International Journal of Theoretical and Applied Finance

COHERENT RISK MEASURES FOR DERIVATIVES UNDER BLACK-SCHOLES ECONOMY

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ABSTRACT

This paper proposes a risk measure for a portfolio of European-style derivative securities over a fixed time horizon under the Black-Scholes economy. The proposed risk measure is scenario-based along the same line as Artzner et al. (1999). The risk measure is constructed by using the risk-neutral probability ($\mathcal{Q}$-measure), the physical probability ($\mathcal{P}$-measure) and a family of subjective probability measures. The subjective probabilities are introduced by using Girsanov's theorem. In this way, we provide risk managers or regulators with the flexibility of adjusting the risk measure according to their risk preferences and subjective beliefs. The advantages of the proposed measure are that it is easy to implement and that it satisfies the four desirable properties introduced in Artzner et al. (1999), which make it a coherent risk measure. Finally, we incorporate the presence of transaction costs into our framework.

Keywords: Coherent risk measure, Black-Scholes model, risk-neutral probability measure, physical probability measure, subjective probability measures.

1. Introduction

Risk management is one of the most important tasks in the insurance and finance industries. Investment banks and financial institutions around the world seek various techniques to manage their risks. Due to the rapid development of derivative markets, the tasks of risk management become more challenging. This accelerates the development of more advanced risk-management techniques. Besides many other issues in the practice of risk management, such as utility functions, decision theory, etc., one of the key issues and the first step is to construct a proper measure of risk. Traditionally, volatility has been a commonly used risk measure by the finance community. Recently, Value-at-Risk (VaR) has also become a very popular risk measure. VaR is an attempt to summarize the total risk of a portfolio by a single number which is a statistical estimation of a portfolio loss with the property that, with a given (small) probability, we stand to incur that loss or more over a given (typically short) holding period. See Embrechts (2000) and J.P. Morgan's Risk Metrics – Technical Document for an introduction and the paper by Duffie and Pan (1997) for a survey. As noted in Duffie and Pan (1997), VaR is not easy to calculate if the portfolio contains derivatives. Perhaps the most effective and easiest way to obtain the VaR when the portfolio contains derivatives is by the Monte Carlo simulation. Furthermore, as noted in Artzner et al. (1999), VaR does not satisfy the subadditivity property in general, especially if the portfolio contains derivatives. This makes the investigation and construction of risk measures for derivative securities an interesting and important issue.

Artzner et al. (1999) studied methods of measuring both market and non-market risk. They presented and justified a set of four desirable properties (translation invariance, positive homogeneity, monotonicity and subadditivity) for risk measures. A risk measure which satisfies the four properties is called a coherent risk measure. Furthermore, they provided a representation form of coherent risk measures as the supremum of the expected loss of a portfolio with respect to a family of probability measures. By interpreting each probability measure as a generalized "scenario", a scenario-based risk measure, which generalises the margin system SPAN (Standard Portfolio Analysis of Risk) developed by the Chicago Mercantile Exchange, has been proposed. Motivated by Artzner et al. (1999), Cvitanic and Karatzas (1999) have studied the dynamic measures of risk which are also discussed in Föllmer and Leukert (1999).

Measuring risk for a portfolio of non-linear instruments, such as derivative securities with non-linear payoff functions, has attracted the attention of both researchers and financial practitioners. In particular, VaR of a portfolio of non-linear instruments has been studied extensively by several authors. See, for example, J.P. Morgan's Risk Metrics-Technical Document, Duffie and Pan (1997) and Jahel et al. (1998). In the literature, there are two commonly used approaches to the calculation of VaR for non-linear instruments. One is the so-called delta-gamma approach. A portfolio of non-linear instruments can be decomposed in terms of the portfolio's delta and gamma. Another approach is to calculate the portfolio's moments and then find a distribution that matches the portfolio's moments as closely as possible. However, in both approaches, the distribution of the portfolio is built from those of the individual instruments. If the portfolio consists of a significant number of non-linear instruments, the task of finding the distribution of the portfolio's return becomes very tedious.

In Siu and Yang (1999), a risk measure for a portfolio of European-type derivative securities over a fixed time horizon in the context of a multiplicative binomial model was proposed. Under the discrete-time binomial model, a risk measure which is easy to implement and satisfies the four coherent properties introduced by Artzner et al. (1999) was provided. This paper can be considered as a continuous counterpart of Siu and Yang (1999). By following the representation form of coherent risk measures introduced by Artzner et al. (1999), our risk measure is also a scenario-based risk measure which involves the use of the risk-neutral probability ($\mathcal{Q}$-measure) , the physical probability ($\mathcal{P}$-measure) and a family of subjective probability measures. Here, the physical probability $\mathcal{P}$, which is also called the statistical/data-generating probability, is the underlying probability law that drives the realization of the stock-price movement. It is objective and unique. In practice, the underlying probability law is not known but we can estimate it through the use of some statistical techniques. A subjective probability measure is assigned according to an agent's subjective beliefs and risk preference. Its assignment needs not be subjected to a general agreement. It is difficult to apply statistical methods for evaluating whether a subjective probability is "well-chosen" or not. See Focardi and Jonas (1997) and Wang (1999) for details. We organize this paper as follows:

Section 2 deals with the classical Black-Scholes model consisting of two primary traded securities: a risky asset (a stock) and a risk-free investment (a bond). We define a risk measure for a European call option to illustrate the idea of our approach. We use the celebrated Black-Scholes formula (see Black and Scholes (1973) or Merton (1973)) to calculate the call price. We use the risk-neutral valuation approach here. The idea of a risk-neutral valuation can be found in Cox and Ross (1976). Duffie (1996) and Musiela and Rutkowski (1997) provided an introduction to the risk-neutral approach as well as other aspects of asset pricing. By the Fundamental Theorem of Asset Pricing, the condition of no arbitrage is essentially equivalent to the existence of a risk-neutral measure (see Harrison and Kreps (1979), Harrison and Pliska (1981) and Dybyig and Ross (1987)). In order to incorporate the risk manager's/regulator's risk preference and subjective beliefs, we introduce a family of subjective probability measures by using Girsanov's theorem. We calculate the expected loss of a portfolio over a fixed time interval with respect to the family of subjective probability measures.

In section 3, we assume that our financial model consists of several risky assets and one risk-free money market. We propose a risk measure for a portfolio of European-type derivative securities with different maturities and written on different underlying assets. It is worth pointing out that the proposed risk measure still satisfies the four coherent properties introduced in Artzner et al. (1999).

Section 4 investigates exotic options. We will use the barrier option as an example. Many other exotic options can be treated similarly.

Section 5 constructs a risk measure for a portfolio of a single call option in the presence of transaction costs. In Leland (1985), an option pricing formula in the presence of small proportional transaction costs was obtained. The risk measure is constructed by making use of Leland's pricing formula.

Finally, in section 6, we summarize the main results from this paper, point out the limitations of our model and suggest some possible topics for further research.

2. Risk Measures for Vanilla European Derivatives in the Univariate Black-Scholes Model

In this section, we deal with plain vanilla European derivatives under the standard Black-Scholes assumptions. We consider a financial market consisting of one risk-free bond B and one risky asset S with the time horizon $[0, T]$, during which all economic activities take place. Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where $\mathcal{P}$ is the physical probability measure. We assume that the physical probability measure $\mathcal{P}$ is given or known. Let r, μ and σ be the risk-free interest rate of the bond B, the expected return, and the volatility of the stock S, respectively. We assume that r, μ and σ are given constants. Let $\{W_t\}$ be a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Then, the evolution of the bond-price process $\{B_t\}$ and the stock-price process $\{S_t\}$ are governed by the following equations:

$$\begin{aligned} d\,B_t &= r\,B_t\,dt\ , \quad B_0 = 1 \\ d\,S_t &= \mu\,S_t\,dt + \sigma\,S_t\,d\,W_t\ , \quad S_0 = s\ . \end{aligned} \tag{1}$$

Let $\{\mathcal{F}_t^W\}$ denote the natural filtration generated by the Brownian motion $\{W_t\}$. That is, $\mathcal{F}_t^W = \sigma\{W_u\,|u \in [0,t]\}$, $t \in [0,T]$. Then, we equip our sample space $(\Omega, \mathcal{F})$ with the filtration $\{\mathcal{F}_t\}$ which is defined as the $\mathcal{P}$-augmentation of $\{\mathcal{F}_t^W\}$.

In order to focus on the main idea of our model, we use the standard European call option with strike price K, maturing at time T and written on the underlying asset S as an illustration. For the valuation of the call option, we use the risk-neutral probability measure $\mathcal{Q}$ which is defined by the following Radon-Nikodym derivative:

$$\left.\frac{d\mathcal{Q}}{d\mathcal{P}}\right|_{\mathcal{F}_t} = \exp\left\{ -\left(\frac{\mu - r}{\sigma}\right)W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t \right\}, \quad t \in [0,T] \tag{2}$$

Then, by Girsanov's theorem,

$$\tilde{W}_t = W_t + \frac{\mu - r}{\sigma}t\ , \quad t \in [0,T] \tag{3}$$

is a standard Brownian motion with respect to $\{\mathcal{F}_t\}$ under $\mathcal{Q}$.

Let $\{Z_t\}$ be the discounted stock-price process $\{e^{-rt}S_t\}$. Then, by using (1), (3) and Itô's formula, it is easy to check that, under $\mathcal{Q}$,

$$d\,Z_t = \sigma\,Z_t\,d\,\tilde{W}_t \tag{4}$$

Thus, $\{Z_t\}$ is a $\mathcal{Q}$-martingale with respect to $\{\mathcal{F}_t\}$. Hence, the current price of the call option at time t is given by

$$C(t) = e^{-r(T-t)}E_{\mathcal{Q}}\{\max(S_T - K, 0)\,|\,\mathcal{F}_t\} \tag{5}$$

Under the usual Black-Scholes assumptions, the expectation (5) can be evaluated to give the celebrated Black-Scholes formula:

$$C(t) = S_t\Phi(d_1) - K\,e^{-r(T-t)}\Phi(d_2) \tag{6}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\ , \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

and $\Phi(\cdot)$ is the c.d.f. of the standard normal distribution.

To simplify the notations, we write the formula for the Black-Scholes call price in (6) as a function $C(S_t, t, K)$ of the current stock price S_t, the current time t and the strike price K.

Given the market (price) information up to the time t ($\mathcal{F}_t$), we define the future net worth of the call option over the future short time horizon $[t, t+h]$ (h is small) as a random variable $C(t+h) - e^{rh}C(t)$, denoted as $\Delta\, C_{t,h}$. For the purpose of measuring risk, it is well-accepted by the finance community that the physical probability measure $\mathcal{P}$ is normally used to calculate the distribution of the future net worth (Profit/Loss distribution) of a portfolio over a fixed future time horizon. However, Ait Sahjlia and Lo (1997) suggested the use of the risk-neutral probability for the calculations of risk measures, in particular the VaR calculations, since they argue that an economic value should be placed on measuring the losses of a portfolio. Here, we use a family of subjective probability measures in order to incorporate the risk manager's/regulator's risk preference and subjective beliefs. We define a family of subjective probabilities equivalent to the $\mathcal{P}$-measure as follows:

Let Λ be an interval $[a, b]$ in $\mathcal{R}$. For each $\lambda \in \Lambda$, we define the subjective probability measure $\mathcal{P}_\lambda \sim \mathcal{P}$ associated with λ by the following Radon-Nikodym derivative:

$$\left.\frac{d\mathcal{P}_\lambda}{d\mathcal{P}}\right|_{\mathcal{F}_t} = \exp\left\{ -\left(\frac{\mu-\lambda}{\sigma}\right) W_t - \frac{1}{2}\left(\frac{\mu-\lambda}{\sigma}\right)^2 t\right\}, \quad t \in [0, T] \tag{7}$$

Then, by Girsanov's theorem,

$$W_t^\lambda = W_t + \frac{\mu-\lambda}{\sigma}\, t\,, \quad t \in [0, T] \tag{8}$$

is a standard Brownian motion with respect to $\{\mathcal{F}_t\}$ under $\mathcal{P}_\lambda$. Hence, under $\mathcal{P}_\lambda$, (1) can be rewritten as

$$d\, S_t = \lambda\, S_t\, dt + \sigma\, S_t\, d\, W_t^\lambda \tag{9}$$

Let P_Λ be a family of subjective probability measures $\{\mathcal{P}_\lambda\}_{\lambda\in\Lambda}$ associated with the index set Λ. Then, by following the representation form of coherent risk measures in Artzner et al. (1999), we define a risk measure for the call option over $[t, t+h]$ with respect to P_Λ and $\mathcal{F}_t$ as follows:

$$\rho_{P_\Lambda}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big) = \sup\big\{ -E_{\mathcal{P}_\lambda}\big(e^{-rh}\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big) \,|\, \lambda \in \Lambda\big\} \tag{10}$$

Note that the risk measure (10) involves a double expectation with the outer expectation taken under the $\mathcal{P}_\lambda$-measure and the inner expectation taken under the $\mathcal{Q}$-measure.

From standard calculations, the conditional expectation $E_{\mathcal{P}_\lambda}\big(C(t+h) \,|\, \mathcal{F}_t\big)$, for each $\lambda \in \Lambda$, can be obtained as follows (see Cox and Rubinstein (1985) or Boyle and Yang (1998)) :

$$E_{\mathcal{P}_\lambda}\big(C(t+h) \,|\, \mathcal{F}_t\big) = C\big(S_t\, e^{\lambda h}, t, K e^{rh}\big) \tag{11}$$

Note that the expression (11) is given by the Black-Scholes formula (6) with the same time to maturity, the same volatility and the same interest rate, but a higher input asset price $S_t\, e^{\lambda h}$ and a higher input strike price $K e^{rh}$. The new asset price $S_t\, e^{\lambda h}$ is equal to the expected value (under $\mathcal{P}_\lambda$) of the asset price at time $(t+h)$, namely $E_{\mathcal{P}_\lambda}(S_{t+h} \,|\, \mathcal{F}_t) = S_t\, e^{\lambda h}$.

Hence, the risk measure (10) can be rewritten in the following form:

$$\rho_{P_\Lambda}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big) = -e^{-rh} C(S_t\, e^{ah}, t, K\, e^{rh}) + C(S_t, t, K) \tag{12}$$

The risk measure (10) can be applied to evaluate the risk of a portfolio consisting of several call and put options with different strike prices and maturities, but written on the same underlying asset S. As long as the components (the numbers of units of call and put options) within the portfolio have been specified, the closed-form expression of the risk measure, which is similar to the expression given in (12), can be obtained. Furthermore, it is worth pointing out that our risk measure is defined from the viewpoint of a buyer. From a writer's point of view, the future net loss of the call option becomes $\Delta\, C_{t,h}$. This means that the future net loss from the writer's viewpoint is the negative of the future net loss from the buyer's perspective. Because of the different views between the writer and the buyer and their risk preferences, they may choose different index sets Λ. For instance, since the writer (buyer) faces the risk due to the upward (downward) stock-price movement, the writer (buyer) may choose the index set Λ containing more upward (downward) stock-price "scenarios". Finally, it is clear that the risk measure (12) satisfies the four coherent properties introduced in Artzner et al. (1999).

Remarks:

- Our risk measure deals with the risk of an unhedged/naked position of options due to the adverse market price movement, while Cvitanic and Karatzas (1999) addressed the problem of measuring the risk of incomplete hedging. Due to the rapid growth in the trading volume of derivatives in the secondary markets, our risk measure is of practical relevance in evaluating the speculative losses from the derivative markets. One common feature between our risk measure and the proposed risk measure in Cvitanic and Karatzas (1999) is that both are scenario-based.
- For each fixed "scenario" $\lambda \in \Lambda$, $E_{\mathcal{P}_\lambda}(-e^{-rh}\Delta C_{t,h} \mid \mathcal{F}_t)$ is the best estimate of the discounted-future-net-loss $-e^{-rh}\Delta C_{t,h}$ in the expected squared-loss-error sense with respect to the measure $\mathcal{P}_\lambda$. Hence, our risk measure is just the best estimate of the loss $-e^{-rh}\Delta C_{t,h}$ under the worst-case "scenario" over Λ. In fact, the concept of VaR and our risk measure are quite different. VaR concerns the statistical estimation of the loss of a portfolio with a certain degree of confidence (or probability level) while our risk measure deals with the estimation of the loss of the portfolio under the worst-case "scenario". Nevertheless, both risk measures express the concept of risk as the portfolio's loss in domestic monetary units. By changing probability measures, we can shift the conditional mean of the discounted-net-loss $-e^{-rh}\Delta\, C_{t,h}$ given $\mathcal{F}_t$

under the $\mathcal{P}_\lambda$-measure to the α-quantile of the conditional distribution of $-e^{-rh}\Delta\, C_{t,h}$ given $\mathcal{F}_t$ under the original $\mathcal{P}$-measure for a given $\alpha \in (0,1)$. That is, there exists an index set Λ_α associated with a given confidence level α such that $\rho_{P_\Lambda}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big) = VaR_{\mathcal{P},\alpha}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big) := \sup\big\{x \in \mathcal{R} \mid \mathcal{P}\big(-e^{-rh}\Delta C_{t,h} > x \mid \mathcal{F}_t\big) > \alpha\big\}$. This somehow suggests a way to relate our risk measure to the VaR measure in the case of the European call option.

- In our model, the index set Λ is chosen according to one's subjective beliefs and risk preference. One may argue that it is not easy to implement our risk measure in some practical situations since the choice of the index set involves human judgment or the practitioner's subjective beliefs. However, we contend that the role of human judgment provides an important way to improve risk measurement. Holton (1997) pointed out the subjective nature of risk and the inappropriateness of neglecting the role of human judgment for measuring risk. In Artzner et al. (1997), it has been mentioned that the only way to improve risk management is to think before calculating risk measure. In Delbaen (1999) and Wang (1999), it is also pointed out that the choice of probability measures for measuring risk is quite subjective. Furthermore, we notice that $E_{\mathcal{Q}}\big(-e^{-rh}\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big) = 0$. This implies that the risk-neutral probability can serve as a reference point for our risk measure. Suppose an agent chooses an index set Λ for applying our risk measure. Then, the risk measure $\rho_{P_\Lambda}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big)$ is positive (zero, negative) if and only if the agent is risk-averse (risk-neutral, risk-taking). This property plays a similar role with the utility functions in financial economics.
- If $\Lambda_1 \subseteq \Lambda_2$, $\rho_{P_{\Lambda_1}}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big) \leq \rho_{P_{\Lambda_2}}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big)$. This means that the more "scenarios" you consider, the more conservative the risk measures obtained are. If an investor is more risk-averse, he/she may choose a set Λ with a smaller a and/or a larger b, and hence enlarge the risk measure.
- $\rho_{P_\Lambda}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big)$ can be interpreted as the margin requirement that should be charged in order to withstand the risk of the portfolio. If $\rho_{P_\Lambda}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big)$ is negative, $-\rho_{P_\Lambda}\big(\Delta\, C_{t,h} \,|\, \mathcal{F}_t\big)$ can be interpreted as the cash amount that can be withdrawn from the current account so that one can still support the expected loss in the portfolio under the worst-case "scenario" over the time horizon $[t, t+h]$.
- The risk measure involves the volatility σ which is assumed to be given in our framework. However, in reality, σ is unknown and hence, so is the risk measure. Since the unknown parameter, σ, is defined under the $\mathcal{P}$-measure, it can be estimated by some statistical techniques which have been provided by some standard texts. For instance, Hull (1997). Then, we can obtain a point estimate for the risk measure by replacing the unknown parameter σ with its estimate $\hat{\sigma}$. By making use of the confidence interval for the unknown parameter σ, we can also obtain a confidence interval for the risk measure. This idea is similar to the idea of confidence intervals for VaR introduced in Chappell and Dowd (1999).

3. Extension of our Risk Measures to the Multivariate Black-Scholes Model

We extend our definition of risk measures for derivatives in the context of the multivariate complete Black-Scholes Model. The extended risk measure is applicable for a portfolio consisting of derivatives written on several underlying assets such as basket options, cross-currency options, exchange options, index options, options on the extremum of several assets, etc. As long as the valuation formulas for the derivatives are available, analytical forms of the risk measures can be obtained. In general, the extended risk measures can be expressed in the expectation form like expression (10). Now we suppose that our financial model consists of one riskless bond B and n risky stocks $S_1, \ldots, S_n$. The time horizon during which all economic activities take place is given by $[0, T]$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with $\mathcal{P}$ being the physical probability measure. As in section two, we assume that the physical probability measure $\mathcal{P}$ is given or known. Let r, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ denote the risk-free interest rate of the bond B, the vector of stock-appreciation rates $(\mu_1, \ldots, \mu_n)'$ (The prime ($'$) denotes transposition) and the volatility matrix $(\sigma_{ij})_{1 \leq i,j \leq n}$ of the stocks, respectively. We suppose that r, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are known or given and that $\boldsymbol{\sigma}$ is non-singular ($\boldsymbol{\sigma}^{-1}$ exists). Let $\{\mathbf{W}_t\}$ denote an n-dimensional standard Brownian motion $(W_{1t}, \ldots, W_{nt})'$ on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Then, the dynamic of the bond-price process $\{B_t\}$ and the stock-price process $\{S_{it}\}$ for the stock S_i $(i = 1, 2, \ldots, n)$ are governed by the following equations:

$$\begin{aligned} d\,B_t &= r\,B_t\,dt\ ; \quad B_0 = 1 \\ d\,S_{it} &= \mu_i\,S_{it}\,dt + \sum_{j=1}^{n} \sigma_{ij}\,S_{it}\,d\,W_{jt}\ ; \quad S_{i0} = s_i; i = 1, 2, \ldots, n\ . \end{aligned} \tag{13}$$

Let $\{\mathcal{F}_t^{\mathbf{W}}\}$ denote the natural filtration generated by the Brownian motion $\{\mathbf{W}_t\}$. That is, $\mathcal{F}_t^{\mathbf{W}} = \sigma\{\mathbf{W}_u\,|u \in [0, t]\}, t \in [0, T]$. Then, we equip our sample space $(\Omega, \mathcal{F})$ with the filtration $\{\mathcal{F}_t\}$ which is defined as the $\mathcal{P}$-augmentation of $\{\mathcal{F}_t^{\mathbf{W}}\}$.

In order to ensure that there is no arbitrage opportunity, we need to define the minimal martingale measure. Let $\boldsymbol{\gamma} := \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - r\,\mathbf{1}_n)$ be the price of unit risk where $\mathbf{1}_n$ is the n-dimensional unit vector. Note that the $\boldsymbol{\gamma}$ defined in this way is not unique. Let $\hat{\gamma} \in \mathcal{R}^n$ be the vector given in Musiela and Rutkowski (1997) (p. 256). We define the minimal martingale measure $\mathcal{Q}$ by the following Radon-Nikodym derivative:

$$\left.\frac{d\mathcal{Q}}{d\mathcal{P}}\right|_{\mathcal{F}_t} = \exp\left\{ -\hat{\gamma}'\mathbf{W}_t - \frac{1}{2}\|\hat{\gamma}\|^2 t \right\}\ ;\ t \in [0, T] \tag{14}$$

By Girsanov's theorem,

$$\tilde{\mathbf{W}}_t = \mathbf{W}_t + \frac{1}{2}\hat{\gamma}t\ ;\ t \in [0, T] \tag{15}$$

is an n-dimensional standard Brownian motion with respect to $\{\mathcal{F}_t\}$ under $\mathcal{Q}$.

Hence, under $\mathcal{Q}$, (13) can be rewritten as

$$d\,S_{it} = r\,S_{it}\,dt + \sum_{j=1}^{n} \sigma_{ij}\,S_{it}\,d\,\tilde{W}_{jt} \tag{16}$$

Now we consider a portfolio V consisting of m contingent claims $V_1, V_2, \ldots, V_m$. Each contingent claim V_k ($k = 1, 2, \ldots, m$) is written on n underlying assets $S_1, \ldots, S_n$ and with maturities at time T_k. Hence, the payoff function for V_k at time T_k can be written as a certain smooth function, $V_k(S_{1,T_k}, S_{2,T_k}, \ldots, S_{n,T_k}, T_k)$, where S_{i,T_k} is the price of the stock S_i at time T_k and $\max\{T_1, T_2, \ldots, T_m\} \leq T$. We assume that the portfolio V consists of $\phi_k(t,h)$ units of the contingent claim V_k over the small time horizon $[t, t+h]$. Note that $\phi_k(t,h)$ remains unchanged over $[t, t+h]$. If $(t+h) > T_k$, we set $\phi_k(t,h) = 0$. If $\phi_k(t,h)$ is negative, $-\phi_k(t,h)$ is interpreted as the number of units selling short for V_k.

Then the no arbitrage price of the portfolio V at time u, where $u \in [t, t+h]$, is as follows:

$$V(u) = \sum_{k=1}^{m} \phi_k(t,h) e^{-r(T_k-u)} E_{\mathcal{Q}}\big\{V_k(S_{1,T_k}, S_{2,T_k}, \ldots, S_{n,T_k}, T_k) \,|\, \mathcal{F}_u\big\} \tag{17}$$

Let $\mathbf{\Lambda}$ be a compact and convex set in $\mathcal{R}^n$. For each $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_n)' \in \mathbf{\Lambda}$, we define the subjective probability measure $\mathcal{P}_{\boldsymbol{\lambda}}$ associated with $\boldsymbol{\lambda}$ by the following Radon-Nikodym derivative:

$$\left.\frac{d\mathcal{Q}}{d\mathcal{P}}\right|_{\mathcal{F}_t} = \exp\left\{ -\boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\lambda})'\mathbf{W}_t - \frac{1}{2}\|\boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\lambda})\|^2 t \right\} \;;\; t \in [0, T] \tag{18}$$

Then, by Girsanov's theorem,

$$\mathbf{W}^{\boldsymbol{\lambda}}_t = \mathbf{W}_t + \frac{1}{2}\boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\lambda})t \;;\; t \in [0, T] \tag{19}$$

is an n-dimensional standard Brownian motion with respect to $\{\mathcal{F}_t\}$ under $\mathcal{P}_{\boldsymbol{\lambda}}$.

Hence, (13) can be rewritten as

$$d\,S_{it} = \lambda_i\,S_{it}\,dt + \sum_{j=1}^{n} \sigma_{ij}\,S_{it}\,d\,W^{\boldsymbol{\lambda}}_{jt} \;;\; i = 1, 2, \ldots, n \tag{20}$$

where $W^{\boldsymbol{\lambda}}_{jt}$ is the j-th component of $\mathbf{W}^{\boldsymbol{\lambda}}_t$.

Let $P_{\mathbf{\Lambda}}$ be a family of subjective probability measures, $\{\mathcal{P}_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda} \in \mathbf{\Lambda}}$, associated with the index set $\mathbf{\Lambda}$. Suppose $\Delta\,V_{t,h}$ denotes the change in the no arbitrage value (measured in terms of the value at time $(t+h)$) of the portfolio V over the time horizon $[t, t+h]$. Then, similar to section two, the risk measure for the portfolio V over the time horizon $[t, t+h]$, with respect to $P_{\mathbf{\Lambda}}$ and $\mathcal{F}_t$, is defined as:

$$\begin{aligned}\rho_{P_{\mathbf{\Lambda}}}(\Delta\,V_{t,h} \,|\, \mathcal{F}_t) \quad = \quad & \sup\Bigg\{ -\sum_{k=1}^{m} \phi_k(t,h) e^{-r(T_k-t)} \Big\{ E_{\mathcal{P}_{\boldsymbol{\lambda}}} \Big[E_{\mathcal{Q}}\big(V_k(S_{1,T_k}, \ldots, S_{n,T_k}, T_k) \\ & |\, \mathcal{F}_{t+h}\big) \,|\, \mathcal{F}_t \Big] - E_{\mathcal{Q}}(V_k(S_{1,T_k}, \ldots, S_{n,T_k}, T_k) \,|\, \mathcal{F}_t \Big\} \,|\, \boldsymbol{\lambda} \in \mathbf{\Lambda} \Bigg\}\end{aligned} \tag{21}$$

An interesting special case of risk measure (21) is obtained by letting $n = 1$. This special case deals with a portfolio of options with different maturities but written on the same underlying asset. It is clear that risk measure (21) satisfies the four coherent properties and hence, is a coherent risk measure. In the following example, we use a portfolio consisting of a single exchange option maturing at time T as an illustration. Closed-form expression of the risk measure for the exchange option over the time horizon $[t, t+h]$ can be obtained.

Example:

We consider a financial model consisting of one risk-free asset, B, and two risky stocks, S_1 and S_2. The governing equations for the evolution of the financial instruments are given in (13) with $n = 2$. The exchange option gives the holder the right to exchange S_2 by S_1 at time T. We can view the exchange option as a call on the underlying stock S_2 with a strike price equal to the price of the stock S_1 at time T, or a put on the stock S_1 with a strike price equal to the price of the stock S_2 at time T. For more detailed discussions on exchange options, see Margable (1978), Rubinstein (1991), Hull (1997), Kwok (1998), etc. Let S_{1T} and S_{2T} denote the prices of the stocks S_1 and S_2 at time T, respectively. Then the payoff function of the exchange option at expiry T is given as:

$$X(T) = \max(S_{2T} - S_{1T}, 0) \tag{22}$$

In Margable (1978), the following pricing formula for the exchange option was derived:

$$X(t) = S_{2t}\Phi(\hat{d}_1) - S_{1t}\Phi(\hat{d}_2)\ , \tag{23}$$

where

$$\begin{aligned} \hat{d}_1 &= \frac{\ln\left(\frac{S_{2t}}{S_{1t}}\right) + \frac{1}{2}\sigma_x^2(T-t)}{\sigma_x\sqrt{T-t}}\ , \\ \hat{d}_2 &= \hat{d}_1 - \sigma_x\sqrt{T-t} \end{aligned}$$

and

$$\sigma_x^2 = \sigma_{11}^2 + \sigma_{12}^2 - 2(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) + \sigma_{21}^2 + \sigma_{22}^2$$

Note that the prices for the American and European exchange options are the same since it is not optimal to exercise an American exchange option prematurely (see Kwok (1998)). To simplify the notations, we write the pricing formula (23) as a function, $X(S_{1t}, t, S_{2t})$, of the current price S_{1t} for the stock S_1, the current time t and the current price S_{2t} for the stock S_2.

From standard calculations, it can be shown that, for each $\boldsymbol{\lambda} := (\lambda_1, \lambda_2) \in \boldsymbol{\Lambda}$, the conditional expectation $E_{\mathcal{P}_{\boldsymbol{\lambda}}}\big(X(t+h)\,|\,\mathcal{F}_t\big)$ is given as follows:

$$E_{\mathcal{P}_{\boldsymbol{\lambda}}}\big(X(t+h)\,|\,\mathcal{F}_t\big) = X\big(S_{1t}\,e^{\lambda_1 h}, t, S_{2t}\,e^{\lambda_2 h}\big) \tag{24}$$

Let $\Delta X_{t,h}$ be $X(t+h) - e^{rh}X(t)$. Suppose λ_1^+ and λ_2^- denote $\max_{\lambda\in\Lambda}\lambda_1$ and $\min_{\lambda\in\Lambda}\Lambda_2$, respectively. Then, the risk measure for the exchange option over the time horizon $[t, t+h]$ is given as follows:

$$\rho_{P_{\mathbf{\Lambda}}}(\Delta X_{t,h} \,|\, \mathcal{F}_t) \quad = \quad -e^{-rh} X(S_{1t}\, e^{\lambda_1^+ h}, t, S_{2t}\, e^{\lambda_2^- h}) + X(S_{1t}, t, S_{2t}) \tag{25}$$

Note that the risk measure (25) is a special case of the risk measure (21) with $n = 2$, $m = 1$ and $V(T) = X(T)$.

4. Risk Measures for Barrier Options

Recently, barrier options have become increasingly popular in over-the-counter options markets. Now a large variety of barrier options are available in equity, foreign exchange and fixed-income markets. For more detailed discussions on barrier options, see Hull (1996), Jarrow and Turnbull (1996), Nelken (1996) and Kunitomo and Ikeda (1992). Basically, one can classify eight types of standard European barrier options. Namely, down-and-out calls, up-and-out calls, down-and-in puts, down-and-out puts, etc. Closed-form valuation formulas for all eight types of barrier options were obtained in Rubinsten and Reiner (1991). Intuitively, one expects barrier options to be cheaper than vanilla contracts. However, the reduction in premium payment requires the holder of barrier options to bear a higher risk since barrier options are extinguished when the price of the underlying asset hits a prespecified price level (barrier). Therefore, it is essential for the holder of barrier options to perform risk measurement for his/her risky portfolio. In this section, we consider the risk-measurement problems with barrier options. We only construct a risk measure for a portfolio consisting of a single down-and-out call option which matures at time T for illustration. An analytical form of the risk measure can be obtained. For other types of barrier options, analytical forms of the risk measures can be calculated by their corresponding valuation formulas in a similar way. Now, let L and K be the barrier and the strike price of the down-and-out option respectively. We assume that $S_0 > L$. Let $I\{A\}$ denote the indicator function of an event A. Then the discounted payoff of the down-and-out call option is given by:

$$e^{-rT} \max\big\{S_T - K, 0\big\}\, I\Big\{\min_{0\le t\le T} S_t > L\Big\} \tag{26}$$

We assume that the evolution of the prices of the risk-free bond B and the risky stock S are the same as in section two. Let $C(S_t, t, K)$ denote the European call option price. Then the European down-and-out call option price at the current time t is given by:

$$\begin{aligned} & bdo(S_t, t, L) \\ = \quad & C(S_t, t, K) - \Big(\frac{L}{S_t}\Big)^{\frac{2r}{\sigma^2}} \Big[L\Phi(d_3) - \frac{KS_t}{L} e^{-r(T-t)} \Phi(d_4)\Big] \end{aligned} \tag{27}$$

where

$$\begin{aligned} d_3 \quad &= \quad \frac{\ln\big(\frac{L^2}{S_t K}\big) + \big(r + \frac{1}{2}\sigma^2\big)(T-t)}{\sigma\sqrt{T-t}}\ , \\ d_4 \quad &= \quad d_3 - \sigma\sqrt{T-t} \end{aligned}$$

Let $\Delta bdo(t,h)$ be $bdo(S_{t+h},t+h,L)I\Big\{\min_{0\le t\le T} S_t > L\Big\} - e^{rh}bdo(S_t,t,L)$. Again, we define the risk measure for holding the long position of a down-and-out call option over the future short time horizon $[t,t+h]$, with respect to the family P_Λ (where P_Λ is defined in section two) and $\mathcal{F}_t$, as follows:

$$\rho_{P_\Lambda}\big(\Delta bdo(t,h)\,|\,\mathcal{F}_t\big) = \sup\big\{-E_{\mathcal{P}_\lambda}\big(e^{-rh}\Delta bdo(t,h)\,|\,\mathcal{F}_t\big)\,|\,\lambda\in\Lambda\big\} \qquad (28)$$

Let $\Phi_2(x,y;\rho_0)$ denote the distribution function of a standardized bivariate normal random vector, where ρ_0 is the coefficient of correlation between the random variables. From standard calculations, for each $\lambda\in\Lambda$, the conditional expectation $E_{\mathcal{P}_\lambda}\big(bdo(S_{t+h},t+h,L)\,|\,\mathcal{F}_t\big)$, denoted as $EB(\lambda,r,\sigma,S_t,t,h,T,K,L)$, is obtained as follows:

$$\begin{aligned}
& EB(\lambda,r,\sigma,S_t,t,h,T,K,L)\\
= \;& bdo\big[e^{\lambda h}S_t,t,K\,e^{rh},L\,e^{rh}\big]\\
& -L\Big(\frac{L}{S_t}\Big)^{\frac{2\lambda}{\sigma^2}}e^{\lambda h}\Phi_2\left(d_3',\frac{\ln(\frac{L}{S_t})+(\lambda+\frac{1}{2}\sigma^2)h}{\sigma\sqrt{h}}\;;\;\sqrt{\frac{h}{T-t}}\right)\\
& +L\Big(\frac{L}{S_t}\Big)^{\frac{2r}{\sigma^2}}e^{\frac{2r(r-\lambda)}{\sigma^2}h+rh}\Phi_2\left(\hat{d}_3',\frac{\ln(\frac{L}{S_t})+(2r-\lambda+\frac{1}{2}\sigma^2)h}{\sigma\sqrt{h}}\;;\;\sqrt{\frac{h}{T-t}}\right)\\
& -S_t\,e^{\lambda h}\Phi_2\left(\hat{d}_1,\frac{\ln(\frac{L}{S_t})-(\lambda+\frac{1}{2}\sigma^2)h}{\sigma\sqrt{h}}\;;\;-\sqrt{\frac{h}{T-t}}\right)\\
& +S_t\Big(\frac{S_t}{L}\Big)^{\frac{2(r-\lambda)}{\sigma^2}}e^{\frac{2r(r-\lambda)}{\sigma^2}h+rh}\Phi_2\left(d_1',\frac{\ln(\frac{L}{S_t})-(2r-\lambda+\frac{1}{2}\sigma^2)h}{\sigma\sqrt{h}}\;;\;-\sqrt{\frac{h}{T-t}}\right)\\
& +K\,e^{-(T-t-h)r}\Phi_2\left(\hat{d}_2,\frac{\ln(\frac{L}{S_t})-(\lambda-\frac{1}{2}\sigma^2)h}{\sigma\sqrt{h}}\;;\;-\sqrt{\frac{h}{T-t}}\right)\\
& -K\,e^{-(T-t-h)r}\Big(\frac{S_t}{L}\Big)^{\frac{2(r-\lambda)}{\sigma^2}}e^{(\lambda+\frac{2r(r-\lambda)}{\sigma^2}-r)h}\Phi_2\left(d_2',\frac{\ln(\frac{L}{S_t})-(2r-\lambda-\frac{1}{2}\sigma^2)h}{\sigma\sqrt{h}}\;;\;-\sqrt{\frac{h}{T-t}}\right)\\
& +K\,e^{-(T-t-h)r}\Big(\frac{L}{S_t}\Big)^{\frac{2\lambda}{\sigma^2}-1}\Phi_2\left(d_4',\frac{\ln(\frac{L}{S_t})+(\lambda-\frac{1}{2}\sigma^2)h}{\sigma\sqrt{h}}\;;\;\sqrt{\frac{h}{T-t}}\right)\\
& -K\,e^{-(T-t-h)r}S_t\,e^{\lambda h}\Big(\frac{L\,e^{rh}}{S_t\,e^{\lambda h}}\Big)^{\frac{2r}{\sigma^2}}e^{rh}\Phi_2\left(\hat{d}_4,\frac{\ln(\frac{L}{S_t})+(2r-\lambda-\frac{1}{2}\sigma^2)h}{\sigma\sqrt{h}}\;;\;\sqrt{\frac{h}{T-t}}\right)
\end{aligned} \qquad (29)$$

where

$$\begin{aligned}
d_1' &= \frac{\ln\big(\frac{S_t\,e^{-\lambda h}}{K\,e^{-rh}}\big)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,,\\
d_2' &= \frac{\ln\big(\frac{S_t\,e^{-\lambda h}}{K\,e^{-rh}}\big)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,,
\end{aligned}$$

$$\begin{aligned}
\hat{d}_3 &= \frac{\ln\left(\frac{L^2\,e^{2rh}}{S_t\,e^{\lambda h}\,K\,e^{rh}}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,, \\
d_3' &= \frac{\ln\left(\frac{L^2\,e^{-2rh}}{S_t\,e^{-\lambda h}\,K\,e^{-rh}}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,, \\
\hat{d}_4 &= \frac{\ln\left(\frac{L^2\,e^{2rh}}{S_t\,e^{\lambda h}\,K\,e^{rh}}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,, \\
d_4' &= \frac{\ln\left(\frac{L^2\,e^{-2rh}}{S_t\,e^{-\lambda h}\,K\,e^{-rh}}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\,,
\end{aligned}$$

Hence, the risk measure (28) can be rewritten as follows:

$$\begin{aligned}
&\rho_{P_\Lambda}(\Delta bdo(t,h)\,|\,\mathcal{F}_t) \\
=\;& \sup\left\{ -e^{-rh} EB(\lambda, r, \sigma, S_t, t, h, T, K, L) + bdo(S_t, t, L)\,|\,\lambda \in \Lambda\right\} \qquad (30)
\end{aligned}$$

where $EB(\lambda, r, \sigma, S_t, t, h, T, K, L)$ is given in (29).

Finally, it is worth pointing out that the analytical form of a risk measure can be obtained in a similar way when the portfolio consists of other types of exotic options for which closed-form valuation formulas are available. Some examples of these are look back options, step options and passport options, etc. For detailed discussions on step options and passport options, see Linetsky (1999) and Delbaen and Yor (1999) respectively.

5. Risk Measure for Vanilla European Derivatives in the presence of Transaction costs

In this section, we incorporate transaction costs into our framework. Under the assumption that proportional transaction costs are incurred when the underlying asset is traded, as well as the other usual assumptions in the Black-Scholes model, Leland (1985) provided a modification of the Black-Scholes model, where the replicating portfolio for the option can be revised only at regular time intervals in such a way that the total transaction costs incurred in the replication are finite. Following his approach, the pricing formulas for the standard European call and put options can be obtained as the modification of the Black-Scholes formulas with the original volatility replaced by the modified volatility which depends on the Leland number. Hence, based on Leland's option pricing model, we can obtain the closed-form expression of the risk measure for a portfolio consisting of standard European call and put options over the short time horizon $[t, t+h]$. Of course, Leland's adjustment for transaction costs is only an approximation. For simplicity, we only deal with a single European call option with a strike price, K, maturing at time T. The general result can be obtained in a similar way. First, we impose the following assumptions:

(1) The dynamics of the bond price and the stock price are the same as in section two, where the volatility is assumed to be constant.

(2) The proportional transaction costs are incurred only when the underlying asset is traded. However, there is no transaction cost in the derivative and bond markets.

(3) The replicating portfolio for the call option can be revised at regular intervals with common length δt.

(4) If α units of the underlying asset are bought (if $\alpha > 0$) or sold (if $\alpha < 0$) at the price S, then the transaction costs incurred is $\frac{1}{2}L|\alpha|S$, where L is the round-trip transaction cost per unit dollar of transaction.

(5) There is no dividend payment in the underlying stock during the life of the option.

Then, from Leland's option pricing model with transaction costs, the price of the call option is given as follows:

$$\tilde{C}(t) = S_t \Phi(\tilde{d}_1) - K\, e^{-r(T-t)} \Phi(\tilde{d}_2) \tag{31}$$

where

$$\begin{aligned} \tilde{d}_1 &= \frac{\ln(\frac{S_t}{K}) + (r + \frac{1}{2}\tilde{\sigma}^2)(T-t)}{\tilde{\sigma}\sqrt{T-t}} \ , \\ \tilde{d}_2 &= \tilde{d}_1 - \tilde{\sigma}\sqrt{T-t} \ , \end{aligned}$$

and

$$\tilde{\sigma}^2 = \sigma^2 \left[1 + \sqrt{\frac{2}{\pi}} \left(\frac{L}{\sigma\sqrt{\delta t}} \right) \right]$$

For details of the proof, see Leland (1985) or Kwok (1998). Note that formula (31) resembles the Black-Scholes formula for the call option except that the original volatility σ is replaced by the modified volatility $\tilde{\sigma}$. Since the modified volatility $\tilde{\sigma}$ is greater than the original volatility σ, Leland's call price is greater than the Black-Scholes call price. It is expected that the presence of transaction costs affects the risk measure only through the option's pricing formula. In the following, we construct the risk measure for the call option based on pricing formula (31). We use notations with the same meaning as those used in section two. For simplicity, we write pricing formula (31) as the function $\tilde{C}(S_t, t, K)$ of the current stock price S_t, the current time t and the strike price K.

Let $\Delta\,\tilde{C}_{t,h}$ be $\tilde{C}(t+h) - e^{rh}\tilde{C}(t)$. $\tilde{C}(S_t\, e^{\lambda h}, t, K\, e^{rh})$ denotes the modification of the expression (11) with the original volatility σ replaced by the modified volatility $\tilde{\sigma}$. Then, by following the same procedure as in section two, the closed-form expression of the risk measure can be obtained as follows:

$$\rho_{P_\Lambda}(\Delta\,\tilde{C}_{t,h}\,|\,\mathcal{F}_t) = -e^{-rh}\tilde{C}(S_t\, e^{ah}, t, K\, e^{rh}) + \tilde{C}(S_t, t, K) \ , \tag{32}$$

It is not difficult to check that risk measure (32) is coherent.

6. Conclusion and future research

In this paper, we considered the well-known Black-Scholes model and proposed a risk measure for a portfolio containing derivative securities. The popular risk measure, Value-at-Risk, is difficult to implement when the portfolio contains derivatives and it is not a coherent risk measure. The advantages of our risk measure are that it is easy to implement and it is coherent. Our method can be applied, not only to standard call or put options, but, as long as the pricing formula is available, to calculate a risk measure as well.

We used the risk-neutral probability $\mathcal{Q}$ to calculate the price of derivatives. We calculated the expected loss of a portfolio over a fixed time period with respect to a family of subjective probability measures in order to incorporate the risk trader's risk preference and subjective beliefs. Our risk measure is scenario-based, which is of the same type as the representation form of coherent risk measures. Transaction costs were included by using the option pricing formula with transaction costs given in Leland (1985).

However, the risk measure defined in this paper also has some limitations. For example, in the presence of transaction costs, we can only measure the risk of a single call option. In this paper, we did not include the case of American options. Although the American options can be treated similarly, the computational problem becomes very difficult when using the method in this paper. We can incorporate the model risk using our model by letting the volatility σ vary on an interval. All the analyses remain the same.

Many related problems can be considered in the future. The relationship between Siu and Yang (1999) and this paper need to be investigated further. In particular, whether or not we can obtain the risk measure of this paper by taking proper limit of the risk measure in Siu and Yang (1999) is an interesting problem. Furthermore, the problem of measuring risk for derivatives could become very challenging if we include the stochastic interest rate models and foreign exchange rate models in our formulation. Some numerical studies could also prove to be enlightening. The problem of implementing efficient numerical procedures and algorithms may also be worth investigating.

Acknowledgements

The authors would like to thank the referee for many valuable suggestions and comments. We would also like to thank Professor Phelim Boyle for many helpful and stimulating discussions on the related subjects. This work was supported by a grant from The Research Grants Council of HKSAR (Project No. HKU 7168/98H).

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