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Characterizing the positive polynomials which are not SOS

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Abstract— **Several analysis and synthesis tools in control systems are based on polynomial sum of squares (SOS) relaxations. However, almost nothing is known about the gap existing between positive polynomials and SOS of polynomials. This paper investigates such a gap proposing a matrix characterization of PNS, that is homogeneous forms that are not SOS. In particular, it is shown that any PNS is the vertex of an unbounded cone of PNS. Moreover, a complete parameterization of the set of PNS is introduced.**

Index Terms— **Optimization in control, Positive polynomial, Hilbert's 17th problem, SOS, LMI.**

I. Introduction

Positive polynomials play a key role in control systems analysis and synthesis as confirmed by the large number of papers appeared in the last years. First of all, this is due to the fact that stability conditions can be reformulated in terms of positivity of a Lyapunov function and negativity of its time derivative. These functions are usually polynomials constituting the natural extension of the classic quadratic Lyapunov functions in the attempt of achieving less conservative results. Another important reason is that the computation of performance indexes as $\mathcal{H}_\infty$ gain and convergence decay rate, can be analogously reformulated.

Unfortunately, to establish whether a polynomial is positive or not, is still a difficult problem that can not be solved systematically because it amounts to solving a nonconvex optimization. In order to deal with this problem, gridding methods have been proposed, for example based on the use of Chebychev points, but their conservativeness and computational burden are generally unacceptable, reason that has motivated the search for alternative approaches.

Such a search recently provided the sum of squares (SOS) relaxation. In this approach, the positivity of a homogeneous form (equivalently of a polynomial) is established by checking if it is a SOS of homogeneous forms, operation which amounts to solving a linear matrix inequality (LMI) feasibility problem, i.e. a convex optimization (see for example [1]). Due to the existence of powerful tools for solving LMIs [2], [3], SOS relaxations quickly became an essential tool in the automatic control field. In robust control, SOS relaxations have been employed to obtain less conservative conditions than those provided by quadratic Lyapunov functions to assess robust stability of linear systems affected by parametric uncertainty, in both cases of time-varying uncertainty [4], [5], [6], [7], [8], [9] and time-invariant uncertainty [10], [11], [12]. An analogous use of SOS has been made to obtain less conservative conditions in the computation of robust performance indexes [13], [14]. SOS have been exploited also in the field of nonlinear systems [15], [16], [17], [18], [19], [20], hybrid systems [21], [22], and time-delay systems [23]. See also [1], [24], [25] for further applications of SOS.

"Can any positive homogeneous form be written as a SOS?" This question was made by Hilbert in his 17th problem and has a negative answer as it is known since some decades. It is hence known that, in spite of their popularity, SOS relaxations can be conservative. However, almost nothing is presently known about the set of homogeneous forms that are positive but not SOS (we will refer to such homogeneous forms as PNS). Only few isolated examples have been found (see [26] for a survey of these examples).

The aim of this paper is to characterize PNS since actually they represent the gap between several fundamental problems in control systems and the corresponding solution tools. First, some remarks about the distance between PNS and SOS are introduced, in particular showing that the set of PNS, when not empty, has a non empty interior. Then, a matrix characterization of PNS is proposed based on eigenvectors and eigenvalues decomposition. Such a characterization is based on the concept introduced in this paper of maximal matrix for the representation of homogeneous forms. It is shown that any PNS is the vertex of an unbounded cone of PNS whose directions correspond to strictly positive SOS. Such a cone can be linearly parameterized in a convex set. Moreover, a complete parameterization of the set of PNS is proposed, providing hence a technique to construct PNS.

The paper is organized as follows. In Section II some preliminaries about the representation and classification of homogeneous forms are reported. Section III presents the main results of the paper about the representation of homogeneous forms and characterization of PNS. Lastly, Sections IV and V conclude with an illustrative example and some remarks.

II. Preliminaries

A. Homogeneous forms representation

Let the notation be as follows:

- $\mathbb{N}, \mathbb{R}$: natural number set (including 0) and real number set;
- $\mathcal{S}_n$: set of symmetric matrices $n \times n$;
- I_n: identity matrix $n \times n$;
- A': transpose of matrix A;
- $A \succ 0$ ($A \succeq 0$): symmetric positive definite (semidefinite) matrix A;
- $\lambda_{min}(A)$: minimum real eigenvalue of A;
- $\ker(A)$: null space of matrix A;

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- $\mathrm{img}(A)$: image of matrix A;
- $\mathrm{diag}(x)$: diagonal matrix $n \times n$ whose diagonal components are the components of $x \in \mathbb{R}^n$;
- x_i or $(x)_i$: i-th component of vector x;
- $x \geq 0$ ($x > 0$): vector with positive (strictly positive) components;
- x^q: $x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$ with $x \in \mathbb{R}^n, q \in \mathbb{N}^n$.

For $n, m \in \mathbb{N}$ define the set

$$\mathcal{Q}_{n,m} = \left\{ q \in \mathbb{N}^n : \sum_{i=1}^{n} q_i = m \right\} \tag{1}$$

whose cardinality is

$$\sigma(n,m) = \frac{(n+m-1)!}{(n-1)!m!}. \tag{2}$$

We say that $f(x)$ is a *homogeneous form* of degree m in $x \in \mathbb{R}^n$ if

$$f(x) = \sum_{q \in \mathcal{Q}_{n,m}} c_q x^q \tag{3}$$

where $c_q \in \mathbb{R}$ are the coefficients of $f(x)$. The set of homogeneous forms of degree m in $x \in \mathbb{R}^n$ is denoted by $\Xi_{n,m}$.

Let $x^{[m]} \in \mathbb{R}^{\sigma(n,m)}$ be a vector whose components constitute a base for the homogeneous forms of degree m in x. Then, $f(x) \in \Xi_{n,m}$ can be represented as

$$f(x) = \vec{f}' x^{[m]} \tag{4}$$

where $\vec{f} \in \mathbb{R}^{\sigma(n,m)}$ is the coefficient vector of $f(x)$. For $f(x) \in \Xi_{n,m}$ we define the norm

$$\|f(x)\|_c = \|\vec{f}\|. \tag{5}$$

Any $g(x) \in \Xi_{n,2m}$ can be written as

$$g(x) = x^{[m]'} \left(G + L(\alpha)\right) x^{[m]} \tag{6}$$

where G is any matrix in $\mathcal{S}_{\sigma(n,m)}$ satisfying $g(x) = x^{[m]'} G x^{[m]}$, and $L : \mathbb{R}^{\tau(n,2m)} \to \mathcal{S}_{\sigma(n,m)}$ is any linear parameterization of the set

$$\mathcal{L}_{n,2m} = \left\{ L \in \mathcal{S}_{\sigma(n,m)} : \; x^{[m]'} L x^{[m]} = 0 \right\} \tag{7}$$

whose dimension is

$$\tau(n,2m) = \frac{1}{2}\sigma(n,m)\left[\sigma(n,m)+1\right] - \sigma(n,2m). \tag{8}$$

The representation (6) is known as *Gram matrix method* [26] and *complete square matricial representation (CSMR)* [24].

In the sequel we will say that the matrix G (resp., $G + L(\alpha)$) in (6) is a SMR (resp., CSMR) matrix of $g(x)$. Moreover, it will be assumed that the vector $x^{[m]} \in \mathbb{R}^{\sigma(n,m)}$ is chosen to satisfy

$$x^{[m]'} x^{[m]} = \|x\|^{2m}. \tag{9}$$

A possible choice for $x^{[m]} \in \mathbb{R}^{\sigma(n,m)}$ guaranteeing (9) is the following. Select $\varphi : \{i \in \mathbb{N} : \; 1 \leq i \leq \sigma(n,m)\} \to \mathcal{Q}_{n,m}$ such that it is a bijective function, and define the i-th component of $x^{[m]} \in \mathbb{R}^{\sigma(n,m)}$ according to

$$\left(x^{[m]}\right)_i = \sqrt{\frac{m!}{(\varphi(i))_1!(\varphi(i))_2! \cdots (\varphi(i))_n!}} \; x^{\varphi(i)}. \tag{10}$$

B. Positive forms, SOS and PNS

We say that $g(x) \in \Xi_{n,2m}$ is *positive* if $g(x) \geq 0$ for all x or, equivalently, if $\mu(g) \geq 0$ where $\mu(g)$ is the *positivity index* of $g(x)$ defined as

$$\mu(g) = \min_{\|x\|=1} g(x). \tag{11}$$

The set of positive homogeneous forms of degree $2m$ in $x \in \mathbb{R}^n$ is denoted by $\Phi_{n,2m}$.

The form $g(x) \in \Xi_{n,2m}$ is a *SOS* if and only if there exist k forms $f_i(x) \in \Xi_{n,m}$, $1 \leq i \leq k$, such that

$$g(x) = \sum_{i=1}^{k} f_i(x)^2. \tag{12}$$

It is straightforward to verify that $g(x)$ is a SOS if and only if there exists α such that $G + L(\alpha) \succeq 0$ or, equivalently, if and only if $\lambda(g) \geq 0$ where $\lambda(g)$ is the *SOS index* of $g(x)$ defined as

$$\lambda(g) = \max_{\alpha} \lambda_{min}\left(G + L(\alpha)\right). \tag{13}$$

The quantity $\lambda(g)$ can be computed by solving the eigenvalue problem (EVP)

$$\begin{aligned} \lambda(g) = \max_{t,\alpha} \; & t \\ \text{s.t.} \;\; & G + L(\alpha) - tI_{\sigma(n,m)} \succeq 0 \end{aligned} \tag{14}$$

that is a convex optimization constrained by LMIs. The set of SOS of degree $2m$ in $x \in \mathbb{R}^n$ is denoted by $\Sigma_{n,2m}$.

The form $g(x) \in \Xi_{n,2m}$ is a *PNS* if and only if $g(x)$ is positive but it is not a SOS or, equivalently, if and only if $\mu(g) \geq 0$ and $\lambda(g) < 0$. The set of PNS of degree $2m$ in $x \in \mathbb{R}^n$ is denoted by $\Delta_{n,2m}$. It has been shown that $\Delta_{n,2m}$ is empty in the following cases [26], [27]:

- $m = 1$ for all n;
- $n \leq 2$ for all m;
- $n = 3$ and $m \leq 2$.

Hence, it turns out that $\Xi_{n,2m} \supset \Phi_{n,2m}$, $\Phi_{n,2m} = \Sigma_{n,2m} \cup \Delta_{n,2m}$, $\Sigma_{n,2m} \cap \Delta_{n,2m} = \emptyset$.

III. Main results

A. The maximal SMR matrix

Let us introduce the following concept, which is the base for the characterization of PNS proposed in this paper. Given $g(x) \in \Xi_{n,2m}$, a SMR matrix G of $g(x)$ is said *maximal* if

$$\lambda_{min}(G) = \lambda(g). \tag{15}$$

The maximal SMR matrices of $g(x)$ are hence be given by

$$G + L(\alpha^*) \tag{16}$$

where α^* is an optimal value of α in (13), that is a value of α for which the maximum $\lambda(g)$ is achieved (α^* exists because $\lambda(g)$ is bounded whenever $\|g(x)\|_c$ is bounded).

In order to characterize the maximal SMR matrices, let us introduce the following matrix decomposition. The quadruplet $\langle \lambda_{min}(G), \beta, V_0, V_p \rangle$ is said a *decomposition* of matrix $G \in \mathcal{S}_{\sigma(n,m)}$ if

$$G = VDV' \tag{17}$$

where $D \in \mathcal{S}_{\sigma(n,m)}$ is the diagonal matrix containing the eigenvalues of G defined by the minimum eigenvalue $\lambda_{min}(G)$ of multiplicity $\sigma(n,m)-r$ and the vector $\beta \in \mathbb{R}^r$, $\beta > 0$, as

$$D = \lambda_{min}(G) I_{\sigma(n,m)} + \begin{bmatrix} 0 & \\ & \mathrm{diag}(\beta) \end{bmatrix}, \tag{18}$$

and $V \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ is an orthonormal eigenvector matrix defined as

$$V = [V_0 \ V_p] \tag{19}$$

where the columns of $V_0 \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)-r}$ are a base of the eigenspace of the minimum eigenvalue, and the columns of $V_p \in \mathbb{R}^{\sigma(n,m)\times r}$ are bases for the eigenspaces of the other eigenvalues.

Let us observe that the introduced decomposition is not unique. In fact, there are multiple choices for β, V_0, V_p that satisfy the required conditions. In particular, if $\langle \lambda_{min}(G), \beta, V_0, V_p \rangle$ is a decomposition of G, it follows that also $\langle \lambda_{min}(G), B\beta, V_0 A, V_p B \rangle$ is for all orthonormal matrices $A \in \mathbb{R}^{\sigma(n,m)-r\times\sigma(n,m)-r}$ and for all permutation matrices $B \in \mathbb{R}^{r\times r}$.

The following result holds.

Theorem 1: Let $\langle \lambda_{min}(G), \beta, V_0, V_p \rangle$ be any decomposition of the matrix $G \in \mathcal{S}_{\sigma(n,m)}$ and define

$$\zeta(V_0) = \max_{\|\alpha\|=1} \lambda_{min}\left(V_0' L(\alpha) V_0\right). \tag{20}$$

Then, G is a maximal SMR matrix if and only if $\zeta(V_0) \leq 0$. Moreover, $\zeta(V_0)$ does not depend on the chosen decomposition.

Proof From (15) it follows that G is maximal if and only if

$$\lambda_{min}\left(G + L(\alpha)\right) \leq \lambda_{min}(G) \ \forall \alpha \neq 0$$

and, hence, if and only if for all $\alpha \neq 0$ there exists $y \in \mathbb{R}^{\sigma(n,m)}$, $\|y\| = 1$, such that

$$y'\left(G + L(\alpha)\right) y \leq \lambda_{min}(G). \tag{21}$$

Let $\langle \lambda_{min}(G), \beta, V_0, V_p \rangle$ be a decomposition of G. Then, (21) can be rewritten as

$$y' V_p \,\mathrm{diag}(\beta) V_p' y \leq -y' L(\alpha) y. \tag{22}$$

Observe that $L(\alpha)$ depends linearly on α. This means that $V_p' y$ must tend to zero as α tends to zero since $\mathrm{diag}(\beta) \succ 0$. Moreover, if (22) holds with the pair $\langle y, \alpha \rangle$, it also holds with the pair $\langle y, c\alpha \rangle$ for all $c \geq 1$. Therefore, it turns out that G is maximal if and only if

$$\forall \alpha \neq 0 \ \forall \varepsilon > 0 \ \exists y, \ \|y\| = 1 : \ \|V_p' y\| < \varepsilon \text{ and (22) holds}$$

or, equivalently, if and only if

$$\forall \alpha \neq 0 \ \exists y, \ \|y\| = 1 : \ V_p' y = 0 \text{ and (22) holds.} \tag{23}$$

Since $\ker(V_p') = \mathrm{img}(V_0)$, it follows that $V_p' y = 0$ if and only if $y \in \mathrm{img}(V_0)$. Hence, (23) can be rewritten as

$$\forall \alpha \neq 0 \ \exists y \in \mathrm{img}(V_0), \ \|y\| = 1 : \ y' L(\alpha) y \leq 0. \tag{24}$$

Write $y \in \mathrm{img}(V_0)$ as $y = V_0 p$ with $p \in \mathbb{R}^{\sigma(n,m)-r}$. Since $y' L(\alpha) y$ depends linearly on α, condition (24) can be rewritten as

$$\forall \alpha, \ \|\alpha\| = 1, \ \exists p, \ \|p\| = 1 : \ p' V_0' L(\alpha) V_0 p \leq 0$$

and, hence, as $\zeta(V_0) \leq 0$.

Lastly, observe that the choice of V_0 in the decomposition of G does not affect $\zeta(V_0)$. In fact, all the matrices whose columns are an orthonormal base of the eigenspace of the minimum eigenvalue of G can be written as $V_0 A$ where $A \in \mathbb{R}^{\sigma(n,m)-r\times\sigma(n,m)-r}$ is an orthonormal matrix. Since it turns out that the eigenvalues of $A' V_0' L(\alpha) V_0 A$ are the same of $V_0' L(\alpha) V_0$, we can conclude that $\zeta(V_0 A) = \zeta(V_0)$. □

Theorem 1 provides a further necessary and sufficient condition to establish if a given matrix G is a maximal SMR matrix. This condition is important because it states that the property of being a maximal SMR matrix is related only to the eigenspace of the minimum eigenvalue, contrary to the condition (15) which involves the whole matrix by exploiting the SOS index. Hence, Theorem 1 provides a way to construct maximal SMR matrices.

Observe that $\zeta(V_0)$ cannot be easily calculated because the set $\{\alpha : \ \|\alpha\| = 1\}$ is non convex. The following result proposes an alternative index for V_0.

Theorem 2: Let $w \in \mathbb{R}^{\tau(n,2m)}$, $w \neq 0$, be any vector and define

$$\eta(V_0) = \max\{\eta(V_0, 1), \eta(V_0, -1)\} \tag{25}$$

where

$$\eta(V_0, k) = \sup_{\alpha: \ w'\alpha = k} \lambda_{min}\left(V_0' L(\alpha) V_0\right). \tag{26}$$

Then, $\zeta(V_0) \leq 0$ if and only if $\eta(V_0) \leq 0$. Moreover, $\eta(V_0)$ does not depend on the chosen decomposition of G.

Proof "$\Rightarrow$" Suppose for contradiction that $\zeta(V_0) \leq 0$ and $\eta(V_0) > 0$. Then, there exists $\tilde{\alpha} \in \mathbb{R}^{\tau(n,2m)}$ such that $|w'\tilde{\alpha}| = 1$ and $\lambda_{min}\left(V_0' L(\tilde{\alpha}) V_0\right) > 0$. Define $\bar{\alpha} = \|\tilde{\alpha}\|^{-1}\tilde{\alpha}$. We have that $\|\bar{\alpha}\| = 1$ and $\lambda_{min}\left(V_0' L(\bar{\alpha}) V_0\right) = \|\tilde{\alpha}\|^{-1}\lambda_{min}\left(V_0' L(\tilde{\alpha}) V_0\right) > 0$. This is impossible because $\zeta(V_0) \leq 0$.

"$\Leftarrow$" Suppose for contradiction that $\eta(V_0) \leq 0$ and $\zeta(V_0) > 0$. Then, there exists $\tilde{\alpha} \in \mathbb{R}^{\tau(n,2m)}$ such that $\|\tilde{\alpha}\| = 1$ and $\lambda_{min}\left(V_0' L(\tilde{\alpha}) V_0\right) > 0$. Suppose $w'\tilde{\alpha} \neq 0$ and define $\bar{\alpha} = |w'\tilde{\alpha}|^{-1}\tilde{\alpha}$. We have that $\|w'\bar{\alpha}\| = 1$ and $\lambda_{min}\left(V_0' L(\bar{\alpha}) V_0\right) = |w'\tilde{\alpha}|^{-1}\lambda_{min}\left(V_0' L(\tilde{\alpha}) V_0\right) > 0$. This is impossible because $\eta(V_0) \leq 0$. Suppose now that $w'\tilde{\alpha} = 0$. Then, for all $\varepsilon > 0$ there exists $\hat{\alpha} \in \mathbb{R}^{\tau(n,2m)}$ such that $\|\hat{\alpha}\| = 1$ and $\|\hat{\alpha} - \tilde{\alpha}\| < \varepsilon$ and $w'\hat{\alpha} \neq 0$. For continuity of the function $\lambda_{min}\left(V_0' L(\tilde{\alpha}) V_0\right)$ with respect to α, such a $\hat{\alpha}$ can be chosen to satisfy also the constraint $\lambda_{min}\left(V_0' L(\hat{\alpha}) V_0\right) > 0$. Repeating the procedure by using $\hat{\alpha}$ instead of $\tilde{\alpha}$, we conclude the proof.

Lastly, the choice of V_0 in the decomposition of G does not affect $\eta(V_0)$ for the same reasoning of Theorem 1. □

Theorem 2 provides an alternative index for V_0 that can be computed through two convex optimizations. In fact, it turns out that $\eta(V_0,k)$ is the solution of the EVP

$$\eta(V_0,k) = \sup_{t,\alpha} \; t \quad \text{s.t.} \left\{ \begin{array}{l} w'\alpha - k = 0 \\ V_0' L(\hat{\alpha}) V_0 - t I_{\sigma(n,m)-r} \succeq 0. \end{array} \right. \tag{27}$$

Observe that the free vector w defines the two planes into which the unit shell $\{\alpha : \;\|\alpha\| = 1\}$ used in Theorem 1 is crushed in order to achieve convexity. Although the sign of $\eta(V_0)$ does not depend on the choice of w, the absolute value does. Another difference between $\zeta(V_0)$ and $\eta(V_0)$ is that the former is bounded whereas the second may be not.

B. PNS characterization

For $f(x) \in \Xi_{n,m}$ define the ball with radius $\delta \in \mathbb{R}$ centered in $f(x)$ as

$$\mathcal{B}_\delta(f) = \left\{ \tilde{f}(x) \in \Xi_{n,m} : \; d(\tilde{f}, f) < \delta \right\} \tag{28}$$

where $d : \Xi_{n,m} \times \Xi_{n,m} \to \mathbb{R}$ is the distance in $\Xi_{n,m}$ defined as

$$d(\tilde{f}, f) = \|\tilde{f}(x) - f(x)\|_c. \tag{29}$$

Let us start by observing that, contrary to $\Xi_{n,2m}$ and $\Sigma_{n,2m}$, $\Delta_{n,2m}$ can be non convex. In fact, consider in $\Delta_{3,6}$ the Motzkin form and the Stengle form (see [26] and references therein):

$$\begin{array}{rcl} g_{Mot}(x) & = & x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2 \quad (30) \\ g_{Ste}(x) & = & x_1^3 x_3^3 + (x_2^2 x_3 - x_1^3 - x_1 x_3^2)^2. \quad (31) \end{array}$$

It can be verified that $\lambda\left(\frac{1}{2}(g_{Mot} + g_{Ste})\right) = 0$, that is $\frac{1}{2}\left(g_{Mot}(x) + g_{Ste}(x)\right)$ is not a PNS.

The following lemma introduces some remarks about the closeness between $\Delta_{n,2m}$ and $\Sigma_{n,2m}$.

Lemma 1: Suppose that $\Delta_{n,2m}$ is not empty. Then:

1) there exists $g(x) \in \Delta_{n,2m}$ such that $\mu(g) > 0$;
2) any $g(x) \in \Delta_{n,2m}$ such that $\mu(g) > 0$ is an interior point of $\Delta_{n,2m}$, that is there exists $\delta > 0$ such that $\mathcal{B}_\delta(g) \subset \Delta_{n,2m}$;
3) for all $g(x) \in \Delta_{n,2m}$ there exists $\delta > 0$ such that $\mathcal{B}_\delta(g) \cap \Phi_{n,2m} \subset \Delta_{n,2m}$.

Proof First, if $\Delta_{n,2m}$ is not empty, there exists $g(x) \in \Delta_{n,2m}$ such that $\mu(g) \geq 0$. Suppose that $\mu(g) = 0$ and define $\tilde{g}(x) = g(x) + \varepsilon\|x\|^{2m}$. From (11) it follows that $\mu(\tilde{g}) = \mu(g) + \varepsilon = \varepsilon$. Moreover, from (9) we have that $G + \varepsilon I_{\sigma(n,m)}$ is a SMR matrix of $\tilde{g}(x)$. Hence, from (13) it follows that $\lambda(\tilde{g}) = \lambda(g) + \varepsilon$. Since $\lambda(g) < 0$ we conclude that, for all $0 < \varepsilon < -\lambda(g)$, $\tilde{g}(x) \in \Delta_{n,2m}$ and $\mu(\tilde{g}) > 0$.

Second, consider $g(x) \in \Delta_{n,2m}$ such that $\mu(g) > 0$. For continuity of $\mu(g)$ and $\lambda(g)$ with respect to $g(x)$, it follows that there exists $\delta > 0$ such that, for all $\tilde{g}(x) \in \Xi_{n,2m}$ satisfying $\|\tilde{g} - g(x)\|_c < \delta$, $\mu(\tilde{g}) > 0$ and $\lambda(\tilde{g}) < 0$, that is $g(x)$ is an interior point of $\Delta_{n,2m}$.

Third, consider $g(x) \in \Delta_{n,2m}$. If $\mu(g) > 0$, $g(x)$ is an interior point of $\Delta_{n,2m}$ and item 3) is clearly satisfied. Suppose hence $\mu(g) = 0$. For the same reasoning of item 2), there exists $\delta > 0$ such that, for all $\tilde{g}(x) \in \Xi_{n,2m}$ satisfying $\|\tilde{g} - g(x)\|_c < \delta$, $\lambda(\tilde{g}) < 0$, that is $\mathcal{B}_\delta(g) \cap \Sigma_{n,2m} = \emptyset$. Hence, item 3) holds. □

Lemma 1 states that the set of PNS, if non empty, contains form with a strictly positive positivity index, that is positive forms that vanish only in the origin. These forms are interior points for $\Delta_{n,2m}$, that is owning a neighborhood included in $\Delta_{n,2m}$. Moreover, it is stated that any PNS form owns a neighborhood where all positive forms are PNS, hence meaning that arbitrary small variations can not change a PNS into a SOS.

As we have seen in Section II-B, to establish whether a form $g(x)$ is a PNS amounts to establishing whether $\mu(g) \geq 0$ and $\lambda(g) < 0$. The following result provides a further characterization of PNS and is the first step toward the construction of such forms.

Lemma 2: Let $G \in \mathcal{S}_{\sigma(n,m)}$ be any maximal SMR matrix of $g(x) \in \Delta_{n,2m}$, and let $\langle \lambda_{min}(G), \beta, \; V_0, V_p \rangle$ be any decomposition of G. Then,

$$\nexists\, x \neq 0 : \; V_p' x^{[m]} = 0. \tag{32}$$

Proof Now, suppose for contradiction that there exists $\tilde{x} \neq 0$ such that $\tilde{x}^{[m]} \in \ker(V_p')$. We have:

$$\begin{array}{rcl} g(\tilde{x}) & = & \tilde{x}^{[m]\prime} [V_0 \; V_p] \left(\lambda_{min}(G) I_{\sigma(n,m)} \right. \\ & & \left. + \begin{bmatrix} 0 & \\ & \operatorname{diag}(\beta) \end{bmatrix} \right) [V_0 \; V_p]' \tilde{x}^{[m]} \\ & = & \lambda_{min}(G) \|V_0' \tilde{x}^{[m]}\|^2. \end{array}$$

Observe that $\lambda_{min}(G) < 0$ since G is a maximal SMR matrix of a PNS. Moreover, $\|V_0' \tilde{x}^{[m]}\| \neq 0$ since $\operatorname{img}(V_0) = \ker(V_p')$. Hence, $g(\tilde{x}) < 0$. This is impossible because $g(x)$ is a PNS. □

Lemma 2 provides a necessary condition for a form to be a PNS: the absence of solutions $x \neq 0$ in the polynomial system $V_p' x^{[m]} = 0$. Observe that this condition is equivalent to the absence of vectors $x^{[m]} \neq 0$ in the linear space $\operatorname{img} V_0$.

The following result presents a way to generate PNS from any PNS.

Theorem 3: Given $g(x) \in \Delta_{n,2m}$, let $G \in \mathcal{S}_{\sigma(n,m)}$ be any maximal SMR matrix of $g(x)$ and let $\langle \lambda_{min}(G), \beta, V_0, V_p \rangle$ be any decomposition of G. For $\gamma \in \mathbb{R}^r$, $\gamma \geq 0$, define the SOS $s(x; V_p, \gamma) \in \Sigma_{n,2m}$

$$s(x; V_p, \gamma) = x^{[m]\prime} V_p \operatorname{diag}(\gamma) V_p' x^{[m]}. \tag{33}$$

and the cone of forms with vertex in $g(x)$

$$\mathcal{C}(g) = \left\{ h(x) \in \Xi_{n,2m} : \; h(x) = g(x) + s(x; V_p, \gamma), \; \gamma \geq 0 \right\}. \tag{34}$$

Then, $\mathcal{C}(g) \subset \Delta_{n,2m}$. Moreover,

$$\exists \delta > 0 : \; \mu\left(g + s(V_p, \gamma)\right) \geq \mu(g) + \delta \min_{1 \leq i \leq r} \gamma_i. \tag{35}$$

Proof First of all, $s(x; V_p, \gamma)$ is a SOS because its SMR matrix $S(V_p, \gamma) = V_p \operatorname{diag}(\gamma) V_p'$ satisfies $S(V_p, \gamma) \succeq 0$ for all $\gamma \geq 0$.

In order to prove that $\mathcal{C}(g)$ contains only PNS, observe that $H = G + S(V_p, \gamma)$ is a maximal SMR matrix of $h(x) =$

$g(x)+s(x;V_p,\gamma)$. In fact:

$$
\begin{array}{rcl}
H & = & [V_0\ V_p]\left(\lambda_{min}(G)I_{\sigma(n,m)}+\left[\begin{array}{cc}0 & \\ & \mathrm{diag}(\beta)\end{array}\right]\right)\\
& & [V_0\ V_p]'+V_p\,\mathrm{diag}(\gamma)V_p'\\
& = & [V_0\ V_p]\left(\lambda_{min}(G)I_{\sigma(n,m)}+\left[\begin{array}{cc}0 & \\ & \mathrm{diag}(\beta+\gamma)\end{array}\right]\right)\\
& & [V_0\ V_p]'
\end{array}
$$

which clearly implies that $\langle\lambda_{min}(G),\beta+\gamma,V_0,V_p\rangle$ is a decomposition of H. Hence, from Theorem 1 it follows that H is a maximal SMR matrix because $\zeta(V_0)\leq 0$ being G a maximal SMR matrix. From the fact that H is a maximal SMR matrix it follows that $\lambda(h)=\lambda_{min}(H)=\lambda_{min}(G)=\lambda(g)$. Moreover, we have that $\mu(h)\geq\mu(g)$ because $s(x;V_p,\gamma)$ is a SOS. Since $g(x)\in\Delta_{n,2m}$ we conclude that $\lambda(h)=\lambda(g)<0$ and $\mu(h)\geq\mu(g)\geq 0$, that is $h(x)\in\Delta_{n,2m}$.

Lastly, observe that $\mu\left(g+s(V_p,\gamma)\right)\geq\mu(g)+\mu\left(s(V_p,\gamma)\right)$. Moreover,

$$
s(x;V_p,\gamma)\geq\|V_p'x^{[m]}\|^2\min_{1\leq i\leq r}\gamma_i\quad\forall x\ \forall\gamma.
$$

According to Lemma 2, $V_p'x^{[m]}\neq 0$ for all $x\neq 0$. Hence, (35) holds with $\delta=\mu(v)>0$ where $v(x)=\|V_p'x^{[m]}\|^2$. □

Theorem 3 states that any PNS is the vertex of a cone of PNS. In particular, the cone is unbounded and its directions correspond to strictly positive SOS that can be linearly parameterized in a convex set. Observe also that, according to (35), there exist PNS whose positivity index μ is arbitrarily large, that is arbitrarily large positive forms that are not SOS.

How to construct PNS? In order to answer to this question, let us define the set

$$
\Theta_{n,2m}=\bigcup_{1\leq r\leq\sigma(n,m)}\Theta_{n,2m}(r)\qquad(36)
$$

where

$$
\begin{array}{rcll}
\Theta_{n,2m}(r) & = & \{\langle\delta,\beta,V_p\rangle:\ \delta\in\mathbb{R},\delta\in(0,1], & (37)\\
& & \beta\in\mathbb{R}^r,\beta>0,\ \text{and}\ V_p\in\mathcal{V}_{n,2m}(r)\} & \\
\mathcal{V}_{n,2m}(r) & = & \{V_p\in\mathbb{R}^{\sigma(n,m)\times r}:\ V_p'V_p=I_r, & (38)\\
& & \zeta(\mathrm{cmp}(V_p))\leq 0,\ \text{and (32) holds}\} &
\end{array}
$$

and $\mathrm{cmp}(V_p)\in\mathbb{R}^{\sigma(n,m)\times\sigma(n,m)-r}$ is a matrix whose columns are an orthonormal base of $\ker(V_p')$. For $\theta\in\Theta_{n,2m}(r)$ define the form

$$
\psi(x;\theta)=s(x;V_p,\beta)-\delta\mu\left(s(V_p,\beta)\right)\|x\|^{2m}.\qquad(39)
$$

The following result provides an answer to the question introducing a parameterization of $\Delta_{n,2m}$.

Theorem 4: For all $g(x)\in\Delta_{n,2m}$, there exists $\theta\in\Theta_{n,2m}$ such that $g(x)=\psi(x;\theta)$. Moreover, $\psi(x;\theta)\in\Delta_{n,2m}$ for all $\theta\in\Theta_{n,2m}$.

Proof Suppose $g(x)\in\Delta_{n,2m}$. Let G be a maximal SMR matrix of G, and let $\langle\lambda_{min}(G),\beta,V_0,V_p\rangle$ be a decomposition of G. We have:

$$
\begin{array}{rl}
g(x) = & x^{[m]'}[V_0\ V_p]\left(\lambda_{min}(G)I_{\sigma(n,m)}+\left[\begin{array}{cc}0 & \\ & \mathrm{diag}(\beta)\end{array}\right]\right)\\
& [V_0\ V_p]'x^{[m]}\\
& =x^{[m]'}\left(\lambda_{min}(G)I_{\sigma(n,m)}+V_p\,\mathrm{diag}(\beta)V_p'\right)x^{[m]}\\
& =\lambda_{min}(G)\|x\|^{2m}+s(x;V_p,\beta).
\end{array}
$$

Hence, $g(x)=\psi(x;\theta)$ where $\theta=\langle\delta,\beta,V_p\rangle$ and

$$
\delta=-\frac{\lambda_{min}(G)}{\mu\left(s(V_p,\beta)\right)}.
$$

Observe that $\delta\in(0,1]$ because $\lambda_{min}(G)=\lambda(g)<0$ and $\lambda_{min}(G)+\mu\left(s(V_p,\beta)\right)=\mu(g)\geq 0$. Moreover, $\beta>0$. Then, from Theorem 1 and Lemma 2, it follows that $V_p\in\mathcal{V}_{n,2m}(r)$ where r is the length of β. Therefore, $\theta\in\Theta_{n,2m}$.

Now, consider $\theta=\langle\delta,\beta,V_p\rangle\in\Theta_{n,2m}$. We have that a SMR matrix of $\psi(x;\theta)$ is given by

$$
\begin{array}{rcl}
\Psi(\theta) & = & V_p\,\mathrm{diag}(\beta)V_p'-\delta\mu\left(s(V_p,\beta)\right)I_{\sigma(n,m)}\\
& = & [\mathrm{cmp}(V_p)\ V_p]\left(\left[\begin{array}{cc}0 & \\ & \mathrm{diag}(\beta)\end{array}\right]\right.\\
& & \left.-\delta\mu\left(s(V_p,\beta)\right)I_{\sigma(n,m)}\right)[\mathrm{cmp}(V_p)\ V_p]'.
\end{array}
$$

Since $V_p'V_p=I_r$ and $\beta>0$, it follows that $\langle-\delta\mu\left(s(V_p,\beta)\right),\beta,\mathrm{cmp}(V_p),V_p\rangle$ is a decomposition of $\Psi(\theta)$. From Theorem 1 we have that $\Psi(\theta)$ is a maximal SMR matrix because $\zeta(\mathrm{cmp}(V_p))\leq 0$. Moreover, from Lemma 2 it follows that $\mu\left(s(V_p,\beta)\right)>0$. Hence, $\lambda\left(\psi(\theta)\right)=-\delta\mu\left(s(V_p,\beta)\right)<0$ and $\mu\left(\psi(\theta)\right)=(1-\delta)\mu\left(s(V_p,\beta)\right)\geq 0$. Therefore, $\psi(x;\theta)\in\Delta_{n,2m}$. □

Theorem 4 states that the set of PNS is the image of $\Theta_{n,2m}$ through the function $\psi(x;\theta)$. This result provides hence a technique to construct all existing PNS that amounts to finding matrices V_p in $\mathcal{V}_{n,2m}(r)$ and calculating the positivity index $\mu\left(s(V_p,\beta)\right)$.

Unfortunately, the set $\mathcal{V}_{n,2m}(r)$ can not be explicitly described at present. A method to find elements in this set consists of looking for matrices V_p with a fixed structure for which the property (32) and the positivity index $\mu\left(s(V_p,\beta)\right)$ can be easily assessed, and using the remaining free parameters to satisfy $\zeta(\mathrm{cmp}(V_p))\leq 0$. The example in Section IV has been found with this method.

IV. Illustrative Example

We show here the construction of a simple PNS by using Theorem 4 for $n=3,m=3$. Choose $x^{[m]}=[x_1^3,\sqrt{3}x_1^2x_2,\sqrt{3}x_1^2x_3,\sqrt{3}x_1x_2^2,\sqrt{6}x_1x_2x_3,\sqrt{3}x_1x_3^2,x_2^3,\sqrt{3}x_2^2x_3,\sqrt{3}x_2x_3^2,x_3^3]'$ and a linear parameterization $L(\alpha)$ of $\mathcal{L}_{n,2m}$. Select $r=3$ and

$$
V_p=\frac{1}{7}\left[\begin{array}{cccccccccc}6 & 0 & 0 & -2 & 0 & -3 & 0 & 0 & 0 & 0\\ 0 & -3 & 0 & 0 & 0 & 0 & 6 & 0 & -2 & 0\\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & -3 & 0 & 6\end{array}\right]'.
$$

Observe that $V_p'V_p=I_r$. Moreover, it can be easily verified from Theorem 2 that $\zeta(\mathrm{cmp}(V_p))<0$. Now, the structure of V_p allows us to easily assess the property (32) and the positivity index $\mu\left(s(V_p,\beta)\right)$. In fact, $V_p' x^{[m]} = \frac{1}{7}[w_1(x), w_2(x), w_3(x)]'$ where

$$\begin{cases} w_1(x) &= x_1\left(6x_1^2 - 2\sqrt{3}x_2^2 - 3\sqrt{3}x_3^2\right) \\ w_2(x) &= x_2\left(-3\sqrt{3}x_1^2 + 6x_2^2 - 2\sqrt{3}x_3^2\right) \\ w_3(x) &= x_3\left(-2\sqrt{3}x_1^2 - 3\sqrt{3}x_2^2 + 6x_3^2\right). \end{cases}$$

It is straightforward to see that (32) holds. Hence, $V_p \in \mathcal{V}_{n,2m}(r)$ and

$$\theta = \langle \delta, \beta, V_p \rangle \in \Theta_{n,2m} \quad \forall \delta \in (0,1]\ \forall \beta > 0.$$

Select $\beta = [49, 49, 49]'$. We have that

$$s(x; V_p, \beta) = w_1(x)^2 + w_2(x)^2 + w_3(x)^2.$$

Through simple calculations that amounts to finding the roots of polynomial equations in one variable up to the fourth degree, we find $\mu\left(s(V_p,\beta)\right) = 0.7433$. Therefore, from Theorem 4 it follows that $\psi(x;\theta)$ is a PNS for all $\delta \in (0,1]$ where

$$\psi(x;\theta) = w_1(x)^2 + w_2(x)^2 + w_3(x)^2 - 0.7433\|x\|^{2m}\delta.$$

Lastly, from Theorem 3 it follows that the cone

$$\begin{aligned} \mathcal{C}(\psi(\theta)) &= \{h(x) \in \Xi_{n,2m} :\ h(x) = \psi(x;\theta) \\ &\quad + s(x; V_p, \gamma),\ \gamma \geq 0\} \\ &= \{h(x) \in \Xi_{n,2m} :\ h(x) = \textstyle\sum_{i=1}^3 (1+\gamma_i) w_i(x)^2 \\ &\quad -0.7433\|x\|^{2m}\delta,\ \gamma \geq 0\} \end{aligned}$$

with vertex $\psi(x;\theta)$ contains only PNS, that is $\mathcal{C}(\psi(\theta)) \subset \Delta_{n,2m}$.

V. Conclusion

A matrix characterization of PNS, that is homogeneous forms that are not SOS, has been proposed. This characterization, based on eigenvectors and eigenvalues decomposition, provides new results about the structure of these forms. In particular, it is shown that any PNS is the vertex of an unbounded cone of PNS. Moreover, a complete parameterization of the set of PNS is introduced. This parameterization, although partially implicit, provides a technique to construct PNS.

The results proposed in this paper represent a first step in the characterization of the existing gap between positive polynomials and SOS of polynomials, about which only few isolated examples were known until now. Since this gap is at the root of the conservativeness of analysis and synthesis tools recently proposed in several areas of control systems as robust control, nonlinear control and hybrid control, it is expected that such a characterization can play a key role in future developments of this field.

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