NECESSARY AND SUFFICIENT CONDITIONS FOR STABLE H_∞ STABILIZABILITY *

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Abstract

In this paper, the stable H_∞ control problem is addressed. It is shown that the stable H_∞ control problem is reduced to finding a unimodular matrix such that a two block H_∞ optimization problem is solvable. And then it can be solved if an unimodular matrix is selected such that a Nehari problem is solvable. In fact, if the unimodular matrix is selected as a constant matrix, then the result of [10] is obtained.

1. Introduction

Recently, the H_∞ optimization theory has been extensively studied for a given linear time-invariant system, see [3, 5, 6, 11, 1] . However, it is well known that these design methods always yield an unstable controller, which is undesirable in practice. For example, even if an unstable controller is used for a stable plant, the system will become unstable when the feedback sensors fail, i.e., the system is open-loop. On the other hand, stabilization using an unstable controller always introduces additional right half-plane zeros into the closed-loop transfer function matrix beyond those of the original plant. As it is known that the right half-plane zeros of a system affect its stability to track reference signals and/or to reject disturbances. It is preferable to use a stable controller whenever possible.

It is well known that the strong stabilization problem is solvable if and only if the plant satisfies the parity interlacing property (PIP) condition [8] . For SISO systems, the strongly stabilizing controller can be constructed by Youla's interpolation approach [9, 8] . For the MIMO systems, Saif *et al.* [7] proposed an H_∞ optimization approach to construct the strongly stabilizing controller. In this paper, we will address the stable H_∞ controller design. We will prove that the stable H_∞ control problem can be reduced to finding an unimodular matrix such that a two-block H_∞ optimization problem is sovable and then it is solvable if a Nehari problem is solvable.

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2. Preliminaries

In the following, a transfer function $G(s)$ with realization (A, B, C, D) will be denoted by

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Lemma 1 [8, 11] *Suppose a plant G is stabilizable and detectable. Select constant matrices F and H such that the matrices $A_F = A+BF$ and $A_H = A+HC$ are both Hurwitz stable such that $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$, where*

$$\begin{bmatrix} M & X \\ N & Y \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & -H \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right],$$

$$\begin{bmatrix} \tilde{Y} & -\tilde{X} \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A+HC & -B+HD & -H \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]$$

satisfying

$$\begin{bmatrix} \tilde{Y} & -\tilde{X} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & X \\ N & Y \end{bmatrix} = I$$

Then all stabilizing controllers for G can be parameterized as

$$\begin{aligned} K(s) &= (X + MQ)(Y + NQ)^{-1} \\ &= (\tilde{Y} + Q\tilde{N})^{-1}(\tilde{X} + Q\tilde{M}) = \mathfrak{F}(J, Q) \end{aligned}$$

where $\mathfrak{F}(J, Q)$ denotes a linear fractional transformation (LFT) on J and Q, with

$$\begin{aligned} J &= \begin{bmatrix} XY^{-1} & \tilde{Y}^{-1} \\ Y^{-1} & -Y^{-1}N \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A+BF+HC+HDF & -H & B+HF \\ \hline F & 0 & I \\ -(C+DF) & I & -D \end{array}\right] \end{aligned}$$

and $Q(s) \in RH_\infty$ such that $(I + Y^{-1}NQ)(\infty)$ is invertible.

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Assume $Q(s)$ is given by

$$Q = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right],$$

then $K(s)$ can be written to

$$K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \hat{A} & (B+HD)C_Q & \hat{B}_2 \\ -B_Q(C+DF) & A_Q & B_Q \\ \hline F - D_Q(C+DF) & C_Q & D_Q \end{array}\right]$$

where

$$\begin{aligned} \hat{A} &= \tilde{A} - (B+HD)D_Q(C+DF) \\ \hat{B}_2 &= -H + (B+HD)D_Q \end{aligned}$$

Obviously, $K(s)$ is stable if and only if the matrix A_K is stable.

Lemma 2 [7] *If $V(s) \in RH_\infty$, $P(s)$ is unimodular in RH_∞ and $\|P^{-1}V\|_\infty < 1$, then $V(s) + P(s)$ is unimodular in RH_∞.*

Lemma 3 *For any proper transfer function $V(s) \in RH_\infty$, there always exists a unimodular transfer function $P(s)$ in RH_∞ such that $V(s)+P(s)$ is unimodular in RH_∞.*

Proof: Let

$$\gamma > \|V(s)\|_\infty, \quad V_1(s) = \gamma^{-1}V(s)$$

then $\|V_1(s)\|_\infty < 1$. So $V_1(s) + I$ is unimodular, i.e. $V(s) + P(s)$ is unimodular in RH_∞, where $P(s) = \gamma I$ ■

Lemma 4 *For any unimodular transfer function $T(s)$ in RH_∞, there always exists a factorization*

$$T(s) = V(s) + P(s)$$

where $V(s)$, $P(s) \in RH_\infty$ and $P(s)$ is unimodular in RH_∞ such that

$$\left\|P^{-1}V\right\|_\infty < 1$$

Proof: Select two scalars γ_0 and γ such that

$$\|T(s) + \gamma_0 I\|_\infty < \gamma \tag{1}$$

Obviously, $\|\gamma^{-1}(T(s) + \gamma_0 I)\|_\infty < 1$. Let $\gamma^{-1}(T(s) + \gamma_0 I) = P^{-1}V = P^{-1}(T - P)$. Then

$$P[T(s) + (\gamma_0 + \gamma)I] = \gamma T$$

If we select γ satisfying (1) such that $T(s) + (\gamma_0 + \gamma)I$ is unimodular, then a unimodular $P(s)$ can be constructed as

$$P(s) = \gamma T(T(s) + (\gamma_0 + \gamma)I)^{-1}$$

It is easy to find that the selection $\gamma > -\gamma_0 > 0$ can guarantee that $T(s) + (\gamma_0 + \gamma)I$ is unimodular. ■

3. Problem Statement

Consider a linear time-invariant system $G(s)$ with the following minimal realizations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t), \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector of $G(s)$, $u \in \mathbb{R}^m$ is the control vector, $w \in \mathbb{R}^p$ contains all the external input signals, such as disturbances, sensor noise, and commands, $z \in \mathbb{R}^q$ is the control output vector, and $y \in \mathbb{R}^q$ is the vector of the measured variables available for feedback control, all matrices in (2) are constant and with appropriate dimensions. Sometimes we denote $G(s)$ as

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

In this paper, without loss of generality, following the formulation in [3, 11] , we assume that the system $G(s)$ satisfies the following standard assumptions:

A1 (A, B_2) is stabilizable and (A, C_2) is detectable;

A2 D_{12} has full column rank and D_{21} has full row rank;

A3 $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

A4 $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has row column rank for all ω.

The *Stable H_∞ Control Problem* is defined as the design of a stabilizing controller

$$u = K(s)y \tag{3}$$

where $K(s) \in RH_\infty$, such that

$$\|T_{zw}\|_\infty < \gamma \tag{4}$$

with $\gamma > 0$ is the given H_∞ performance level, T_{zw} as the closed-loop transfer function from w to z subject to the feedback control (3). It is well known that T_{zw} can be described by a linear fractional transformation (LFT) on $G(s)$ and $K(s)$

$$T_{zw}(s) = \mathfrak{F}(G(s), K(s))$$

The standard H_∞ control problem (that is, when $K(s)$ is not restricted to be stable) is solvable if and only if two algebraic Riccati equations (AREs) have unique positive semidefinite solutions, and a spectral radius condition is satisfied. Then all stabilizing controllers satisfying $\|T_{zw}\|_\infty < \gamma$ can be parameterized as

$$K(s) = \mathfrak{F}(J(s), Q(s)), \quad Q(s) \in RH_\infty, \quad \|Q\|_\infty < \gamma \tag{5}$$

where $J(s)$ is of the form

$$J(s) = \begin{bmatrix} J_{11}(s) & J_{12}(s) \\ J_{21}(s) & J_{22}(s) \end{bmatrix} \tag{6}$$

$$= \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right] \tag{7}$$

such that $\hat{D}_{12}$ and $\hat{D}_{21}$ are invertible and $\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1$ and $\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$ are both stable, i.e., J_{12}^{-1}, $J_{21}^{-1} \in$

RH_∞.

From Youla's paramterization, (5) can be written as the following alternative representation [11]

$$\begin{aligned}\mathfrak{F}(J,Q) &= (X_J + M_J Q)(Y_J + N_J Q)^{-1} \\ &= (\tilde{Y}_J + Q\tilde{N}_J)^{-1}(\tilde{X}_J + Q\tilde{M}_J) \quad (8)\end{aligned}$$

where

$$\begin{bmatrix} M_J & X_J \\ N_J & Y_J \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 & \hat{B}_2 - \hat{B}_1\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{B}_1\hat{D}_{21}^{-1} \\ \hline \hat{C}_1 - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 & \hat{D}_{12} - \hat{D}_{11}\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{D}_{11}\hat{D}_{21}^{-1} \\ -\hat{D}_{21}^{-1}\hat{C}_2 & -\hat{D}_{21}^{-1}\hat{D}_{22} & \hat{D}_{21}^{-1} \end{array}\right],$$

$$\begin{bmatrix} \tilde{M}_J & \tilde{N}_J \\ \tilde{X}_J & \tilde{Y}_J \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1 & \hat{B}_1 - \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} & -\hat{B}_2\hat{D}_{12}^{-1} \\ \hline \hat{C}_2 - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{21} - \hat{D}_{22}\hat{D}_{12}^{-1}\hat{D}_{11} & -\hat{D}_{22}\hat{D}_{12}^{-1} \\ \hat{D}_{12}^{-1}\hat{C}_1 & \hat{D}_{12}^{-1}\hat{D}_{11} & \hat{D}_{12}^{-1} \end{array}\right]$$

Assume $Q(s)$ is given by

$$Q = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]$$

then $K(s)$ can be written as

$$K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \hat{A} + \hat{B}_2 D_Q \hat{C}_2 & \hat{B}_2 C_Q & \hat{B}_1 + \hat{B}_2 D_Q \hat{D}_{21} \\ B_Q \hat{C}2 & A_Q & B_Q \hat{D}_{21} \\ \hline \hat{C}_1 + \hat{D}_{12} D_Q \hat{C}_2 & \hat{D}_{12} C_Q & \hat{D}_{11} + \hat{D}_{12} D_Q \hat{D}_{21} \end{array}\right]$$

Obviously, $K(s) \in RH_\infty$ if and only if the matrix A_K is stable, that is to make $K(s)$ stable is equivalent to find a stable compensator $Q(s) \in RH_\infty$ to stabilize the associated plant $J_{22}(s)$.

Lemma 5 [10] *Given a linear time-invariant plant $G(s)$ satisfying assumptions (A1-A4) and the H_∞ performance index $\gamma > 0$, the set of all H_∞ controllers are parameterized as (5). Then there exists a stable H_∞ controller if and only if there exists a stable controller $Q(s) \in RH_\infty$ with $\|Q(s)\|_\infty < \gamma$ stabilizes the associated plant $J_{22}(s)$.*

The above lemma means that the stable H_∞ controller problem is reduced to a strong stabilization problem with the controller satisfying the H_∞-norm constraint. It is well known that the strong stabilization problem is solvable if and only if the plant satisfies the parity interlacing property (PIP) condition [8] . For SISO systems, the strongly stabilizing controller can be constructed by Youla's interpolation approach [9, 8] . For MIMO systems, Saif *et al.* [7] proposed a H_∞ optimization approach to construct the strongly stabilizing controller. In the following, we will propose a design method of the stable H_∞ controller design.

4. Main Results

Assume that $J(s)$ has been computed and that $J_{22}(s)$ is unstable. A necessary condition for the system to be strongly H_∞ stabilizable is that $J_{22}(s)$ satisfys the PIP condition. For any coprime factorization

$$J_{22}(s) = N_J M_J^{-1}$$

where $N_J(s)$, $M_J(s) \in RH_\infty$, there exists a $Q(s) \in RH_\infty$ such that $K(s) \in RH_\infty$ if and only if there exists a $Q(s) \in RH_\infty$ such that $M_J + QN_J$ is unimodular in RH_∞, i.e., $(M_J + QN_J)^{-1} \in RH_\infty$.

Define

$$U_J(s) = M_J(s) + QN_J(s)$$

Then there exists a factorization

$$U_J(s) = R(s) + \gamma_0 T(s)$$

such that $\|T^{-1}R\|_\infty < \gamma_0$, where $R(s), T(s) \in RH_\infty$, γ_0 is a constant and $T(s)$ is unimodular in RH_∞. So the stable H_∞ control problem is solvable if there exists a $Q(s) \in RH_\infty$ such that

$$\|T^{-1}(M_J + QN_J - \gamma_0 T)\|_\infty < \gamma_0, \quad \|Q(s)\|_\infty < \gamma$$

Let $\gamma_0 = \gamma \|T^{-1}\|_\infty$, the above condition is satisfied if

$$\|(M_J + QN_J - \gamma_0 T)\|_\infty < \gamma, \quad \|Q(s)\|_\infty < \gamma$$

It holds if

$$\begin{aligned} &\left\| \begin{bmatrix} M_J + QN_J - \gamma_0 T & Q \end{bmatrix} \right\|_\infty < \gamma \\ - \quad &\left\| \hat{M} + Q\hat{N} \right\|_\infty < \gamma, \quad Q(s) \in RH_\infty \end{aligned}$$

where $\hat{M} = \begin{bmatrix} M_J - \gamma_0 T & 0 \end{bmatrix}$, $\hat{N} = \begin{bmatrix} N_J & I \end{bmatrix}$. This means that the stable H_∞ control problem can be reduced to select a unimodular transfer matrix $T(s) \in RH_\infty$ and a transfer function $Q(s) \in RH_\infty$ such that

$$\left\| \hat{M} + Q\hat{N} \right\|_\infty < \gamma$$

Theorem 1 *The stable suboptimal H_∞ control problem is solvable if there exists a unimodular matrix $T(s)$ such that $\min_{Q \in RH_\infty} \left\| \hat{M} + Q\hat{N} \right\|_\infty < \gamma$.*

Remark 1 *After T is selected, it reduces to a one-side model matching problem and it can be solved using the inner-outer factorization approach* [2, 4] . *On the other hand, if we set $\gamma_0 = 1 + \varepsilon \geq \gamma$, $T = I$, then we can obtain a similar result of* [10] .

Let $\hat{N}$ have a co-inner/co-outer factorization as $\hat{N} = \hat{N}_o \hat{N}_i$, (see [4]), and define

$$\begin{aligned} M &= \begin{bmatrix} M_1 & M_2 \end{bmatrix} = \hat{M} \begin{bmatrix} \hat{N}_i^* & (\hat{N}_i^\perp)^* \end{bmatrix} \\ \hat{Q} &= Q\hat{N}_o \end{aligned}$$

where $\hat{N}_i^\perp$ is a complementary co-inner factor such that $\begin{bmatrix} \hat{N}_i \\ \hat{N}_i^\perp \end{bmatrix}$ is square and inner, then

$$\min_{Q \in RH_\infty} \left\| \hat{M} + Q\hat{N} \right\|_\infty = \min_{\hat{Q} \in RH_\infty} \left\| \begin{bmatrix} M_1 + \hat{Q} & M_2 \end{bmatrix} \right\|_\infty$$

which is a two-block H_∞ optimization problem. Define

$$\gamma_{\text{opt}} \triangleq \min_{\hat{Q}\in RH_\infty} \left\| \begin{bmatrix} M_1+\hat{Q} & M_2 \end{bmatrix} \right\|_\infty$$

γ_{opt} can be obtained using the "γ−iteration" approach [2, 4] . So for the stable H_∞ control problem, it is solvable if $\gamma \geq \gamma_{\text{opt}} \geq \|M_2\|_\infty$ [2, 4] . When $\gamma > \|M_2\|_\infty$, to find a $\hat{Q} \in RH_\infty$ such that

$$\left\| \begin{bmatrix} M_1+\hat{Q} & M_2 \end{bmatrix} \right\|_\infty < \gamma$$

if and only if

$$\left\| S^{-1}(M_1+\hat{Q}) \right\|_\infty < 1 \qquad (9)$$

where S is a spectral factor of the para-Hermitian matrix $(\gamma^2 I - M_2 M_2^*)$, i.e.,

$$SS^* = \gamma^2 I - M_2 M_2^*$$

Define

$$Y = S^{-1}M_1, \;\; X = S^{-1}\hat{Q}$$

we have following result.

Theorem 2 *The stable H_∞ control problem is solvable if there exists a unimodular matrix $T(s)$ such that $\|M_2\|_\infty < \gamma$ and*

$$\|Y+X\|_\infty < 1 \qquad (10)$$

where $X \in RH_\infty$.

To find a matrix $X \in RH_\infty$ such that (10) holds is a Nehari problem and it can be solved using the method in [4] .

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