Cutoff frequency of quasi-vector mode of optical waveguide with arbitrary refractive index profile

Kai-Ming Lo and E. Herbert Li

Department of Electrical and Electronic Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong

ABSTRACT

Based on the Galerkin's method, a numerical method is developed to analysis the cutoff frequencies of guided modes of optical waveguides with arbitrary refractive index profile. Solutions are presented in the quasi-vector regime. Optical waveguide structures with single core of arbitrary shape are considered in this paper. The calculated results are compared favorably with exact vector solution and circular-harmonic expansion method.

Keywords: moment methods, optical waveguide theory, optical propagation in nonhomogeneous media, optical polarization, optical waveguides, semiconductor waveguides, optical directional couplers

1. INTRODUCTION

For a complete study of guiding properties of optical waveguides, not only propagation constants and modal fields but also cutoff frequencies are needed. This is especially important for designing single mode devices. While the full-vector modal field solutions are most informative, the calculation is known to be tedious.[1] It is found more promising to use the quasi-vector solutions which including polarization correction and is essential for designing polarized waveguides.[2] In this paper, a numerical method is proposed to calculate the quasi-vector modal solutions of optical waveguide with arbitrary refractive index profile. While this paper is concentrated in the calculation of cutoff frequencies, the method developed here can calculate propagation constants and modal fields as well. The numerical method used is the Galerkin's method which transform the quasi-vector wave equation into an eigenvalue problem. The problem is readily solved using LAPACK subroutines.[3] Eigenvalues of the problem are cutoff frequencies of guided modes which can be used to determine the single mode operating region of an optical waveguide.

The use of Galerkin's method in solving the scalar wave equation for an optical waveguides with arbitrary refractive index profiles was first proposed by Henry and Verbeek.[4] Same method was used by Marcuse[5] in solving the full-vector wave equation. A mapping scheme was employed by Hewlett and Ladouceur[6] to eliminate the need of enclosing waveguide structures within a rectangle whose size affect the accuracy of calculations near modal cutoff. Alternately, that elimination can be done by using Hermite-Gauss functions[2] instead of sine functions[4–6] as basis functions. However, such elimination is only valid for waveguides with homogeneous cladding. For inhomogeneous cladding waveguides like rib waveguides, the cladding have to be truncated if Hermite-Gauss basis functions are used. On the other hand, no truncation of cladding is needed if sine basis functions are used in a mapped infinite domains.[6]

While the mapping scheme was first developed for the modal cutoff calculations of optical waveguides in the scalar regime,[6] it is employed in this paper to found the cutoff frequencies of quasi-vector modes of optical waveguide structures. In the next section, we will derive the quasi-vector wave equation and establish the details of solving this equation using the Galerkin's method. In Section 3, we compare numerical results of present method with those of other authors. The summary section anticipates the range of potential applications of present method.

Further author information –
K.M.L.(correspondence): Email: kmlo@eee.hku.hk; Telephone: (852) 2857 8487; Fax: (852) 2559 8738
E.H.L.: Email: ehli@eee.hku.hk; WWW: http://www-herbert.eee.hku.hk; Telephone: (852) 2859 7091

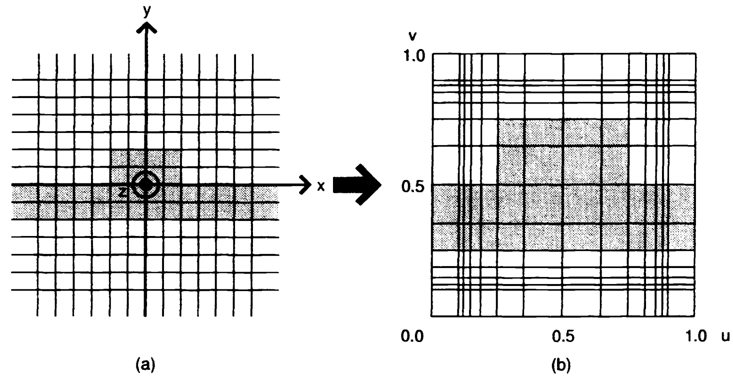

Figure 1. An optical structure in (a) x-y plane and (b) transformed u-v domains. A rib waveguide is used as an example.

2. MATHEMATICAL FORMULATION

2.1. Quasi-vector wave equation

Maxwell's equations are employed to calculate the spatial variation of electric field $\mathbf{E}(x,y,z)$ and magnetic field $\mathbf{H}(x,y,z)$ of an optical waveguide. The dielectric constant $\varepsilon(x,y,z)$ of a waveguide is related to its refractive index $n(x,y,z)$ by $\varepsilon = n^2\varepsilon_0$, where ε_0 is the free space electric permittivity. The magnetic permeability is taken to have its free space value ($\mu = \mu_0$) everywhere. The field vectors are taken to depend on time through the implicit factor $\exp(-i\omega t)$. Under these conditions and regions are free of charges and current, Maxwell's equations are written as

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{E} &= i(\mu_0/\varepsilon_0)^{1/2}k\mathbf{H} , & (1)\\ \boldsymbol{\nabla}\times\mathbf{H} &= -i(\varepsilon_0/\mu_0)^{1/2}kn^2\mathbf{E} & (2)\end{aligned}$$

where $k = 2\pi/\lambda$ is the free space wavenumber and λ is the wavelength of light in free space.

If we eliminate the magnetic field from (1) and (2) by $\boldsymbol{\nabla}\times(1)$ and substitute $\boldsymbol{\nabla}\times\mathbf{H}$ from (2) into, we obtain the vector wave equation

$$(\boldsymbol{\nabla}^2 + k^2n^2)\mathbf{E} = -\boldsymbol{\nabla}(\mathbf{E}\cdot\boldsymbol{\nabla}\ln n^2) \tag{3}$$

by using two vector identities

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \boldsymbol{\nabla}^2\mathbf{E} \tag{4}$$

and

$$\boldsymbol{\nabla}\cdot(n^2\mathbf{E}) = n^2\boldsymbol{\nabla}\cdot\mathbf{E} + \mathbf{E}\cdot\boldsymbol{\nabla}n^2 = 0 . \tag{5}$$

The $\boldsymbol{\nabla}^2$ in (3) is a vector operator. However, if the field vectors have components referred to fixed cartesian directions x, y and z as indicated in Fig. 1(a), the vector operator $\boldsymbol{\nabla}^2$ is replaced by the scalar Laplacian ∇^2. Moreover, if an optical waveguide with refractive index profile that does not change with distance z along the waveguide, i.e. $n = n(x,y)$. The electric field of the waveguide can be written in separable form as

$$\mathbf{E}(x,y,z) = \mathbf{e}(x,y)\exp(i\beta z) \tag{6}$$

where β is the propagation constant. Thus if we set

$$\mathbf{e} = e_x\hat{\mathbf{x}} + e_y\hat{\mathbf{y}} + e_z\hat{\mathbf{z}} \tag{7}$$

in (6) where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are unit vector parallel to the axes in Fig. 1(a) and using

$$\boldsymbol{\nabla}^2 = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2 , \tag{8}$$

(3) is reduced to two equations coupling the field components e_x and e_y as follows:

$$\frac{\partial^2 e_x}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2} + (k^2 n^2 - \beta^2) e_x + 2\frac{\partial}{\partial x}\left(e_x \frac{\partial \ln n}{\partial x} + e_y \frac{\partial \ln n}{\partial y}\right) = 0 , \tag{9}$$

$$\frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_y}{\partial y^2} + (k^2 n^2 - \beta^2) e_y + 2\frac{\partial}{\partial y}\left(e_x \frac{\partial \ln n}{\partial x} + e_y \frac{\partial \ln n}{\partial y}\right) = 0 . \tag{10}$$

If the coupling terms in (9) and (10) are neglected, we have

$$\frac{\partial^2 e_x}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2} + (k^2 n^2 - \beta^2) e_x + 2\frac{\partial}{\partial x}\left(e_x \frac{\partial \ln n}{\partial x}\right) = 0 , \tag{11}$$

$$\frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_y}{\partial y^2} + (k^2 n^2 - \beta^2) e_y + 2\frac{\partial}{\partial y}\left(e_y \frac{\partial \ln n}{\partial y}\right) = 0 . \tag{12}$$

These are in fact the scalar wave equation with polarization correction which are referred here as the quasi-TE wave equation and the quasi-TM wave equation.

2.2. Galerkin's method

The quasi-TE wave equation (11) will be solved below using the Galerkin's method. Since the formulation of the quasi-vector wave equation (11) and (12) are the same, the procedures developed here are applicable for solving the quasi-TM wave equation (12).

To eliminate the need of enclosing optical waveguide structures within a rectangle,[4,5] the whole x-y plane is mapped onto a unit square in u-v space as shown in Fig. 1 using the transformation functions[6]:

$$x = \alpha_x \tan\left[\pi\left(u - \frac{1}{2}\right)\right] , \tag{13}$$

$$y = \alpha_y \tan\left[\pi\left(v - \frac{1}{2}\right)\right] \tag{14}$$

where α_x and α_y are scaling parameters in the x and y directions respectively. The same change of variables is applied to the quasi-TE wave equation (11) and in the u-v space it is written as

$$\left(\frac{du}{dx}\right)^2 \frac{\partial^2 e_x}{\partial u^2} + \frac{d^2 u}{dx^2}\frac{\partial e_x}{\partial u} + \left(\frac{dv}{dy}\right)^2 \frac{\partial^2 e_x}{\partial v^2} + \frac{d^2 v}{dy^2}\frac{\partial e_x}{\partial v}$$
$$+ (k^2 n^2 - \beta^2) e_x + 2\left(\frac{du}{dx}\right)^2 \frac{\partial}{\partial u}\left(e_x \frac{\partial \ln n}{\partial u}\right) + 2\frac{d^2 u}{dx^2} e_x \frac{\partial \ln n}{\partial u} = 0 \tag{15}$$

where $e_x = e_x(u, v)$, $n = n(u, v)$. The unknown electric field component e_x is expanded as

$$e_x = \sum_{i}^{N_m N_n} a_i \phi_i(u, v) = \sum_{m_i=1}^{N_m} \sum_{n_i=1}^{N_n} a_{m_i, n_i} \phi_i(u, v) \tag{16}$$

where the index i and spatial frequencies m_i and n_i are related to each other through integer quotient function div and remainder on division function mod as follow:

$$m_i = (i - 1) \text{ div } N_n + 1 , \tag{17}$$

$$n_i = (i - 1) \text{ mod } N_n + 1 . \tag{18}$$

The expansion functions $\phi_i(u, v)$ are chosen as the complete set of orthonormal sine basis functions as

$$\phi_i(u, v) = 2 \sin(m_i \pi u) \sin(n_i \pi v) . \tag{19}$$

The field expansion (16) is substituted into (15), it is then multiplied by $\phi_j(u,v)$ and integrated over the unit square shown in Fig. 1(b) and yield the result:

$$\sum_i^{N_m N_n} (S_{j,i} + P_{j,i} - W^2 \delta_{j,i}) a_i = 0 \tag{20}$$

where

$$S_{j,i} = V^2 A_{j,i} + B_{j,i} \tag{21}$$

correspond to the scalar wave equation[6] with

$$A_{j,i} = \int_{u=0}^{1} \int_{v=0}^{1} g(u,v)\phi_i(u,v)\phi_j(u,v)\,du\,dv \ , \tag{22}$$

$$B_{j,i} = \rho^2 (I_1 + I_2 + I_3 + I_4) \ , \tag{23}$$

and

$$g(u,v) = \frac{n^2(u,v) - n_{\text{cl}}^2}{n_{\text{co}}^2 - n_{\text{cl}}^2} \ . \tag{24}$$

The delta function $\delta_{j,i}$ in (20) is defined as

$$\delta_{j,i} = \begin{cases} 1 & \text{if } j = i \ , \\ 0 & \text{if } j \neq i \ . \end{cases} \tag{25}$$

The waveguide parameter V and cladding parameter W are defined as follows[7]:

$$V = k\rho (n_{\text{co}}^2 - n_{\text{cl}}^2)^{1/2} \ , \tag{26}$$

$$W = \rho (\beta^2 - k^2 n_{\text{cl}}^2)^{1/2} \ . \tag{27}$$

The core and cladding refractive index, n_{co} and n_{cl}, and normalization parameter ρ are selected base on the refractive index profile of an optical waveguide structure. Moreover,

$$P_{j,i} = \rho^2 (I_5 + I_6) \tag{28}$$

correspond to the polarization correction. The six integrals I_1 to I_6 in (23) and (28) are defined in the Appendix as (31) to (36) and can be evaluated analytically in terms of sum of trigonometric functions if the refractive index profile $n(x,y)$ is approximated by rectangles of uniform refractive index.

2.2.1. Modal propagation constants

The double summation series in (20) can be written as a matrix eigenvalue equation, $\mathbf{M}\,\mathbf{a} = W^2\,\mathbf{a}$, by defining a vector $\mathbf{a}$ consisting of the elements a_i and a matrix $\mathbf{M}$ composed of the elements $S_{j,i} + P_{j,i}$. LAPACK subroutines are used to solve this equation,[3] the propagation constants of the bound modes of an optical waveguide are calculated from the real, positive eigenvalues W^2 and the corresponding modal field is calculated via the Fourier coefficients of associated eigenvectors $\mathbf{a}$.

2.2.2. Modal cutoff frequencies

For a given optical waveguide structure, its guided mode will become cutoff when the operating wavelength is longer than its cutoff wavelength λ_{co}. A cutoff value V_{co} is also defined as the waveguide parameter V at modal cutoff. During the cutoff of guided mode, its modal propagation constant $\beta = kn_{\text{cl}}$ or cladding parameter $W = 0$. Combining those cutoff conditions, (20) becomes

$$\sum_i^{N_m N_n} (V_{\text{co}}^2 A_{j,i} + B_{j,i} + P_{j,i}) a_i = 0 \ . \tag{29}$$

Same as in the last section, the double summation series in (29) can be written as a matrix eigenvalue equation, $\mathbf{M}\,\mathbf{a} = (1/V_{\text{co}})^2\,\mathbf{a}$, by defining a vector $\mathbf{a}$ consisting of the elements a_i and a matrix $\mathbf{M}$ composed of the elements $-(B_{j,i} + P_{j,i})^{-1} A_{j,i}$. LAPACK subroutines are used to solve this equation,[3] the cutoff wavelength or cutoff frequency $\nu_{\text{co}} = c/\lambda_{\text{co}}$, c is the velocity of light in free space, of the guided mode of a waveguide is calculated from the real, positive eigenvalue $(1/V_{\text{co}})^2$ and the corresponding modal field at cutoff is calculated via the Fourier coefficient of associated eigenvector $\mathbf{a}$.

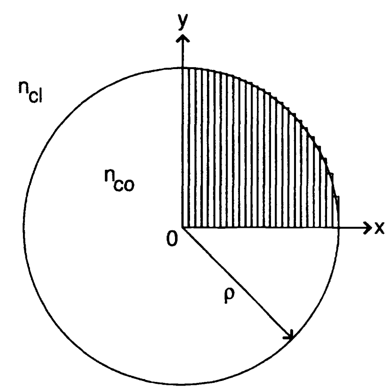

Figure 2. Structure of a circular core optical fiber. The core is represented by 25 rectangles.

Table 1. Cutoff values V_{co} of quasi-TE E_{21} mode for different core refractive index n_{co}. The calculation was done with $n_{cl} = 1$, $\alpha_x = \alpha_y = \rho$, and $N_m^o = N_n^e = N$. The parameters N_m^o and N_n^e indicate the number of odd and even spatial frequencies used in the x and y direction respectively.

n_{co}	N				Exact	%
	10	15	20	25	vector	error
1.01	2.4078	2.4072	2.4069	2.4069	2.4048	0.09
1.02	2.4111	2.4105	2.4106	2.4107	2.4048	0.25
1.05	2.4210	2.4208	2.4217	2.4224	2.4048	0.73

3. NUMERICAL RESULTS

3.1. Circular core optical fiber

Step-index circular core optical fibers are considered in this section. Figure 2 depicts its structure. The cutoff values $V_{co} = k_{co}\rho(n_{co}^2 - n_{cl}^2)^{1/2}$, $k_{co} = 2\pi/\lambda_{co}$, of the first higher quasi-TE E_{21} mode for different core refractive index is given in Table 1. Notice that the guided mode of an optical waveguide is denoted by the E_{mn} mode (m, n are both positive integers with $m-1$ and $n-1$ field zeros in the x and y directions, respectively.

From the table, V_{co} is converging as the number of spatial frequencies N are increased. By comparing with the exact vector solution,[7] the first zero of J_0—Bessel function of first kind of order zero, i.e. $V_{co} \approx 2.4048$, we found that the accuracy of present method is decrease as the core refractive index is increased. This is expected as the neglecting of minor field e_y (in quasi-TE case) is only valid for small difference in refractive index between core and cladding.[1] Nevertheless, percentage error of cutoff values are less than 1% for practical optical fibers with difference ($< 5\%$) in refractive index between core and cladding.

3.2. Elliptical core optical fiber

The next waveguide structure to be considered is an elliptical core optical fiber. Its structure is shown in Fig. 3. Table 2 gives the modified cutoff values $V_{co}^\star$ for different core refractive index n_{co} and core aspect ratio a/b. $V_{co}^\star$ is defined as

$$V_{co}^\star = V_{co}(a/b)^{1/2} \; ; \; V_{co} = k_{co}b(n_{co}^2 - n_{cl}^2)^{1/2} \; . \tag{30}$$

Both quasi-vector solutions (TE and TM) are given together with full-vector numerical results using circular-harmonic expansion method by Su.[8]

Among the quasi-TE modes, the larger the n_{co}, the larger the difference of $V_{co}^\star$ between present method and full-vector solutions. The explanation for those results have been given in the last section. On the other hand, the larger the aspect ratio a/b, the better the agreement of $V_{co}^\star$ between present quasi-vector solutions and full-vector

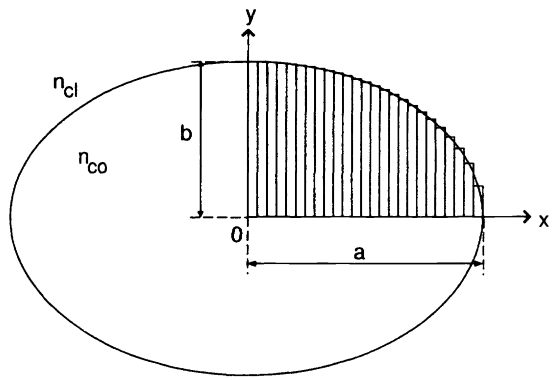

Figure 3. Structure of an elliptical core optical fiber. The core is represented by 25 rectangles.

Table 2. Modified cutoff values $V_{co}^{\star}$ of elliptical core optical fibers for different core refractive index n_{co} and core aspect ratio a/b. The calculation was done with $n_{cl} = 1$, $\alpha_x = a$; $\alpha_y = b$, and $N_m^o = N_n^e = 25$ for E_{21} mode; $N_m^e = N_n^o = 25$ for E_{12} mode; $N_m^e = N_n^e = 25$ for E_{31} mode; $N_m^o = N_n^o = 25$ for E_{22} mode. The parameters $N_m^{e,o}$ and $N_n^{e,o}$ indicate the number of odd and even spatial frequencies used in the x and y direction respectively.

Mode	n_{co}	$a/b = 1.5$		$a/b = 2.0$		$a/b = 1.5$		$a/b = 2.0$	
		TE	Su[8]	TE	Su[8]	TM	Su[8]	TM	Su[8]
E_{21}	1.001	2.192	2.193	2.076	2.084	2.192	2.193	2.076	2.084
	1.020	2.198	2.201	2.081	2.132	2.203	2.206	2.090	2.139
	1.500	2.353	2.294	2.206	2.241	2.454	2.389	2.405	2.399
E_{12}	1.001	2.686	2.686	2.936	2.935	2.686	2.686	2.937	2.935
	1.020	2.691	2.692	2.940	2.947	2.695	2.696	2.946	2.954
	1.500	2.800	2.896	3.015	3.090	2.939	2.976	3.212	3.233
E_{31}	1.001	3.383	3.351	3.120	3.104	3.383	3.351	3.120	3.104
	1.020	3.387	3.355	3.123	3.126	3.391	3.360	3.131	3.135
	1.500	3.504	3.408	3.223	3.127	3.554	3.483	3.363	3.300
E_{22}	1.001	3.899	3.898	4.026	3.976	3.899	3.898	4.027	3.976
	1.020	3.905	3.905	4.030	4.042	3.911	3.912	4.041	4.053
	1.500	4.044	4.043	4.126	4.116	4.221	4.183	4.414	4.379

solutions by Su.[8] This can be explained by the fact that the minor field (e_y in quasi-TE mode) is getting smaller as the aspect ratio is increased. In other words, the assumption of minor field equals zero in quasi-vector solution is more convincing for optical waveguide with an elongated or slab like cross section.[1] Similar features of the calculated results are seen for the quasi-TM modes.

3.3. Rectangular core optical waveguide

The last structure to be considered is optical waveguide with a rectangular core. Its structure is show in Fig. 4. Same as in the last section for elliptical core, the modified cutoff values $V_{co}^{\star}$ for different core refractive index n_{co} and core aspect ratio a/b are given in Table 3.

The calculated results carry the same trend as those in elliptical core. However, the agreement of present method with full-vector solution by Su[8] is more profound as the aspect ratio is increased when compared with results for elliptical core in last section. This is expected as rectangular cores have a more slab like cross section.

4. CONCLUSION

A numerical method for solving the problem of finding the cutoff frequencies of guided quasi-vector modes of optical waveguide with arbitrary refractive index is described. The problem was solved using the Galerkin's method in a

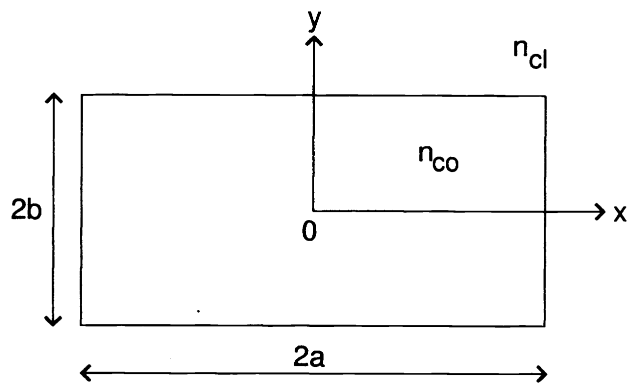

Figure 4. Structure of a rectangular core optical fiber.

Table 3. Modified cutoff values $V_{\rm co}^{\star}$ of rectangular core optical waveguide for different core refractive index $n_{\rm co}$ and core aspect ratio a/b. The calculation was done with $n_{\rm cl} = 1$, $\alpha_x = a$; $\alpha_y = b$, and $N_m^o = N_n^e = 25$ for E_{21} mode; $N_m^e = N_n^o = 25$ for E_{12} mode; $N_m^e = N_n^e = 25$ for E_{31} mode; $N_m^o = N_n^o = 25$ for E_{22} mode. The parameters $N_m^{e,o}$ and $N_n^{e,o}$ indicate the number of odd and even spatial frequencies used in the x and y direction respectively.

Mode	$n_{\rm co}$	$a/b = 1.0$		$a/b = 1.5$		$a/b = 2.0$		$a/b = 1.0$		$a/b = 1.5$		$a/b = 2.0$	
		TE	Su[8]	TE	Su[8]	TE	Su[8]	TM	Su[8]	TM	Su[8]	TM	Su[8]
E_{21}	1.001	2.136	2.137	1.929	1.929	1.816	1.849	2.137	2.137	1.929	1.929	1.816	1.849
	1.020	2.146	2.143	1.937	1.953	1.823	1.857	2.142	2.146	1.937	1.953	1.826	1.860
	1.500	2.370	2.097	2.125	2.026	1.983	1.988	2.257	2.175	2.113	2.094	2.054	2.096
E_{12}	1.001	2.137	2.137	2.413	2.408	2.660	2.704	2.137	2.137	2.414	2.408	2.660	2.704
	1.020	2.412	2.161	2.417	2.443	2.663	2.705	2.146	2.173	2.423	2.443	2.670	2.715
	1.500	2.257	2.327	2.457	2.519	2.711	2.745	2.370	2.592	2.679	2.754	2.943	2.987
E_{31}	1.001	3.383	3.325	3.042	3.011	2.793	2.936	3.383	3.325	3.042	3.011	2.794	2.836
	1.020	3.385	3.366	3.046	3.044	2.798	2.838	3.385	3.366	3.048	3.045	2.802	2.841
	1.500	3.421	3.425	3.161	3.032	2.916	2.838	3.421	3.425	3.149	3.098	2.966	2.917
E_{22}	1.001	3.199	3.196	3.263	3.220	3.387	3.471	3.199	3.196	3.263	3.220	3.388	3.471
	1.020	3.209	3.237	3.270	3.311	3.393	3.477	3.209	3.237	3.276	3.318	3.403	3.487
	1.500	3.434	3.286	3.422	3.422	3.503	3.594	3.434	3.286	3.582	3.648	3.762	3.855

mapped infinite domains, the mode field is expanded into a two dimensional Fourier sine series and resulting in a matrix eigenvalue equation which is solved using the LAPACK subroutines.

The accuracy of present method is compared with the circular-harmonic expansion method for elliptical core optical fibers and rectangular core optical waveguides. Moreover, solutions for step-index circular core optical fibers are compared with the exact vector solution. Results shown that the present quasi-vector solutions provide a good approximation of the full-vector solution.

The assumption used in cutoff frequencies calculation for quasi-vector modes i.e. $e_x \gg e_y$ in quasi-TE mode and $e_y \gg e_x$ in quasi-TM mode, is accurate for two classes of waveguides. First, optical waveguides with arbitrary core shape and small difference in refractive index between core and cladding. Second, arbitrary refractive index profile waveguides with an elongated or slab like cross section.

Finally, the proposed method is easy to implement but extremely useful in analyzing optical waveguides with arbitrary refractive index profile.

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APPENDIX

The integrals I_1 to I_6 in (23) and (28) are given as follows:

$$\begin{aligned} I_1 &= -m_i^2\pi^2 \int_{u=0}^{1}\int_{v=0}^{1} \left(\frac{du}{dx}\right)^2 \phi_i(u,v)\phi_j(u,v)dudv \\ &= -\frac{m_i^2}{2\alpha_x^2}\left\{\frac{3\delta_{m_i,m_j}}{4} - \frac{\delta_{m_i,m_j-2}}{2} - \frac{\delta_{m_i,m_j+2}}{2} + \frac{\delta_{m_i,2-m_j}}{2} + \frac{\delta_{m_i,m_j-4}}{8} + \frac{\delta_{m_i,m_j+4}}{8} - \frac{\delta_{m_i,4-m_j}}{8}\right\}\delta_{n_i,n_j}\ , \end{aligned} \tag{31}$$

$$\begin{aligned} I_2 &= m_i\pi \int_{u=0}^{1}\int_{v=0}^{1} \left(\frac{d^2u}{dx^2}\right) \frac{1}{\tan(m_i\pi u)}\phi_i(u,v)\phi_j(u,v)dudv \\ &= \frac{m_i}{\alpha_x^2}\left\{\frac{\delta_{m_i,2-m_j}}{4} + \frac{\delta_{m_i,m_j-2}}{4} - \frac{\delta_{m_i,m_j+2}}{4} - \frac{\delta_{m_i,4-m_j}}{8} - \frac{\delta_{m_i,m_j-4}}{8} + \frac{\delta_{m_i,m_j+4}}{8}\right\}\delta_{n_i,n_j}\ , \end{aligned} \tag{32}$$

$$\begin{aligned} I_3 &= -n_i^2\pi^2 \int_{u=0}^{1}\int_{v=0}^{1} \left(\frac{dv}{dy}\right)^2 \phi_i(u,v)\phi_j(u,v)dudv \\ &= -\frac{n_i^2}{2\alpha_y^2}\left\{\frac{3\delta_{n_i,n_j}}{4} - \frac{\delta_{n_i,n_j-2}}{2} - \frac{\delta_{n_i,n_j+2}}{2} + \frac{\delta_{n_i,2-n_j}}{2} + \frac{\delta_{n_i,n_j-4}}{8} + \frac{\delta_{n_i,n_j+4}}{8} - \frac{\delta_{n_i,4-n_j}}{8}\right\}\delta_{m_i,m_j}\ , \end{aligned} \tag{33}$$

$$\begin{aligned} I_4 &= n_i\pi \int_{u=0}^{1}\int_{v=0}^{1} \left(\frac{d^2v}{dy^2}\right) \frac{1}{\tan(n_i\pi v)}\phi_i(u,v)\phi_j(u,v)dudv \\ &= \frac{n_i}{\alpha_y^2}\left\{\frac{\delta_{n_i,2-n_j}}{4} + \frac{\delta_{n_i,n_j-2}}{4} - \frac{\delta_{n_i,n_j+2}}{4} - \frac{\delta_{n_i,4-n_j}}{8} - \frac{\delta_{n_i,n_j-4}}{8} + \frac{\delta_{n_i,n_j+4}}{8}\right\}\delta_{m_i,m_j}\ , \end{aligned} \tag{34}$$

$$\begin{aligned} I_5 &= m_i\pi \int_{u=0}^{1}\int_{v=0}^{1} 2\left(\frac{du}{dx}\right)^2 \frac{1}{\tan(m_i\pi u)}\phi_i(u,v)\phi_j(u,v)\frac{\partial \ln(n)}{\partial u}dudv \\ &\quad + \int_{u=0}^{1}\int_{v=0}^{1} 2\left(\frac{du}{dx}\right)^2 \phi_i(u,v)\phi_j(u,v)\frac{\partial^2 \ln(n)}{\partial u^2}dudv \\ &= \frac{1}{\alpha_x^2}\int_{u=0}^{1} du \int_{v=0}^{1} dv\, \{\ln(n)\ \{\{16[c(2u)-c(4u)] - m_j^2[c(4u)-4c(2u)+3]s_i(u)s_j(u)s_i(v)s_j(v)\} \\ &\quad + m_im_j[c(4u)-4c(2u)+3]c_i(u)c_j(u)s_i(v)s_j(v) - 4m_i[s(4u)-2s(2u)]c_i(u)s_j(u)s_i(v)s_j(v) \\ &\quad - 8m_j[s(4u)-2s(2u)]s_i(u)c_j(u)s_i(v)s_j(v)\}\}\ , \end{aligned} \tag{35}$$

$$\begin{aligned} I_6 &= \int_{u=0}^{1}\int_{v=0}^{1} 2\left(\frac{d^2u}{dx^2}\right) \phi_i(u,v)\phi_j(u,v)\frac{\partial \ln(n)}{\partial u}dudv \\ &= \frac{2}{\alpha_x^2}\int_{u=0}^{1} du \int_{v=0}^{1} dv\, \{\ln(n)\{4[c(4u)-c(2u)]s_i(u)s_j(u)s_i(v)s_j(v) + m_i[s(4u)-2s(2u)]c_i(u)s_j(u)s_i(v)s_j(v) \\ &\quad + m_j[s(4u)-2s(2u)]s_i(u)c_j(u)s_i(v)s_j(v)\}\}\ . \end{aligned} \tag{36}$$

In (35) and (36), sine related functions $s(2u) = \sin(2u)$, $s_i(u) = \sin(m_i\pi u)$, $s_i(v) = \sin(n_i\pi v)$, etc. For cosine related functions, they are abbreviated by the symbol c.

REFERENCES

1. K. S. Chiang, "Review of numerical and approximate methods for the modal analysis of general optical dielectric waveguides," *Opt. Quantum Electron.* **26**, pp. S113–S134, 1994.
2. A. Weisshaar, J. Li, R. L. Gallawa, and I. C. Goyal, "Vector and quasi-vector solutions for optical waveguide modes using efficient Galerkin's method with Hermite-Gauss basis functions," *J. Lightwave Technol.* **13**, pp. 1795–1800, 1995.

3. E. Anderson, Z. Bai, C. Bischof, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, and D. Sorensen, *LAPACK Users's Guide*, S.I.A.M., Philadelphia, 2nd. ed., 1995.
4. C. H. Henry and B. H. Verbeek, "Solution of the scalar wave equation for arbitrarily shaped dielectric waveguides by two-dimensional Fourier analysis," *J. Lightwave Technol.* **7**, pp. 308–313, 1989.
5. D. Marcuse, "Solution of the vector wave equation for general dielectric waveguides by the Galerkin method," *IEEE J. Quantum Electron.* **28**, pp. 459–465, 1992.
6. S. J. Hewlett and F. Ladouceur, "Fourier decomposition method applied to mapped infinite domains: scalar analysis of dielectric waveguides down to modal cutoff," *J. Lightwave Technol.* **13**, pp. 375–383, 1995.
7. A. W. Snyder and J. D. Love, *Optical Waveguide Theory*, Chapman and Hall, London, 1983.
8. C. C. Su, "Cutoff frequency of a homogeneous optical fiber with arbitrary cross section," *IEEE Trans. Microwave Theory Tech.* **33**, pp. 1101–1105, 1985.