Holomorphic Maps from Ratioanl Homogeneous Spaces of Picard Number 1 onto Projective Manifolds

Jun-Muk Hwang [1] **and Ngaiming Mok**

The aim of this article is to prove the following, which gives an affirmative answer to a question raised by Lazarsfeld ([La]).

Main Theorem *Let S be a rational homogeneous space of Picard number 1, and $f : S \to X$ be a surjective holomorphic map to a projective manifold X of positive dimension. Then either X is biholomorphic to the projective space $\mathbf{P}_n, n = \dim(S)$, or X is biholomorphic to S and f is a biholomorphic automorphism.*

Lazarsfeld had proved this when S is the projective space ([La]), as an application of Mori's proof of Hartshorne conjecture ([Mr]). He used Mori's result that if the restriction of the tangent bundle to any rational curve of minimal degree through a generic point is ample, the manifold is $\mathbf{P}_n$. By this result of Mori, the key problem in proving the Main Theorem lies in understanding certain curves on S on which the restrictions of the tangent bundle of S deviate from being ample. The case when S is the hyperquadric was proved by Paranjape and Srinivas ([PS]), and independently by Cho and Sato ([CS]). Tsai proved the case of compact irreducible Hermitian symmetric space S ([Ts]). In these works, certain curves with the above mentioned property were classified using the global geometry of S.

The major difference in our approach from these earlier works is that we view it as an extension problem of holomorphic maps. Namely, we deduce the Main Theorem from the following. Note that the condition that S is of Picard number 1 implies that f is a finite morphism.

Theorem 1 *Let S be as above and $f : S \to X$ be a finite morphism to a projective manifold different from $\mathbf{P}_n$. Let $s, s' \in S$ be an arbitrary pair of distinct points such that $f(s) = f(s')$ and f is unramified at s and s'. Write φ for the unique germ of holomorphic map at s, with target space S, such that $\varphi(s) = s'$ and $f \circ \varphi = f$. Then φ extends to a biholomorphic automorphism of S.*

Once Theorem 1 is obtained, it is easy to get the Main Theorem, as explained in Section 4. By Tanaka-Yamaguchi theory of differential systems on S, and its refinement formulated in Proposition 10, the proof of Theorem 1 is reduced to an infinitesimal study of curves on S on which the restriction of the tangent bundle deviates from being ample. This leads one to study deformation theory of such curves in combination with the isotropic representation of the parabolic group. For Hermitian symmetric spaces, this part was done in [Ts] using the fine structure theory of Hermitian symmetric spaces. But some of the key properties of the isotropic

[1]supported by S.N.U. Research Fund, by RIBS, and by the KOSEF through the GARC at Seoul National University

representation of the parabolic subgroup for the Hermitian symmetric cases do not hold for general S. Most notably, the finiteness of number of orbits of the isotropic representation, which was used in a crucial way in [Ts], no longer holds in general (see [Ro] for examples). One of the key ingredient of our proof is a general deformation theoretic result in Section 1, which replaces a good deal of Lie theory. This enables us to carry out the proof with only a small amount of information regarding the isotropy representation. We expect that this result will be useful in the study of general finite morphisms over Fano manifolds.

Our proof is independent of the previous results [CS], [PS] and [Ts]. For Hermitian symmetric spaces, the proof is quite simple and consists of the following: Section 1, Propositions 4, 5, 6 in Section 2, Proposition 9 and Corollary in Section 3, and Section 4.

In this paper, all varieties and morphisms are defined over the complex numbers. A variety needs not be irreducible, but has finitely many irreducible components.

1 Varieties of distinguished tangents and varieties of minimal rational tangents

In this section, we will discuss a general result concerning a finite morphism from a projective manifold to a Fano manifold, i.e. a projective manifold X with K_X^{-1} ample. To state our result, it is convenient to introduce the concept of varieties of distinguished tangents.

Let $g : M \to Z$ be a regular map between two quasi-projective complex algebraic varieties. We consider a stratification of M into finitely many irreducible quasi-projective nonsingular subvarieties $M = M_1 \cup \cdots \cup M_k$, which is canonically associated to g, such that for each i, the reduced image $g(M_i)$ is nonsingular and the holomorphic map $g|_{M_i} : M_i \to g(M_i)$ is of constant rank. It will be called the **g-stratification of** M and is defined as follows.

Any constructible set can be decomposed into its smooth loci and singular loci. Each component of the singular loci can be viewed as a reduced constructible set and can be decomposed further into its smooth loci and singular loci. Repeating this procedure, we have the singular loci stratification of any constructible set into a finite number of nonsingular subvarieties.

For the given regular map g, consider the singular loci stratification of the image $g(M)$. Then consider the singular loci stratification of the inverse image of each component of this stratification of $g(M)$. Each component of this stratification of M can be stratified further by the rank of the map g restricted to each component. Replace g by the restriction of g to one component of this stratification of M and apply the same procedure again. Repeating these finitely many times, certainly we get a stratification $M = M_1 \cup \cdots \cup M_k$ with the properties mentioned above.

The following two statements are immediate from the construction.

- Any tangent vector to $g(M_i)$ can be realized as the image of the tangent vector to a local holomorphic arc in M_i.

- When a connected Lie group acts on M and Z, and g is equivariant, M_i and $g(M_i)$ are invariant under this group action.

Let Y be a projective manifold and $y \in Y$ be a point. Let $\mathcal{M}_y$ be an irreducible component of the Hilbert scheme of curves in Y passing through y, whose generic point corresponds to an irreducible reduced curve smooth at y. Let $\mathcal{M}'_y \subset \mathcal{M}_y$ be the (reduced) quasi-projective subvariety corresponding to curves smooth at y. We have the tangent map $\Phi_y : \mathcal{M}'_y \to \mathbf{P}T_y(Y)$. Let $\{M_i\}$ be the Φ_y-stratification of $\mathcal{M}'_y$. A subvariety of $\mathbf{P}T_y(Y)$ will be called a **variety of distinguished tangents** in $\mathbf{P}T_y(Y)$, if it is the closure of the image $\Phi_y(M_i)$ for some choice of $\mathcal{M}_y$ and M_i. Note that there exist only countably many subvarieties in $\mathbf{P}T_y(Y)$ which can serve as varieties of distinguished tangents, because the Hilbert scheme has only countably many components.

Given an irreducible reduced curve l in Y and a smooth point $y \in l$, consider $\mathcal{M}_y$ which parametrizes deformations of l fixing y. $[l]$ is contained in $\mathcal{M}'_y$ where the tangent map is well-defined. Let M_1 be the component of the stratification of $\mathcal{M}'_y$ associated to the tangent map, so that $[l] \in M_1$. The variety of distinguished tangents corresponding to M_1 is called **the variety of distinguished tangents associated to l at y** and is denoted by $\mathcal{D}_y(l)$. It is an irreducible subvariety and $\mathbf{P}T_y(l)$ is a smooth point on it.

Let $N_l^* = \mathcal{I}/\mathcal{I}^2$ be the conormal sheaf of l, where $\mathcal{I}$ denotes the ideal sheaf of l. We have a natural map $j : N_l^* \to \Omega(Y)|_l$, where $\Omega(Y) = \mathcal{O}(T^*(Y))$. j is injective if l is an immersed curve. In general, $Ker(j)$ is a sheaf supported on finitely many points. Let N'_l be the image of j in $\Omega(Y)$. The following is a generalization of Lemma 1.3 of [Ts].

Lemma 1 *Let l_t be a 1-dimensional deformation of $l = l_0$ fixing y so that $[l_t] \in M_1$. Let $\nu(t) \in \mathcal{D}_y(l) \subset \mathbf{P}T_y(Y)$ be the tangent vectors to l_t, and $\nu'(0) \in T_{\nu(0)}(\mathbf{P}T_y(Y))$ be the derivative. Then any $w \in H^0(l, N'_l)$ annihilates $\nu'(0)$.*

Proof. From $H^1(l, Ker(j)) = 0$, w can be lifted to a section of N_l^*. The Kodaira-Spencer class κ of the deformation lies in $\mathrm{Hom}(N_l^*, \mathcal{O}_l)$. Since $\kappa(y) = 0$ and the pairing $< \kappa, w >$ should be constant on l, $0 = d < \kappa, w > (\nu(0)) =< d\kappa(\nu(0)), w(y) >=< \nu'(0), w(y) >$. □

For a generic member l of any family of irreducible reduced curves covering an open subset of Y, their Kodaira-Spencer classes generate the sheaf $Hom(N_l^*, \mathcal{O}_l)$ at generic points of l. In this case, N'_l cannot have a non-zero section which vanishes at some point on l. From the countability of the number of components of the Hilbert scheme, there exist countably many proper subvarieties of Y such that for any irreducible reduced curve l passing through a point y outside them, N'_l cannot have a non-zero section which vanishes at some point of l. We say that a point $y \in Y$ is **very general**, if it lies outside these countably many proper subvarieties of Y.

Proposition 1 *Let $y \in Y$ be a very general point and l be a curve smooth at [illegible] the tangent space of $\mathcal{D}_y(l)$ at the point $\mathbf{P}T_y(l)$ must have dimension $\leq n - 1 - h^0($[illegible]* $n = \dim(Y)$.

Proof. From the property of the stratification M_i, any tangent vector to $\mathcal{D}_y(l)$ at $\mathbf{P}T_y(l)$ can be realized as $\nu'(0)$ for some $[l_t] \in M_1$. Since N'_l cannot have non-zero sections vanishing at y, $H^0(l, N'_l)$ imposes $h^0(l, N'_l)$ conditions on the tangent vectors to $\mathcal{D}_y(l)$ by Lemma 1. □

A natural example of varieties of distinguished tangents appear in the study of Fano manifolds. Let X be a Fano manifold of Picard number 1 of dimension n. We can summarize the result of [Mr] and (2.4) of [Mk] as follows.

Proposition 2 *There exists a component $\mathcal{K}$ of the Hilbert scheme of rational curves on X, whose generic point corresponds to a rational curve C which is an immersed $\mathbf{P}_1$ and*

$$T(X)|_C = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-1-q} \oplus \mathcal{O}^q$$

for $q = h^0(C, N_C^*)$. For a generic point $x \in X$ let $\mathcal{K}_x \subset \mathcal{K}$ be the (not necessarily irreducible) subvariety corresponding to curves through x. Let $\Phi_x : \mathcal{K}_x \to \mathbf{P}T_x(X)$ be the tangent map, which is only a rational map. Let $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ be the strict image of $\mathcal{K}_x$. Then $\dim(\mathcal{C}_x) = n - 1 - q$.

Suppose X is different from $\mathbf{P}_n$. Then one of the following cases holds.

Case (i) *There exists $\mathcal{K}$ with $q > 0$.*

Case (ii) *There exists a (not necessarily irreducible) hypersurface $\mathcal{K}'_x \subset \mathcal{K}_x$ whose generic points correspond to rational curves C', which are immersed $\mathbf{P}_1$ through x, and*

$$T(X)|_{C'} = [\mathcal{O}(2)]^2 \oplus [\mathcal{O}(1)]^{n-3} \oplus \mathcal{O}.$$

In Case (ii), let $\mathcal{C}'_x \subset \mathbf{P}T_x(X)$ be the strict image of $\mathcal{K}'_x$ under the tangent map. Then $\dim(\mathcal{C}'_x) = n - 2$.

If there exists $\mathcal{K}$ so that (i) holds, we will say that X is **of type (i)**. Otherwise, we say that X is **of type (ii)**. We will call generic members of $\mathcal{K}$ minimal rational curves.

Remark It has been conjectured that X of type (ii) does not exist. We understand that some proposed proofs are being circulated in preprint form. Since no proof has been published yet, we will include type (ii) in our discussion.

Note that $\mathcal{C}_x$ and $\mathcal{C}'_x$ need not be irreducible. They are called **varieties of minimal rational tangents**. Each component of them is an example of the variety of distinguished tangents. These special subvarieties of the tangent spaces are very useful in the study of finite morphisms to Fano manifolds of Picard number 1, by the following

Proposition 3 *Let $f : Y \to X$ be a finite morphism from a projective manifold Y to a Fano manifold X of Picard number 1, different from $\mathbf{P}_n$. Choose $x \in X$ and $y \in f^{-1}(x)$ so that y is very general and $df : T_y(Y) \to T_x(X)$ is an isomorphism. Then each irreducible component of $df^{-1}(\mathcal{C}_x) \subset \mathbf{P}T_y(Y)$ for X of type (i) (resp. $df^{-1}(\mathcal{C}'_x)$ for X of type (ii)) is a variety of distinguished tangents $\mathcal{D}_y(l)$ for a suitable choice of a curve l through y.*

Proof. First, consider X of type (i). Choose a generic point $x \in X$ and a component $\mathcal{C}_1$ of $\mathcal{C}_x$. Choose a minimal rational curve C through x as in Proposition 2 so that $\mathbf{P}T_x(C)$ is a generic point of $\mathcal{C}_1$. Let l be an irreducible component of $f^{-1}(C)$ through $y \in f^{-1}(x)$. We define a map $H^0(C, N_C^*) \to H^0(l, N_l')$ as follows. Given $w \in H^0(C, N_C^*)$, we can regard it as a section of $T^*(X)|_C$ since C is an immersed $\mathbf{P}_1$. By pulling it back, we get a section of $T^*(Y)|_l$ which annihilates tangent vectors to l at generic points. From the exact sequence $0 \to N_l' \to \Omega(Y) \to \Omega(l) \to 0$, where $\Omega(l)$ is the sheaf of Kähler differentials of l, we see that this corresponds to an element in $H^0(l, N_l')$. We note that this map is clearly injective. Thus $h^0(l, N_l') \geq h^0(C, N_C^*)$.

Obviously $\mathbf{P}df_y^{-1}(T_x(C)) \in \mathcal{D}_y(l)$. Thus each generic point of $df_y^{-1}(\mathcal{C}_1)$ is contained in some $\mathcal{D}_y(l)$ for a suitable choice of a curve l, depending on C, satisfying $h^0(l, N_l') \geq h^0(C, N_C^*)$. Since there are only countably many subvarieties in $\mathbf{P}T_y(Y)$, which can serve as a variety of distinguished tangents, we can assume that $df_y^{-1}(\mathcal{C}_1) \subset \mathcal{D}_y(l)$, by choosing l generically. We have $\dim(\mathcal{C}_1) = n - 1 - h^0(C, N_C^*)$. Applying Proposition 1,

$$\begin{aligned} n - 1 - h^0(C, N_C^*) &= \dim(df_y^{-1}(\mathcal{C}_1)) \\ &\leq \dim(\mathcal{D}_y(l)) \\ &\leq n - 1 - h^0(l, N_l') \\ &\leq n - 1 - h^0(C, N_C^*) \end{aligned}$$

which implies $df_y^{-1}(\mathcal{C}_1) = \mathcal{D}_y(l)$.

For type (ii) , arguing in the same way, replacing C by C', we get $df_y^{-1}(\mathcal{C}_1') = \mathcal{D}_y(l)$. □

2 Pull-backs of varieties of minimal rational tangents to rational homogeneous spaces of Picard number 1

In this article, S denotes a rational homogeneous space of Picard number 1, of dimension n. We always assume that S is different from $\mathbf{P}_n$, unless stated otherwise. Let us recall some basic facts concerning S (see section 3 of [Ya] for details).

After fixing a base point $o \in S$, we can write $S = G/P$ where G is a connected and simply connected simple complex Lie group and P is a maximal parabolic subgroup. Let $\mathbf{g}$ be the Lie algebra of G and $\mathbf{p}$ be the parabolic subalgebra corresponding to P. Fix a Levi decomposition $\mathbf{p} = \mathbf{u} + \mathbf{l}$ where $\mathbf{u}$ is nilpotent and $\mathbf{l}$ is reductive. The center $\mathbf{z}$ of $\mathbf{l}$ is one dimensional. We fix a Cartan subalgebra $\mathbf{h} \subset \mathbf{l}$, which is also a Cartan subalgebra of $\mathbf{g}$. We have the root system $\Phi \subset \mathbf{h}^*$ of $\mathbf{g}$ with respect to $\mathbf{h}$. We can choose a set Φ^+ of positive roots uniquely by requiring that $\mathbf{u}$ is contained in the span of negative root spaces. Fix a system of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. The maximality of $\mathbf{p}$ implies that there is a unique simple root α_i satisfying $\alpha_i(\mathbf{z}) \neq 0$. We say that $\mathbf{p}$ is **the maximal parabolic subalgebra associated to the simple root** α_i, and S is **of type** $(\mathbf{g}, \alpha_i)$.

Conversely, given a Cartan subalgebra $\mathbf{h}$, a simple root system of $(\mathbf{g}, \mathbf{h})$ and a distinguished simple root α_i, we can recover $\mathbf{p}$ as follows.

Given an integer k, $-m \leq k \leq m$, we define Φ_k as the set of all roots $\sum_{q=1}^{r} m_q \alpha_q$ with $m_i = k$. Here m is the largest integer such that $\Phi_m \neq 0$. For $\alpha \in \Phi$, let $\mathbf{g}_\alpha$ be the corresponding root space. Define

$$
\begin{aligned}
\mathbf{g}_0 &= \mathbf{h} \oplus \bigoplus_{\alpha \in \Phi_0} \mathbf{g}_\alpha \\
\mathbf{g}_k &= \bigoplus_{\alpha \in \Phi_k} \mathbf{g}_\alpha, \quad k \neq 0.
\end{aligned}
$$

Then $\mathbf{g}_k$ is an eigenspace for the adjoint representation of $\mathbf{z}$. In fact, there exists an element $\theta \in \mathbf{z}$ such that $[\theta, v] = kv$ for $v \in \mathbf{g}_k$. The eigenspace decomposition $\mathbf{g} = \bigoplus_{k=-m}^{m} \mathbf{g}_k$ gives a graded Lie algebra structure on $\mathbf{g}$. We have

$$
\begin{aligned}
\mathbf{p} &= \mathbf{g}_0 \oplus \mathbf{g}_{-1} \oplus \cdots \oplus \mathbf{g}_{-m} \\
\mathbf{l} &= \mathbf{g}_0 \\
\mathbf{u} &= \mathbf{g}_{-1} \oplus \cdots \oplus \mathbf{g}_{-m}
\end{aligned}
$$

Remark Our choice of $\mathbf{p}$ has a different sign from the choice in some references, e.g. [Ya]. As many geometers do, we prefer this choice for the reason that positive roots correspond to positive line bundles.

Now we go to the situation of the Main Theorem.

Let S be our rational homogeneous space of Picard number 1 of dimension n, different from $\mathbf{P}_n$, and $f : S \to X$ be a surjective holomorphic map onto a projective manifold X. From the Picard number condition, f is a finite morphism. Let $R \subset S$ be the ramification divisor. From $K_S = f^*K_X + R$, we see that K_X^{-1} is ample. Thus X is a Fano manifold of Picard number 1. We assume that X is different from $\mathbf{P}_n$. We can apply the result of Section 1, to this case with $Y = S$. We have the following strengthened form of Proposition 3.

Proposition 4 *Set $Y = S = G/P$ in the statement of Proposition 3 and let $y \in Y$ be a very general point. Then each irreducible component of $df_y^{-1}(\mathcal{C}_x)$ for X of type (i) (resp. $df_y^{-1}(\mathcal{C}'_x)$ for X of type (ii)) is a P_y-invariant subvariety, where $P_y \subset G$ is the isotropy subgroup at y, acting on $\mathbf{P}T_y(S)$ in a natural way. Furthermore, there exists a Zariski dense open set $\tilde{S} \subset S$, such that for any two points $s_1, s_2 \in \tilde{S}$, the P_{s_1}-invariant subvariety $df_{s_1}^{-1}(\mathcal{C}_{f(s_1)})$ and P_{s_2}-invariant subvariety $df_{s_2}^{-1}(\mathcal{C}_{f(s_2)})$ are conjugate by an element of G, when X is of type (i). The same statement holds for $df_{s_1}^{-1}(\mathcal{C}'_{f(s_1)})$ and $df_{s_2}^{-1}(\mathcal{C}'_{f(s_2)})$ for X of type (ii).*

Proof. From the property of the stratification M_i for an equivariant map, any variety of distinguished tangent in $\mathbf{P}T_y(S)$ is P_y-invariant. So the first statement is a direct consequence of Proposition 3.

Over a Zariski dense open subset $\tilde{X} \subset X$, $\{\mathcal{C}_x, x \in \tilde{X}\}$ (resp. $\{\mathcal{C}'_x, x \in \tilde{X}\}$) form a flat family, hence their pull-backs to S form a flat family over a Zariski dense open subset of S. The second statement follows from the fact that only countably many P-invariant subvarieties in $\mathbf{P}T_o(S)$ can serve as the union of finitely many varieties of distinguished tangents. □

To refine this result further, we need more informations about the structure of the isotropy representation of P. The next simple Lemma regarding the graded Lie algebra structure, will be quite useful.

Lemma 2 (Lemma 3.2 of [Ya]) *If $X \in \mathbf{g}_k, k < m$ and $[X, \mathbf{g}_1] = 0$, then $X = 0$. For $k < m$, $\mathbf{g}_k = [\mathbf{g}_{k+1}, \mathbf{g}_{-1}]$.*

The tangent space $T_o(S)$ can be identified with $\mathbf{g}/\mathbf{p}$ canonically, which can be identified with $\mathbf{g}_{-1} \oplus \cdots \oplus \mathbf{g}_{-m}$ by a choice of a Levi factor $L \subset P$. The isotropy representation of P on $T_o(S)$ corresponds to the adjoint action on $\mathbf{g}/\mathbf{p}$. The action of the reductive group L on $\mathbf{g}_k$ is irreducible for any $k \neq 0$. Let $W^k \subset \mathbf{P}\mathbf{g}_k$ be the orbit of highest weight vectors of L-action on $\mathbf{P}\mathbf{g}_k$.

Proposition 5 *Choose an identification $T_o(S) = \mathbf{g}_1 \oplus \cdots \oplus \mathbf{g}_m$, by choosing a Levi factor $L \subset P$.*

1. *For any $v \in \mathbf{P}(\oplus_{j=1}^{k} \mathbf{g}_j) - \mathbf{P}(\oplus_{j=1}^{k-1} \mathbf{g}_j), 1 \leq k \leq m$, the closure of the P-orbit of v contains W^k.*

2. *For any $v \in \mathbf{P}T_o(S)$, the closure of the P-orbit of v contains $W^1 \subset \mathbf{P}\mathbf{g}_1$.*

3. *For any $v \in \mathbf{P}T_o(S) - \mathbf{P}\mathbf{g}_1$, the closure of the P-orbit of v contains W^2.*

Proof. Let $\mathbf{C}^* \subset P$ be the group corresponding to $\mathbf{z}$, the center of the Lie algebra $\mathbf{l}$ of L. $t \in \mathbf{C}^*$ acts on $v \in T_o(S), v = v_1 + \cdots + v_k, v_j \in \mathbf{g}_j$ by $t \cdot v = t^1 v_1 + \cdots + t^k v_k$. It follows that the closure of $\mathbf{C}^*$-orbit of $[v] \in \mathbf{P}T_o(S)$ contains an element in $\mathbf{P}\mathbf{g}_k$ if $v_k \neq 0$. Given an irreducible representation V of a reductive Lie group, applying the same argument to various $\mathbf{C}^*$-subgroups of the maximal algebraic torus, we can see that the closure of any orbit in $\mathbf{P}V$ contains the highest weight orbit. It follows that the closure of any L-orbit in $\mathbf{P}\mathbf{g}_k$ contains W^k, which gives the first statement.

Let $\mathcal{U} \subset P$ be the unipotent radical of P. $\mathcal{U}$-action on $\mathbf{P}T_o(S)$ has a unique fixed point component $\mathbf{P}\mathbf{g}_1$ by Lemma 2. Since the closure of any $\mathcal{U}$-orbit must contain a $\mathcal{U}$-fixed point, the closure of any P-orbit intersects $\mathbf{P}\mathbf{g}_1$. Now, the second statement of Propostion follows from the first statement with $k = 1$.

The last statement follows from the first one, if we show that the closure of the P-orbit of a vector $v_k, \mathbf{P}v_k \in W^k$, contains a vector in $\mathbf{P}\mathbf{g}_{k-1}$ for $2 \leq k \leq m$. From $[v_k, \mathbf{g}_{-1}] \neq 0$, the P-orbit of $\mathbf{P}v_k$ contains an element of the form $\mathbf{P}(v_k + \zeta_{k-1})$ with $0 \neq \zeta_{k-1} \in \mathbf{g}_{k-1}$. Then the closure of the $\mathbf{C}^*$-orbit of $\mathbf{P}(v_k + \zeta_{k-1})$ must contain $\mathbf{P}\zeta_{k-1} \in \mathbf{P}\mathbf{g}_{k-1}$. □

The number m is called the **depth** of S. It is equal to the coefficient of α_i in the expression for the maximal root of $\mathbf{g}$. When the depth is 1, S is an irreducible Hermitian symmetric space of the compact type. When the depth is 2 and $\dim(\mathbf{g}_2) = 1$, S is a homogeneous contact manifold. For convenience, we will say that S is **of symmetric type**, and **of contact type**, respectively.

The homogeneous vector bundle on G/P associated with the adjoint representation of P on $\mathbf{g}/\mathbf{p}$ is the tangent bundle $T(S)$. The subspace $(\mathbf{p}\oplus\mathbf{g}_1\oplus\cdots\oplus\mathbf{g}_k)/\mathbf{p} \subset \mathbf{g}/\mathbf{p}$ is P-invariant, which defines a subbundle D^k of the tangent bundle, namely, a distribution. We have a fiber subbundle $\mathcal{W} \subset \mathbf{P}D^1 \subset \mathbf{P}T(S)$ defined by the highest weight orbit of the isotropy representation on $\mathbf{g}_1$, i.e. $\mathcal{W}_o$, the fiber at o, is equivalent to W^1.

Proposition 6 *Let $f : S \to X$ be a finite holomorphic map from our rational homogeneous space S to a Fano manifold X of Picard number 1, different from $\mathbf{P}_n$. Let $s, s' \in S$ be an arbitrary pair of distinct points such that $f(s) = f(s')$ and f is unramified at s and s'. Write φ for the unique germ of holomorphic map at s, with target space S, such that $\varphi(s) = s'$ and $f \circ \varphi = f$. Then φ preserves the distribution D^k, for some $k < m$. If S is of symmetric type, φ preserves $\mathcal{W}$.*

Proof. Suppose φ preserves D^k (resp. $\mathcal{W}$ when S is of symmetric type) at generic points of a small neighborhood of s where φ is defined. Then it preserves D^k (resp. $\mathcal{W}$ when S is of symmetric type) at every point of that neighborhood. Hence we may assume that $s, s' \in \tilde{S}$ of Proposition 4.

Identify $\mathbf{P}T_s(S)$ and $\mathbf{P}T_{s'}(S)$ with $\mathbf{P}T_o(S)$ by G-action. Such identifications need not be unique, but we fix one. Let $\mathcal{C} \subset \mathbf{P}T_o(S)$ be the (not necessarily irreducible) P-invariant subvariety corresponding to $df_s^{-1}(\mathcal{C}_{f(s)})$ and $df_{s'}^{-1}(\mathcal{C}_{f(s')})$ for X of type (i) (or $df_s^{-1}(\mathcal{C}'_{f(s)})$ and $df_{s'}^{-1}(\mathcal{C}'_{f(s')})$ for X of type (ii)). Then $d\varphi$ corresponds to an element $\sigma \in PGL(T_o(S))$ which preserves $\mathcal{C}$. We need to show that σ preserves D_o^k, and W^1 when S is of symmetric type.

Let $A \subset PGL(T_o(S))$ be the algebraic subgroup generated by σ and the image of P in $PGL(T_o(S))$ under the isotropy representation. By Proposition 5, any A-invariant subvariety contains W^1. It follows that any A-invariant subvariety contains $A \cdot W^1$, the orbit of W^1 under A. Certainly, $A \cdot W^1$ is a constructible set and its Zariski closure $\overline{A \cdot W^1}$ is again A-invariant. The complement B of $A \cdot W^1$ in $\overline{A \cdot W^1}$ is A-invariant. B is constructible and its Zariski closure $\overline{B} \subset \overline{A \cdot W^1}$ is a proper A-invariant subvariety. If B is non-empty, then $\overline{B}$ must contain $A \cdot W^1$ so that $\overline{B} = \overline{A \cdot W^1}$, a plain contradiction. Thus B is empty and $A \cdot W^1$ is a closed subvariety in $\mathbf{P}T_o(S)$. Since it is homogeneous under A, it must be smooth. Each component of $A \cdot W^1$ is P-invariant and contains W^1 by Proposition 5. We conclude that $A \cdot W^1$ is an irreducible homogeneous submanifold of $\mathbf{P}T_o(S)$. Since $A \cdot W^1 \subset \mathcal{C}$, $A \cdot W^1$ is a proper submanifold of $\mathbf{P}T_o(S)$.

Suppose that $A \cdot W^1$ is linearly degenerate in $\mathbf{P}T_o(S)$. Then σ preserves its linear span, hence σ must preserve some D_o^k, $k < m$, because they are the only P-invariant linear subspaces.

Suppose that $A \cdot W^1$ is linearly nondegenerate in $\mathbf{P}T_o(S)$. Since A is a linear group, we know that $A \cdot W^1$ is a rational homogeneous space equivariantly embedded in $\mathbf{P}T_o(S)$. Let H be the connected component of its automorphism group in $PGL(T_o(S))$. By Borel fixed point theorem, the solvable radical R of H has fixed points on $A \cdot W^1$, which must be preserved by H since R is normal in H. Since $A \cdot W^1$ is homogeneous under H, the solvable radical must be the identity and H is semisimple. Let $\hat{H} \subset GL(T_o(S))$ be the inverse image of H. Then $A \cdot W^1$ must be the highest weight orbit of this irreducible representation of $\hat{H}$. It follows that S has an irreducible reductive G-structure with G $= \hat{H}$. (Here G is a symbol for a general group, not to be confused with the simple group G acting on S.) From [HM2], only S of symmetric type can have such a structure and $A \cdot W^1$ must agree with W^1 (Proposition 10 of [HM2]). Hence σ must preserve W^1. □

When S is of contact type, i.e., when D^1 defines a distribution of rank $n-1$ on S, we need to strengthen Proposition 4. We start with a few lemmas about S of contact type. Note that the distribution D^1 is not integrable. In fact, the following can be easily checked (e.g. Section 2 of [Hw]).

Lemma 3 *Let S be of contact type. The Frobenius bracket $F^1 : \wedge^2 D^1_o \to \mathbf{g}_2 = T_o(S)/D^1_o$ defines a symplectic form on D^1_o, and the homogeneization of W^1 in D^1_o is Lagrangian with respect to F^1. In other words, $\mathcal{T} \subset \mathbf{P} \wedge^2 D^1_o$, the variety of tangential lines to W^1, is contained in $\mathbf{P}Ker(F^1)$. Furthermore $\mathcal{T}$ spans $\mathbf{P}Ker(F^1)$.*

A direct consequence is

Lemma 4 *Let $\mathcal{C} \subset \mathbf{P}D^1_o$ be a P-invariant subvariety different from W^1. Then the variety of tangential lines to $\mathcal{C}$ is nondegenerate in $\mathbf{P} \wedge^2 D^1_o$.*

Proof. By Proposition 5, $\mathcal{C}$ contains W^1 and $\dim(\mathcal{C}) > \dim(W^1)$. From Lemma 3, if the variety of tangential lines to $\mathcal{C}$ is degenerate in $\wedge^2 D^1_o$, then the tangent spaces of homogeneization of $\mathcal{C}$ in D^1_o must be isotropic with respect to the symplectic form F^1. But $\dim(\mathcal{C}) > \dim(W^1)$ and the homogeneization of W^1 is Lagrangian, a contradiction. □

Lemma 5 *When S is of contact type, the P-action on $\mathbf{P}T_o(S) - \mathbf{P}D^1_o$ is transitive.*

Proof. Let $\rho \in \Phi$ be the maximal root. Then $\mathbf{g}_2 = \mathbf{g}_\rho$ is one-dimensional. Let $\mathcal{U} \subset P$ be the unipotent radical with the Lie algebra $\mathbf{g}_{-1} \oplus \mathbf{g}_{-2}$. We have $[\mathbf{g}_2, \mathbf{g}_{-1}] = \mathbf{g}_1$ from Lemma 2. This means that $\mathcal{U}$-orbit of $\mathbf{P}\mathbf{g}_2$ is dense in $\mathbf{P}T_o(S) - \mathbf{P}D^1_o$. Since orbits of unipotent groups in affine varieties are closed ([Bo], Prop. 4.10, p.88), the P-orbit of $\mathbf{P}\mathbf{g}_2$ is $\mathbf{P}T_o(S) - \mathbf{P}D^1_o$. □

Proposition 7 *Let $f : S \to X$ be a finite morphism from S of contact type to a Fano manifold X of Picard number 1, different from $\mathbf{P}_n$. Then X must be of type (i), and for a generic point $s \in S$, $df_s^{-1}(\mathcal{C}_{f(s)})$ coincides with $\mathcal{W}_s \subset \mathbf{P}D^1_s$.*

Proof. Suppose X is of type (ii) and let $s \in S$ be a generic point. From Lemma 5 and P_s-invariance, we have $df_s^{-1}(\mathcal{C}'_{f(s)}) \subset \mathbf{P}D^1_s$. Since $\mathcal{C}'_{f(s)}$ is of codimension 1 in $\mathbf{P}T_{f(s)}$, we have $df_s^{-1}(\mathcal{C}'_{f(s)}) = \mathbf{P}D^1_s$. Let l be the component of $f^{-1}(C')$ through s. From $H^0(C', T^*(X)|_{C'}) \neq 0$, we have a non-zero section $\omega \in H^0(l, T^*(S)|_l)$. ω annihilates D_s^1 from Lemma 1. It is well-known that $T(S)/D^1$ is an ample line bundle (e.g. Section 1 of [Hw]). This implies that ω must be identically zero on $T(S)/D^1|_l$. Annihilating both D_s^1 and $T_s(S)/D_s^1$, ω vanishes at s, a contradiction to the fact that no section of $T^*(S)|_l$ has zero ($T(S)$ is semipositive).

Now assume X is of type (i). By exactly the same arguement as above, $df_s^{-1}(\mathcal{C}_{f(s)})$ is a proper subvariety of $\mathbf{P}D_s^1$. Each irreducible component of $df_s^{-1}(\mathcal{C}_{f(s)})$ is nondegenerate in $\mathbf{P}D_s^1$, because any P_s-invariant subvariety of $\mathbf{P}D_s^1$ contains $\mathcal{W}_s$. It follows that $df(D^1)$ defines a non-integrable meromorphic distribution D on X with the property that $\mathcal{C}_x \subset \mathbf{P}D_x$ at a generic point $x \in X$ and each irreducible component of $\mathcal{C}_x$ is linearly nondegenerate in $\mathbf{P}D_x$. Now we recall the following result proved in [HM1].

Proposition 8 (Proposition 9 of [HM1]) *Let D be a non-integrable meromorphic distribution on a Fano manifold X of Picard number 1, so that each component of $\mathcal{C}_x$ at generic x is contained and nondegenerate in $\mathbf{P}D_x$. Then the variety of tangential lines to any component of $\mathcal{C}_x$ is degenerate in $\mathbf{P} \wedge^2 D_x$ at generic x.*

It follows that each component of $df_s^{-1}(\mathcal{C}_{f(s)})$ must be equal to $\mathcal{W}_s$ from Lemma 4. □

Corollary *Let $f : S \longrightarrow X$ be a finite holomorphic map from S of contact type to a Fano manifold X of Picard number 1, different from $\mathbf{P}_n$. Let $s, s' \in S$ be an arbitrary pair of distinct points such that $f(s) = f(s')$ and f is unramified at s and s'. Write φ for the unique germ of holomorphic map at s, with target space S, such that $\varphi(s) = s'$ and $f \circ \varphi = f$. Then φ preserves $\mathcal{W}$.*

3 Differential systems on G/P

The purpose of this section is to prove the following extension theorem using the results of the previous section.

Theorem 1 *Let $f : S \longrightarrow X$ be a finite holomorphic map from our rational homogeneous space S to a Fano manifold X of Picard number 1, different from $\mathbf{P}_n$. Let $s, s' \in S$ be an arbitrary pair of distinct points such that $f(s) = f(s')$ and f is unramified at s and s'. Write φ for the unique germ of holomorphic map at s, with target space S, such that $\varphi(s) = s'$ and $f \circ \varphi = f$. Then φ can be extended to a biholomorphic automorphism of S.*

The key ingredient is the following result of Tanaka and Yamaguchi ([Ta], [Ya]) concerning differential systems on S.

Proposition 9 *Let $U \subset S$ be a connected open set. Then a holomorphic vector field on U can be extended to a global holomorphic vector field on S if it preserves $\mathcal{W}|_U$. Furthermore, if S is neither of symmetric type nor of contact type, then a holomorphic vector field on U can be extended to a global holomorphic vector field on S if it preserves $D^1|_U$.*

Proof. When S is neither of symmetric type nor of contact type, this is Corollary 5.4 of [Ya]. When S is of symmetric type, this is Lemma 11.10 of [Oc], or in Theorem 5.2 of [Ya]. Now assume S is of contact type. We may assume that U is a small neighborhood of $o \in S$. We have $T_o(S) \cong \mathbf{g}/\mathbf{p} \cong \mathbf{g}_1 + \mathbf{g}_2$. The adjoint representation of $\mathbf{g}_0$ on $\mathbf{g}_1$ is faithful by Lemma 2. Let $\mathbf{g}'_0 \subset End(\mathbf{g}_1)$ be the image. Since a choice of n global vector fields on S can generate $T_o(S)$, to show that a local vector field can be extended to a global vector field, we may assume that it vanishes at o. Theorem 5.2 of [Ya] says that a local holomorphic vector field on S preserving D^1 and vanishing at o can be extended to a global holomorphic vector field on S, if the endomorphism of D^1_o induced by its first order part lies in $\mathbf{g}'_0 \subset End(\mathbf{g}_1)$. From the description of W^1 for S of contact type (e.g. section 2 of [Hw]), $\mathbf{g}'_0$ consists of elements of $End(\mathbf{g}_1)$ preserving $W^1 \subset \mathbf{P}\mathbf{g}_1$. Thus any local holomorphic vector field on S preserving $\mathcal{W}$ and vanishing at o can be extended to a global vector field on S. □

Corollary *Let $U_1, U_2 \subset S$ be two connected open sets and $f_{12} : U_1 \longrightarrow U_2$ be a biholomorphic map preserving the distribution $D^1 \subset T(S)$. If S is neither of symmetric type nor of contact type, then f_{12} can be extended to a biholomorphic automorphism of S. When S is of symmetric type or of contact type, f_{12} can be extended to a biholomorphic automorphism of S, if f_{12} preserves the fiber subbundle $\mathcal{W} \subset \mathbf{P}D^1$.*

Proof. f_{12} sends holomorphic vector fields on U_1 preserving D^1 (resp. $\mathcal{W}$) to holomorphic vector fields on U_2 preserving D^1 (resp. $\mathcal{W}$). It follows that f_{12} induces an automorphism of $\mathbf{g}$ sending $\mathbf{p}$ to its conjugates. Thus f_{12} defines an automorphism of S, which naturally extends the map from U_1 to U_2. □

This Corollary, combined with Proposition 6 and Corollary to Proposition 7 gives a proof of Theorem 1 when $m \leq 2$. To prove Theorem 1 when $m > 2$, it suffices to have the following refinement of Tanaka-Yamaguchi theory.

Proposition 10 *Suppose the depth of S is > 2. Let $f_{12} : U_1 \longrightarrow U_2$ be any biholomorphic map between two open subsets of S, preserving one of the distribution $D^k, 1 < k < m$. Then f_{12} preserves D^1.*

We want to show that D^1 can be recovered from D^k by considering the rank of Frobenius bracket for D^k. We will use the following lemma which is a direct consequence of the fact that when elements of $\mathbf{g}$ are regarded as global holomorphic vector fields on S, the Lie bracket of two elements corresponds to the Frobenius bracket of two vector fields.

Lemma 6 *Let $F^k : D^k_o \otimes D^k_o \longrightarrow T_o(S)/D^k_o$ be the Frobenius form for the distribution D^k at $o \in S$. Under an identification of $T_o(S)$ with $\mathbf{g}_1 \oplus \cdots \mathbf{g}_m$, we have*

$$F^k(\xi, \zeta) \;=\; [\xi, \zeta] \bmod \mathbf{g}_1 \oplus \cdots \oplus \mathbf{g}_k$$

for any $\xi, \zeta \in D^k_o$.

Lemma 6 reduces the proof of Proposition 10 to the following.

Proposition 11 *Fix $k, 2 \leq k < m$. Write F for F^k. For $\xi \in D_o^k$, denote by $F_\xi : D_o^k \rightarrow T_o(S)/D_o^k$ the linear map defined by $F_\xi(\zeta) = F(\xi, \zeta)$. Let L be a Levi factor of P so that $L(\mathbf{g}_j) = \mathbf{g}_j, 1 \leq j \leq m$. Let $\eta_1 \in \mathbf{g}_1 = D_o^1$ be a non-zero highest weight vector of $\mathbf{g}_1$ as an irreducible L-module. Then for any $\xi \in D_o^k$ which is not in $\mathbf{g}_1$, we have*

$$rank\ F_\xi \quad > \quad rank\ F_{\eta_1}.$$

The rank of F_ξ is constant along the P-orbit of ξ and is lower-semicontinuous on $\xi \in D_o^k$. From Proposition 5, the proof of Proposition 11 is reduced to checking the following inequality for a non-zero highest weight vector $\eta_j \in \mathbf{g}_j$.

$$\dim([\eta_1, \bigoplus_{j=1}^{k} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{k} \mathbf{g}_j) \quad < \quad \dim([\eta_2, \bigoplus_{j=1}^{k} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{k} \mathbf{g}_j).$$

The rest of this section will be devoted to the proof of this inequality.

Since S has depth > 2, only exceptional simple Lie algebras matter (see e.g. p.454 diagrams in [Ya] for a list of depths). First, we will prove the case of E_6, E_7 and E_8, by interpretting both sides of the inequality as certain Chern numbers.

The quotient map $G \rightarrow G/P$ defines a P-principal bundle on $S = G/P$. The left action of P on the reductive group $L = P/\mathcal{U}$ where $\mathcal{U}$ is the unipotent radical of P, induces an L-principal bundle on S. Since S has Picard number 1, there exists a unique ample generator $\mathcal{L}$ of the Picard group. This line bundle $\mathcal{L}$ is homogeneous and is associated to the L-principal bundle by a 1-dimensional representation of L. This representation can be described as follows. Let α_i be the simple root defining P. Let $H_{\alpha_i} \in \mathbf{h}$ be its coroot, i.e. the element of $[\mathbf{g}_{\alpha_i}, \mathbf{g}_{\alpha_i}]$ satisfying $\alpha_i(H_{\alpha_i}) = 2$. Let λ_i be the i-th fundamental weight, namely the one defined by $\lambda_i(H_{\alpha_j}) = \delta_{ij}$. Consider the irreducible representation of G with lowest weight $-\lambda_i$. P is the stabilizer of the lowest weight line. The representation of P on the lowest weight line gives rise to a 1-dimensional representation of L, whose dual representation is the one inducing $\mathcal{L}$. In other words, the weight of the representation inducing $\mathcal{L}$ is λ_i.

Although we have assumed that S is different from $\mathbf{P}_n$, the above argument works for $S = \mathbf{P}_1$. In this case, $\mathbf{g} = \mathbf{sl}_2$ has a unique simple root and corresponding coroot. L is isomorphic to $\mathbf{C}^*$ and the L-principal bundle is $\mathcal{O}(1)^*$, the complement of the zero section in the hyperplane bundle $\mathcal{O}(1)$. The weight having integer value k on the coroot gives rise to the line bundle $\mathcal{O}(k)$ on $\mathbf{P}_1$.

On our rational homogeneous space $S = G/P$, we have rational curves associated to positive roots in the following way. Let α be any positive root and $H_\alpha \in \mathbf{h}$ be its coroot. Let $\mathbf{s}_\alpha \subset \mathbf{g}$ be the subalgebra isomorphic to $\mathbf{sl}_2$ such that $\mathbf{s}_\alpha \cap \mathbf{h} = \mathbf{C}H_\alpha$ and H_α is the coroot for $\mathbf{s}_\alpha$. The orbit of $o \in S$ under the subgroup $S_\alpha \subset G$ with Lie algebra $\mathbf{s}_\alpha$ is a rational curve and will be denoted by C_α.

Lemma 7 *Suppose all roots of the simple Lie algebra* $\mathbf{g}$ *are of equal length. For each positive root* $\alpha \in \mathbf{g}_k, 1 \leq k \leq m$, C_α *has degree* k *with respect to* $\mathcal{L}$.

Proof. Since the root system is self-dual, $H_\alpha = \sum_{j=1}^m m_j H_{\alpha_j}$ with $m_i = k$. Hence the i-th fundamental weight defining $\mathcal{L}$ has value k on H_α. □

For $\mathbf{g} = E_6, E_7$, or E_8, the inequality we want to establish is a simple consequence of the next lemma.

Lemma 8 *Suppose all roots of* $\mathbf{g}$ *are of equal length. Let* $\mathcal{V}^k, 1 \leq k \leq m$ *be the quotient bundle* $T(S)/D^k$. *Then the first Chern number of the bundle* $\mathcal{V}^k$ *restricted to* C_α, $\alpha \in \Phi_1 \cup \cdots \cup \Phi_k$ *is equal to*

$$\dim([\mathbf{g}_\alpha, \bigoplus_{j=1}^k \mathbf{g}_j] \bmod \bigoplus_{j=1}^k \mathbf{g}_j).$$

In particular, from Lemma 7,

$$2\dim([\eta_1, \bigoplus_{j=1}^k \mathbf{g}_j] \bmod \bigoplus_{j=1}^k \mathbf{g}_j) \quad = \quad \dim([\eta_2, \bigoplus_{j=1}^k \mathbf{g}_j] \bmod \bigoplus_{j=1}^k \mathbf{g}_j).$$

Proof. $\mathcal{V}^k$ is the homogenous vector bundle on S associated to the representation $\mathbf{g}/\mathbf{p}$ of P. This bundle has a composition series $0 \subset D^{k+1}/D^k \subset \cdots \subset D^{m-1}/D^k \subset D^m/D^k = \mathcal{V}^k$. To calculate the Chern number of $\mathcal{V}^k$, we may add up the Chern numbers of the L-bundles $D^{k+1}/D^k, \ldots, D^m/D^{m-1}$. We recall Grothendieck's splitting theorem for principal bundles on $\mathbf{P}_1$ with reductive structure groups and associated vector bundles([Gr]).

Proposition 12 *Let* $\mathcal{O}(1)^*$ *be the* $\mathbf{C}^*$*-principal bundle on* $\mathbf{P}_1$. *Let* L *be a connected reductive complex Lie group. Up to conjugation, any* L*-principal bundle on* $\mathbf{P}_1$ *is associated to* $\mathcal{O}(1)^*$ *by a group homomorphism from* $\mathbf{C}^*$ *to a maximal torus of* L. *Let* $\mathbf{h}$ *be a Cartan subalgebra of* L *and* $H \in \mathbf{h}$ *be the image of the coroot of* $\mathbf{sl}_2$, *under this homomorphism. Given a representation of* L *with weights* $\mu_1, \ldots, \mu_l \in \mathbf{h}^*$, *the associated vector bundle on* $\mathbf{P}_1$ *splits as* $\mathcal{O}(\mu_1(H)) \oplus \cdots \oplus \mathcal{O}(\mu_l(H))$.

Hence the Chern number of $\mathcal{V}^k$ restricted to C_α is $\sum_{\beta \in \Phi_{k+1} \cup \cdots \cup \Phi_m} \beta(H_\alpha)$. Since all roots have the same length, for any $\alpha, \beta \in \Phi$, $\beta(H_\alpha)$ is 1 if $\beta - \alpha \in \Phi$ and is -1 if $\beta + \alpha \in \Phi$. From $\alpha \in \Phi_1 \cup \cdots \cup \Phi_k$, the Chern number is

$$\begin{aligned}\sum_{\beta \in \Phi_{k+1} \cup \cdots \cup \Phi_m} \beta(H_\alpha) &= \sharp\{\beta \in \Phi_{k+1} \cup \cdots \cup \Phi_m, \beta - \alpha \in \Phi\} - \sharp\{\beta \in \Phi_{k+1} \cup \cdots \cup \Phi_m, \beta + \alpha \in \Phi\} \\ &= \sharp\{\beta \in \Phi_{k+1} \cup \cdots \cup \Phi_m, \beta - \alpha \in \Phi_1 \cup \cdots \cup \Phi_k\}.\end{aligned}$$

On the other hand, we have $[\mathbf{g}_\alpha, \mathbf{g}_\gamma] = \mathbf{g}_{\alpha+\gamma}$ if $\alpha + \gamma \in \Phi$, and 0 if $\alpha + \gamma$ is not a root. Hence

$$\dim([\mathbf{g}_\alpha, \bigoplus_{j=1}^k \mathbf{g}_j] \bmod \bigoplus_{j=1}^k \mathbf{g}_j) \quad = \quad \sharp\{\gamma \in \Phi_1 \cup \cdots \cup \Phi_k, \alpha + \gamma \in \Phi_{k+1} \cup \cdots \cup \Phi_m\}.$$

By putting $\alpha + \gamma = \beta$, we can see that the last line is equal to the Chern number. $\square$

Remark The geometric meaning of Lemma 8 is the following. The bundle $\mathcal{V}^k$ splits over C_α in the form $[\mathcal{O}(1)]^a \oplus [\mathcal{O}]^b$ where a is the dimension given above. This follows from the fact that each quotient bundle of the composition series splits with degrees $-1, 0$, or 1, and D^m/D^l is semi-positive for each $l, k \le l \le m$.

Now let us look at the cases of G_2 and F_4. There is one case of depth > 2 for $\mathbf{g} = G_2$, two cases of depth > 2 for $\mathbf{g} = F_4$. Using the convention of [Ti] for the numbering of roots, they are $(G_2, \alpha_1), (F_4, \alpha_2)$, and (F_4, α_3). For these three cases, we can prove the desired inequality in a way similar to above by interpretting both sides as certain Chern numbers with suitable modifications. But for each of the remaining 3 cases, we need a different sort of modification of the argument. For this reason, it seems better just to check the inequality explicitly by writing down all the positive roots. We use the notations of [Ti] for positive roots.

For (G_2, α_1),

$$\begin{aligned} \Phi_1 &= \{10, 11 = \eta_1\} \\ \Phi_2 &= \{21 = \eta_2\} \\ \Phi_3 &= \{31, 32\} \end{aligned}$$

We need to consider only $k = 2$ case.

$$\begin{aligned} \dim([\eta_1, \bigoplus_{j=1}^{2} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{2} \mathbf{g}_j) &= 1, \\ \dim([\eta_2, \bigoplus_{j=1}^{2} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{2} \mathbf{g}_j) &= 2. \end{aligned}$$

For (F_4, α_2),

$$\begin{aligned} \Phi_1 &= \{0100, 0110, 0111, 1100, 1110, 1111 = \eta_1\} \\ \Phi_2 &= \{0210, 0211, 0221, 1210, 1211, 1221, 2210, 2211, 2221 = \eta_2\} \\ \Phi_3 &= \{1321, 2321\} \\ \Phi_4 &= \{2421, 2431, 2432\} \end{aligned}$$

For $k = 2$,

$$\begin{aligned} \dim([\eta_1, \bigoplus_{j=1}^{2} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{2} \mathbf{g}_j) &= 2, \\ \dim([\eta_2, \bigoplus_{j=1}^{2} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{2} \mathbf{g}_j) &= 3. \end{aligned}$$

For $k = 3$,

$$\begin{aligned}
\dim([\eta_1, \bigoplus_{j=1}^{3} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{3} \mathbf{g}_j) &= 1, \\
\dim([\eta_2, \bigoplus_{j=1}^{3} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{3} \mathbf{g}_j) &= 2.
\end{aligned}$$

For (F_4, α_3),

$$\begin{aligned}
\Phi_1 &= \{0010, 0011, 0110, 0111, 0210, 0211, \\
&\quad 1110, 1111, 1210, 1211, 2210, 2211 = \eta_1\} \\
\Phi_2 &= \{0221, 1221, 1321, 2221, 2321, 2421 = \eta_2\} \\
\Phi_3 &= \{2431, 2432\}
\end{aligned}$$

We only need to consider $k = 2$.

$$\begin{aligned}
\dim([\eta_1, \bigoplus_{j=1}^{2} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{2} \mathbf{g}_j) &= 1, \\
\dim([\eta_2, \bigoplus_{j=1}^{2} \mathbf{g}_j] \bmod \bigoplus_{j=1}^{2} \mathbf{g}_j) &= 2.
\end{aligned}$$

This finishes the proof of Proposition 10, and thus Theorem 1.

Remark Just to prove Theorem 1, we need not consider the case (G_2, α_1), because the underlying complex manifold S is a 5-dimensional hyperquadric corresponding to (B_3, α_1), which is covered by Corollary to Proposition 9. We checked this case to make the statement of Proposition 10 in a complete form.

4 Reduction to a finite group action

We will show Theorem 1 implies the Main Theorem. It suffices to prove the following.

Theorem 2 *Let $f : S \to X$ be a finite morphism from our rational homogeous space of Picard number 1 different from $\mathbf{P}_n$ to a Fano manifold X of Picard number 1, with a nontrivial ramification $R \subset S$. Suppose for an arbitrary pair of distinct points $s, s' \in S - R$ with $f(s) = f(s')$, the unique germ of holomorphic map φ at s, with target space S, satisfying $\varphi(s) = s'$ and $f \circ \varphi = f$ can be extended to a biholomorphic automorphism of S. Then S is a hyperquadric and X is a projective space.*

We will prove it by showing that f is the quotient map by a finite group action.

Proposition 13 *Let $f : S \to X$ be as above. Write $R \subset S$ for the ramification divisor of f and $B = f(R)$. Then, $f^{-1}(f(R)) = R$, $f|_{S-R} : S - R \to X - B$ is a Galois covering, and the covering transformations extend to automorphisms of S.*

Proof. Suppose $f^{-1}(f(R)) \neq R$. Then, there exist s_1, s_2 such that $f(s_1) = f(s_2) := x$, f is unramified at s_1 and ramified at s_2. Choose a small neighborhood U of x so that the component U_1 of $f^{-1}(U)$ containing s_1 is biholomorphic to U by f. Let U_2 be the component of $f^{-1}(U)$ containing s_2. Choose $s_1' \in U_1, s_2' \in U_2$ such that $f(s_1') = f(s_2') := x'$ and f is unramified at both s_1' and s_2'. Let φ be the germ of holomorphic map at s_1', with target space S, such that $\varphi(s_1') = s_2'$ and $f \circ \varphi \equiv f$. By assumption, there exists a global automorphism $\tilde{\varphi}$ on S such that φ is the germ of $\tilde{\varphi}$ at s_1'. From $f \circ \varphi \equiv f$ we have $f \circ \tilde{\varphi} \equiv f$. Since $f(U_1) = U$, $\tilde{\varphi}(s_1') = s_2' \in U_2$, and U_1, U_2 are connected components of $f^{-1}(U)$, it follows that $\tilde{\varphi}(U_1) \subset U_2$. Define $\psi : U \to U_2$ by $\psi = \tilde{\varphi} \circ (f|_{U_1}^{-1})$. Then, $f \circ \psi = (f \circ \tilde{\varphi}) \circ (f|_{U_1}^{-1}) = f \circ (f|_{U_1}^{-1}) = id$ on U. Since $\psi(x) = s_2$, we have $df_{s_2} d\psi_x = id$, contradicting with the fact that f is ramified at s_2.

Since $f^{-1}(f(R)) = R$ we obtain a covering map $f|_{S-R} : S - R \to X - B$. Given any $s, s' \in S - R$ with $f(s) = f(s')$ we have $\tilde{\varphi} \in Aut(S)$ such that $\tilde{\varphi}(s) = s'$ and $f \circ \tilde{\varphi} \equiv f$. It follows that the covering transformations of $f|_{S-R} : S - R \to X - B$ act transitively on the fiber $f^{-1}(x)$ for any $x \in X - B$, so that $f|_{S-R}$ is a Galois covering and the covering transformations extend to automorphisms of S. □

Proposition 14 *Let $f : S \to X$ be as above. Then there exists a finite cyclic group $F \subset Aut(S)$ which fixes a hypersurface $E \subset S$ pointwise.*

Proof. Let $Gal(f) \subset Aut(S)$ be the finite subgroup consisting of the covering transformations of $f|_{S-R}$ obtained in Proposition 13. Let $E \subset R$ be an irreducible branch. Pick a generic point $s_0 \in E$. Then, there exists some open neighborhood U_0 of s_0 in S and U of $x = f(s_0)$ in X such that $f|_{U_0} : U_0 \to U$ is a branched covering ramified exactly on $U_0 \cap E$ and such that $f|_{U_0 \cap E} : U_0 \cap E \to X$ is injective. Let $s, s' \in U_0 - E$ be such that $f(s) = f(s')$. We have $\tilde{\varphi} \in Gal(f)$ such that $\tilde{\varphi}(s) = s'$ and $f \circ \tilde{\varphi} \equiv f$. Then, $\tilde{\varphi}(U_0) = U_0$, $U_0 \cap R = U_0 \cap E$ and $\tilde{\varphi}(U_0 \cap E) = U_0 \cap E$. Since $f|_{U_0 \cap E} : U_0 \cap E \to X$ is injective and $f(\tilde{\varphi}(s_0)) = f(s_0)$ it follows that $\tilde{\varphi}(s_0) = s_0$. We have therefore found some $\tilde{\varphi} \in Gal(f)$, $\tilde{\varphi} \neq id$, such that $\tilde{\varphi}$ fixes every point on E. Now set F to be the cyclic group generated by $\tilde{\varphi}$. □

Now Theorem 2 follows from the following.

Proposition 15 *Let S be a rational homogeneous space of Picard number 1 of dimension $n \geq 3$, different from $\mathbf{P}_n$. Suppose there exists a nontrivial finite cyclic group $F \subset Aut(S)$ which fixes a hypersurface $E \subset S$ pointwise. Then S is the hyperquadric, E is equal to an $\mathcal{O}(1)$-hypersurface, and the quotient of S by F, endowed with the standard normal complex structure, is a projective space.*

Proof. First, we will show that S is a hyperquadric and E is an $\mathcal{O}(1)$-hypersurface. This can be checked easily for $n = 3$. We will prove it by induction on n. So assume that it is true for rational homogeneous spaces of Picard number 1 of dimension $< n$ and $3 < n$.

The finite cyclic group F acts on $V := H^0(S, \mathcal{O}(E))$ and V^*, which decomposes into 1-dimensional weight spaces. S can be embedded in $\mathbf{P}V^*$, and E is a hyperplane section. Since F fixes E pointwise, we see that F acts on $\mathbf{P}V^*$ with one isolated fixed point Q and a fixed point hyperplane h so that $E = S \cap h$. The action of F on $\mathbf{P}V$ has an isolated fixed point $h^\perp$ and a fixed point hyperplane $Q^\perp$.

Since E is the fixed point component of a finite group action, E is smooth. As E is of dimension ≥ 3, E is simply connected and of Picard number 1 by Lefshetz hyperplane section theorem. Moreover from the complete reducibility of linear representations of F, $T(S)|_E$ splits as a direct sum of $T(E)$ and the normal bundle N_E. This implies that $T(E)$ is generated by global sections, because $T(S)$ is generated by global sections. It follows that E is a rational homogeneous space of Picard number 1. Let $\mathcal{Y} \subset \mathbf{P}V$ be the set of points corresponding to hyperplane sections of $S \subset \mathbf{P}V^*$ which are rational homogeneous spaces. Since smoothness of hyperplane sections is an open condition and the complex structures of rational homogeneous spaces are infinitesimally rigid, $\mathcal{Y}$ is a Zariski open subset, which is nonempty from $h^\perp \in \mathcal{Y}$.

The group G of $S = G/P$, also acts on $\mathbf{P}V$. Clearly $\mathcal{Y}$ is G-invariant. Since G acts irreducibly on $\mathbf{P}V$, the G-orbit of the hyperplane $Q^\perp$ must intersect the Zariski dense open set $\mathcal{Y}$. It follows that $\mathcal{Y}$ intersects $Q^\perp$. This means that we have an F-invariant hyperplane section $E' \subset S, E \neq E'$ so that E' is rational homogeneous of Picard number 1. By induction, E' is isomorphic to a hyperquadric, with $E \cap E'$ equal to an $\mathcal{O}(1)$-hyperplane section. It follows that $c_1(S) = n$ and S itself is biholomorphic to a hyperquadric by Kobayashi-Ochiai, with E equal to an $\mathcal{O}(1)$-hyperplane section.

We have a homogeneous coordinate system $(x_0, \ldots, x_{n+1})$ on $\mathbf{P}V^*$, coming from the weight space decomposition, so that F acts by $(\zeta x_0, \ldots, \zeta x_n, x_{n+1})$ for a suitable character ζ of F. Since a smooth hyperquadric in $\mathbf{P}V^*$ is being preserved by F, $\zeta^2 = 1$ and the hyperquadric must be of the form $Q(x_0, \ldots, x_n) + x_{n+1}^2$ where $Q(x_0, \ldots, x_n)$ is a nondegenerate quadratic form. It follows that F is of order 2 and the quotient map is simply the projection $(x_0, \ldots, x_{n+1}) \to (x_0, \ldots, x_n)$ whose image is $\mathbf{P}_n$. $\square$

Acknowledgment The first author would like to thank the University of Hong Kong for the partial support for numerous visits in 1997. He is grateful to Prof. K. Yamaguchi for valuable informations.

References

[Bo] Borel, A. : *Linear algebraic groups.* Graduate texts in math. **126**, Springer, New York, 1991

[CS] Cho, K. and Sato, E.: Smooth varieties dominated by G/P. preprint (1989) and Smooth projective varieties dominated by smooth quadric hypersurfaces in any characteristic. Math. Zeit. **217** (1994) 553-565

[Gr] Grothendieck, A.: Sur la classification des fibrés holomorphes sur la sphère de Riemann. Amer. J. Math. **79** (1957) 121-138

[Hw] Hwang, J.-M.: Rigidity of homogeneous contact manifolds under Fano deformation. J. reine angew. Math. **486** (1997) 153-163

[HM1] Hwang, J.-M. and Mok, N.: Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation. to appear in Invent. Math.

[HM2] Hwang, J.-M. and Mok, N.: Uniruled projective manifolds with irreducible reductive G-structures. J. reine angew. Math. **490** (1997) 55-64

[La] Lazarsfeld, R.: Some applications of the theory of positive vector bundles. Lect. Notes Math. **1092** *Complete intersections.* (1984) 29-61

[Mk] Mok, N.: The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. J. Diff. Geom. **27** (1988) 179-214

[Mr] Mori, S.: Projective manifolds with ample tangent bundles. Ann. Math. **110** (1979) 593-606

[Oc] Ochiai, T.: Geometry associated with semi-simple flat homogeneous spaces. Trans. A.M.S. **152** (1970) 1-33

[PS] Paranjape, K. H. and Srinivas, V.: Self maps of homogeneous spaces. Invent. Math. **98** (1989) 425-444

[Ro] Röhrle, G.: Parabolic subgroups of positive modality. Geom. Dedicata **60** (1996) 163-186

[Ta] Tanaka, N.: On the equivalence problems associated with simple graded Lie algebras. Hokkaido Math. J. **8** (1979) 23-84

[Ti] Tits, J.: *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen.* Lect. Notes Math. **40** (1967)

[Ts] Tsai, I.-H.: Rigidity of holomorphic maps from compact Hermitian symmetric spaces to smooth projective varieties. J. Alg. Geom. **2** (1993) 603-634

[Ya] Yamaguchi, K.: Differential systems associated with simple graded Lie algebras. Adv. Stud. Pure Math. **22** *Progress in differential geometry.* (1993) 413-494

Jun-Muk Hwang
Department of Mathematics
Seoul National University
Seoul 151-742, Korea
e-mail: jmhwang@math.snu.ac.kr

Ngaiming Mok
Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong
e-mail: nmok@hkucc.hku.hk