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THEOREM JUSTIFICATION AND ACQUISITION IN DYNAMIC GEOMETRY: A CASE OF PROOF BY CONTRADICTION

ABSTRACT. Theorem acquisition and deductive proof have always been core elements in the study and teaching of Euclidean geometry. The introduction of dynamic geometry environments, DGE (e.g. Cabri-Géomètre, Geometer's Sketchpad), into classrooms in the past decade has posed a challenge to this praxis. Students can experiment through different dragging modalities on geometrical objects that they construct, and consequently infer properties (generalities, theorems) about the geometrical artifacts. Because of the inductive nature of the DGE, the experimental-theoretical gap that exists in the acquisition and justification of geometrical knowledge becomes an important pedagogical and epistemological concern. In this paper, we will describe and study a 'Cabri proof by contradiction' of a theorem on cyclic quadrilaterals given by a pair of 16 year-old students in a Hong Kong secondary school. We will discuss how their construction motivates a visual-cognitive scheme on 'seeing' proof in DGE, and how this scheme could fit into the theoretical construct of cognitive unity of theorems proposed by Boero, Garuti and Mariotti (1996). The issue of a cognitive duality and its relation to visualization will be raised and discussed. Finally, we propose a possible perspective to bridge the experimental-theoretical gap in DGE by introducing the idea of a dynamic template as a visualizer to geometrical theorem justification and acquisition.

KEY WORDS: dynamic geometry, proof, justification, visualization.

INTRODUCTION

Theorem acquisition and proof have traditionally been at the heart of the study and teaching of Euclidean geometry. In our mathematical culture, Euclidean geometry has been regarded as a formal system and proofs in it are deductive in nature. A geometrical theorem or statement is justified by deducing it from known theorems and axioms in manners that are consistent with a formal axiomatic system. This has given rise to a tradition in which the teaching of Euclidean geometry is largely about teaching deductive reasoning by means of proving theorems. However, the introduction of dynamic geometry environments, DGE (e.g. Cabri-Géomètre, Geometer's Sketchpad), into the classrooms in the past decade has posed a challenge to this praxis (see, for example, Chazan, 1993; Hölzl, 1996; Noss & Hoyles, 1996; King & Schattschneider, 1997; Mariotti & Bartolini Bussi, 1998). In a dynamic geometry environment, teacher and students can experiment through different dragging modalities on geometrical objects that they have constructed, and consequently infer properties (generalities, theorems) about the geometrical artifacts. The ideal synthetic view of Euclidean geometry is approximated by a computer model that might consequently give rise to a different geometry and suggest new styles of reasoning:

"…dynamic geometry should not be treated as if it is merely a new interface to Euclidean construction. Line segments that stretch and points that move relative to each other are not trivially the same objects that one treats in the familiar synthetic geometry, and this suggests new styles of reasoning." (Goldenberg 1995, p.220)

On the one hand, this possible new geometry is shaped by anomalies due to 'computational transposition' (Balacheff 1993) from abstract idealism to concrete computational graphic images, while on the other hand it has evolved from pedagogical rationales.

"*Cabri's* drag-mode may be *axiomatically* neutral but certainly not *heuristically* neutral. Thus, dragging suggest new styles of consideration and reasoning which are in a way characteristic of *Cabri* geometry…..not in an axiomatic sense but in a didactic one." (Hölzl, 1996, p.177)

Drag-mode in dynamic geometry seems to be a kernel that is potent with rich didactic possibilities. Studies have been conducted to investigate the effect of different dragging behaviours of students. Hölzl observed that some students favored a "drag & link" strategy and did not "simply want to fix a solution but to create new knowledge" (Hölzl, 1996, p.182). Arzarello, Micheletti, Olivero and Robutti (1998) analysed the types of dragging strategy (wandering dragging, dragging test, lieu meut dragging) that students employed in arriving at correspondingly different conjectural statements for an open geometrical problem. Leung and Lopez-Real (2000) analysed students' Cabri solutions to a geometrical construction problem and proposed a dragging scheme that could have guided them through their Cabri exploration. This dragging scheme seemed to open up a 'zone of proximal solutions' (to paraphrase Vygotsky, 1978) between the students and the Cabri environment in which insight and understanding could be developed via open investigation and experimentation (Leung & Lopez-Real, 2000, p.150). Arzarello (2000) further commented on students' dragging 'tempos' (slow-fast) that seems to reflect a synchronization between visual perception and cognition, and hence suggested that "a conscious use of dragging…..can support the subject in the processes of generating generalities." (Arzarello, 2000, p.29)

When this empirical and inductive dimension is to be added to a pedagogical structure that is traditionally rooted in deductive logic, careful examination is needed on how to combine these two seemingly opposite perspectives: that is, to deal with acquisition and justification (proof) of geometrical knowledge in a pedagogical situation embedded in a DGE. The passage from 'intuitive' geometry to 'theoretical' geometry in the evolution of a justification in a proof is neither simple nor spontaneous. The possibility of modifying the system of relations among statements in geometrical knowledge mediated by DGE is entertained by educational researchers (see for example, Mariotti, 1997). Hoyles and Healy (1999) investigated how visual reasoning using Cabri, in particular through robust and soft construction, can motivate students to explain their empirical conjectures using formal proof. Their findings indicated that there is a disparity in students' perceptions between Cabri constructions and Euclidean formal proofs.

"… Cabri-Géomètre helps students in defining and identifying geometrical properties and the dependencies between them, but not in proving them…after starting on the writing of the proof, the computer interactions were suspended" (Healy, 2000, p.114).

A connection seems to be missing to bridge the empirical and the theoretical cognitive domains. This breach may be due to the traditional teaching emphasis on accepting something is true (T) only if it can be proved (P) (i.e. P$\Rightarrow$ T). Students might see the proof of a theorem (hence accepting the truth of it) as independent from exploratory activities in which the content of the theorem can be experimentally verified. However, deVilliers has argued that "in actual mathematical research, the forward implication (T$\Rightarrow$ P), ……..often plays a far greater role [in conjecturing and proving] in motivating and guiding our action" (de Villiers, 1997, p.20). It is the conviction that something is true that drives us to seek a proof. In DGE, we can easily be convinced of:

"the general validity of a conjecture by seeing its truth displayed while objects undergo continuous transformation across the screen [but] this provides no personally satisfactory *explanation* of *why it may be true*. … There is no insight or understanding into how it is the consequence of other familiar results" (de Villiers, 1997, p.22).

Boero, Garuti, Lemut and Mariotti (Garuti et al, 1996; Boero et al, 1996; Mariotti et al, 1997; Garuti et al, 1998; Boero et al, 1999) conducted a body of research into students' behaviour in the linkage between the process of producing conjectures (or generating conditionality of statements) and the process of proving theorems. In particular, Boero et al proposed a hypothesis on conjecture production as follows:

"the conditionality of the statement can (authors' emphasis) be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process, that will be subsequently detached from it and then "crystallize" from a logic point of view ("if….., then…..")" (Boero et al, 1996, p.121).

The transformation from detachment to crystallization seems to be a critical process that could bridge the intuitive-formal epistemological gap. Harel used the term "transformational proof scheme" to describe such a process when "students' justifications attend to the generality aspects of a conjecture and involve mental operations that are goal oriented and intended-anticipatory" (Harel, 1996, p.62). A theoretical construct called the cognitive unity of theorems was proposed (Garuti et al, 1996; Mariotti et al, 1997) as an attempt to fill the cognitive gap between empirical postulation and formal reasoning. It is expressed in the following terms:

"during the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of the previously produced arguments according to a logical chain." (Garuti et al, 1998, p.345)

In DGE, intensive argumentative activities involve intelligent interaction between students and a virtual microworld. Instead of visual activities in DGE that focus mainly on empirical verification (evidence), we should seek to design structured activities that may lead to formation of conjectures and have the potential to bring about insight and understanding. These structured conjecture-forming activities in DGE should generate an argumentative reasoning process, like that of Simon's (1996) "transformational reasoning".

"Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated." (Simon, 1996, p.201)

He goes on to suggest that:

"…..transformational reasoning is a natural inclination of the human learner who seeks to understand and to validate mathematical ideas. [It] involves envisioning the transformation of a mathematical situation and the results of that transformation. The affective consequence of transformational reasoning is often a sense *of understanding how it works*." (Simon, 1996, p.207)

DGE fits naturally with transformational reasoning because in it a figural continuum of geometrical states can be visually generated. This genre of reasoning mediates (or is a hybrid) between inductive and deductive reasoning and "may not only produce a different way of thinking about mathematical situations, it may also involve a different set of questions" (Simon, 1996, p.203). Furthermore, mathematical understanding may be achieved through a "realization of the appropriateness" (Simon, 1996, p.203) of a dynamic argumentative process (conducted through transformational reasoning) situated in a particular geometrical context. In DGE, the objects upon which transformational reasoning acts usually possess a dual nature. On the one hand they are spatial figures (drawings) depicted on the computer screen while on the other hand, when properly constructed, they represent ideal geometrical concepts that are formally constrained under an axiomatic system. This duality was the essence of Fischbien's theory of figural concepts:

"In this symbiosis between concept and figure, as it is revealed in geometrical entities, it is the image component which stimulates new directions of thought, but there are the logical, conceptual constraints which control the formal rigour of the process." (Fischbein, 1993, p.139)

Hence figural concepts are holistic cognitive structures that embrace the simultaneous interpretation of sensorial images (figural properties) and abstract symbolic constraints (conceptual properties) of geometrical entities. They "constitute only the ideal limit of a process of fusion and integration between the logical and the figural facets" (Fischbein, 1993, p.150).

In summary, we outline a possible framework on theorem acquisition and justification in DGE:

Theorem acquisition and justification in DGE is a schematic cognitive-visual dual process potent with structured conjecture-forming activities, in which dynamic visual explorations through different dragging modalities are applied on geometrical entities. These activities stimulate argumentative/transformational reasoning, which enables the process to converge towards integrated figural concepts that could bring about formal mathematical proofs, hence producing a cognitive unity in acquiring and proving geometrical theorems.

The focus of the above framework is on the epistemic (i.e. knowledge producing) process that brings about the security of an 'integrated knowledge', rather than the formulation of a 'rational proof'. Rodd (2000) distinguishes justification (formal proof) and warrant as "rationale for a belief" and "that which secures knowledge" respectively. She argued philosophically that "proof does not always warrant, and a warrant may be other than a proof". In particular, she discussed the issue of visualization as a mathematical warrant in the context of DGE. In this sense, our framework on theorem acquisition and justification in DGE is about a warrant on geometrical theorems. This warrant embodies what Rota called "the exchangeability of theorem and proof" (Rota, 1997, p.190). In brief, this refers to a common phenomenon in research mathematics, in which during the process of developing a proof for a particular theorem, new significant mathematical possibilities often arise, which sometimes even overshadow the original intended theorem. Rota proposed that:

"a rigorous version of the notion of possibility be added to the formal baggage of metamathematics….A realistic look at the development of mathematics shows that the reasons for a theorem are found only after digging deep and focusing upon the possibilities of the theorem." (Rota, 1997, p.191)

Hanna & Jahnke (1993) also called for a shift to a pragmatic view of proof in which meaning, new aspects of the theorems proved, and potentiality for future applications are emphasied rather than merely the logical deduction of a formal proof. In the light of this open approach to mathematics, the visual dynamic nature of DGE makes DGE an ideal laboratory to explore the richness of geometrical knowledge, in particular, the nature of geometrical theorems.

In this paper, we will first describe and analyse a case of "Cabri proof by contradiction" of a theorem on cyclic quadrilaterals given by a pair of 16 year-old students in a Hong Kong secondary school. We will then discuss how their construction motivated us to begin to put together a scheme for "seeing" proof by contradiction in DGE, and to discuss how this scheme could fit into our framework. The issue of a cognitive duality and its relation to visualization will be discussed, and we will propose a possible perspective to bridge the experimental-theoretical gap in DGE by introducing the idea of a dynamic template as a visualizer to geometrical theorem justification and acquisition. Finally, we will point out two directions of research implied by our proposed scheme.

A CASE STUDY

Background

Hilda and Jane were Form 4 (Grade 10) students in a band one secondary school in Hong Kong. (Hong Kong's secondary schools are streamed according to students' ability. A band one school is for the most able students.) They were introduced to deductive proof in Euclidean geometry in Form 3 and became quite proficient at it. Hilda and Jane's Form 4 mathematics teacher acquainted them with the Cabri computer environment and since then, they were treating it as part of their mathematics toolkit, using it to explore mathematics whenever they felt the need. We were researching students' problem solving strategies with and without the use of computers, and decided to run regular after-school problem-solving workshops for Form 3 and Form 4 students on a voluntary basis in Hilda and Jane's school. In the workshops, Cabri was introduced to students for the benefit of those who were not familiar with it, and subsequently students were asked to use it to solve some geometrical problems. Hilda and Jane joined the workshops, and they always worked together as a pair.

One of the most difficult problems that was set in the workshops was the following:

> *Let ABCD be a quadrilateral such that each pair of interior opposite angles adds up to 180°. Find a way to prove that ABCD must be a cyclic quadrilateral.*

This is the converse of a theorem with which students were familiar and which they had already proved in their normal coursework. They were also familiar with this converse statement, although no proof had been given in the textbook or by their teacher. The 'traditional' proof of this converse is by contradiction and it is for this reason that it is omitted in the textbook. In any case, it is important to note that the students had never experienced this type of proof.

Hilda and Jane's Proof

Hilda and Jane worked on the problem using Cabri and the diagram for their solution is shown in Figure 1 along with their written proof.

PROOF:

Assume that for a quadrilateral with each pair of interior opposite angles adding up to 180°, the four vertices can be on different circles.
From the diagram we see that it has a contradiction as the sum of the opposite angles of the blue quadrilateral (EBFD) is 360°, which is impossible.
Therefore, for a quadrilateral with each pair of interior opposite angles adding up to 180°, the four vertices must be on the same circle.

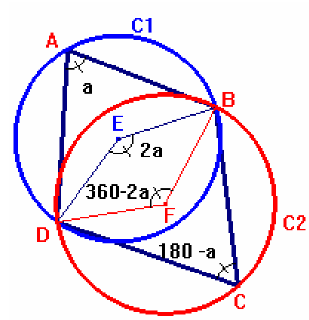


Figure 1

Hilda and Jane's Cabri proof

The Interview

We interviewed Hilda and Jane after the workshops. The following is an excerpt from the interview. (H is Hilda, J is Jane, I_1 and I_2 were the interviewers.)

1. I_1: Did you start with the quadrilateral or did you start with a circle?
2. J: To start we draw a quadrilateral.
3. I_1: So this quadrilateral is such that this angle plus this one makes 180, right? I can see you got 'a' marked here and (180 – a) here.
4. J: We assumed that the sum is 180.
5. I_1: When you say you assumed, in fact you drew any quad and then you just said we'll assume this is 180-a. OK. What did you do then?
6. H: Angle at centre is twice angle at circumference. We used this property to say that this angle is 2a and this one is 360-2a.
7. I_1: So before you did that presumably you first of all drew a circle through 3 of the points and then you did the same for these 3 points.
8. H: Yes.
9. I_1: So then you marked these 2 centres. What did you say after that?
10. H: Because the angle sum of a quadrilateral is 360 and these two (*referring to* $\angle E$ *and* $\angle F$) already add up to 360 so this is not possible.
11. I_2: So this is impossible. But do you think you have proved this?
12. H: Not yet.
13. I_2: Why not yet?

14. H: Mmm…. When this point and the whole circle is moved to … then it is not a quadrilateral …. This one for example, if we move it up to here.

15. I_1: Actually I'm interested in the assumptions that you're making. Because you started off by saying let's assume that this angle and this one make 180, and you didn't actually draw it like that. You drew any quadrilateral and you just marked these two adding up to 180.

16. J: If I draw a quadrilateral with these two angles is 180, then if I draw a circle it goes through the 4 points.

17. I_1: Exactly. So this was a problem right? Is that what you did to begin with? Did you try to draw a quadrilateral that did have the opposite angles supplementary?

18. J: No. We learned that theorem before. We knew already that it would give one circle so we didn't think about drawing it.

From Hilda and Jane's written proof, the interview and our observation notes of their interaction with Cabri, we attempt to reconstruct the process they went through to reach their conclusion. For what follows, H & J stands for Hilda and Jane.

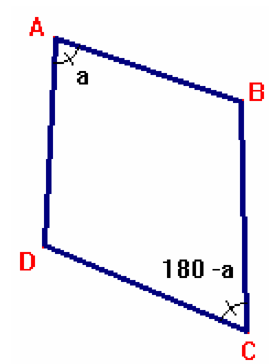


Figure 2

A wrongly labelled Cabri quadrilateral ABCD was drawn intentionally to force a visual perception that opposite angles add up to 180°.

Analysis of Hilda and Jane's Proof

H & J started their process of seeking a proof by insisting that any quadrilateral ABCD constructed in Cabri must satisfy the condition that each pair of opposite angles add up to 180°. They forced this assumption (presupposition) onto the Cabri world by marking the angles arbitrarily in a symbolic fashion (Figure 2). In other words, they deliberately drew a 'wrong' picture in Cabri and assumed that it is correct (lines 4, 5, 15). This tag-on labelling kept reminding them of the extra meaning that they gave to quadrilateral ABCD. In fact, this is simply equivalent to drawing a 'sketch' in paper-and-pencil geometry. We should note that their use of the word 'assume' in this context is therefore quite different to the use in their written proof. In the first line of the written proof, the stated assumption is precisely the kind of statement needed to initiate a proof by contradiction in the traditional sense; that is, as a starting point from which to investigate the *consequences* of the assumption. Returning to their labelling in the Cabri diagram, H & J thus conjured up a *biased Cabri world* that existed as a kind of hybrid between their visual cognition and the actual Cabri environment[1].

H & J's goal was to prove that such a quadrilateral ABCD must be cyclic. This goal motivated them to start to construct circles that would pass through the vertices of ABCD. H & J knew from prior knowledge that a unique circle could be drawn through any three given points. They chose to construct circles C1 (passing through A, B, D) with centre E and C2 (passing through B, D, C) with centre F, not expecting that circles C1 and C2 would coincide (lines 1-2, 7-9, see Figure 3). As a consequence of their forced presupposition, they observed, using a property of the circle that they were

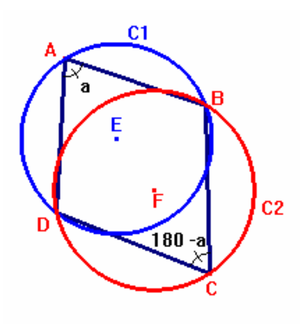


Figure 3

Construction of circles C1 and C2.

[1] It is interesting to note that in a questionnaire on the advantages and disadvantages of using computer software in geometry, H commented that one of the disadvantages of DGE is that it 'cannot draw a wrong picture'.

familiar with, that $\angle$DEB = 2a and $\angle DFB = 360^\circ - 2a$ (line 6). As before, they labelled these two angles symbolically (see Figure 4). In doing so, H & J literally 'saw' that the sum of a pair of opposite interior angles in the (convex) quadrilateral EBFD equals 360° (line 10). This, of course, contradicted the Euclidean property of a quadrilateral concerning the sum of its interior angles. H & J's *forced presupposition* resulted in an 'impossible Euclidean quadrilateral' EBFD. We call this quadrilateral EBFD a *pseudo-quadrilateral* in H & J 's *biased Cabri world*. At this point, H & J concluded that if their forced presupposition were to result in a true Euclidean configuration, then A, B, C and D must lie on the same circle because otherwise, an impossibility would occur (see their written proof). However, when they were asked to revive their experience in reaching this conclusion during the interview, they were not completely convinced that they had proved what they wanted (lines 11-14). We try to speculate about their worries and delve further into their biased Cabri world.

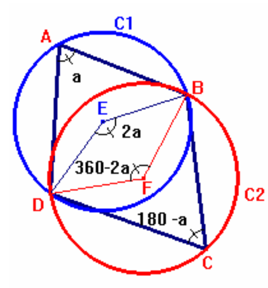


Figure 4
Construction of an "impossible" quadrilateral EBFD.

When C is being dragged sufficiently far inside the circle C1, the angle values 2a and $360^\circ - 2a$ correspond to the exterior angles (instead of interior angles as depicted in Figure 4) $\angle$DEB and $\angle$DFB respectively (see Figure 5). During the session when H & J were trying to come up with a Cabri proof, we recall that this scenario made them uncomfortable due to the sudden "flip" of the quadrilateral EBFD. This flip resulted in

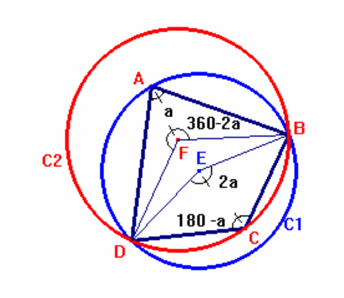


Figure 5

The "flipping" of quadrilateral EBFD.

the interior angles $\angle$DEB and $\angle$DFB, that give rise to the contradiction, becoming exterior angles, and the original convex quadrilateral EBFD became a re-entrant quadrilateral. It would not be difficult for H & J to figure out that the contradiction still holds in this case since:

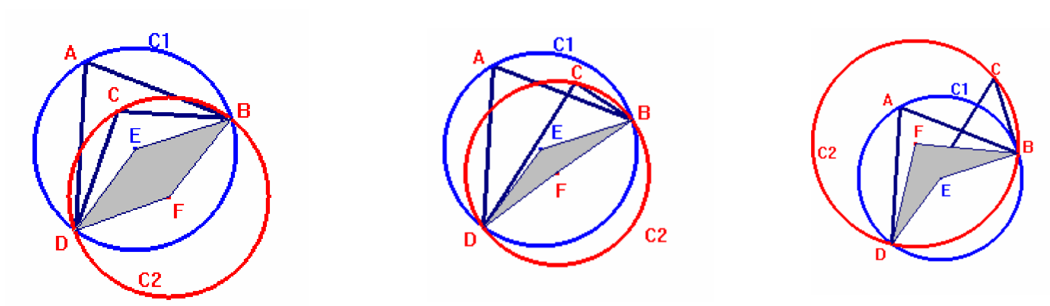
$$\text{interior } \angle \text{DEB} + \text{interior } \angle \text{DFB} = (360^\circ - 2a) + 2a = 360^\circ.$$

However, they did not consider this in their written proof. We speculate that this ‘flipping’ produced some kind of visual uncertainty (or even a conflict) for H & J, which prevented them being fully convinced of their conclusion. This could have disturbed H & J and led them to doubt the validity of their construction. If they had put the two situations depicted in Figure 4 and Figure 5 together, and commented on the contradictions that these configurations entailed, they would have produced a complete formal proof by contradiction of the theorem.

Conjecture-forming Activities

Even though their written proof was not a complete one, H & J’s construction essentially captured the “validity” of the theorem. In retrospect, they did not really need the Cabri environment to arrive at the proof that they had produced. However, the Cabri world did inspire them to construct the pseudo-quadrilateral EBFD that acted as a *visual guide*, helping them to structure their geometrical reasoning. We could hence regard DGE as a catalytic agent that visually promotes transformational reasoning. This would motivate argumentative (conjecture-forming) dragging activities that foster insight and understanding. In the following, we will suggest how such dragging activities could have taken place in H & J's situation.

Figure 5 was not the only scenario that contributed to H & J's uncertainty. H mentioned in the interview (Line 14) that when C is dragged to different positions, ABCD is no longer a quadrilateral and this prevented her being confident about her proof. However, it is exactly this aberration that could open up the situation from a specific consideration to a more general scenario.



Case 1 Case 2 Case 3

Figure 6

The above cases show how the shape of the pseudo-quadrilateral EBFD changes as C is being dragged to different positions.

In Figure 6, we shade the pseudo-quadrilateral EBFD and hide the marked angles to make the diagrams more suggestive in the following discussion. In Case 1, quadrilateral ABCD is not convex whereas in Case 2 and Case 3, we cannot form a quadrilateral with A, B, C, D in this order. These cases contributed further to the visual conflict (a type of cognitive conflict) that H & J experienced and that hindered them from finalizing their proof. The situation in Case 1 could easily be dealt with since upon careful calculation, the contradiction $\angle$ DEB = 2a and $\angle \text{DFB} = 360^{\circ} - 2\text{a}$ continues to hold. Moreover, visually it seems obvious that A, B, C and D cannot lie on the same circle since a circle is a convex object while the quadrilateral ABCD here is not. Case 2 and Case 3 posit a new configuration for A, B, C and D which is the result of folding a convex quadrilateral ABCD along the diagonal BD. If we insist that the pseudo-quadrilateral EBFD possesses the same contradiction (i.e. $\angle$ DEB + $\angle$ DFB = 360°) as before in these two cases, then the forced presupposition (assumption) for the biased Cabri world needs to be changed to $\angle$DAB = $\angle$DCB instead of $\angle$DAB + $\angle$DCB = 180°. In fact, this new configuration is actually validating another familiar Euclidean theorem concerning concyclic points. Therefore, even in Case 2 and Case 3, we are essentially still in H & J's biased Cabri world though the two forced presuppositions seem to be different on the surface. This dragging episode thus opens up intensive arguments on the plausible geometrical meanings that different positions of the dragged point C might entail. In particular, Case 2 and Case 3 together seem to suggest a different theorem (conjecture). Furthermore, the pseudo- quadrilateral EBFD plays an important role in organizing the cognitive-visual process that would eventually lead to the acquisition and justification of an integrated theorem. EBFD is a visual object that measures the degree of anomaly of the biased Cabri world with respect to the different positions of the vertices A, B, C and D. There are positions where the pseudo quadrilateral EBFD vanishes when a vertex of ABCD is being dragged. Figure 7 depicts a sequence of snapshots in a dragging episode when C is being dragged until EBFD vanishes. The last picture in the sequence shows that when C lies on the circumcircle C1 of A, B, and D, then E and F coincide. Furthermore, at this instance, $\angle \text{DEB} + \angle \text{DFB} = 360^{\circ}$ (which has been a contradiction arising from the

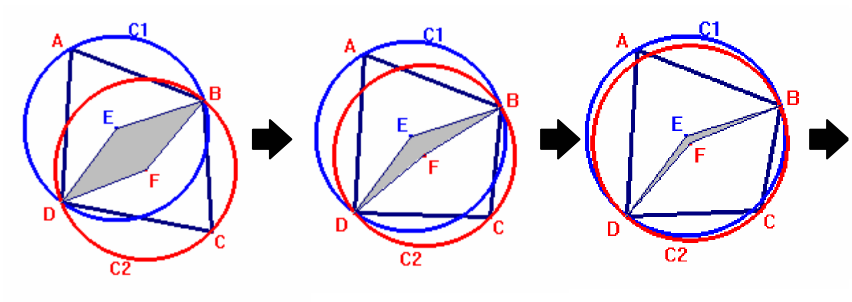


Figure 7

Figure 7 depicts a sequence of snapshots in a dragging episode when C is being dragged until the pseudo-quadrilateral EBFD vanishes.

pseudo- quadrilateral EBFD) is now a valid Euclidean statement. In fact, this condition holds only when C lies on C1. That is, when A, B, C and D are concyclic. We call C1 a *locus of validity* associated with the pseudo-quadrilateral EBFD. By this we mean the path that C traces out when it is being dragged intentionally to keep EBFD vanished, that is, maintaining the final configuration illustrated in Figure 7. This locus confines the types of configuration that A, B, C and D can assume under which the forced presupposition is valid.

It seems reasonable (in the sense of Simon's transformational reasoning) to conclude from the above dynamic visual process that if pairs of opposite interior angles of a quadrilateral add up to 180°, then the quadrilateral is cyclic. The drag-until-vanish activity described above is an example of an intensive argumentative activity in DGE, in which a dual process of conjecture-forming and justification seems to take place.

A SCHEME OF DYNAMIC THEOREM JUSTIFICATION AND ACQUISITION

In the above analysis, we saw that the content of a formal proof by contradiction was essentially captured in an episode of ‘moving pictures’ in the Cabri world, driven by H & J. To what extent do we accept this dynamic construction as a kind of ‘picture proof’? Does this episode of ‘moving pictures’ constitute a figural concept that is a hybrid between inductive and deductive thinking? After experiencing a strategic dragging episode, how possible is it that the structure of a formal proof can emerge from the dynamic variation of some inter-dependent constructions in the Cabri world? The pseudo-quadrilateral in the biased Cabri world that H & J constructed seems to be the key artifact that gave the insight to H & J to form their Reductio ad Absurdum proof. We also saw how the dynamic variation of this pseudo-quadrilateral via dragging captured those locations where the imposed condition is Euclidean valid, and in turn suggested the geometrical theorem that was aimed to be proved. Thus the dual role that this pseudo-quadrilateral plays might bring about the cognitive unity of a theorem bridging the empirical-theoretical gap between inductive acquisition and formal justification (in particular, proof by contradiction) of a geometrical statement in DGE. We try to schematize the cognitive-visual process that composes this cognitive unity of a theorem in which a proof by contradiction could be ‘visualized’. We will put in parentheses the ideas and Cabri objects, that Hilda and Jane used, corresponding to the constructs in our scheme.

Suppose A is some type of geometrical configuration, e.g. a quadrilateral, in DGE. We begin by assuming that A (the quadrilateral ABCD) satisfies a certain condition C(A) (interior opposite angles are supplementary) and impose it on all geometrical configurations of type A in DGE. This *forced presupposition* evokes a kind of 'mental labelling' (the arbitrary labelling of $\angle$DAB=2a and $\angle$DCB=180-2a) in our minds which *acts* on DGE. The forced presupposition makes an object of type A in DGE biased with extra meaning. This *biased DGE* exists as a kind of hybrid state between the *visual-true* DGE (a virtual representation of the Euclidean world) and a *pseudo-true* interpretation, C(A), insisted on by us. Depending on our disposition, an object depending on A can be constructed which inherits a local (i.e. depending on the location of the object) property that is not necessarily consistent with the Euclidean world because of C(A), hence resulting in a (local) contradiction. We call such an object associated with A in the biased DGE a *pseudo object* and denote it by O(A) (the quadrilateral EBFD). When part of A (the point C) is being dragged to different positions, O(A) might vanish (or degenerate, i.e., a plane figure to a line, a line to a point). The path or locus on which this happens gives a constraint under which the forced presupposition C(A) is Euclidean valid, i.e., where the biased microworld is being realized in the Euclidean world. We call this path the *locus of validity* of C(A) associated with O(A) (the circle C1). On the one hand, the locus of validity restricts A to certain special configurations resulting in A possessing a certain property which either we are trying to verify (prove) or are consequently led to discover (making conjecture). On the other hand, the essence of a formal proof by contradiction is encapsulated in the pseudo object O(A) and the locus of validity of C(A) associated with O(A) which convince us of the validity of a geometrical theorem. In either case, these artifacts are potent with insight and understanding that could lead to both formulation of formal proofs and acquisition of theorems, and hence serve as visual tools for argumentative activities in the cognitive unity of a theorem (Figure 8).

DISCUSSION

Duality and Visualization

We suggest from our analysis that an important feature in the above cognitive scheme is the critical dependence on the interaction between the person engaging in the mathematical task and DGE. In particular, the role of visualization is pivotal in the development of epistemic behaviour during the process that we described in the scheme. The construction of a pseudo object relies on the person's visual interpretation of configurations in the biased microworld. The determination of the locus of validity is the result of episodes of intelligent dragging. In the person's cognitive realm, there can exist a duality in interpreting the dynamic visual information simulated by the drag-mode. During any dragging episode, the boundary between exploring new geometrical situations and justifying a theorem is a blurred one. A pseudo object and its locus of validity could be instrumental in making a conjecture and proving the conjecture at the same time. The holistic nature of the dynamic visual representation in DGE allows variation in meaning when a DGE entity is observed (via dragging) from different points of view. Hence the dragging modality can be interpreted as a kind of "random access" to different cognitive modes (making conjecture, formulating proof) in the mind of the person who is interacting with DGE. This duality in interpretation in DGE is somewhat similar to Sfard's (1991) discussion on the dual nature of mathematical conceptions, that is, the complementary unity between operational (process, algorithm) conception and structural (abstract object) conception of a mathematical entity. It facilitates the acquisition of deeper insight into the task at hand that could lead to further generalization.

Argumentative Stages of a Proof by Contradiction in DGE

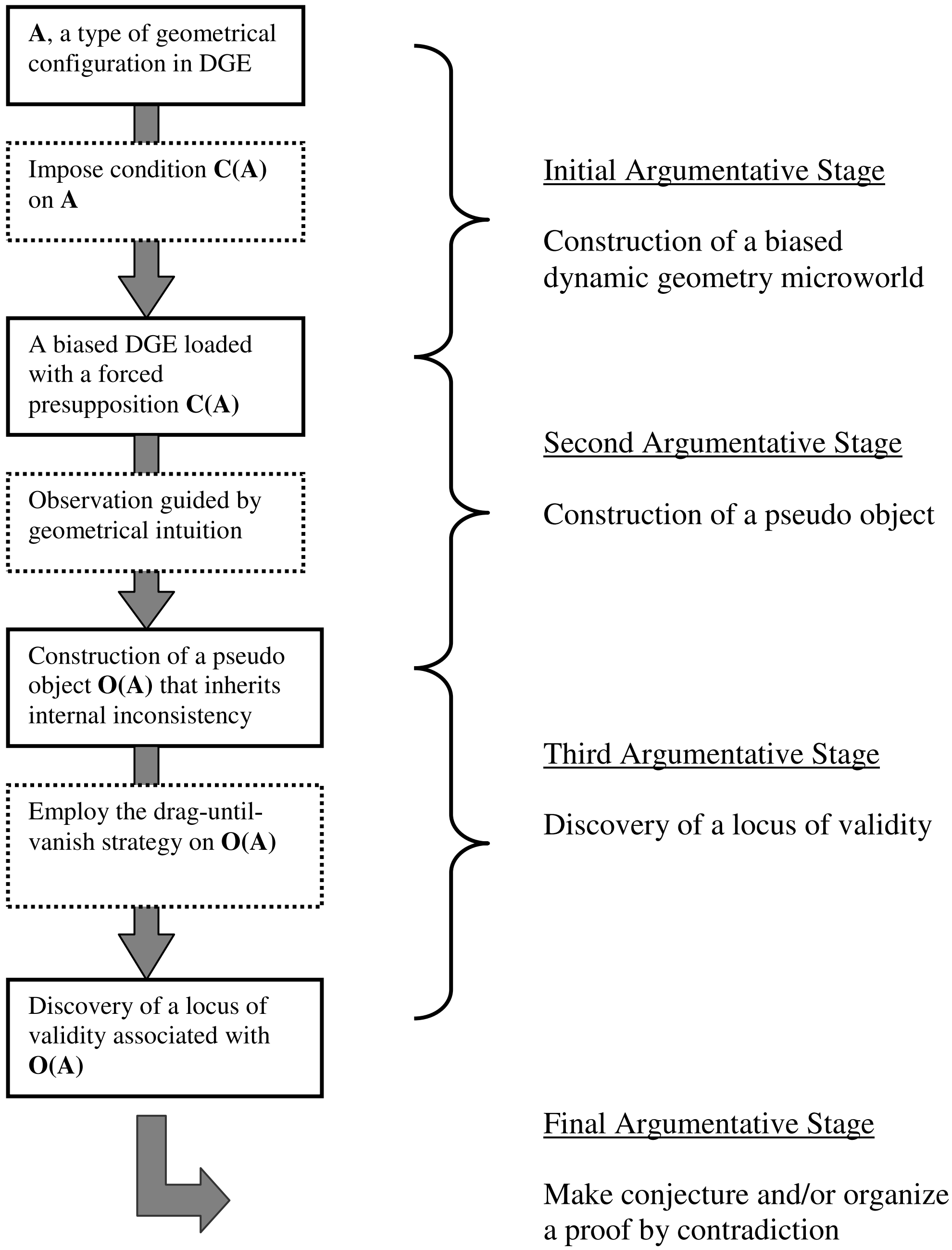


Figure 8

A schematic flow on how the cognitive unity of a theorem could be broken down into different stages of argumentative activities in the context of seeking a proof by contradiction.

This kind of interaction between a person and DGE thus creates a web of meaning in which "the learners can draw upon *and reconstruct* (authors' emphasis) for support – in ways that they choose as appropriate for their struggle to construct meaning for some mathematics." (Noss & Hoyles, 1996, p.108)

In light of the duality discussed above, instead of asking whether one can prove a geometrical theorem in DGE, perhaps the question should be how much *content value* of a traditional formal proof is being *carried* by the microworld in a particular construction-dragging episode. By content we mean here the potentiality of the *emergence* (this happens in the person's mind) of a formal proof structure. Appropriate and suggestive visual effects increase the magnitude of content value. For example, the drag-until-vanish Reductio ad Absurdum visualization procedure described in the scheme played an important role in realizing a proof. In this sense, the sequence of snapshots depicted in Figure 7 represents a potentiality that has a "high" content value. We might want to call such a visualization procedure a *potential proof* if our mind-set is on theorem justification. A traditional symbolic logical proof could then be an interpretation of such a visual potential proof. This opens up the question of equivalence between a visual potential proof in DGE and a formal Euclidean proof. To what extent do we accept a visual potential proof as a 'proof' in the traditional sense of the word? Is there a kind of visual logic system for DGE that is structurally isomorphic with a formal logical system on which the concept of a proof can be based? These are questions that deserve further investigation. Rather than addressing this problem of equivalence, let us recall the concept of visual theorems proposed by Davis:

“Briefly, a visual theorem is the graphical or visual output from a computer program – usually one of a family of such outputs – which the eye organizes into a coherent, identifiable whole and which is able to inspire mathematical questions of a traditional nature or which contributes in some way to our understanding or enrichment of some mathematical or real world situation.” (Davis, 1993, p.333)

“It [visual theorem] is the passage from the mathematical iteration to the perceived figure grasped and intuited in all its stateable and unstateable visual complexities.” (Davis, 1993, p.339)

This idea of visual theorem agrees quite nicely with our visualization scheme in DGE that plays the role of argumentative activity in the theory of cognitive unity of a theorem. Using Resnick’s (1998) idea of abstract mathematical objects as positions in patterns and templates as concrete representations of patterns, we can think of our visualization scheme as a *dynamic template* and the computer environment as a vehicle that carries it. A special position resulting from a dragging episode when this dynamic template is activated represents an abstract geometrical object (a figural concept) and possibly a theorem associated with it. With this interpretation we can then think of constructions (e.g. a pseudo object) and their associated dragging modalities as means to discover these positions. For example, in the case studied above, points on a locus of validity (circle in this case) are positions in which a forced presupposition on a quadrilateral is Euclidean valid. These positions specify the general pattern of a quadrilateral into those locations we call cyclic quadrilaterals (a mathematical object) and at the same time, verifying a theorem about such quadrilaterals. We could think of proof as *meta-pattern finding* among patterns. In DGE, these meta-patterns might emerge via intelligent construction and dragging by the engaging person. In H & J's Reductio ad Absurdum proof, the behaviour of the pseudo-quadrilateral can be regarded as a meta-pattern that essentially captured a proof of a theorem concerning quadrilaterals. In this sense, a dynamic template constructed by a person engaging in a mathematical task is a kind of *visualizer* of abstract geometrical objects (figural concepts). Interaction between the person and the DGE plays a key role in activating this visualizer to look for insight and understanding. Once a meta-pattern is recognized, it might then be possible to formulate a proof. We suggest that the idea of such a visualizer might be helpful to the understanding of proof in DGE and the construction of visualizers relevant to a geometrical problem may become a core pedagogical content of a dynamic geometry classroom.

Implications for Further Research

The cognitive-visual scheme on proof by contradiction in DGE that we have proposed was inspired by the work of a pair of students. In structuring it we have made a number of speculations on the cognitive processes that might have taken place in the students' minds, especially when the pair of students were not able to articulate their reasoning in logically precise fashion. Even though the theoretical constructs in the scheme appear to be able to capture the essence of a geometrical proof by contradiction in a coherent way and open up a possible new type of geometrical reasoning in DGE, we still have to address the mathematical validity of this hypothetical proposal. An implication on the nature of Euclidean geometry is the following question: What kind of theorem in Euclidean geometry can be written in the following form?

If C(A) is assumed, then an O(A) can always be constructed such that there exists a locus of validity associated with O(A) on which O(A) degenerates and C(A) holds true.

The existence of a locus of validity is the crucial determinant that fosters conjecture and justification. We do not know yet whether there is mathematical analysis that can guarantee its existence in a given geometrical situation. However, one thing we could certainly do is to apply the scheme for different geometrical theorems and test the feasibility of constructing these virtual geometrical objects in DGE. We hope that this line of research will produce more evidence on the validity of the proposed scheme.

One of the aims of our work is to contextualize the theory of cognitive unity of theorems in DGE. We studied a pair of students' behaviour on how they linked up their visual experience in DGE to construct a formal proof to a geometrical theorem. The hypothetical dynamic visualization model in DGE that we constructed to simulate their cognitive processes carries the idea of holistic figural concept with an inherited duality. This duality suggests a type of non-linear reasoning in contrast with the traditional linear deductive reasoning in a symbolic formal system: that is, depending on the disposition of the person interacting with DGE, a dragging activity can randomly access different facets of a geometrical scenario activating various logical modes (e.g. inductive, deductive). This opens up a new didactic discourse in the study of Euclidean geometry supported by

DGE. How should pedagogy be structured in order to nurture this type of non-linear *figural reasoning* that could bring about intensive argumentative activities in the cognitive unity of theorems in DGE? In the context of proof by contradiction, our proposed scheme suggests such a possible pedagogy. This work was motivated by a special case and there were only limited student data for us to work on. Further research in the applicability of the scheme in different didactic situations is needed to investigate the nature of this new discourse in DGE, and the extent to which this scheme can bridge the experimental-theoretical gap.

REFERENCES

Arzarello, F. (2000). Inside and Outside: Spaces, Times and Language in Proof Production. *Proceedings of PME 24: Psychology of Mathematics Education 24th International Conference*, 1 (pp.23-38). Hiroshima, Japan.

Arzarello, F., Micheletti, C., Olivero, F. & Robutti, O. (1998). Dragging in Cabri and Modalities of Transition from Conjectures to Proofs in Geometry. *Proceedings of PME 22: Psychology of Mathematics Education 22nd International Conference*, 2 (pp.32-39). Stellenbosch, South Africa.

Balacheff, N. (1993). Artifical intelligence and real teaching. In C. Keitel and K. Ruthven (Eds.), *Learning from Computers: Mathematics Education and Technology* (pp. 131-158). Berlin: Springer.

Boero, P., Garuti, R. & Mariotti, M. A. (1996). Some dynamic mental process underlying producing and proving conjectures. *Proceedings of PME 20: Psychology of Mathematics Education 20th International Conference*, 2 (pp.121-128). Valencia, Spain.

Boero, P., Garuti, R. & Lemut, E. (1999). About the generation of conditionality of statements and its links with proving. *Proceedings of PME 23: Psychology of Mathematics Education 23rd International Conference*, 2 (pp.137-144). Haifa, Israel.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24, 359-387.

Davis, P. J. (1993). Visual Theorems. *Educational Studies in Mathematics*, 24, 333-144.

de Villiers, M. (1997). The role of proof in investigative, computer-based geometry: Some personal reflection. In King, J. & Schattschneider, D. (Eds.) *Geometry Turned On: Dyamic software in Learning, Teaching, and Research* (*MAA Notes Series 41)* (pp.15-28). Washington. D.C.: Mathematical Association of America.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, 24, 139-162.

Garuti, R., Boero, P., Lemut, E. & Mariotti, M. A. (1996). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. *Proceedings of PME 20: Psychology of Mathematics Education 20th International Conference*, 2 (pp.121-128). Valencia, Spain.

Garuti, R., Boero, P., & Lemut, E. (1998). Cognitive unity of theorems and difficulty of proof. *Proceedings of PME 22: Psychology of Mathematics Education 22nd International Conference*, 2 (pp.345-352). Stellenbosch, South Africa.

Goldenberg, E. P. (1995). Ruminations about dynamic imagery (and a Strong Plea for Research). In R. Sutherland and J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematics Education* (pp. 202-224). Berlin: Springer.

Hanna, G. & Jahnke, H. N. (1993). Proof and application. *Educational Studies in Mathematics*, 24, 421-438.

Harel, G. (1996). Classifying processes of proving. *Proceedings of PME 22: Psychology of Mathematics Education 22nd International Conference*, 3 (pp.59-65). Stellenbosch, South Africa.

Healy, L. (2000). Identifying and explaining geometrical relationship: Interactions with robust and soft Cabri construction. *Proceedings of PME 24: Psychology of Mathematics Education 24th International Conference*, 1 (pp. 103-117). Hiroshima, Japan.

Hölzl, R. (1996). How does 'dragging' affect the learning of geometry. *International Journal of Computers for Mathematical Learning*, 1, 169-187.

Hoyles, C. & Healy, L. (1999). Linking informal argumentation with formal proof through computer-integrated teaching experiments. *Proceedings of PME 23: Psychology of Mathematics Education 23rd International Conference*, 3 (pp.105-112). Haifa, Israel.

King, J. & Schattschneider, D. (Eds.) (1997). *Geometry Turned On: Dynamic Software in Learning, Teaching, and Research* (*MAA Notes Series 41).* Washington. D.C.: Mathematical Association of America.

Leung, A. & Lopez-Real, F. (2000). An analysis of students' explorations and constructions using Cabri geometry. In Clements, M.A., Tairab, H., & Yoong, W.K. (Eds.*) Science, Mathematics and Technical Education in the 20th and 21st Centuries* (pp. 144-154). Universiti Brunei Darussalam.

Mariotti, M. A. (1997). Justifying and proving in geometry: the mediation of a microworld. In Hejny M. & Novotona J. (Eds.) *Proceedings of European Conference on Mathematical Education* (pp. 21-26). Prague: Prometheus Publishing House.

Mariotti, M. A. & Bartolini Bussi, M. G. (1998). From drawing to construction: teacher's mediation within the Cabri environment. *Proceedings of PME 22: Psychology of Mathematics Education 22nd International Conference*, 3 (pp.247-254). Stellenbosch, South Africa.

Mariotti, M. A., Bartolini Bussi, M., Boero, P., Ferri, F. & Garuti, R. (1997). Approaching geometry theorems in contexts: from history and epistemology to cognition. *Proceedings of PME 21: Psychology of Mathematics Education 21st International Conference*, 1 (pp.180-195). Lahti, Finland.

Noss, R. & Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrescht, The Netherlands: Kluwer Academic Publishers.

Resnick, M. D. (1998). Proof as a source of truth. In T. Tymoczko (Ed.), *New Directions in the Philosophy of Mathematics* (pp.317-336). New Jersey: Princeton.

Rodd, M. M. (2000). On mathematical warrants: Proof does not always warrant, and a warrant may be other than a proof. *Mathematical Thinking and Learning*, 2(3), 221-224.

Rota, G-C. (1997). The phenomenology of mathematical proof. *Synthese*, 111, 183-196.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Simon, M. (1996). Beyond inductive and deductive reasoning: the search for a sense of knowing, *Educational Studies in Mathematics*, 30, 197-210.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, Massachusetts: Harvard University Press.