Math. Proc. Camb. Phil. Soc. (1998), **123**, 75
Printed in the United Kingdom

Fixed points of endomorphisms of a free metabelian group

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(*Received* 31 *October* 1995; *revised* 17 *April* 1996)

Abstract

We consider IA-endomorphisms (i.e. Identical in Abelianization) of a free metabelian group of finite rank, and give a matrix characterization of their fixed points which is similar to (yet different from) the well-known characterization of eigenvectors of a linear operator in a vector space. We then use our matrix characterization to elaborate several properties of the fixed point groups of metabelian endomorphisms. In particular, we show that the rank of the fixed point group of an IA-endomorphism of the free metabelian group of rank $n \geqslant 2$ can be either equal to 0, 1, or greater than $(n-1)$ (in particular, it can be infinite). We also point out a connection between these properties of metabelian IA-endomorphisms and some properties of the Gassner representation of pure braid groups.

1. *Introduction*

Let $F = F_n$ be the free group of a finite rank $n \geqslant 2$ with a set $X = \{x_i\}, 1 \leqslant i \leqslant n$, of free generators. Then, let $F' = [F, F]$ be the commutator subgroup of the group F. The group $M = M_n = F/F''$ is called a free metabelian group.

Let φ be an endomorphism of the group F given by $\varphi(x_k) = y_k, 1 \leqslant k \leqslant n$. It also induces an endomorphism of the group M in the natural way. We denote elements of a free group and their natural images in a free metabelian group by the same letters when there is no ambiguity. The same applies to endomorphisms.

Birman [**3**] and Bachmuth [**1**] have used matrices over group rings to study endomorphisms of a free and a free metabelian group, respectively.

Birman [**3**] has given a matrix characterization of automorphisms of a free group among arbitrary endomorphisms (the 'inverse function theorem') as follows. Define the matrix $J_\varphi = (d_j(y_i))_{1 \leqslant i,j \leqslant n}$ (the 'Jacobian matrix' of φ), where d_j denotes partial Fox derivation (with respect to x_j) in the free group ring ZF (see [**6**]). Then φ is an automorphism if and only if the matrix J_φ is invertible.

Bachmuth [**1**] has obtained an inverse function theorem of the same kind on replacing the Jacobian matrix J_φ by its image J_φ^a over the abelianized group ring $Z(F/F')$.

Recently, matrix methods have been used by a number of authors to produce new interesting results on endomorphisms of free and free metabelian groups. Umirbaev [**12**] has generalized Birman's result to partial generating systems (so-called *primitive systems*) of a free group. In [**7**] and [**10**], similar results are obtained for primitive systems of a free metabelian group. In [**11**], *monomorphisms* (i.e. injective endomorphisms) of a free group are characterized by the property of the Jacobian matrix to have left independent rows.

All these results show a remarkable parallelism between the theory of endomorphisms of a free (or a free metabelian) group and the theory of linear operators in a vector space. In this paper, we obtain a matrix characterization of IA-endomorphisms with non-trivial fixed points ('eigenvectors') which, although is similar to the corresponding well-known characterization in linear algebra, also reveals a subtle difference.

IA-endomorphism of a group G is an endomorphism which is Identical in the Abelianization G/G' of the group G. As usual, by the *rank* of a matrix A over a commutative ring S we mean the maximal number of rows of A independent over S.

THEOREM 1·1. *Let φ be an IA-endomorphism of the free metabelian group $M = M_n$ given by $\varphi(x_i) = y_i$, $1 \leqslant i \leqslant n$, and let J_φ^a be the corresponding abelianized Jacobian matrix. Then:*

(i) $\det(J_\varphi^a - I) = 0$, *where I is the $n \times n$ identity matrix;*

(ii) *If* $\operatorname{rank}(J_\varphi^a - I) \leqslant n-2$, *then φ has a non-trivial fixed point inside the commutator subgroup M';*

(iii) *If* $\operatorname{rank}(J_\varphi^a - I) = n-1$, *then φ has a non-trivial fixed point inside the commutator subgroup M' if and only if* $[x_1, x_2]^{u_1} \cdot [x_2, x_3]^{u_2} \cdot \ldots \cdot [x_{n-1}, x_n]^{u_{n-1}} = [y_1, y_2]^{u_1} \cdot [y_2, y_3]^{u_2} \cdot \ldots \cdot [y_{n-1}, y_n]^{u_{n-1}}$ *for some elements $u_k \in Z(M/M')$, not all of them zero.*

The situation is most subtle when $\operatorname{rank}(J_\varphi^a - I) = n-1$. In this case, anything can happen. For example, if φ is an IA-endomorphism of the group M_2 given by $\varphi(x_1) = x_1 s$; $\varphi(x_2) = x_2 s^{-1}$, then φ has no non-trivial fixed points for any $s \in M_2'$, $s \neq 1$ (see Proposition 3·2). On the other hand, if φ is an inner automorphism induced by an element $g \notin M_2'$, then φ has a non-trivial fixed point (outside M_2'). In both situations, $\operatorname{rank}(J_\varphi^a - I) = 1 = 2 - 1$.

Although I was not able to distinguish these possibilities 'by matrix means', an algorithm for detecting fixed points does exist:

THEOREM 1·2. *There is an algorithm for detecting the presence of non-trivial fixed points of an arbitrary IA-endomorphism of a free metabelian group M_n. Also, there is an algorithm for detecting the presence of those non-trivial fixed points that belong to the commutator subgroup M_n'. In both cases, if an algorithm reveals the presence of non-trivial fixed points, then at least one of them can be actually found.*

In case of a free group, these questions remain open even when restricted to detecting fixed points of *automorphisms*.

When all non-trivial fixed points of φ are inside the commutator subgroup M' (which is the case, for example, when φ is the conjugation by a non-trivial element from M'), the fixed point group Fix φ has *infinite rank*:

PROPOSITION 1·3. *Let φ be an IA-endomorphism of a free metabelian group M. Then the intersection of the fixed point group* Fix φ *with M' is a normal subgroup of M. In particular, if this intersection is non-trivial, then it is infinitely generated.*

Furthermore, if φ has the fixed point group of finite rank, then there is a 'generation gap'

THEOREM 1·4. *Let φ be an IA-endomorphism of a free metabelian group M_n. If the rank of the fixed point group* Fix φ *is finite, then it is equal to 0, 1, or is greater than* $(n-1)$.

Then, we have

PROPOSITION 1·5. *For an arbitrary $n \geqslant 3$, there are non-inner (IA-)automorphisms of the free metabelian group M_n that have infinitely generated fixed point group.*

Note that every IA-automorphism of the group M_2 is inner [**1**]. Then, if $u \in M_n'$, the conjugation by u is an automorphism of M_n whose fixed point group coincides with M_n' and therefore is infinitely generated. The same argument proves the existence of *inner* automorphisms with infinitely generated fixed point group for any group of the form $F/[R,R]$ with F/R infinite.

We should also mention here an example of a *finitely presented* group [**5**] whose automorphisms can have infinitely generated fixed point group.

All this makes contrast to the situation in a free group: the fixed point group of any free endomorphism has rank $n = \operatorname{rank} F$ or less. This has been proved in [**2**] for automorphisms, and in [**9**] – for arbitrary endomorphisms.

In general, it is an interesting and important question – how far is the similarity between free and free metabelian endomorphisms extended? The desire to 'lift' properties of metabelian endomorphisms to those of free endomorphisms leads to several interesting questions of which the following one seems particularly attractive:

Problem 1. Suppose $\varphi \in \operatorname{Aut} F$ is a non-inner IA-automorphism with a non-trivial fixed point. Is it true that φ has a fixed point inside $[F, F]$?

Another interesting question is – how can our Theorem 1·1 be strengthened when restricted to *automorphisms*; in particular, we ask:

Problem 2. Is it true that every IA-automorphism of the group M_n has a non-trivial fixed point?

Informally speaking, endomorphisms with non-trivial fixed points are rare. On the other hand, endomorphisms which are automorphisms, are also rare (in contrast to the situation in linear algebra). Whether or not automorphisms with non-trivial fixed points are rare, remains a mystery.

Finally, we point out a connection between the properties of metabelian IA-endomorphisms described above, and some properties of the Gassner representation of pure braid groups. For defenitions of braids, braid groups, and for a background material, we refer to [**4**].

Let $\sigma \in P_n$ be a pure braid on n strands, and $\hat{\sigma}$ the corresponding link. Then, let $\Gamma_{n-1}(\sigma)$ be the image of σ under the reduced Gassner representation of the pure braid group P_n, and $A_{\hat{\sigma}}(t_1, \ldots, t_n)$ the Alexander polynomial of $\hat{\sigma}$. Then (see [**4**, theorem 3·11]):

$$A_{\hat{\sigma}}(t_1, \ldots, t_n) = 0 \text{ if and only if } \det(\Gamma_{n-1}(\sigma) - I) = 0.$$

To explain how this is connected to the subject of the present paper, we mention that the pure braid group P_n is isomorphic to a subgroup of the group of IA-automorphisms of F_n, and the Gassner representation maps every IA-automorphism in this subgroup onto its abelianized Jacobian matrix. Then, the matrix $\Gamma_{n-1}(\sigma)$ that corresponds to the *reduced* Gassner representation, is an $(n-1)\times(n-1)$ matrix which is obtained from that abelianized Jacobian matrix by applying a suitable conjugation, and deleting the last row and the last column of the form $(0,\ldots,0,1)$.

Therefore, if we denote the free IA-automorphism corresponding (under the *unreduced* Gassner representation) to the braid σ by the same letter, the above condition takes the form rank $(J_\sigma^a - I) \leqslant n-2$, which points to part (ii) of our Theorem 1·1.

Thus, we have

COROLLARY 1·6. *The Alexander polynomial of a link $\hat{\sigma}$ is zero if and only if the free automorphism which corresponds to the (pure) braid σ, has a non-trivial fixed point $g \in F'$ modulo F'', i.e. $\sigma(g) = g(\text{mod } F'')$.*

2. *Preliminaries*

Let ZF be the integral group ring of the group F and Δ its augmentation ideal, that is, the kernel of the natural homomorphism $\epsilon\colon ZF \to Z$. More generally, when $R < F$ is a normal subgroup of F, we denote by Δ_R the ideal of ZF generated by all elements of the form $(r-1)$, $r \in R$. It is the kernel of the natural homomorphism $\epsilon_R\colon ZF \to Z(F/R)$.

The ideal Δ is a free left ZF-module with a free basis $\{(x_i - 1)\}$, $1 \leqslant i \leqslant n$, and left Fox derivations d_i are projections to the corresponding free cyclic direct summands. Thus any element $u \in \Delta$ can be uniquely written in the form $u = \sum_{i=1}^{n} d_i(u)(x_i - 1)$.

One can extend these derivations linearly to the whole ZF by setting $d_i(1) = 0$.

The next lemma is an immediate consequence of the definitions.

LEMMA 2·1. *Let J be an arbitrary right ideal of ZF, and let $u \in \Delta$. Then $u \in J\Delta$ if and only if $d_i(u) \in J$ for each $i, 1 \leqslant i \leqslant n$.*

Proof of the next lemma can be found in [**6**].

LEMMA 2·2. *Let R be a normal subgroup of F, and let $y \in F$. Then $y - 1 \in \Delta_R\Delta$ if and only if $y \in R'$.*

We also need the 'chain rule' for Fox derivations (see [**6**]):

LEMMA 2·3. *Let ϕ be an endomorphism of F (it can be linearly extended to ZF) defined by $\phi(x_k) = y_k, 1 \leqslant k \leqslant n$, and let $v = \varphi(u)$ for some $u, v \in ZF$. Then:*

$$d_j(v) = \sum_{k=1}^{n} \varphi(d_k(u))d_j(y_k).$$

Lemma 2·3 implies the following product rule for the Jacobian matrices which looks exactly the same as in the 'usual' situation of analytic functions and Leibnitz derivations: if φ and ψ are two endomorphisms of F, then

$$J_{\varphi(\psi)} = \psi(J_\varphi)\cdot J_\psi.$$

If furthermore φ and ψ are IA-endomorphisms, then considering abelianization of this product rule yields:

$$J^a_{\varphi(\psi)} = J^a_\varphi \cdot J^a_\psi.$$

In particular, there is a (faithful) representation of metabelian IA-automorphisms by matrices from $GL_n(Z(F/F'))$ cf. [**1**].

Another version of Lemma 2·3 is: if $u, v \in ZF$ and $v = \varphi(u)$, then

$$(d_1(v), \ldots, d_n(v)) = (\varphi(d_1(u), \ldots, \varphi(d_n(u))) \cdot J_\varphi. \tag{1}$$

Finally, we recall a well-known action via conjugation of a group ring $Z(F/R)$ on abelian group R/R' under which R/R' becomes the (left) relation module of the group F/R. Namely, if $g \in F/R$ and $r \in R$, then g acts on rR' by taking it to $grg^{-1}R'$; then this action is extended to the whole group ring $Z(F/R)$ by linearity. The result of the action of $u \in Z(F/R)$ on rR' is denoted by $r^u R'$. Then (see [**8**]):

LEMMA 2·4. *For* $r \in R$, $u \in Z(F/R)$, *one has* $d_i(r^u) = u \cdot d_i(r)$ (mod Δ_R).

3. *Proofs*

Proof of Theorem 1·1. (i) Let φ be an IA-endomorphism of the group M; $\varphi(x_i) = y_i$, $1 \leqslant i \leqslant n$.

Then, by the definition of Fox derivatives, we have $y_i - 1 = \sum_{k=1}^n d_k(y_i)(x_k - 1)$ for any i, $1 \leqslant i \leqslant n$.

Since φ is IA, abelianizing this equality gives $x_i^a - 1 = \sum_{k=1}^n d_k^a(y_i)(x_k^a - 1)$, or, in the matrix form:

$$((x_1^a - 1), \ldots, (x_n^a - 1))^t = J^a_\varphi \cdot ((x_1^a - 1), \ldots, (x_n^a - 1))^t,$$

where ^{t} means taking the transpose (i.e. we consider a column, not a row). This is equivalent to

$$(J^a_\varphi - I) \cdot ((x_1^a - 1), \ldots, (x_n^a - 1))^t = 0,$$

hence the columns of the matrix $(J^a_\varphi - I)$ are dependent, so $\det(J^a_\varphi - I) = 0$.

(ii) Let φ take x_i to $x_i s_i$, $s_i \in M'$, $1 \leqslant i \leqslant n$. For notational convenience, consider the endomorphism φ being lifted to an endomorphism of F. We shall use the same letter φ for this lifted endomorphism. When $u \in F$, we denote the abelianization of u (i.e., the image in the group $A = F/F'$) by u^a. This agreement also applies to elements of the group ring ZF.

If $\det(J^a_\varphi - I) = 0$, then the rows of the matrix $(J^a_\varphi - I)$ are dependent over the group ring $ZA = Z(F/F')$. Make a new matrix J' upon multiplying the kth row of the matrix $(J^a_\varphi - I)$ by x_k^a, $1 \leqslant k \leqslant n$. It is clear that the rows of J' are dependent over the group ring ZA, too.

Now, J' is the abelianized Jacobian matrix of the endomorphism φ' that takes x_i to s_i, $1 \leqslant i \leqslant n$. Indeed, $d_j(x_i s_i) = \delta_{ij} + x_i d_j(s_i)$, so that $x_i d_j(\varphi'(x_i)) = d_j(\varphi(x_i)) - \delta_{ij}$.

Thus, the abelianized Jacobian matrix $J^a_{\varphi'}$ has dependent rows (over the group ring ZA). We can write this dependence as follows:

$$\sum_{k=1}^n u_k \cdot d_i^a(s_k) = 0 \tag{2}$$

in ZA for some $u_k \in ZA$, not all of them 0, and i runs from 1 through n.

By Lemmas 2·1, 2·2, 2·4, the system (2) is equivalent to the following relation:

$$\prod_{k=1}^{n} s_k^{u_k} = 1$$

in the group M.

If rank $J^a_{\varphi'} = m \leqslant n-2$, then there are precisely m independent elements (say, $s_1, \ldots, s_m$) among s_k (in other words, these $s_1, \ldots, s_m$ generate a free submodule of the relation module of the group A), and for any j, $m+1 \leqslant j \leqslant n$, we have a relation of the form

$$s_j^{w_j} = \prod_{k=1}^{m} s_k^{v_{kj}} \tag{3}$$

for some $w_j,\ v_{kj} \in ZA$.

Now, we are going to find a non-trivial fixed point $g \in M'$ of the endomorphism φ, in the form

$$g = [x_1, x_2]^{z_1} \cdot [x_2, x_3]^{z_2} \cdot \ldots \cdot [x_{n-1}, x_n]^{z_{n-1}} \tag{4}$$

for $z_k \in ZA$ (to be found).

We need a couple of simple observations:

LEMMA 3·1. (i) *Let* $\varphi(g) = g$ *for some* $g \in M'$. *Then* $\varphi(g^u) = g^u$ *for any* $u \in ZA$.

(ii) *For any element* $h \in M'$, *one has* $h^w = g$ *for some* $w \in ZA$ *and for* $g \in M'$ *of the form* (4).

Proof. (i) Since φ is IA, we have $\varphi(u) = u$ in ZA. The result follows.

(ii) It is clear that the elements $[x_i, x_j]$, $1 \leqslant i < j \leqslant n$, generate M' as a normal subgroup of M. It is well-known that the elements $[x_1, x_2], \ldots, [x_{n-1}, x_n]$ generate a free submodule (of rank $(n-1)$) of the relation module of the group A. On the other hand, there are no free submodule of rank n in this relation module. This means for any pair i, j of indices, one has $[x_i, x_j]^{v_{ij}} = [x_1, x_2]^{u_1} \cdot [x_2, x_3]^{u_2} \cdot \ldots \cdot [x_{n-1}, x_n]^{u_{n-1}}$ for some $v_{ij}, u_k \in$ ZA. The result follows.

We continue now with part (ii) of Theorem 1·1. From $\varphi(g) = g$, we get

$$[s_1, x_2]^{z_1} \cdot [x_1, s_2]^{z_1} \cdot \ldots \cdot [s_{n-1}, x_n]^{z_{n-1}} \cdot [x_{n-1}, s_n]^{z_{n-1}} = 1,$$

or, equivalently:

$$s_1^{(x_2-1)z_1} \cdot s_2^{(1-x_1)z_1} \cdot \ldots \cdot s_{n-1}^{(x_n-1)z_{n-1}} \cdot s_n^{(1-x_{n-1})z_{n-1}} = 1. \tag{5}$$

Conjugate both sides of (5) by $w = \prod_{k=m+1}^{n} w_j$, where $w_j \in ZA$ come from (3). Then, replace every $s_j^{w_j}$, $j \geqslant m+1$, in (5), with the corresponding element on the right-hand side of (3). This gives

$$s_1^{z'_1} \cdot \ldots \cdot s_m^{z'_m} = 1 \tag{6}$$

for some $z'_k \in ZA$, each of which is a ZA-linear combination of z_i, $1 \leqslant i \leqslant n-1$.

Since we have chosen $s_1, \ldots, s_m$ so that they generate a free submodule of the relation module of the group A, the equation (6) is equivalent to a system of equations

$$z'_k = 0, \quad 1 \leqslant k \leqslant m. \tag{7}$$

This is a system of m *homogeneous* ZA-linear equations in $(n-1) > m$ unknowns $z_1, \ldots, z_{n-1}$. It is well-known that a system like that has a non-trivial solution over ZA (since ZA is a commutative domain). This completes the proof.

(iii) The 'if' part is obvious. The 'only if' part follows from Lemma 3·1.

The following example shows how subtle a situation might be in the case when $\operatorname{rank}(J^a_\varphi - I) = n-1$.

PROPOSITION 3·2. *In the group* M_2, *let* φ *take* x_1 *to* $x_1 s$, x_2 *to* $x_2 s^{-1}$ *for some* $s \in M_2'$, $s \neq 1$. *Then* φ *has no non-trivial fixed points.*

Proof. First we show that φ has no non-trivial fixed points in M_2'. Suppose $\varphi(g) = g$; $g \in M_2'$. Since g has the form $[x_1, x_2]^z$ for some $z \in ZA$, we have (cf. (5)):

$$[x_1, x_2]^z = [x_1 s,\ x_2 s^{-1}]^z = [x_1, x_2]^u \cdot s^{(x_2-1)u+(x_1-1)z}. \tag{8}$$

It follows that $s^{(x_2-1)z+(x_1-1)z} = 1$, hence $(x_2-1)\cdot z + (x_1-1)\cdot z = (x_2+x_1-2)\cdot z = 0$ in the group ring ZA, which is only possible if $z = 0$. Thus, $g = 1$.

Now let $g \in M_2$ be an arbitrary element. Then g has the form $x_1^m x_2^n [x_1, x_2]^z$ for some $z \in ZA$. Let us see first what the image of $x_1^m x_2^n$ looks like.

We have: $\varphi(x_1^m x_2^n) = x_1 \cdot s \cdot \ldots \cdot x_1 \cdot s \cdot x_2 \cdot s^{-1} \cdot \ldots \cdot x_2 \cdot s^{-1} = h$. Now we apply to this element h the following 'collecting process': first we collect all the x_1 on the left by permuting them with s. This gives: $h = x_1^m \cdot s^{x_1^{m-1}} \cdot \ldots \cdot s^{x_1} \cdot s \cdot x_2 \cdot s^{-1} \cdot \ldots \cdot x_2 \cdot s^{-1}$. Then, in the same manner, we collect all the x_2 on the right of x_1^m:

$$h = x_1^m x_2^n \cdot s^{x_1^{m-1} x_2^n} \cdot \ldots \cdot s^{x_1 x_2^n} \cdot s^{x_2^n} \cdot s^{-x_2^{n-1}} \cdot \ldots \cdot s^{-x_2} \cdot s^{-1},$$

or, equivalently:

$$h = x_1^m x_2^n \cdot s^{x_1^{m-1} x_2^n + \ldots + x_1 x_2^n + x_2^n - x_2^{n-1} - \ldots - x_2 - 1}. \tag{9}$$

Now write down the whole image of g: $\varphi(g) = \varphi(x_1^m x_2^n \cdot [x_1, x_2]^z) = h \cdot [x_1, x_2]^z \cdot s^{(x_2-1)z+(x_1-1)z}$ (cf. (8)). If $\varphi(g) = g$, then combining this with (9) yields:

$$s^{x_1^{m-1} x_2^n + \ldots + x_1 x_2^n + x_2^n - x_2^{n-1} - \ldots - x_2 - 1 + (x_2-1)z + (x_1-1)z} = 1,$$

or, equivalently:

$$x_1^{m-1} x_2^n + \ldots + x_1 x_2^n + x_2^n - x_2^{n-1} - \ldots - x_2 - 1 + (x_2 + x_1 - 2)\cdot z = 0. \tag{10}$$

All we have to do now is to show that $w = x_1^{m-1} x_2^n + \ldots + x_1 x_2^n + x_2^n - x_2^{n-1} - \ldots - x_2 - 1$ is not divisible by $v = x_2 + x_1 - 2$ in the ring ZA. This is easy to see upon setting $x_1 = -1$; then $w' = (-1)^m x_2^n - x_2^{n-1} - \ldots - x_2 - 1$; $v' = x_2 - 3$. Since $x_2 = 3$ is not a root of the polynomial w', the result follows. This completes the proof of Proposition 3·2.

Proof of Theorem 1·2. Let φ be an IA-endomorphism of the group $M = M_n$.

First of all, we compute the rank of the matrix J^a_φ. If it is not equal to $(n-1)$, then we just refer to Theorem 1·1 (i), (ii).

Suppose $\operatorname{rank}(J^a_\varphi - I) = n-1$. To find out if there is a non-trivial fixed point of φ inside the commutator subgroup M', we consider a system (7); but this time, it is a system of $(n-1)$ homogeneous ZA-linear equations in $(n-1)$ unknowns $z_1, \ldots, z_{n-1}$. To find out if it has a non-trivial solution (which happens if and only if φ has a non-trivial fixed point inside M'), we just compute the corresponding determinant and see if it is equal to 0.

A somewhat more difficult problem is to find out if there is a non-trivial fixed point of φ *outside* M'. In this case, we proceed as in the proof of Proposition 3·2, but instead of having just one equation of the form (10), we'll have a system of $(n-1)$ ZA-linear equations (they are no longer homogeneous!) in $(n-1)$ unknowns.

Again, since ZA is a commutative domain, we can resolve this system (by using Kramer's formula). To apply Kramer's formula, all we need is to be able to find out for a given pair of polynomials, whether or not one of them is divisible by another. But this is equivalent to asking whether or not one of them belongs to the ideal generated by the other. Algorithms like that do exist (in particular, for ZA); they are based on what is known as Gröbner reduction process.

In any case, the existing algorithms in (Laurent) polynomial algebras not only tell us whether or not a given system of ZA-linear equations has a solution, but if it does, they give a solution (although we may not be able to find *all* of them). This completes the proof of Theorem 1·2.

Proof of Proposition 1·3. First of all, we note that the group $\mathrm{Fix}\varphi \cap M'$ is abelian; therefore, if it were finitely generated, then its every subgroup would be finitely generated, too.

Now suppose $g \in \mathrm{Fix}\,\varphi \cap M'$; $g \neq 1$. Then, by Lemma 3·1 (i), $g^u \in \mathrm{Fix}\,\varphi \cap M'$ for any $u \in ZA$ (i.e. $\mathrm{Fix}\,\varphi \cap M'$ is a normal subgroup of M). We are going to show that the subgroup of M' generated by all $g^{x_1^k}$, $k \in Z$, is not finitely generated.

By way of contradiction, suppose it is generated by $g^{x_1}, \ldots, g^{x_1^m}$, $m > 0$. But every element from this finitely generated group has a form g^p for some Laurent polynomial $p = p(x_1)$, whose degree does not exceed m. Therefore, we don't have the element $g^{x_1^{m+1}}$ in this group, hence a contradiction.

Proof of Theorem 1·4. By way of contradiction, suppose $H = \mathrm{Fix}\,\varphi$ is a non-cyclic subgroup of M generated by $h_1, \ldots, h_r$, $r < n$. Then there is a generator of the group M, say x_1, such that $x_1^m \notin H \cdot M'$ for any $m \geqslant 1$.

By Lemma 3·1 (i), the group $S = H \cap M'$ is normal in M. If H is non-cyclic, then S is non-trivial since it contains a non-trivial subgroup H' (note that $H \nsubseteq M'$ since otherwise, H would be infinitely generated by Proposition 1·3). Let $s \in S$, $s \neq 1$. Then for any $m \geqslant 1$, we have an equality of the form

$$\prod_j h_{i_j}^{c_{i_j,m}} = s^{x_1^m}$$

for some integers $c_{i_j,m}$. Let $d_k^a(s) \neq 0$; then (by Lemma 2·4) applying the derivation d_k^a to both sides of the last equality gives (in the group ring ZA):

$$\sum_q \prod_j h_{i_j}^{c_{i_j,m}} \cdot c_{i_q,m} \cdot d_k^a(h_{i_q}) = x_1^m \cdot d_k^a(s) \tag{11}$$

for some collection of indices j, q (of no particular importance to us).

When m runs through 1 to ∞, we are encountering representatives of infinitely many distinct cosets of the group $H \cdot M'/M'$ (as a subgroup of M/M') in the supports of elements on the right-hand side of (11). This is due to the condition $x_1^m \notin H \cdot M'$ for any $m \geqslant 1$ – see above. (By support of a group ring element $u \in ZG$, $u = \sum c_g \cdot g$, we mean the set $\{g,\ c_g \neq 0\}$.)

At the same time, the collection of coset representatives of $H \cdot M'/M'$ in the supports of elements on the left-hand side of (11) does not depend on m, and is therefore finite. This contradiction completes the proof of Theorem 1·4.

Proof of Proposition 1·5. Consider an (IA-)automorphism φ of the group M_n, $n \geqslant 3$, given by $\varphi(x_1) = x_1[x_2, x_3, x_1]$; $\varphi(x_i) = x_i,\ i \geqslant 2$. We are going to prove that the group $\mathrm{Fix}\,\varphi$ is infinitely generated.

It is clear that $\mathrm{Fix}\,\varphi$ contains a subgroup of M_n generated by $x_2, \ldots, x_n$. Now we are going to look for (other) fixed points in the form

$$g = x_1^k \cdot [x_1, x_2]^{u_1} \cdot [x_2, x_3]^{u_2} \cdot \ldots \cdot [x_{n-1}, x_n]^{u_{n-1}} \cdot x_2^{m_1} \cdot \ldots \cdot x_n^{m_{n-1}}.$$

Starting with $\varphi(g) = g$ and arguing along the same lines as in the proof of Theorem 1·1 (ii) and Proposition 3·2, we finally arrive at

$$x_1^k - 1 + (x_1 - 1)(x_2 - 1) \cdot u_1 = 0,$$

which is only possible if $k = u_1 = 0$.

It follows that $x_1^m \notin \mathrm{Fix}\,\varphi \cdot M'$ for any $m \geqslant 1$, and applying the argument from the proof of Theorem 1·4 yields the result.

Acknowledgement. I am grateful to A. Krasilnikov and V. Roman'kov for useful discussions, and to the referee for helpful comments.

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