket with three edges, either a, d or b, c can be reached by guards. Either pair of vertices will cover the entire R_1. By examining the two different ways to triangulate the pocket, it can be shown that the pocket is dominated by the two guards too (figure 3b).

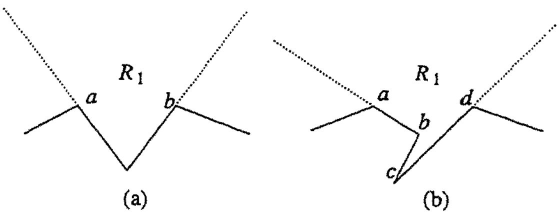

Figure 3

Since every other hull vertex of R_2 regions can be reached by a guard, all R_2 regions will be covered. Note that we only need to ensure that a guard can reach the two end points of each guard edge.

Lemma 2 [4]: After the pockets are triangulated, among all pockets of more than 3 edges, there exists a diagonal d which cuts off a region with exactly 4, 5, or 6 edges.

Proof [4]: Suppose d is the diagonal that cuts off a minimum number of k edges with $k > 3$ (figure 4). Let ABC be a triangle supported by d as shown where C is the vertex in the region of the pocket cut off by d. Due to the minimality of k, the number of edges cut off by AC or BC must be less than 4, so $6 \geq k \geq 4$.

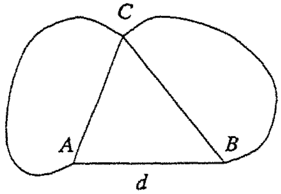

Figure 4

Theorem 1: $\lceil n/3 \rceil$ edge guards are sometimes necessary and always sufficient to cover the exterior of a polygon with n vertices and the guards can be chosen to dominate all triangulation graphs of the pockets. Visibility is required only at the end points of each edge guard

The necessity is shown by a convex polygon. Guards are required at every three edges of the polygon.

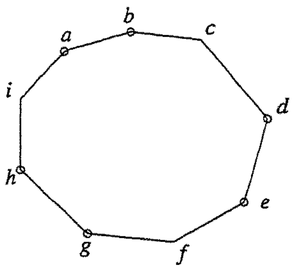

Figure 5

The sufficiency will be proved by induction on n. Lemma 1 establishes the induction basis. If the polygon has no pocket with more than 3 edges, lemma 1 applies. By lemma 2, there is a diagonal d which cuts off a small region from the pocket, so the hypothesis can be applied to the remaining part. Three cases need to be considered in covering the region cut off.

Case 1: d cuts off 4 edges.

The region cut off from the pocket can be covered and dominated by two edge guards with end points at A and B or one edge guard at position g in figures 6a and 6b.

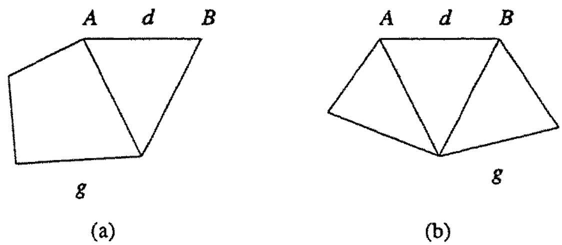

Figure 6

Having replaced the four edges by the diagonal d, the polygon will have $(n - 3)$ edges. By the hypothesis, $\lceil n/3 \rceil - 1$ edge guards are sufficient to cover the exterior and dominate all triangulation graphs of the pockets. However, if a guard is placed at d when applying the induction hypotheses, it can always be replaced by two edge guards with end points at A and B. Since the construction only requires visibility at the end points of the guard edges, the result follows.

Case 2: d cuts off 5 edges.

Again, the region that is cut off can be dominated and covered by one edge guard (position g in figure 7) or two edge guards with end points at A and B. The result is established similar to case 1.

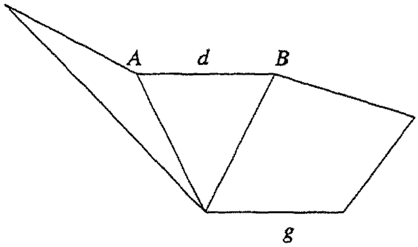

Figure 7

Case 3: d cuts off 6 edges.

There are four configurations in which the region that is cut off can be triangulated (figure 8). In case (a), one edge guard at C suffices to dominate and cover the cut-off region. In case (b), one edge guard at CD suffices, and in case (c), an edge guard at CE suffices. And in case (d), one edge guard at CD and one at vertex B, or one edge guard at CE and one at vertex A can dominate and cover the region. In all four cases, two edge guards with end points at A and B will be sufficient to dominate and cover the region.

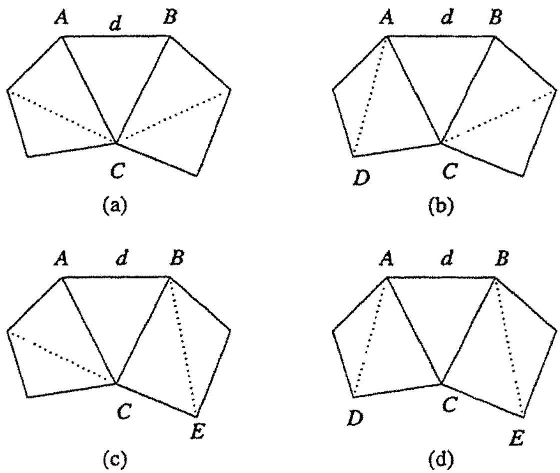

Figure 8

Instead of cutting through d, cut along diagonals AC and BC. The resulting polygon will have $(n-4)$ edges. By hypothesis, $\lceil n/3 \rceil - 1$ guards are sufficient to cover the exterior and dominate all triangulation graphs of the pockets. Also, by hypothesis, triangle ABC is dominated and at least one of the vertices of ABC will be covered. In all cases, one more edge guard suffices to cover and dominate the cut-off region.

3 Rectilinear Polygons

To show that $\lfloor n/4 \rfloor + 1$ edge guards are sufficient for a rectilinear polygon, it is enclosed in a rectangle and its horizontal edges are extended from its convex vertices to partition the region between it and the rectangle as shown in figure 9a. This partitioning method is the same as the ***rectangular decomposition*** used in [3]. The region is decomposed into rectangles. To cover the exterior of the polygon is equivalent to covering all resulting rectangles since the enclosing rectangle can be arbitrarily large. Assume the polygon is in general position (that is, no two nonadjacent convex vertices have the same y coordinates), the total number of such rectangles will be $n/2 + 2$. We can assume the given rectilinear polygon is in general position without loss of generality (see [3, 4]).

This set of rectangles has the following properties (see also [3]). Two rectangles are *adjacent* if they share a horizontal chord. Any two adjacent rectangles can be covered by a vertex guard (or an edge guard). The dual of the partition (the graph in which each rectangle is represented by a node and two nodes are connected if the corresponding rectangles share a horizontal chord, see figure 9b and [3]) is simply a cycle with attached trees. If all the attached trees have only one node, we say that the graph is in *reduced configuration* (cf. reduced triangulation in section 5.2 of [4]). Nodes on the cycle are called *cycle nodes*; all other nodes are called *tree nodes.* A cycle node which has a tree attached to it is called a *root* node. A node such as node 12 is called the parent of nodes 13 and 14. The topmost (or bottommost) edge of the polygon can cover at least three rectangles. For example, the topmost edge of the polygon in figure 9(a) covers rectangles 1, 2, and 9.

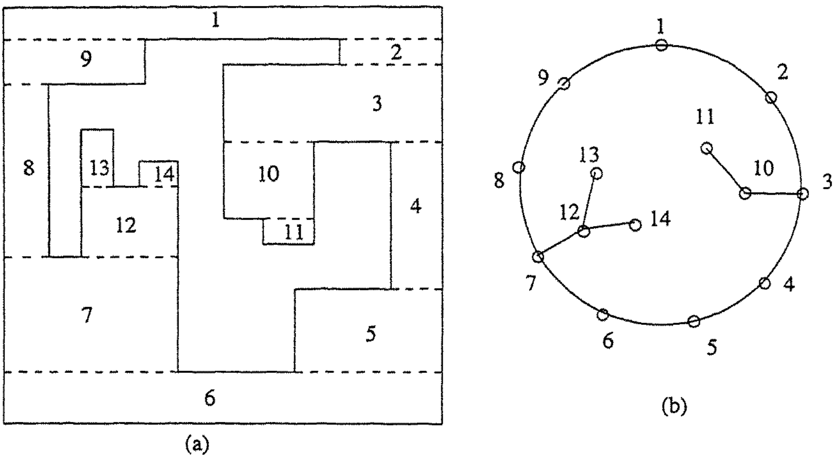

Figure 9

To prove the main result of this section, an example of a family of polygons will be given to establish the necessary number of guards. For sufficiency, we first show that for any rectilinear polygon whose partition is in reduced configuration, $\lfloor n/4 \rfloor + 1$ guards suffice to cover the exterior. Then it will be shown that a rectilinear polygon whose partition is in general configuration can be reduced to one in reduced configuration while maintaining the correct number of guards required.

Theorem 2: $\lfloor n/4 \rfloor + 1$ edge guards are sometimes necessary and always sufficient to cover the exterior of a rectilinear polygon with n vertices.

The necessity is demonstrated by the following figures. In figure 10a, a polygon with 12 vertices requires 4 edge guards. In figure 10b, an addition of 4 vertices necessitate one additional guard. Figure 10c generalizes to polygons with number of edges equal to $4k$ for all values of $k > 2$. The number of guards necessary, $\lfloor n/4 \rfloor + 1$, can be easily established.

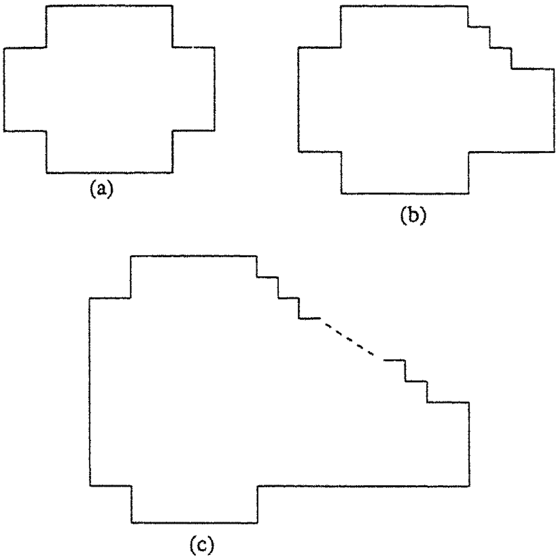

Figure 10

Lemma 3: Each root node is adjacent to at most two tree nodes.

Proof: Each root node is adjacent to two other cycle nodes and the corresponding rectangles of these three cycle nodes must share the same vertical edge of the enclosing rectangle. Only two more rectangles can be attached to it as shown in figures 11a and 11b because the polygon is in general position. (In the figures, A is the root node, D and E are the adjacent cycle nodes and B and C are the adjacent tree nodes.)

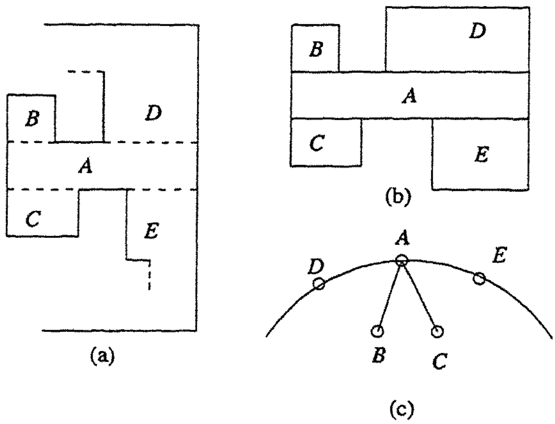

Figure 11

Note also that A, B, and C must share the same vertical edge of the polygon because the polygon is in general position. For example, if B does not share any vertical edge with A as shown in figure 12, x and y are not in general position. In fact, any two adjacent rectangles must share a vertical edge (for a similar argument, refer to [3]).

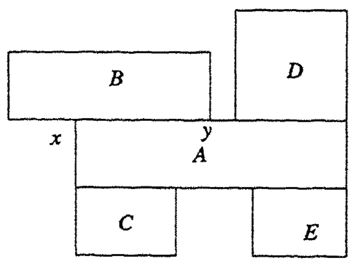

Figure 12

Lemma 4: If a root node is adjacent to two tree nodes, one guard is sufficient to cover all three corresponding rectangles. If a root node is adjacent to exactly one tree node, one guard suffices to cover the two corresponding rectangles and can in addition cover the rectangle corresponding to either of the two adjacent cycle nodes.

Proof: The proof enumerates the possible positions of the guard in figure 13. If a root node is adjacent to two tree nodes, there are only two possible configurations depending on the cycle nodes are on the right or on the left, but they are symmetric. Figure 13a shows one of the configurations where the position of the guard is marked by g and A is the root node, D and E are the adjacent cycle node, and B, C are the adjacent tree nodes. If the root node is adjacent to only one tree node, there are several possible configurations. Three cases result after removal of symmetric ones. Figure 13b, 13c, and 13d show the configurations where the cycle nodes are on the right and the tree node is on top. To cover A, B and E, the position of the guard is shown in figure 13b or 13c depending on the shape of E and to cover A, B and D, the position of the guard is shown in figure 13d.

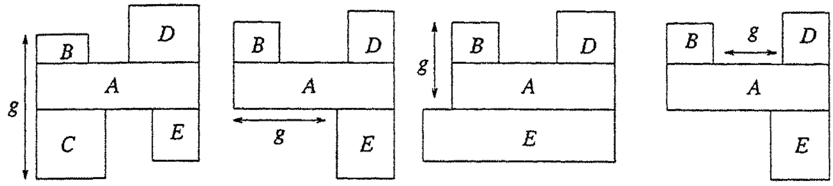

Figure 13

Lemma 5: To cover the exterior of any rectilinear polygon whose partition is in reduced configuration, $\lfloor n/4 \rfloor + 1$ edge guards are sufficient.

Proof: The proof assigns guards to cover the rectangles while maintaining that the average number of rectangles covered by each guard is at least two. All root nodes are grouped with their own tree nodes if the latter exist. By lemma 4, one guard can be assigned to cover each group of rectangles. Then, starting from any root node, consider the cycle nodes between two consecutive root nodes in clockwise order. If there are an even number of cycle nodes, each two of them can be grouped. Otherwise, depending on the number of attached tree nodes of the first root node, choose a different method to group the nodes as follows.

If the root node is adjacent to only one tree node, include the next cycle node into the group of this root node as shown in figure 14. By lemma 4, this group can be covered by a single edge guard.

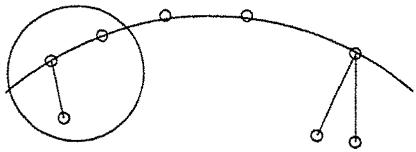

Figure 14

If the root node is adjacent to two tree nodes, group the next cycle node in its own group. On average, each guard will still cover at least two rectangles.

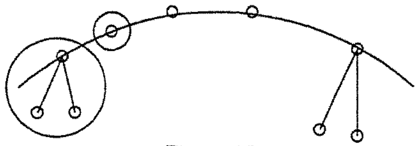

Figure 15

After this procedure, an even number of cycle nodes will remain between two consecutive root nodes. Group these in groups of two. Since rectangles in the same group can be covered by one guard, each guard is responsible for at least two rectangles. If the number of rectangles is odd, since every group of one rectangle is paired with a group of three rectangles, there must be an extra group of three rectangles, so the total number of guards required is $\lfloor (n/2+2)/2 \rfloor$ which is equal to $\lfloor n/4 \rfloor + 1$. There is a special case in which no root node exists. If the number of cycle nodes is even, group every two of them. Otherwise, group the three rectangles which can be covered by the topmost edge as a group, and group the rest of the cycle nodes two by two. Again, $\lfloor n/4 \rfloor + 1$ guards are used. The result follows.

Lemma 6: Every rectilinear polygon whose partition is in general configuration can be changed to one in reduced configuration, while maintaining the invariant that every assigned guard covers at least two rectangles.

Proof: In a manner similar to a root node, a tree node can be adjacent to at most four nodes of which one is a parent node which lies on the path back to the cycle. To reduce the polygon, the following procedure is repeated. Starting at the leaves, a leaf is grouped with its parent if the parent is of degree 2. The group is removed. Continue until we cannot proceed further. At this point, either the tree is reduced or a parent has degree more than two. The rectangles are then grouped as follows depending on the degree of the parent. Let A be the parent of the leaf or leaves.

(a) If A is a root node, then stop. This tree is reduced.

(b) If A is not a root node and is of degree three, group A with the two attached leaves as shown in figures 16a, 16b, 16c, and 16d (four possible configurations for the two leaves assume the parent node D is on the right top corner) and remove the whole group. One guard suffices to cover all removed rectangles. In the figures, B and C are leaves. The guard is placed at g to cover the removed part.

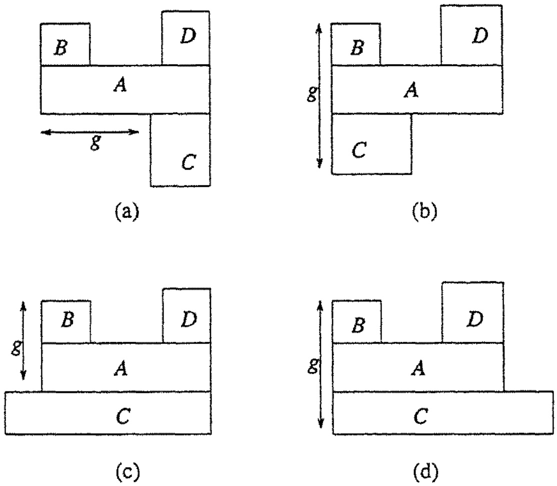

Figure 16

However, in figure 16a, the position of g may not be an edge of the original polygon because of previously removed rectangles. In this case, we must reshuffle the grouping of A, B, and C together with rectangles originally connected to the position of g in the original polygon. Let R denote the rectangle connected immediately to A originally (figure 17). R will have degree less than four as can be easily verified. In fact, no two nodes with degree four can be next to each other. So, two subcases have to be considered depending on the number of rectangles in the group of R.

If the number of rectangles is 2 where R and P are in the same group, there are three subcases depending on the size and the position of P. In figures 17a and 17b, regroup the rectangles into two groups, A, B, R, and P as one group and C as another. In figure 17c, R, P, and C will be one group while A and B will be another. The guards for the two groups can be placed at the positions g_1 and g_2 in each case.

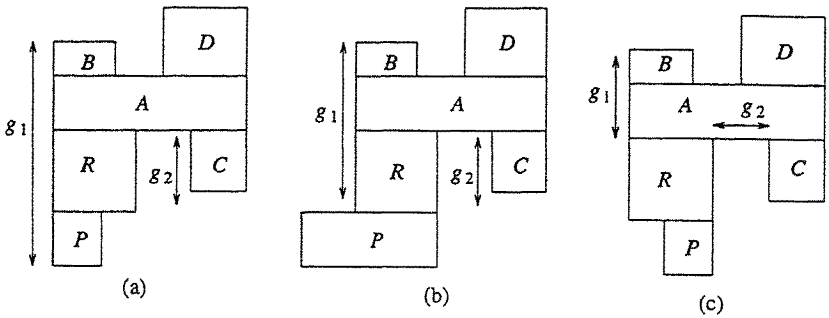

Figure 17

If the number of rectangles is 3, group the rectangles as shown figure 18. B, A, R and P will be in the same group while Q and C will be another. The required guards can be placed at positions g_1 and g_2. In each of the above cases, two groups requires two guards and a total of at least five rectangles are covered, so the invariant of the procedure is still maintained.

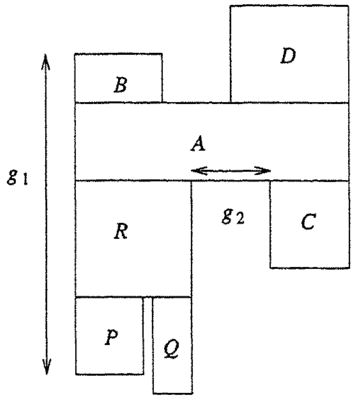

Figure 18

(c) If A is not a root node and is of degree four, group A with the three attached leaves into two groups (A, B, and C form one group while E forms the other, see figure 19) Again two groups require two guards and four rectangles are covered, so the invariant of the procedure is still maintained.

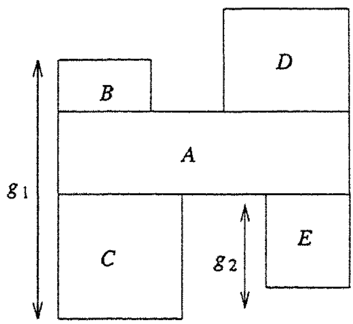

Figure 19

The whole procedure is repeated until all trees are reduced. Then lemma 5 is applied to the resulting polygon. If lemma 5 requires that a guard be placed at a position which is not an edge of the original polygon (only possible in figure 13b), the grouping can be reshuffled as in figure 17. This completes the proof of theorem 2.

4 Related Problems

In this paper, two results concerning edge guards were obtained: to cover the exterior of a general polygon, $\lceil n/3 \rceil$ edge guards are sometimes necessary and sufficient, while for a rectilinear polygon, $\lfloor n/4 \rfloor + 1$ edge guards suffice and the bound is tight.

Most other results for edge guards solved the problem of covering the interior of a polygon instead of the exterior. These results include [1], [2], and [7]. These papers investigated the number of edge guards required to cover the interior of polygons with specific shapes such as monotone, rectilinear monotone, and spiral polygons. Several tight bounds were obtained. However, for general rectilinear polygons and simple polygons, tight bounds of the number of edge guards are not known until now [4, 5]. Other related open problems include finding a minimum number of edge guards sufficient for any star-shaped polygons. In [6], it was shown that $\lfloor n/5 \rfloor$ edge guards are not sufficient.

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