## 1. Introduction

In a recent series of papers, Milgrom and Roberts (1990; 1995), Milgrom and Shannon (1994), and Topkis (1995) have investigated the comparative statics of models undergoing discrete changes in parameter values. These authors extended the earlier results, which were confined to instantaneous rates of change, to comparative statics results for discrete changes. In particular, the extension of local results to discrete changes in the parameters were shown to depend on strong assumptions about complementarity of factors. These papers also discussed the effects of additional constraints on the magnitude of the changes in the dependent variable—the "LeChatelier effects" (Milgrom and Roberts 1997). These authors relied on the new mathematics of lattice theory for their results. In this paper, using simpler standard mathematical techniques, we present some new results and derive the general conditions that are required for these LeChatelier effects in the large. We show, for example, that the short run demand for a factor is always less responsive to price changes than is the long run demand, provided that factor of production and the fixed factor do not switch from being substitutes to being complements (or vice versa) over the relevant range of the price change. The absence of a sign change in the complementarity/substitutability relation holds under conditions that are considerably more general than supermodularity of the production function, as posited by Milgrom et al. For example, Quirk (1997) has recently analyzed comparative statics results that guarantee no change of signs.

Paul Samuelson recognized that the mathematical structure of the LeChatelier principle in classical thermodynamics was similar to the maximization models used in economics. His first written discussion appeared in his *Foundations* (1947), in the context of unconstrained maximization of profits. In his 1949 paper, "The LeChatelier Principle in Linear Programming" (published in Samuelson 1966), he explained the effect in terms of changing the relative concavity of what we now refer to as the indirect objective function. In these and other works, Samuelson recognized the "local" nature of his results. In a "1965 Postscript" inserted in the 1949 paper, he added, "Shortly after I wrote this, I proved that the LeChatelier is indeed not true in the large: in any neighborhood where a complementarity term $\partial^2 U/\partial x_i \partial x_j$ changes sign, we can contrive a counterexample where a finite increase in $P_i$ leads to a smaller finite change with fewer constraints than with more!"

The standard neoclassical comparative statics results are of course all local, consisting of the signs of partial derivatives at some point. If the usual global curvature assumptions are made, e.g., strictly increasing quasiconcave utility functions or strictly concave production functions, then the implied demand curves—the factor demands for the competitive profit maximizing firm and the Hicksian demand functions—are everywhere downward sloping, even for finite changes in the parameters (prices). Indeed, the theory of revealed preference indicates the method of deriving these signed implications with finite methods only. These strong global assumptions are *not* sufficient, however, to imply that when a price changes by a finite amount, the "long run" demand functions will be more elastic than the "short run" demands. The purpose of this note is to reveal more clearly the structure of the LeChatelier results, using the calculus and duality techniques.

## 2. Profit Maximization

Consider the profit maximization problem

$$\max_x f(x) - wx, \tag{1}$$

where $x = (x_1, \ldots, x_n)$, $w = (w_1, \ldots, w_n)$, and the product $wx$ is an inner product. Assume the usual neoclassical first-order necessary and second-order sufficient conditions hold. We denote the resulting factor demand functions by $x = x^*(w)$, and let $\pi^*(w) = f(x^*(w)) - wx^*(w)$ be the indirect profit function.

Suppose now an additional constraint $g(x) = 0$ is added to the maximization problem. To be specific, consider the case where the auxiliary constraint consists of holding one factor, $x_n$, constant. Denote the remaining $n-1$ inputs by the vector $x_{-n}$, and let the resulting "short run" choice functions be $x^s_{-n}(w, x_n)$. As before, let the short run indirect profit function be $\pi^s(w, x_n) = f(x^s_{-n}(w, x_n), x_n) - wx$.

Let $x_n^o = x_n^*(w^o)$, so that the constraint $x_n = x_n^o$ is just binding at $w = w^o$. The usual envelope theorem argument (Silberberg 1990) establishes that the difference between long run and short run profits, $\Delta(w, x_n^o) = \pi^*(w) - \pi^s(w, x_n^o)$, is locally convex at $w = w^o$. Differentiate with respect to $w_i$ ($i \neq n$),

$$\frac{\partial^2 \Delta(w^o, x_n^o)}{\partial w_i^2} = \frac{\partial x_i^s(w^o, x_n^o)}{\partial w_i} - \frac{\partial x_i^*(w^o)}{\partial w_i} \geq 0. \tag{2}$$

This is the local LeChatelier result. To extend this result to finite changes in $w$, we need a condition stronger than the local convexity of $\Delta(\cdot, x_n^o)$. The following proposition establishes that short run factor demand is less responsive to finite price changes than is long run factor demand if and only if the difference between long run and short run profits grows when the factor price moves further and further away from the original level.

**Proposition 1.** Suppose the price of factor $i$ changes from $w_i^o$ to $w_i'$, while the prices of other factors remain unchanged. Then $|x_i^*(w') - x_i^*(w^o)| > |x_i^s(w', x_n^o) - x_i^s(w^o, x_n^o)|$ if and only if $\partial\Delta(w', x_n^o)/\partial w_i$ has the same sign as $w_i' - w_i^o$.

**Proof.** From Hotelling's lemma, $\partial\Delta(w, x_n^o)/\partial w_i = x_i^s(w, x_n^o) - x_i^*(w)$. The proposition follows immediately because $x_i^s(w, x_n^o)$ and $x_i^*(w)$ are both decreasing in $w_i$, and because $x_i^s(w^o, x_n^o) = x_i^*(w^o)$. *Q.E.D.*

A corollary of Proposition 1 is that global convexity of $\Delta(w, x_n^o)$ in $w_i$ implies the global LeChatelier result for factor $i$. This is because $\Delta(w, x_n^o)$ reaches a minimum of zero at $w_i = w_i^o$. If the function is globally convex in $w_i$, it will be increasing in $w_i$ for $w_i > w_i^o$, and decreasing in $w_i$ for $w_i < w_i^o$. In other words, global convexity of $\Delta(w, x_n^o)$ in $w_i$ implies that $\partial\Delta(w', x_n^o)/\partial w_i$ has the same sign as $w_i' - w_i^o$. By Proposition 1, then, the change in long run factor demand resulting from a finite change in price from $w_i^o$ to $w_i'$, $|x_i^*(w') - x_i^*(w^o)|$, will exceed the change in short run factor demand, $|x_i^s(w', x_n^o) - x_i^s(w^o, x_n^o)|$.

A second corollary to Proposition 1 is that the global LeChatelier result will hold for factor $i$ if the signs of $\partial x_i^s/\partial x_n$ and of $\partial x_n^*/\partial w_i$ remain unchanged from $w_i^o$ to $w_i'$ and from $x_n^o$ to $x_n'$. To see this, let $x_n' = x_n^*(w')$. Since $\partial\pi^*(w)/\partial w_i = -x_i^*(w) =$

$-x_i^s(w, x_n^*(w)) = \partial\pi^s(w, x_n)/\partial w_i|_{x_n = x_n^*(w)}$, then

$$\frac{\partial\Delta(w', x_n^o)}{\partial w_i} = \frac{\partial\pi^s(w', x_n')}{\partial w_i} - \frac{\partial\pi^s(w', x_n^o)}{\partial w_i} = \int_{x_n^o}^{x_n'} \frac{\partial^2\pi^s(w', x_n)}{\partial w_i \partial x_n} dx_n. \tag{3}$$

Since $\Delta(w, x_n^o)$ is locally convex at $w_i = w_i^o$, evaluating the second derivative at this point and again using $\partial\pi^*(w)/\partial w_i = \partial\pi^s(w, x_n)/\partial w_i|_{x_n = x_n^*(w)}$ gives

$$\frac{\partial^2\Delta(w^o, x_n^o)}{\partial w_i^2} = \frac{\partial^2\pi^s(w^o, x_n^o)}{\partial w_i \partial x_n} \frac{\partial x_n^*(w^o)}{\partial w_i} \geq 0. \tag{4}$$

So if $\partial^2\pi^s/\partial w_i\partial x_n$ and $\partial x_n^*/\partial w_i$ do not change signs from $w_i^o$ to $w_i'$ and from $x_n^o$ to $x_n'$, they will either be both positive or be both negative in the relevant region. When both derivatives are positive, then $w_i' > w_i^o$ implies $x_n' > x_n^o$, and the expression in (3) is positive. When both derivatives are negative, then $w_i' > w_i^o$ implies $x_n' < x_n^o$, and the expression in (3) is again positive. Similarly, when $w_i' < w_i^o$, (3) can be shown to be negative. Thus, $\partial\Delta(w', x_n^o)/\partial w_i$ has the same sign as $w_i' - w_i^o$. By Proposition 1, then, we have $|x_i^*(w') - x_i^*(w^o)| > |x_i^s(w', x_n^o) - x_i^s(w^o, x_n^o)|$. In the context of the profit maximization problem, $\partial^2\pi^s/\partial w_i\partial x_n = -\partial x_i^s/\partial x_n$. Thus no sign change for $\partial^2\pi^s/\partial w_i\partial x_n$ is the same as no sign change for $\partial x_i^s/\partial x_n$, and the proof of the corollary is complete.

A few examples will clarify the relationship between our results and the results in the existing literature.

*Example 1.*

Suppose $w_i' - w_i^o$ is sufficiently small. Then the partial derivatives $\partial x_i^s/\partial x_n$ and $\partial x_n^*/\partial w_i$ will not change signs locally. Therefore, $|x_i^*(w') - x_i^*(w^o)| > |x_i^s(w', x_n^o) - x_i^s(w^o, x_n^o)|$. Dividing by $w_i' - w_i^o$ and taking the limit as $w_i' - w_i^o$ goes to zero, we get $|\partial x_i^*(w^o)/\partial w_i| > |\partial x_i^s(w^o, x_n^o)/\partial w_i|$, the standard local LeChatelier result.

*Example 2.*

Suppose the Hessian matrix of the production function, $f_{xx}$, is negative definite with nonnegative off-diagonal entries. Then its inverse $f_{xx}^{-1}$ is nonpositive entrywise (e.g., Takayama 1985). Standard comparative statics analysis gives $\partial x^*/\partial w_i = f_{xx}^{-1}e_i$, where $e_i$ is a column vector with 1 in the $i$-th entry and 0 elsewhere. Therefore $\partial x_n^*/\partial w_i < 0$. For the short run problem, let $f_{x_{-n}x_{-n}}$ be the matrix of cross derivatives for the $n-1$ variable inputs. Obviously, $f_{x_{-n}x_{-n}}$ inherits the properties of $f_{xx}$. In particular, the inverse matrix $(f_{x_{-n}x_{-n}})^{-1}$ is nonpositive. Comparative statics analysis gives $\partial x^s_{-n}/\partial x_n = -(f_{x_{-n}x_{-n}})^{-1} f_{x_{-n}x_n}$, where $f_{x_{-n}x_n}$ is a vector of cross derivatives, $\partial^2 f/\partial x_j \partial x_n$, $j \neq n$. Since each element of $f_{x_{-n}x_n}$ is nonnegative, we have $\partial x^s_i/\partial x_n > 0$. Thus both $\partial x^*_n/\partial w_i$ and $\partial x^s_i/\partial x_n$ never change signs, and the global LeChatelier result will hold.

Notice that $f_{xx}$ is negative definite with nonnegative off-diagonal elements implies that the function $f$ is supermodular (Milgrom and Roberts 1990). In this case, $x_i$ and $x_n$ will be complements both in the sense that $\partial x^s_i/\partial x_n > 0$ and in the sense that $\partial x^*_n/\partial w_i < 0$. Another interpretation of a negative definite Hessian matrix with nonnegative off-diagonal elements is that it implies a "generalized converse" of diminishing marginal product. We say that a generalized converse of diminishing marginal product holds when $f_x(x) \geq f_x(y)$ implies $x \leq y$. Then if the marginal product of each input has (weakly) increased, we can infer that the level of each input has (weakly) decreased. To see why this is true, let $g$ denote the inverse function of $f_x$, i.e., $w = f_x(x)$ if and only if $x = g(w)$. Then, for $w = f_x(x)$ and $z = f_x(y)$, a Taylor expansion of $g$ yields

$$x - y = g(f_x(x)) - g(f_x(y)) = \left[ \int_0^1 f_{xx}^{-1} \left( g(f_x(y) + t(f_x(x) - f_x(y))) \right) dt \right] (f_x(x) - f_x(y)),$$

where the inverse matrix $f_{xx}^{-1}$ is defined and is nonpositive entrywise. Consequently, $f_x(x) - f_x(y) \geq 0$ would imply $x - y \leq 0$.

*Example 3.*

Supermodularity of $f$ is sufficient but not necessary for no sign change in $\partial x^s_i/\partial x_n$ and $\partial x^*_n/\partial w_i$. Quirk (1997), for example, derives conditions which allow one to unambiguously sign partial derivatives (and hence guarantee no sign change) in comparative statics. Suppose the matrix of second derivatives of the production function, in addition to being negative definite, has a sign pattern as follows (Quirk 1997, p. 138):

$$f_{xx} = \begin{pmatrix} - & - & - & 0 & 0 \\ - & - & - & 0 & 0 \\ - & - & - & + & 0 \\ 0 & 0 & + & - & + \\ 0 & 0 & 0 & + & - \end{pmatrix}. \tag{5}$$

Then, the $i$-th row of the inverse of $f_{xx}$ has the sign pattern $(?, ?, ?, -, -)$, where the "?" symbol indicates ambiguous sign. Therefore,

$$\frac{\partial x_5^*}{\partial w_4} = (?, ?, ?, -, -) \cdot (0, 0, 0, +, 0)' < 0. \tag{6}$$

Similarly,

$$\frac{\partial x_4^s}{\partial x_5} = (?, ?, ?, -) \cdot (0, 0, 0, -)' > 0. \tag{7}$$

Thus the signs of the partial derivatives are unambiguous and the LeChatelier result for input $x_4$ will hold for all values of $w_4$.

## 3. Cost Minimization

Consider the cost minimization problem with fixed cost coefficients $w$:

$$C^*(y) = \min_x \{wx | f(x) = y\}, \tag{8}$$

where $\partial C^* / \partial y = \lambda^*(y)$ is the marginal cost function. Moreover let $x_n^o = x_n^*(y^o)$ and $x_n' = x_n^*(y')$ be the cost minimizing levels of $x_n$ at output levels $y^o$ and $y'$, respectively.

Define the short run problem:

$$C^s(y, x_n^o) = \min_{x_{-n}} \{wx | f(x) = y, x_n = x_n^o\}. \tag{9}$$

Let $\partial C^s(y, x_n^o)/\partial y = \lambda^s(y, x_n^o)$ be the short run marginal cost.

We know $\Delta(y, x_n^o) = C^*(y) - C^s(y, x_n^o)$ attains a local maximum at $y = y^o$. Therefore

$$\frac{\partial^2 \Delta(y^o, x_n^o)}{\partial y^2} = \frac{\partial^2 C^s(y^o, x_n^o)}{\partial x_n \partial y} \frac{\partial x_n^*(y^0)}{\partial y} \leq 0. \tag{10}$$

In the context of the cost minimization problem, $\partial^2 C^s / \partial x_n \partial y = \partial \lambda^s / \partial x_n$. The above inequality (10) means that $\partial \lambda^s / \partial x_n$ and $\partial x_n^* / \partial y$ must be of opposite signs at $y^o$.

By imposing an additional assumption that these two partial derivatives do not change signs when output changes from $y^o$ to $y'$, we obtain a global LeChatelier result for long run and short run marginal costs.

**Proposition 2.** If $\partial\lambda^s/\partial x_n$ and $\partial x_n^*/\partial y$ do not change signs in the range $[y^o, y']$ and from $x_n^o$ to $x_n'$, then $\lambda^*(y') < \lambda^s(y', x_n^o)$ for $y' > y^o$.

**Proof.** Direct calculations yield

$$\lambda^*(y') - \lambda^s(y', x_n^o) = \lambda^s(y', x_n') - \lambda^s(y', x_n^o) = \int_{x_n^o}^{x_n'} \frac{\partial\lambda^s(y', x_n)}{\partial x_n} dx_n. \tag{11}$$

Case i. $\partial\lambda^s/\partial x_n > 0$ and $\partial x_n^*/\partial y < 0$. Since $y' > y^o$, we have $x_n' < x_n^o$. Therefore (11) is negative.

Case ii. $\partial\lambda^s/\partial x_n < 0$ and $\partial x_n^*/\partial y > 0$. Since $y' > y^o$, we have $x_n' > x_n^o$. Again (11) is negative. *Q.E.D.*

Proposition 2 implies that, when output rises, the short run increase in marginal cost is higher than the long run increase in marginal cost. Obviously, the argument also implies that, when output falls, the savings in long run marginal costs are greater than the savings in short run marginal costs.

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# The LeChatelier Principle: the Long and the Short of It

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*Abstract*: Using ordinary calculus techniques, we investigate the conditions under which LeChatelier effects are signable for finite changes in parameter values. We show, for example, that the short run demand for a factor is always less responsive to price changes than the long run demand, provided that the factor of production and the fixed factor do not switch from being substitutes to being complements (or vice versa) over the relevant range of the price change. The absence of a sign change in the complementarity/substitutability relation holds under conditions that are considerably more general than supermodularity of the production function.

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