Exponential stability for Variable Coefficients Rayleigh Beams under Boundary Feedback Control ¶

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Abstract

The spectrum of a variable coefficients Rayleigh beam with boundary feedback control is discussed in present paper. By using the asymptotic technique, the explicit asymptotic formula of eigenvalues of the closed loop system is given. With help of the result in [13], it is concluded that the closed loop system is a Riesz system. As a result, the spectrum determined growth condition and exponential stability are deduced. In particular, a conjecture in [3] is completely settled.

KeyWords: Rayleigh beam, eigenvalue distributions, Riesz basis, Exponential stability.

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1 Introduction

It is well known that the analysis of the eigenvalue problem of variable coefficient ordinary differential equation with parameter is usually difficult because explicit solution formula is hard to come by. However, in practice, we often have to consider this problem, for instance, non-homogeneous material in engineering and smart material etc, which lead to variable coefficient differential equations. In the present paper, we study one of the beam models— non-homogeneous Rayleigh beam model under the boundary feedback control,

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the motion governed by partial differential equation:

$$\begin{cases} \rho(x)\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x}\left(I_\rho(x)\dfrac{\partial^3 u}{\partial t^2 \partial x}\right) + \dfrac{\partial^2}{\partial x^2}\left(EI(x)\dfrac{\partial^2 u}{\partial x^2}\right) = 0, & 0 < x < 1, t > 0, \\ u(0,t) = \dfrac{\partial u}{\partial x}(0,t) = 0, & t > 0, \\ EI\dfrac{\partial^2 u}{\partial x^2}(1,t) + \alpha \dfrac{\partial^2 u}{\partial x \partial t}(1,t) = 0, & t > 0, \\ \dfrac{\partial}{\partial x}\left(EI\dfrac{\partial^2 u}{\partial x^2}\right)(1,t) - I_\rho \dfrac{\partial^3 u}{\partial x \partial t^2}(1,t) - \beta \dfrac{\partial u}{\partial t}(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x), \end{cases} \tag{1.1}$$

Here, $u(t,x)$ is the transverse displacement and x, t stand respectively for the position and time, and $\rho(x) > 0$ is the mass density, $EI(x) > 0$ is the stiffness of the beam, $I_\rho(x) > 0$ is the mass moment of inertia and $\alpha, \beta \geq 0$ are constant feedback gains that can be tuned. Other details of this model can also be found in [1].

This problem was treated firstly in [2], in which the coefficients of the equation are constants and the exponential stability of the system was obtained under the condition $\alpha = 1, \beta \geq 0$. Recently, [3] consider the Riesz basis property of the same system. As a result, the system satisfies spectrum determined growth condition. In this paper, we assume that

$$\rho(x), I_\rho(x), EI(x) \in C^4[0,1], \tag{1.2}$$

and try to obtain the Riesz basis property and exponential stability of the system.

Here the main difficulty we encountered is to calculate asymptotic eigenvalues of the system. Although there is a suitly complete method to calculate asymptotic fundamental solution for instance see [10], it seems that the eigenvalue problem led from Rayleigh beam is unfit this modality. To settle it, we employ the method of operator pencil used in [7] [9] and [8] to obtain the asymptotic expansion of the fundamental solutions of the eigenvalue boundary problem, and then use them to expand the characteristic determinant and obtain asymptotic expansions for the eigenvalues.

The rest of the paper is organized as follows. In §2 we convert system (1.1) into an evolution equation in an appropriate Hilbert space, then we prove that the evolutionary system associates a C_0 semigroup whose generator has compact resolvent. Therefore the eigenvalue problem leads the eigenvalue boundary problem of an variable coefficient ordinary differential equation. In order to solve the eigenvalue boundary problem, we shall use a space-scaling transformation to derive an equivalent boundary problem that is ready to expand asymptotically. In §3 an asymptotic distribution of the eigenvalues of the system is obtained via expanding the characteristic determinant. In the last section, further property of semigroup and exponential stability of the system are indicated and a conjecture in [3] is discussed and settled.

2 Basic State Space Setup and Eigenvalue Problem

We start our investigation by formulating the problem in the following Hilbert spaces:

$$V := \{f \in H^1(0,1) \mid f(0) = 0\},$$

endowed with norm

$$\|f\|_V^2 := \int_0^\ell [\rho(x)|f(x)|^2 + I_\rho(x)|f'(x)|^2]dx$$

and

$$W = \{f \in H^2(0,1) \mid f(0) = f'(0) = 0, f'(x) = df(x)/dx\},$$

endowed with norm

$$\|f\|_W^2 = \int_0^\ell EI(x)|f''(x)|^2dx.$$

Easy to see that

$$W \subset V \subset L^2[0,1] \subset V' \subset W',$$

where W' and V' are the dual spaces of W and V respectively.

Now we define linear operators $A, D \in \mathcal{L}(W, W')$ and $B, C \in \mathcal{L}(V, V')$ by

$$\langle A\phi, \psi\rangle = \int_0^1 EI(x)\phi''(x)\overline{\psi''(x)}dx, \quad \forall\ \phi, \psi \in W, \tag{2.1}$$

$$\langle D\phi, \psi\rangle = \phi'(1)\overline{\psi'(1)}, \quad \forall\ \phi, \psi \in W, \tag{2.2}$$

$$\langle B\phi, \psi\rangle = \phi(1)\overline{\psi(1)}, \quad \forall\ \phi, \psi \in V, \tag{2.3}$$

$$\langle C\phi, \psi\rangle = \int_0^1 \left[\rho(x)\phi(x)\overline{\psi(x)} + I_\rho\phi'(x)\overline{\psi'(x)}\right] dx, \quad \forall\ \phi, \psi \in V. \tag{2.4}$$

Lax-Milgram Theorem [6, pp.92] says that A (resp. C) is a canonical isomorphism of W (resp. V) onto W' (resp. V'). With these operators, the equation (1.1) can be written into a variational equation

$$\langle Cu_{tt}, \psi\rangle + \langle Au, \psi\rangle + \alpha\langle Du_t, \psi\rangle + \beta\langle Bu_t, \psi\rangle = 0, \quad \forall\ \psi \in W. \tag{2.5}$$

Let

$$\mathcal{H} := W \times V$$

with norm $\|(f,g)\|_{\mathcal{H}}^2 = \|f\|_W^2 + \|g\|_V^2$. Then we can define a linear operator $\mathcal{A}$ on $\mathcal{H}$ by:

$$\mathcal{D}(\mathcal{A}) := \{(f,g) \in \mathcal{H} \mid g \in W, Af + \alpha Dg + \beta Bg \in V'\}, \tag{2.6}$$

$$\mathcal{A}(f,g) := \left(g, -C^{-1}(Af+\alpha Dg+\beta Bg)\right), \quad \forall\ (f,g)\in\mathcal{D}(\mathcal{A}). \tag{2.7}$$

Thus, (2.5) can be formulated into an evolution equation in $\mathcal{H}$ as

$$\begin{cases} \frac{dY(t)}{dt} = \mathcal{A}Y(t), \quad t>0, \quad (\text{where } Y(t)=(u,u_t)) \\ Y(0)=Y_0=(u_0,u_1). \end{cases} \tag{2.8}$$

Lemma 2.1 *Let $\mathcal{A}$ be defined by (2.6) and (2.7). Then $\mathcal{A}$ is a densely defined closed dissipative operator with $0\in\rho(\mathcal{A})$, and so $\mathcal{A}$ generates a C_0 semigroup of contraction.*

PROOF For any $(f,g)\in\mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}(f,g),(f,g)\rangle &= \int_0^1 \left[EI(x)g''(x)\overline{f''(x)} - \rho(x)C^{-1}[Af+\alpha Dg+\beta Bg]\overline{g(x)}\right]dx \\ &\quad -\int_0^1 I_\rho(x)(C^{-1}[Af+\alpha Dg+\beta Bg])'(x)\overline{g'(x)}dx \\ &= \int_0^1 EI(x)g''(x)\overline{f''(x)}dx - \int_0^1 [Af+\alpha Dg+\beta Bg]\overline{g(x)}dx \\ &= \langle Ag,f\rangle - \langle Af,g\rangle - \alpha\langle Dg,g\rangle - \beta\langle Bg,g\rangle \\ &= \langle Ag,f\rangle - \langle Af,g\rangle - \alpha|g'(1)|^2 - \beta|g(1)|^2. \end{aligned}$$

So

$$Re\langle \mathcal{A}(f,g),(f,g)\rangle = -\alpha|g'(1)|^2 - \beta|g(1)|^2 \le 0.$$

To show that $0\in\rho(\mathcal{A})$, we let $(y,z)\in\mathcal{H}$ and consider the resolvent equation

$$\mathcal{A}(f,g) = (y,z).$$

So $y=g$ and

$$z = -C^{-1}[Af+\alpha Dg+\beta Bg].$$

Therefore, for any $\psi\in W$,

$$\langle Af+\alpha Dg+\beta Bg,\psi\rangle = -\langle Cz,\psi\rangle.$$

Substituting $g=y$ into the above equation yields

$$\langle Af,\psi\rangle = -\langle Cz+\alpha Dy+\beta By,\psi\rangle, \quad \forall\ \psi\in W. \tag{2.9}$$

Since for any $\psi\in W$, we have

$$\langle A\psi,\psi\rangle = \|\psi\|_W^2.$$

So from the Lax-Milgram Theorem, there exists a unique $f\in W$ so that (2.9) holds and $0\in\rho(\mathcal{A})$. The last assertion is a direct consequence from the theory of semigroup (cf. [12, pp.3, Theorem 1.2.4]). $\square$

Lemma 2.2 *Let $(f,g)\in\mathcal{H}$. Then $(f,g)\in\mathcal{D}(\mathcal{A})$ if and only if $f\in W\cap H^3$ and $g\in W$ such that*

$$\Big(EI(x)f''(x)\Big)'\Big|_{x=1}+I_\rho\Big[C^{-1}(Af,\alpha Dg+\beta Bg)\Big]'\Big|_{x=1}-\beta g(1)=0,\quad EI(1)f''(1)+\alpha g'(1)=0.$$

From this we see that $\mathcal{A}^{-1}$ is compact.

PROOF The sufficiency is obvious. To prove the necessity, let $(f,g)\in\mathcal{D}(\mathcal{A})$ and $\mathcal{A}(f,g)=(y,z)\in\mathcal{H}$. Then we have $g=y\in W$ and

$$-C^{-1}[Af+\alpha Dg+\beta Bg]=z.$$

Since $z\in V$ and $C:V\to V'$ is an isomorphism, so we have

$$Af+\alpha Dy+\beta By=-Cz,\quad \text{in}\quad V'\subset W',$$

and hence

$$\begin{aligned}&\int_0^1 EI(x)f''(x)\overline{\psi''(x)}dx+\alpha y'(1)\overline{\psi'(1)}+\beta y(1)\overline{\psi(1)}\\ &+\int_0^1\Big[\rho(x)z(x)\overline{\psi(x)}+I_\rho(x)z'(x)\overline{\psi'(x)}\Big]dx=0,\quad\forall\,\psi\in W.\end{aligned}\tag{2.10}$$

Now for any $\phi\in C_0^\infty(0,1)$, let $\psi(x)=\int_0^x\phi(s)ds$ and substitute it into (2.10) yields

$$\begin{aligned}&\int_0^1 EI(x)f''(x)\overline{\phi'(x)}dx+\beta y(1)\int_0^1\overline{\phi(x)}dx\\ &+\int_0^1\overline{\phi(x)}dx\int_x^1\rho(s)z(s)ds+\int_0^1 I_\rho(x)z'(x)\overline{\phi(x)}dx=0\end{aligned}$$

and

$$\int_0^1 EI(x)f''(x)\overline{\phi'(x)}dx=-\int_0^1\Big[\beta y(1)+\int_x^1\rho(s)z(s)ds+I_\rho(x)z'(x)\Big]\overline{\phi(x)}dx=0,$$

for all $\phi\in W$. Thus

$$\Big(EI(x)f''(x)\Big)'=\beta y(1)+\int_x^1\rho(s)z(s)ds+I_\rho(x)z'(x)\in L^2[0,1].\tag{2.11}$$

Since $EI\in C^4[0,1]$ (see. (1.2)), so we have $f\in H^3(0,1)\cap W$. In particular,

$$\Big(EI(x)f''(x)\Big)'\Big|_{x=1}=\beta y(1)+I_\rho(x)z'(x)\Big|_{x=1}.$$

Inserting $g=y$ and $z=-C^{-1}(Af+\alpha Dg+\beta Bg)$ into the above yields

$$\Big(EI(x)f''(x)\Big)'\Big|_{x=1}+I_\rho(x)\Big[C^{-1}(Af+\alpha Dg+\beta Bg)\Big]'\Big|_{x=1}-\beta g(1)=0.$$

Again, for $\phi \in V$ with $\phi(1) = 1$, we let $\psi := \int_0^x \phi(s)ds$, and insert it into (2.10) and to conclude from (2.11) that

$$EI(1)f''(1) + \alpha y'(1) = 0.$$

Since $g = y$, the necessity is proven because

$$EI(1)f''(1) + \alpha g'(1) = 0.$$

By Lemma 2.1, $\mathcal{A}^{-1}$ exists and is bounded on $\mathcal{H}$. From The Sobolev Embedding Theorem, $\mathcal{A}^{-1}$ is compact. $\square$

We are now in a position to investigate the eigenvalue problem of $\mathcal{A}$. Let $\lambda \in \sigma(\mathcal{A})$ and $(\phi, \psi) \in \mathcal{H}$ be such that

$$\mathcal{A}(\phi, \psi) = \lambda(\phi, \psi).$$

Then, $\psi = \lambda\phi$ and ϕ satisfies

$$\begin{cases} \lambda^2\rho(x)\phi(x) - \lambda^2\Big(I_\rho(x)\phi'(x)\Big)' + \Big(EI(x)\phi''(x)\Big)'' = 0, \quad 0 < x < 1, \\ \phi(0) = \phi'(0) = 0, \\ EI(1)\phi''(1) + \alpha\lambda\phi'(1) = 0, \\ \Big(EI\phi''\Big)'(1) - \lambda^2 I_\rho(1)\phi'(1) - \beta\lambda\phi(1) = 0. \end{cases} \tag{2.12}$$

Lemma 2.3 *Let $h_1(x), h_2(x)$ be two linearly independent solutions for the second order linear homogeneous differential equation*

$$\Big(I_\rho(x)\phi'(x)\Big)' - \rho(x)\phi(x) = 0, \tag{2.13}$$

then we have

$$D := h_1(0)h_2'(1) - h_1'(1)h_2(0) \neq 0. \tag{2.14}$$

PROOF. Assume not, then the following system of linear equations in t_1 and t_2

$$\begin{cases} t_1h_1(0) + t_2h_2(0) = 0, \\ t_1h_1'(1) + t_2h_2'(1) = 0, \end{cases}$$

is singular because the determinant of the coefficient matrix is $h_1(0)h_2'(1) - h_1'(1)h_2(0) = 0$. So there exists a non-trivial solution, say $\binom{c_1}{c_2}$. Let $z := c_1y_1 + c_2y_2$, then z is a solution of the following initial problem:

$$\begin{cases} \Big(I_\rho(x)z'(x)\Big)' - \rho(x)z(x) = 0, \\ z(0) = z'(1) = 0. \end{cases}$$

By the uniqueness theorem, $z \equiv 0$ and so y_1 and y_2 are linearly dependent, which contradicts the assumption of the lemma. $\square$

Lemma 2.4 *If $\alpha+\beta>0$, then*

$$\mathrm{Re}(\lambda)<0. \tag{2.15}$$

PROOF. We go back to the eigenvalue equation (2.12). Multiplying $\bar{\phi}$, the conjugate of ϕ, on both side of the first equation in (2.12) and integrating from 0 to 1 with respect to x, we obtain

$$\begin{aligned}&\lambda^2\int_0^1 \rho(x)|\phi(x)|^2dx+\lambda^2\int_0^1 I_\rho(x)|\phi'(x)|^2dx+\beta\lambda|\phi(1)|^2\\&+\alpha\lambda|\phi'(1)|^2+\int_0^1 EI(x)|\phi''(x)|^2dx=0.\end{aligned} \tag{2.16}$$

Write $\lambda=\mathrm{Re}\lambda+i\mathrm{Im}\lambda$, then

$$\begin{aligned}&\big((\mathrm{Re}\lambda)^2-(\mathrm{Im}\lambda)^2\big)\int_0^1\Big(\rho(x)|\phi(x)|^2+I_\rho(x)|\phi'(x)|^2\Big)dx+\beta(\mathrm{Re}\lambda)|\phi(1)|^2\\&+\alpha(\mathrm{Re}\lambda)|\phi'(1)|^2+\int_0^1 EI(x)|\phi''(x)|^2dx=0,\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}&2(\mathrm{Re}\lambda)(\mathrm{Im}\lambda)\int_0^1\Big(\rho(x)|\phi(x)|^2+I_\rho(x)|\phi'(x)|^2\Big)dx\\&+\beta(\mathrm{Im}\lambda)|\phi(1)|^2+\alpha(\mathrm{Im}\lambda)|\phi'(1)|^2=0.\end{aligned} \tag{2.18}$$

If $\mathrm{Im}\lambda=0$, then $\mathrm{Re}\lambda<0$ by (2.17). If $\mathrm{Im}\lambda\neq 0$, then $\mathrm{Re}\lambda<0$ by (2.18) and the proof is completed. $\square$

To further simplify (2.12), we expand it to yield:

$$\begin{cases}\phi^{(4)}(x)+2\dfrac{EI'(x)}{EI(x)}\phi'''(x)+\dfrac{EI''(x)}{EI(x)}\phi''(x)\\ \qquad -\lambda^2\left(\dfrac{I_\rho(x)}{EI(x)}\phi''(x)+\dfrac{I_\rho'(x)}{EI(x)}\phi'(x)-\dfrac{\rho(x)}{EI(x)}\phi(x)\right)=0,\\ \phi(0)=\phi'(0)=0,\\ EI(1)\phi''(1)+\alpha\lambda\phi'(1)=0,\\ EI(1)\phi'''(1)+EI'(1)\phi''(1)-\lambda^2I_\rho(1)\phi'(1)-\beta\lambda\phi(1)=0.\end{cases} \tag{2.19}$$

Introducing a space-scaling transformation

$$\phi(x):=f(z),\quad z:=\frac{1}{h}\int_0^x\left(\frac{I_\rho(\zeta)}{EI(\zeta)}\right)^{1/2}d\zeta, \tag{2.20}$$

where

$$h:=\int_0^1\left(\frac{I_\rho(\zeta)}{EI(\zeta)}\right)^{1/2}d\zeta, \tag{2.21}$$

then equation (2.19) can be rewritten as

$$\begin{cases} f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) \\ \qquad - h^2\lambda^2\Big[f''(z) + d(z)f'(z) - e(z)f(z)\Big] = 0, \\ f(0) = f'(0) = 0, \\ b_{21}f''(1) + b_{22}f'(1) + b_{23}\alpha\lambda f'(1) = 0, \\ b_{11}f'''(1) + b_{12}f''(1) + b_{13}f'(1) - \lambda^2 b_{14}f'(1) - \beta\lambda f(1) = 0. \end{cases} \tag{2.22}$$

Let $f' := df/dz$, $z_x := dz/dx$ and

$$a(z) := 6\frac{z_{xx}}{z_x^2} + 2\frac{1}{z_x}\frac{EI'(x)}{EI(x)}, \quad z_x = \frac{1}{h}\left(\frac{I_\rho(x)}{EI(x)}\right)^{1/2}, \tag{2.23}$$

$$b(z) := 3\frac{z_{xx}^2}{z_x^4} + 4\frac{z_{xxx}}{z_x^3} + 6\frac{z_{xx}}{z_x^3}\frac{EI'(x)}{EI(x)} + \frac{1}{z_x^2}\frac{EI''(x)}{EI(x)}, \tag{2.24}$$

$$c(z) := \frac{z_{xxxx}}{z_x^4} + 2\frac{z_{xxx}}{z_x^4}\frac{EI'(x)}{EI(x)} + \frac{z_{xx}}{z_x^4}\frac{EI''(x)}{EI(x)}, \tag{2.25}$$

$$d(z) := \frac{z_{xx}}{z_x^2} + \frac{1}{h^2 z_x^3}\frac{I_\rho'(x)}{EI(x)}, \quad e(z) := \frac{1}{h^2 z_x^4}\frac{\rho(x)}{EI(x)}, \tag{2.26}$$

$$\begin{aligned} &b_{11} := z_x^3(1)EI(1), \quad b_{12} := 3z_x(1)z_{xx}(1)EI(1) + z_x^2(1)EI'(1), \\ &b_{14} := I_\rho(1)z_x(1), \quad b_{13} := z_{xxx}(1)EI(1) + z_{xx}(1)EI'(1), \\ &b_{21} := z_x^2(1)EI(1), \quad b_{22} := z_{xx}(1)EI(1), \quad b_{23} := z_x(1). \end{aligned} \tag{2.27}$$

If we replace λ by $\mu := h\lambda$, then (2.22) changes to

$$\begin{cases} f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) \\ \qquad - \mu^2\Big[f''(z) + d(z)f'(z) - e(z)f(z)\Big] = 0, \\ f(0) = 0, \\ f'(0) = 0, \\ b_{21}f''(1) + b_{22}f'(1) + b_{23}\alpha h^{-1}\mu f'(1) = 0, \\ b_{11}f'''(1) + b_{12}f''(1) + b_{13}f'(1) - h^{-2}\mu^2 b_{14}f'(1) - \beta h^{-1}\mu f(1) = 0, \end{cases} \tag{2.28}$$

which is equivalent to equation (2.19). In summary, we have the following result.

Theorem 2.1 *$\lambda \in \sigma(\mathcal{A})$ iff equation (2.28) has a nonzero solution $f(z)$ for $\mu := h\lambda$. In addition, the function $\phi(x)$ in the corresponding eigenfunction $(\phi, \lambda\phi)$ of $\mathcal{A}$ is given by (2.20).*

3 Asymptotic Expressions of Eigenfrequencies

In this section, we shall obain asymptotic expansions for the eigenvalues of $\mathcal{A}$. The main trick is to treat the fundamental solutions of (2.28) first, and then use them to expand the characteristic determinant of $\mathcal{A}$ and obtain the asymptotic eigen-frequency.

To begin, we use a standard technique of Naimark [10] and divide the complex plane into four sectors

$$\mathcal{S}_k := \left\{ z \in \mathbb{C} : \frac{k\pi}{2} \leq \arg z \leq \frac{(k+1)\pi}{2} \right\}, \quad k = 0, 1, 2, 3 \tag{3.1}$$

and for each $\mathcal{S}_k$, we will pick ω_1 and ω_2 (both square roots of -1) so that

$$\mathrm{Re}(\rho\omega_1) \leq \mathrm{Re}(\rho\omega_2), \quad \forall\ \rho \in \mathcal{S}_k. \tag{3.2}$$

In particular, we will choose

$$\omega_1 := e^{i\frac{\pi}{2}}, \quad \omega_2 := e^{i\frac{3}{2}\pi}$$

in sector $\mathcal{S}_0$ and re-shuffle them in each the remaining sectors so that (3.2) holds. Writing $\mu := \rho\omega_1$ for ρ in each sector $\mathcal{S}_k$, we have the following result on the fundamental solutions of (2.28) from [8, Theorem 3] (see also [7]).

Lemma 3.1 *In each sector $\mathcal{S}_k$, for $\rho \in \mathcal{S}_k$ with $|\rho|$ sufficiently large, the equation*

$$f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) + \rho^2\Big[f''(z) + d(z)f'(z) - e(z)f(z)\Big] = 0 \tag{3.3}$$

has four linearly independent fundamental solutions $y_s(z;\rho)$ $(s = 1, 2, 3, 4)$ and they possess the following asymptotic expressions (for $j = 0, 1, 2, 3$)

$$y_s^{(j)}(z;\rho) = h_s^{(j)}(z) + \mathcal{O}(\lambda_1^{-2}), \qquad s = 1, 2, \tag{3.4}$$

$$y_s^{(j)}(z;\rho) = (\rho\omega_{s-2})^j e^{\rho\omega_{s-2}x}\left[y_0(z) + \mathcal{O}(\rho^{-1})\right], \quad s = 3, 4, \tag{3.5}$$

$$y_0(z) := e^{-\frac{1}{2}\int_0^z (a(t) - d(t))dt}. \tag{3.6}$$

Here, $h_1(z) := h_1(x(z)), h_2(z) := h_2(x(z))$ are the two linearly independent solutions of (2.2) after the transformation $x(z) := z(x)^{-1}$. That is, they are two linearly independent solutions of

$$f''(z) + d(z)f'(z) - e(z)f(z) = 0.$$

From (3.4) and (3.5), we can obtain asymptotic expansion of the boundary conditions of system (2.28). For brevity, we shall use the following notation in the sequel

$$[a]_1 := a + \mathcal{O}(\rho^{-1}).$$

Theorem 3.1 *Denote the boundary conditions of the system (2.28) respectively by U_1, U_2, U_3 and U_4. Then, for $\rho \in \mathcal{S}_0$ with $|\rho|$ sufficiently large, we have the following asymptotic expansions,*

$$U_4(y_s;\rho) \;= y_s(0;\rho) = \begin{cases} h_s(0) + \mathcal{O}(\rho^{-2}) := [h_s(0)]_1, & s = 1,2, \\ 1 + \mathcal{O}(\rho^{-1}) := [1]_1, & s = 3,4, \end{cases} \tag{3.7}$$

$$U_3(y_s;\rho) \;= y_s'(0;\rho) = \begin{cases} x_z(0)h_s'(0) + \mathcal{O}(\rho^{-2}) := [x_z(0)h_s'(0)]_1, & s = 1,2, \\ \rho\omega_{s-2}\Big(1 + \mathcal{O}(\rho^{-1})\Big) := \rho\omega_{s-2}[1]_1, & s = 3,4, \end{cases} \tag{3.8}$$

$$\begin{aligned} U_2(y_s;\rho) &= y_s''(1;\rho) + \frac{b_{22}}{b_{21}}y_s'(1;\rho) + i\frac{b_{23}}{b_{21}}\alpha h^{-1}\rho y_s'(1;\rho) \\ &= \begin{cases} \rho\left(i\dfrac{b_{23}}{b_{21}}\alpha h^{-1}x_z(1)h_s'(1) + \mathcal{O}(\rho^{-1})\right), & s = 1,2, \\ \rho^2 e^{\rho\omega_{s-2}}\left(y_0(1)\omega_{s-2}^2 + i\dfrac{b_{23}}{b_{21}}\alpha h^{-1}y_0(1)\omega_{s-2} + \mathcal{O}(\rho^{-1})\right), & s = 3,4, \end{cases} \\ &:= \begin{cases} \rho\left[i\dfrac{b_{23}}{b_{21}}\alpha h^{-1}x_z(1)h_s'(1)\right]_1, & s = 1,2, \\ \rho^2 e^{\rho\omega_{s-2}}\left[y_0(1)\omega_{s-2}^2 + i\dfrac{b_{23}}{b_{21}}\alpha h^{-1}y_0(1)\omega_{s-2}\right]_1, & s = 3,4, \end{cases} \end{aligned} \tag{3.9}$$

$$\begin{aligned} U_1(y_s;\rho) &= y_s'''(1;\rho) + \frac{b_{12}}{b_{11}}y_s''(1;\rho) + \frac{b_{13}}{b_{11}}y_s'(1;\rho) + h^{-2}\rho^2\frac{b_{14}}{b_{11}}y_s'(1;\rho) - i\beta h^{-1}b_{11}^{-1}\rho y_s(1,\rho) \\ &= \begin{cases} \rho^2\left(\dfrac{b_{14}}{b_{11}}h^{-2}x_z(1)h_s'(1) + \mathcal{O}(\rho^{-1})\right), & s = 1,2, \\ \rho^3 e^{\rho\omega_{s-2}}\left(y_0(1)\omega_{s-2}^3 + \dfrac{b_{14}}{b_{11}}h^{-2}y_0(1)\omega_{s-2} + \mathcal{O}(\rho^{-1})\right), & s = 3,4, \end{cases} \\ &:= \begin{cases} \rho^2\left[x_z(1)h_s'(1)\right]_1, & s = 1,2, \\ \rho^3 e^{\rho\omega_{s-2}}\left[y_0(1)\omega_{s-2}^3 + y_0(1)\omega_{s-2}\right]_1, & s = 3,4. \end{cases} \end{aligned} \tag{3.10}$$

PROOF. The proof is just a direct substitution of the fundamental solutions (3.4)-(3.5) into the boundary conditions and makes use of the fact that in (3.10),

$$\frac{b_{14}}{b_{11}} = \frac{I_\rho(1)z_x(1)}{z_x^3(1)EI(1)} = h^2.$$

□

Since the zeros of $\Delta(\rho)$ are the eigenvalues of (2.1) (cf. [10, pp.13-15]), to estimate the eigenvalues, we substitute (3.7)-(3.10) into the characteristic determinant

$$\Delta(\rho) := \begin{vmatrix} U_4(y_1,\rho) & U_4(y_2,\rho) & U_4(y_3,\rho) & U_4(y_4,\rho) \\ U_3(y_1,\rho) & U_3(y_2,\rho) & U_3(y_3,\rho) & U_3(y_4,\rho) \\ U_2(y_1,\rho) & U_2(y_2,\rho) & U_2(y_3,\rho) & U_2(y_4,\rho) \\ U_1(y_1,\rho) & U_1(y_2,\rho) & U_1(y_3,\rho) & U_1(y_4,\rho) \end{vmatrix} \tag{3.11}$$

and obtain the following asymptotic expansion for it.

Theorem 3.2 *In sector $\mathcal{S}_0$, the characteristic determinant $\Delta(\rho)$ of the characteristic equation (2.28) has an asymptotic expansion*

$$\Delta(\rho) = -i\rho^5 y_0(1)x_z(1)D\left\{e^{-i\rho}\left(1-\alpha\gamma\right) + e^{i\rho}\left(1+\alpha\gamma\right) + \mathcal{O}(\rho^{-1})\right\}, \tag{3.12}$$

where $\gamma := \Big(I_\rho(1)EI(1)\Big)^{-1/2}$, $D := \Big(h_2'(1)h_1(0) - h_1'(1)h_2(0)\Big)$ the nonzero determinant defined in (2.3). Furthermore, the boundary problem (2.28) is strongly regular in the sense of [9, p.259] iff the following condition holds:

$$1-\alpha\Big(I_\rho(1)EI(1)\Big)^{-1/2} \neq 0 \quad (i.e.\ 1-\alpha\gamma \neq 0). \tag{3.13}$$

PROOF. In sector $\mathcal{S}_0$, with $\omega_1 := i, \omega_2 := -i$, we conclude that

$$U_1(y_s,\rho) \;= \rho^3 e^{\rho\omega_{s-2}}[0]_1, \quad s = 3,4, \tag{3.14}$$

$$U_2(y_s,\rho) \;= \rho^2 e^{\rho\omega_{s-2}}\left[-y_0(1) + (-1)^s\frac{b_{23}}{b_{21}}\alpha h^{-1}y_0(1)\right]_1, \quad s=3,4. \tag{3.15}$$

Substituting (3.7), (3.8), (3.9), (3.10), (3.14) and (3.15) into the characteristic determinant, we have

$$\begin{aligned}\Delta(\rho) \;&=\; \begin{vmatrix} [h_1(0)]_1 & [h_2(0)]_1 & [1]_1 & [1]_1 \\ [x_z(0)h_1'(0)]_1 & [x_z(0)h_2'(0)]_1 & i\rho[1]_1 & -i\rho[1]_1 \\ \rho\left[i\frac{b_{23}}{b_{21}}\alpha h^{-1}x_z(1)h_1'(1)\right]_1 & \rho\left[i\frac{b_{23}}{b_{21}}\alpha h^{-1}x_z(1)h_2'(1)\right]_1 & \rho^2 e^{\rho\omega_1}\left[-y_0(1)-\frac{b_{23}}{b_{21}}\alpha h^{-1}y_0(1)\right]_1 & \rho^2 e^{\rho\omega_2}\left[-y_0(1)+\frac{b_{23}}{b_{21}}\alpha h^{-1}y_0(1)\right]_1 \\ \rho^2[x_z(1)h_1'(1)]_1 & \rho^2[x_z(1)h_2'(1)]_1 & \rho^3 e^{\rho\omega_1}[0]_1 & \rho^3 e^{\rho\omega_2}[0]_1 \end{vmatrix} \\ &=\; -i\rho^5 y_0(1)x_z(1)D\left\{e^{\rho\omega_2}\left[1-\frac{b_{23}}{b_{21}}\alpha h^{-1}\right]_1 + e^{\rho\omega_1}\left[1+\frac{b_{23}}{b_{21}}\alpha h^{-1}\right]_1\right\}.\end{aligned}$$

Combining with (2.20),(2.27), we have

$$\frac{b_{23}}{b_{21}} = \frac{z_x(1)}{z_x^2(1)EI(1)} = h\Big(I_\rho(1)EI(1)\Big)^{-1/2} = h\gamma, \tag{3.16}$$

which yields (3.12). The strong regularity defined in [9, Def.2.7] can be verified directly from the fact that $y_0(1), x_z(1) > 0$ and (2.14),(3.13).

From [10, pp.56-74] we know that expression (3.12) also holds in the remaining sectors $\mathcal{S}_k$ under the exact same arguments as in sector $\mathcal{S}_0$ and conclude that the set of eigenvalues in sectors $\mathcal{S}_1$ and $\mathcal{S}_3$ are the same as those in $\mathcal{S}_0$ and $\mathcal{S}_2$. $\square$

Theorem 3.3 *Suppose that condition (3.13) is fulfilled, then the eigenvalues λ_k of the problem (2.12) have the following asymptotic behavior*

$$\lambda_k = \frac{1}{h}\left(\frac{1}{2}\xi_0 + k\pi i\right) + \mathcal{O}(k^{-1}), \quad k = \pm 1, \pm 2, \dots, \tag{3.17}$$

where $h := \int_0^1 \left(\frac{I_\rho(\zeta)}{EI(\zeta)}\right)^{1/2} d\zeta$ *and*

$$\xi_0 := \begin{cases} \ln \dfrac{\alpha\gamma - 1}{\alpha\gamma + 1}, & \alpha\gamma > 1, \\ \ln \dfrac{1-\alpha\gamma}{1+\alpha\gamma} + \pi i, & \alpha\gamma < 1. \end{cases} \tag{3.18}$$

Also,

$$\mathrm{Re}\xi_0 = \ln\left|\frac{\alpha\gamma - 1}{\alpha\gamma + 1}\right| < 0 \quad \text{and} \quad \mathrm{Re}\lambda_k \to \frac{1}{2h}\mathrm{Re}\xi_0 < 0, \quad k \to \infty. \tag{3.19}$$

PROOF. Since equation (2.12) is equivalent to (2.22), in sector $\mathcal{S}_0$, we obtain from (3.12) and (2.14) that equation $\Delta(\rho) = 0$ becomes

$$e^{-i\rho}(1-\alpha\gamma) + e^{i\rho}(1+\alpha\gamma) + \mathcal{O}(\rho^{-1}) = 0. \tag{3.20}$$

Solving the equation with lower order terms,

$$e^{-i\rho}(1-\alpha\gamma) + e^{i\rho}(1+\alpha\gamma) = 0,$$

we get solutions

$$\tilde{\mu}_k = i\rho_k = \frac{1}{2}\xi_0 + k\pi i, \quad k = 1, 2, \dots, \tag{3.21}$$

where ξ_0 defined in (3.18). Applying Rouche's theorem to (3.20), its solutions are

$$\mu_k = \frac{1}{2}\xi_0 + k\pi i + \mathcal{O}(k^{-1}), \quad k = 1, 2, \dots. \tag{3.22}$$

In sector $\mathcal{S}_2$, we can use the same argument to the asymptotic eigen-distribution. First, in order to satisfy (3.2), we take $\omega_1 := -i$ and $\omega_2 := i$. Then equations (3.8)-(3.10) and (3.14)-(3.15) change to (recall that $\mu = \rho\omega_1 = -i\rho$)

$$U_3(y_s;\rho) = \begin{cases} [x_z(0)h_s'(0)]_1, & s=1,2, \\ (-1)^s i\rho[1]_1, & s=3,4, \end{cases} \tag{3.23}$$

$$U_2(y_s;\rho) = \begin{cases} \rho\left[-i\dfrac{b_{23}}{b_{21}}\alpha h^{-1}x_z(1)h_s'(1)\right]_1, & s=1,2, \\ \rho^2 e^{\rho\omega_{s-2}}\left[-y_0(1)+(-1)^s\dfrac{b_{23}}{b_{21}}\alpha h^{-1}y_0(1)\right]_1, & s=3,4, \end{cases} \tag{3.24}$$

$$U_1(y_s;\rho) = \begin{cases} \rho^2\left[x_z(1)h_s'(1)\right]_1, & s=1,2, \\ \rho^3 e^{\rho\omega_{s-2}}\left[0\right]_1, & s=3,4. \end{cases} \tag{3.25}$$

Thus we have,

$$\Delta(\rho) = i\rho^5 y_0(1)x_z(1)D\left\{e^{i\rho}(1-\alpha\gamma)+e^{-i\rho}(1+\alpha\gamma)+\mathcal{O}(\rho^{-1})\right\} \tag{3.26}$$

and the characteristic determinant $\Delta(\rho)=0$ becomes

$$e^{i\rho}(1-\alpha\gamma)+e^{-i\rho}(1+\alpha\gamma)+\mathcal{O}(\rho^{-1}) = 0. \tag{3.27}$$

So

$$\bar{\mu}_k = \frac{1}{2}\xi_0 - k\pi i + \mathcal{O}(k^{-1}), \quad k=1,2,\ldots. \tag{3.28}$$

Hence, we can conclude from (3.22) and (3.28) that

$$\mu_k = \frac{1}{2}\xi_0 + k\pi i + \mathcal{O}(k^{-1}), \quad k=\pm1,\pm2,\ldots. \tag{3.29}$$

Since $\mu = h\lambda$, so

$$\lambda_k = \frac{1}{h}\mu_k = \frac{1}{h}\left(\frac{1}{2}\xi_0 + k\pi i\right) + \mathcal{O}(k^{-1}), \quad k=\pm1,\pm2,\ldots. \tag{3.30}$$

Note that the set of eigenvalues in $\mathcal{S}_3$ and $\mathcal{S}_4$ are exactly the same as those in $\mathcal{S}_0$ and $\mathcal{S}_2$, so all eigenvalues of $\mathcal{A}$ satisfy (3.30). The proof is then completed. □

All the above discussions can be summarized into the following result on the spectrum of $\mathcal{A}$.

Theorem 3.4 *Let $\mathcal{A}$ be defined before. Then each $\lambda \in \sigma(\mathcal{A})$ is an eigenvalue and is simple when $|\lambda|$ is large enough, and has asymptotic expression given by (3.30).*

4 Completeness of Generalized Eigenfunction System and Riesz basis

In this section we will discuss the completeness of the generalized eigenfunctions of $\mathcal{A}$, which is necessary for discussing of Riesz basis property of system (2.8). We begin with the following lemma.

Lemma 4.1 *Let $\mathcal{A}$ be defined as in (2.6) and (2.7) and λ_k $(k = \pm1, \pm2, \ldots)$ be eigenvalues given in (3.17) and $\delta > 0$. Then there exists a constant $M > 0$ such that, for any $\lambda \in \rho(\mathcal{A})$ with $|\lambda - \lambda_k| > \delta, k = \pm1, \pm2, \ldots,$ we have*

$$\|R(\lambda, \mathcal{A})\| \leq M|\lambda|^2. \tag{4.1}$$

PROOF. Let $\lambda \in \rho(\mathcal{A})$ and $(\phi, \psi) \in \mathcal{H}$, we consider the resolvent equation

$$[\lambda I - \mathcal{A}](f, g) = (\phi, \psi),$$

i.e.,

$$\begin{cases} \lambda f - g = \phi, \\ \lambda g + C^{-1}(Af + \alpha Dg + \beta Bg) = \psi. \end{cases} \tag{4.2}$$

Simplifying the second equation in (4.2), we have

$$\lambda\Big[\rho(x)g(x) - \Big(I_\rho(x)g'(x)\Big)'\Big] + \Big(EI(x)f''(x)\Big)'' = \Big[\rho(x)\psi(x) - \Big(I_\rho(x)\psi'(x)\Big)'\Big]$$

with the boundary conditions $f(0) = f'(0) = 0$,

$$\Big(EI(x)f''(x)\Big)'\Big|_{x=1} + I_\rho(x)\Big(\psi(x) - \lambda g(x)\Big)'\Big|_{x=1} - \beta g(1) = 0$$

and

$$EI(x)f''(x)\Big|_{x=1} + \alpha g'(1) = 0.$$

Thus

$$g(x) = \lambda f(x) - \phi(x)$$

and $f(x)$ satisfies the following equations:

$$\lambda^2\Big[\rho(x)f(x) - \Big(I_\rho(x)f'(x)\Big)'\Big] + \Big(EI(x)f''(x)\Big)'' = F(x, \lambda), \tag{4.3}$$

$$f(0) = f'(0) = 0, \quad EI(x)f''(x)\Big|_{x=1} + \alpha\lambda f'(1) = \alpha\phi(1), \tag{4.4}$$

$$\Big(EI(x)f''(x)\Big)'(1) - \lambda^2 I_\rho(1)f'(1) - \beta\lambda f(1) = -v(\lambda), \tag{4.5}$$

where

$$\begin{cases} F(x,\lambda) := \Big[\rho(x)(\psi(x)+\lambda\phi(x)) - \Big(I_\rho(x)[\psi'(x)+\lambda\phi'(x)]\Big)'\Big], \\ v(\lambda) := I_\rho(x)\Big(\psi(x)+\lambda\phi(x)\Big)'(1)+\beta\phi(1)=0. \end{cases} \tag{4.6}$$

Let $y_j(x,\lambda)(j=1,2,3,4)$ be the fundamental solutions of the homogenous equation of (4.3). Then any solution $f(x)$ of (4.3)–(4.5) can be expressed by the formula (see [10, pp.31, Theorem 2])

$$f(x,\lambda) = \int_0^1 G(x,\xi,\lambda)F(\xi,\lambda)d\xi, \tag{4.7}$$

where $G(x,\xi,\lambda)$ is the Green's function given by

$$G(x,\xi,\lambda) := \frac{1}{\Delta(\rho)}H(x,\xi,\lambda) \tag{4.8}$$

with

$$H(x,\xi,\rho) := \begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) & y_4(x,\lambda) & \eta(x,\xi,\lambda) \\ U_1(y_1) & U_1(y_2) & U_1(y_3)(x) & U_1(y_4) & U_1(\eta) \\ U_2(y_1) & U_2(y_2) & U_2(y_3)(x) & U_2(y_4) & U_2(\eta) \\ U_3(y_1) & U_3(y_2) & U_3(y_3)(x) & U_3(y_4) & U_3(\eta) \\ U_4(y_1) & U_4(y_2) & U_4(y_3)(x) & U_4(y_4) & U_4(\eta) \end{vmatrix}, \tag{4.9}$$

$$\eta(x,\xi,\lambda) := \frac{1}{2}\mathrm{sign(x-\xi)}\sum_{j=1}^{4} y_j(x,\lambda)z_j(\xi,\lambda), \tag{4.10}$$

$z_j(x,\lambda) := \frac{W_j(x,\lambda)}{W(x,\lambda)}$, $W(x,\lambda)$ the Wronskian determinant of $\{y, y_2, y_3, y_4\}$ and $W_j(x,\lambda)$ the cofactors of $y_j(x,\lambda)$ in $W(x,\lambda)$. Substituting (3.4)–(3.5) and (3.7)–(3.10) into (4.9) and (4.10) respectively, we obtain that for $\lambda \in \rho(\mathcal{A})$ with $|\lambda|$ large enough, there exists a constant M independent of $x,\xi \in [0.1]$ so that

$$\begin{aligned} &|H(x,\xi,\lambda)| \le M|\lambda|^5 e^{|\rho|}, \rho\omega_1 = h\lambda \in \mathbb{C} \\ &|\tfrac{\partial}{\partial x}H(x,\xi,\lambda)| \le M|\lambda|^6 e^{|\rho|}, \\ &|\tfrac{\partial^2}{\partial x^2}H(x,\xi,\lambda)| \le M|\lambda|^7 e^{|\rho|}. \end{aligned} \tag{4.11}$$

Also, by (3.12) we have

$$\begin{aligned} &|G(x,\xi,\lambda)| \le M_1, \\ &|\tfrac{\partial}{\partial x}G(x,\xi,\lambda)| \le M_1|\lambda|, \\ &|\tfrac{\partial^2}{\partial x^2}G(x,\xi,\lambda)| \le M_1|\lambda^2|, \end{aligned} \tag{4.12}$$

where M_1 is a constant which is independent of $x,\xi \in [0.1]$. From these, we obtain estimates for $f(x)$ and its derivatives, for $j=0,1,2$,

$$|f^{(j)}(x)| \le \int_0^1 \Big|\frac{\partial^j}{\partial x^j}G(x,\xi.\lambda)F(\xi,\lambda)\Big|d\xi \le M_1|\lambda^j|\int_0^1 |F(\xi,\lambda)|d\xi. \tag{4.13}$$

Eventually, we have the following estimate on the resolvent operator

$$\begin{aligned}
\|(f,g)\|^2 &= \int_0^1 EI(x)|f''(x)|^2dx + \int_0^1 \rho(x)|g(x)|^2 + I_\rho(x)|g'(x)|^2dx \\
&= \int_0^1 EI(x)|f''(x)|^2dx + \int_0^1 \rho(x)|\lambda f(x)-\phi(x)|^2 + I_\rho(x)|\lambda f'(x)-\phi'(x)|^2dx \\
&\le \int_0^1 EI(x)|f''(x)|^2dx + |\lambda|^2\int_0^1 [\rho(x)|f(x)|^2 + I_\rho(x)|f'(x)|^2]dx + 2\|\phi\|_V \\
&\le M_2|\lambda^4|\Big[\|\phi\|_W^2 + \|\psi\|_v^2\Big],
\end{aligned} \tag{4.14}$$

where M_2 is some constant. So $\|R(\lambda,\mathcal{A})\| \le M_2|\lambda|^2$. The proof is then completed. □

Corollary 4.1 *Let $\Gamma(\theta)$ be a ray with at original point and direction θ and assumptions be given in Theorem 4.1, then estimates (4.1) for $\|R(\lambda,A)\|$ are also true on the rays $\Gamma(-\pi/4)$, $\Gamma(\pi/4)$ and $\Gamma(\pi)$.*

PROOF. From Lemma 2.4, Theorem 3.3 and the conjugate property for eigenvalues, we obtain that there is no eigenvalue on the right complex half plan and ray $\Gamma(\pi)$. Thus we can choose rays $\Gamma(-\pi/4)$, $\Gamma(\pi/4)$ and $\Gamma(\pi)$ and estimates are also true on them. □

Theorem 4.1 *Let $\mathcal{A}$ be defined as in (2.6) and (2.7). If condition (3.13) is fulfilled, then the generalized eigenfunctions of operator $\mathcal{A}$ are complete in Hilbert space $\mathcal{H}$.*

POOF. Let $\sigma(\mathcal{A}) = \{\lambda_n, n\in\mathbb{N}\}$ and P_n be the Riesz projection associated with λ_n. Denote

$$Sp(\mathcal{A}) = \left\{\sum_{k=1}^N P_k y, \quad y\in\mathcal{H},\ \forall\ N\in\mathbb{N}\right\}$$

and

$$\mathcal{Q}_\infty = \{y\in\mathcal{H}; \quad P_k^* y = 0,\ \forall\ k\in\mathbb{N}\}.$$

Easy to see that $\mathcal{H}$ has an orthogonal decomposition

$$\mathcal{H} = \overline{Sp(\mathcal{A})} \oplus \mathcal{Q}_\infty.$$

So the generalized eigenfunctions of operator $\mathcal{A}$ are complete in $\mathcal{H}$ if and only if $\mathcal{Q}_\infty = \{0\}$.

Now for any $Z\in\mathcal{Q}_\infty$, $R(\lambda,\mathcal{A}^*)Z$ is an entire function on complex plane $\mathbb{C}$ valued in $\mathcal{H}$. Since $\|R(\lambda,\mathcal{A}^*)\| = \|R^*(\lambda,\mathcal{A})\| = \|R(\lambda,\mathcal{A})\|$, the conclusions of Lemma 4.1 and Corollary 4.2 are also true. According to the Theorem of Phragmén-Lindelöf (see [15]), we see that $R(\lambda,\mathcal{A}^*)Z$ is at most a polynomial of degree two in λ, that is,

$$(\lambda I-\mathcal{A}^*)^{-1}Z = c_0 + c_1\lambda + c_2\lambda^2,$$

then

$$\begin{aligned} Z &= (\lambda I - \mathcal{A}^*)(c_0 + c_1\lambda + c_2\lambda^2) \\ &= -c_0\mathcal{A}^* + (c_0 - c_1\mathcal{A}^*)\lambda + (c_1 - c_2\mathcal{A}^*)\lambda^2 + c_2\lambda^3. \end{aligned}$$

Comparing coefficients, we see that

$$c_0 = c_1 = c_2 = 0.$$

Therefore $Z = 0$ and $\mathcal{Q}_\infty = \{0\}$. □

To obtain the Riesz property of generalized eigenfunction system of $\mathcal{A}$, we need the following result from [13].

Theorem 4.2 *Let X be a separable Hilbert space, and $\mathcal{A}$ be the generator of a C_0 semi-group $T(t)$. Suppose that the following three conditions hold:*

1) $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$ and $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k=1}^\infty$ consists of only isolated eigenvalues of finite multiplicity;

2) for $m_a(\lambda_k) := \dim E(\lambda_k, \mathcal{A})X$ and $E(\lambda_k, \mathcal{A})$ is the Riesz projector associated with λ_k, we have

$$\sup_{k\geq 1} m_a(\lambda_k) < \infty;$$

3) there is a constant α such that

$$\sup\{Re\lambda \mid \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{Re\lambda | \lambda \in \sigma_2(\mathcal{A})\}$$

and

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0. \tag{4.15}$$

Then the following assertions are true:

i) There exist two $T(t)$-invariant closed subspaces X_1 and X_2 such that $\sigma(\mathcal{A}|_{X_1}) = \sigma_1(\mathcal{A})$, $\sigma(\mathcal{A}|_{X_2}) = \sigma_2(\mathcal{A})$, and $\{E(\lambda_k, \mathcal{A})X_2\}_{k=1}^\infty$ forms a Riesz basis of subspaces for X_2. Furthermore,

$$X = \overline{X_1 \oplus X_2}.$$

ii) If $\sup_{k\geq 1} ||E(\lambda_k, \mathcal{A})|| < \infty$, then

$$D(\mathcal{A}) \subset X_1 \oplus X_2 \subset X.$$

iii) X has the topological direct sum decomposition

$$X = X_1 \oplus X_2$$

if and only if $\sup_{n\geq 1} ||\sum_{k=1}^n E(\lambda_k, \mathcal{A})|| < \infty$.

Combining Theorem 4.1, 4.2 together with Theorem 3.4, we have the following result.

Theorem 4.3 *assume that (3.13) be fulfilled. Then system (2.8) is a Riesz system (in the sense that its eigenfunctions form a Riesz basis in $\mathcal{H}$) and hence it satisfies the spectrum determined growth condition.*

PROOF For system (2.8), from Theorem 3.3 and 3.4, we may take $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$, $\sigma_1(\mathcal{A}) = \{\infty\}$. Theorem 3.4 shows that conditions 2) and 3) in Theorem 4.2 are true. Finally, Lemma 4.1 implies that $X_1 = \{0\}$. Therefore, the first assertion of Theorem 4.2 says that there is a sequence of generalized eigenfunctions of $\mathcal{A}$ that forms a Riesz basis for $\mathcal{H}$. Since the spectrum determined growth condition is a direct consequence of the existence of a Riesz basis, the proof is completed. □

As a consequence of Theorem 4.3, we have a stability result for system (2.8).

Corollary 4.2 *Let condition (3.13) be fulfilled with $\alpha > 0$ and $\beta \geq 0$. Then the system (2.8) is exponentially stable. The decay rate is given by*

$$\omega(\mathcal{A}) = \sup\{Re\lambda : \lambda \in \sigma(\mathcal{A})\} < 0,$$

which is negative.

PROOF. Theorem 4.3 ensures $\omega(\mathcal{A}) = \sup\{Re\lambda, \lambda \in \sigma(\mathcal{A})\}$. Lemma 2.4 says that $Re\lambda < 0$ provided $\lambda \in \sigma(\mathcal{A})$ and Theorem 3.3 shows that imaginary axis is not an asymptote of $\sigma(\mathcal{A})$. Therefore $\sup\{Re\lambda : \lambda \in \sigma(\mathcal{A})\} < 0$. □

Remark 4.1 *The special case that $\rho(x) = EI(x) \equiv 1$ and $I_\rho(x) \equiv \gamma_1 > 0$ was discussed in [2], [3]. In this constant case, expression (3.17) then becomes (for $k = \pm 1, \pm 2, \ldots$)*

$$\lambda_k = \frac{1}{\sqrt{\gamma_1}}\left(\frac{1}{2}\xi_1 + k\pi i\right) + \mathcal{O}(k^{-1}), \tag{4.16}$$

with

$$\xi_1 = \begin{cases} \ln \frac{\alpha-\sqrt{\gamma_1}}{\alpha+\sqrt{\gamma_1}}, & \alpha > \sqrt{\gamma_1}, \\ \ln \frac{\sqrt{\gamma_1}-\alpha}{\alpha+\sqrt{\gamma_1}} + \pi i, & \alpha < \sqrt{\gamma_1} \end{cases} \tag{4.17}$$

and

$$\mathrm{Re}\lambda_k \to \frac{1}{2\sqrt{\gamma_1}} \ln \left| \frac{\alpha - \sqrt{\gamma_1}}{\alpha + \sqrt{\gamma_1}} \right| < 0, \quad k \to \infty. \tag{4.18}$$

So the closer α to $\alpha^* := \sqrt{\gamma_1}$ the larger the damping rate for the system (1.1) which is the conjecture made in [3]. However, we cannot achieve the largest damping rate by setting the control gain $\alpha = \sqrt{\gamma_1}$ because then $\Delta(\rho)$ in (3.12) will never be zero and the eigenvalue problem (2.12) is degenerate in the sense that there are no more eigenvalues except finite number at all (cf. [14]).

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