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RESAMPLING METHODS FOR HOMOGENEITY TESTS OF COVARIANCE MATRICES

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Abstract: Testing hypotheses on covariance matrices has long been of interest in statistics. The test of homogeneity is very often a preliminary step in discriminant analysis, cluster analysis, MANOVA, etc. In this article we propose non-parametric tests which are based on the eigenvalues of the differences among the sample covariance matrices after a common rescaling. Three resampling techniques for calculating p-values are shown to be asymptotically valid: bootstrap, random symmetrization and permutation. Monte Carlo simulations show that the bootstrap performs less satisfactorily than the others in adhering to the nominal level of significance. Some theoretic ground for this phenomenon is given. The simulation results also suggest that the homogeneity tests proposed in this article performs better than the bootstrap version of Bartlett's test.

Key words and phrases: Bartlett homogeneity test, bootstrap, non-parametric tests, permutation test, random symmetrization.

1. Introduction

Under a multinormality assumption, hypotheses testing for homogeneity in the k-sample problem can be handled by the likelihood ratio test (LRT). The exact distribution of the LRT is very complicated. When the sample size is sufficiently large, one usually employs the chi-square distribution, the limiting null distribution, for the LRT. Box (1949) obtained a correction factor for Bartlett's LRT and proposed his M statistic with the same chi-square distribution for testing homogeneity in the k-sample problem.

Without the multinormality assumption, likelihood ratios would be different. If one still uses the statistics obtained under normality then, for example, the asymptotic null distribution for Bartlett's homogeneity test is no longer chi-square, but a linear combination of chi-squares as pointed out by Zhang and Boos (1992). The lack of correct null distribution for traditional statistics forces researchers to look for other means of implementing the tests. Resampling techniques such as the bootstrap represent one resolution. Beran and Srivastava (1985) considered bootstrap implementation of tests based on functions of eigenvalues of a covariance matrix in a one-sample problem. Zhang, Pantula and

Boos (1991) proposed a pooled bootstrap methodology. For the k-sample problem, Zhang and Boos (1992, 1993) studied bootstrap procedures to obtain the asymptotic critical values for Bartlett's statistic for homogeneity without the multinormality assumption. Among other things, Zhang and Boos (1993) developed bootstrap theory for quadratic-type statistics and demonstrated the idea using Bartlett's test as an example.

An alternative approach to constructing multivariate tests is Roy's (1953) union-intersection principle. One uses the fact that a random vector is multivariate normal if and only if every non-zero linear function of its elements is univariate normal. This leads to viewing the multivariate hypothesis as the joint statement (intersection) of univariate hypotheses of all linear functions of univariate components, and a joint rejection region consisting of the union of all corresponding univariate rejection regions if they are available. The two-sample Roy test is in terms of the largest and smallest eigenvalues of one Wishart matrix in the metric of the other. But, so far, there is no Roy test for the problem of more than two samples. One reason may be the difficulty of extending the idea of comparison of variances in terms of ratio to more than two samples. We briefly describe the difficulty. In a two-sample case, we may use either σ_1^2/σ_2^2 or σ_2^2/σ_1^2, as they are the reciprocal. It is not so simple otherwise. If we want an aggregate statistic of pairwise ratios, one way is to sum up σ_i^2/σ_j^2, $1 \le i \ne j \le k$. In case we sum up the ratios over all $i < j$, as the ratios are not permutation invariant with i and j, we may obtain conflicting conclusion if we use the sum of the ratios over $j > i$ as a test statistic. Furthermore if we sum up all ratios over $i \ne j$, although it will be invariant with i and j,there is some confounding. This can be demonstrated for $k = 2$ with the statistic $\sigma_1^2/\sigma_2^2 + \sigma_2^2/\sigma_1^2$. When the first ratio is large, the second will be small, the average will be moderate and vice versa. It is similar in the general case. However, the absolute values of the differences $(\sigma_i^2 - \sigma_j^2)/(\sigma_1^2 + \cdots + \sigma_k^2)$, $1 \le i \ne j \le k$, are invariant with respect to i and j. The sum of the absolute values over $i < j$ can be used as a test statistic without a confounding effect, and so can the maximum of those absolute values. In this article, we consider both the maximum and the sum (average), and find in simulations that the sum test statistic works better.

Without multinormality and without reference to the likelihood ratio or union intersection principle, we obtain homogeneity tests for more than two samples based on eigenvalues of differences of the sample covariance matrices subject to a common re-scaling. The asymptotic distributions of the test statistics are identified. We also consider the validity of some resampling techniques, namely the bootstrap, random symmetrization and permutation procedures, for calculating the critical values and p-values for these tests. All of the techniques are asymptotically valid for the problem. There is theory supporting the conclusion that permutation procedures and random symmetrization perform better than the bootstrap in adhering to the nominal level of significance in some cases. Our Monte Carlo studies indicate that the permutation test generally has higher power than the bootstrap test and that random symmetrization is compatible to the bootstrap in power performance. Random symmetrization, if applicable, is easy to implement. Simulation results also suggest that the test proposed here is better than the bootstrapped Bartlett test studied by Zhang and Boos (1992).

The article is organized as follows: the construction of tests is in Section 2 and the resampling approximations are presented in Section 3. Proofs are postponed to the Appendix.

2. Construction of Tests

Let $\boldsymbol{X}_1^{(i)}, \boldsymbol{X}_2^{(i)}, \ldots, \boldsymbol{X}_{m_i}^{(i)}$, $i = 1, \ldots, k$, be an iid sample from a d-dimensional distribution with finite fourth moments, mean $\boldsymbol{\mu}^{(i)}$ and covariance matrix $\boldsymbol{\Sigma}^{(i)}$. We are interested in the homogeneity hypothesis

$$H_0 : \boldsymbol{\Sigma}^{(1)} = \boldsymbol{\Sigma}^{(2)} = \cdots = \boldsymbol{\Sigma}^{(k)} \text{ vs } H_1 : \boldsymbol{\Sigma}^{(i)} \neq \boldsymbol{\Sigma}^{(j)} \text{for some } i \neq j \ . \tag{2.1}$$

Denote the sample covariance matrix for the ith sample by

$$\hat{\boldsymbol{\Sigma}}^{(i)} = \frac{1}{m_i} \sum_{j=1}^{m_i} (\boldsymbol{X}_j^{(i)} - \hat{\boldsymbol{\mu}}^{(i)})(\boldsymbol{X}_j^{(i)} - \hat{\boldsymbol{\mu}}^{(i)})^T \ , \tag{2.2}$$

where $\hat{\boldsymbol{\mu}}^{(i)}$ is either $\boldsymbol{\mu}^{(i)}$ or the sample mean, depending on whether $\boldsymbol{\mu}^{(i)}$ is known or not. The pooled sample covariance matrix is

$$\hat{\boldsymbol{\Sigma}} = 1/N \sum_{i=1}^{k} m_i \hat{\boldsymbol{\Sigma}}^{(i)} \ , \quad \text{where} \quad N = \sum_{i=1}^{k} m_i \ . \tag{2.3}$$

Based on the idea of multiple comparison, (e.g., see Dunnett (1994), O'Brien (1979, 1981)), we construct tests by combining pairwise comparisons. The pairwise comparison between the lth and ith samples is based on

$$\begin{aligned} M_{li} &= \max\left\{\text{absolute eigenvalues of } \sqrt{\frac{m_l m_i}{N}} \hat{\boldsymbol{\Sigma}}^{-1/2} (\hat{\boldsymbol{\Sigma}}^{(l)} - \hat{\boldsymbol{\Sigma}}^{(i)}) \hat{\boldsymbol{\Sigma}}^{-1/2}\right\}, \\ A_{li} &= \text{average } \left\{\text{absolute eigenvalues of } \sqrt{\frac{m_l m_i}{N}} \hat{\boldsymbol{\Sigma}}^{-1/2} (\hat{\boldsymbol{\Sigma}}^{(l)} - \hat{\boldsymbol{\Sigma}}^{(i)}) \hat{\boldsymbol{\Sigma}}^{-1/2}\right\}. \end{aligned} \tag{2.4}$$

We propose using the average of the $k(k-1)/2$ pairwise comparisons as the test statistic,

$$LM = \frac{2}{k(k-1)} \sum_{i<l} M_{li}, \tag{2.5a}$$

$$LA = \frac{2}{k(k-1)} \sum_{i<l} A_{li} \ . \tag{2.5b}$$

The null hypothesis is rejected if LM (LA) is greater than the critical value which is to be determined. We first identify the limiting distribution of L in the following lemma.

To state results, we need some notation for vectorization of a symmetric matrix. For a symmetric $d \times d$ matrix $\boldsymbol{S}$, let vech($\boldsymbol{S}$) be the column vector obtained by stacking up the $d(d+1)/2$ distinct elements of $\boldsymbol{S}$ in the order of the first column vector, then the second column vector omitting the first element, etc.

Lemma 2.1. *Assume $m_i/N \to \lambda_i$, $0 < \lambda_i < 1$, as $m_i \to \infty$ for $i = 1, \ldots, k$, and that the distributions of samples are continuous and have finite fourth moments. Under* (3.1)*, the asymptotic joint distribution of $\sqrt{m_l m_i/N}\hat{\boldsymbol{\Sigma}}^{-1/2}(\hat{\boldsymbol{\Sigma}}^{(l)} - \hat{\boldsymbol{\Sigma}}^{(i)})\hat{\boldsymbol{\Sigma}}^{-1/2}$, $1 \le i, l \le k$, is identical with the asymptotic joint distribution of $\sqrt{\lambda_i}\boldsymbol{W}_l - \sqrt{\lambda_l}\boldsymbol{W}_i$, $1 \le i, l \le k$, where $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_k$ are independent and vech($\boldsymbol{W}_i$) is multivariate normal with zero mean vector and covariance matrix*

$$\boldsymbol{V}_i = COV(vech((\boldsymbol{X}_1^{(i)} - \boldsymbol{\mu}^{(i)})(\boldsymbol{X}_1^{(i)} - \boldsymbol{\mu}^{(i)})^T)) \ . \tag{2.6}$$

Furthermore, the asymptotic distributions of LM and LA are, respectively, the distributions of the random variables

$$\tfrac{2}{k(k-1)} \textstyle\sum_{i<l} \max\{\textit{absolute eigenvalues of } \sqrt{\lambda_i}\boldsymbol{W}_l - \sqrt{\lambda_l}\boldsymbol{W}_i\}, \tag{2.7a}$$

$$\tfrac{2}{k(k-1)} \textstyle\sum_{i<l} \textit{average } \{\textit{absolute eigenvalues of } \sqrt{\lambda_i}\boldsymbol{W}_l - \sqrt{\lambda_l}\boldsymbol{W}_i\} \ . \tag{2.7b}$$

Under the alternative in (2.1)*, LM and LA diverge to infinity.*

Although the conclusion above does not lend itself to the calculation of p-values, we may employ resampling techniques for implementation.

3. Resampling Approximation

We consider three sampling techniques in this section, including the bootstrap, random symmetrization and the permutation test.

3.1. Bootstrap

We follow the pooled re-sampling procedure suggested by Zhang and Boos (1992) and let

$$(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_N) = (\boldsymbol{X}_1^{(1)} - \hat{\boldsymbol{\mu}}^{(1)}, \ldots, \boldsymbol{X}_{m_1}^{(1)} - \hat{\boldsymbol{\mu}}^{(1)}, \ldots, \boldsymbol{X}_1^{(k)} - \hat{\boldsymbol{\mu}}^{(k)}, \ldots, \boldsymbol{X}_{m_k}^{(k)} - \hat{\boldsymbol{\mu}}^{(k)}) \ , \tag{3.1}$$

where $\hat{\boldsymbol{\mu}}^{(i)}$ is either $\boldsymbol{\mu}^{(i)}$ or the sample mean, depending on whether $\boldsymbol{\mu}^{(i)}$ is known or not. Let $(\boldsymbol{Z}_1^*, \ldots, \boldsymbol{Z}_N^*)$ be drawn with replacement from the given sample

$(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_N)$, and let

$$\hat{\boldsymbol{\Sigma}}_i^* = 1/m_i \sum_{j=1+N_{i-1}}^{N_i} (\boldsymbol{Z}_j^* - \bar{\boldsymbol{Z}}^{i*})\,(\boldsymbol{Z}_j^* - \bar{\boldsymbol{Z}}^{i*})^T\ , \quad i = 1, \ldots, k\ , \tag{3.2}$$

where $\bar{\boldsymbol{Z}}^{i*}$ is the sample mean of $\boldsymbol{Z}^*{}_j$ for $N_{i-1} + 1 \leq N_i$, $N_i = \sum_{l=1}^{i} m_l$ for $i = 1, \ldots, k$, and $N_0 = 0$. The bootstrap counterparts of (2.5a) and (2.5b) are then

$$LM_B = \frac{2}{k(k-1)} \sum_{i<l} \max \Big\{ \text{absolute eigenvalues of } \sqrt{\frac{m_l m_i}{N}} \hat{\boldsymbol{\Sigma}}^{\frac{-1}{2}} (\hat{\boldsymbol{\Sigma}}_l^* - \hat{\boldsymbol{\Sigma}}_i^*) \hat{\boldsymbol{\Sigma}}^{\frac{-1}{2}} \Big\}, \tag{3.3a}$$

$$LA_B = \frac{2}{k(k-1)} \sum_{i<l} \text{average} \Big\{ \text{absolute eigenvalues of } \sqrt{\frac{m_l m_i}{N}} \hat{\boldsymbol{\Sigma}}^{\frac{-1}{2}} (\hat{\boldsymbol{\Sigma}}_l^* - \hat{\boldsymbol{\Sigma}}_i^*) \hat{\boldsymbol{\Sigma}}^{\frac{-1}{2}} \Big\}. \tag{3.3b}$$

The asymptotic equivalence of LM_B and LM and of LA_B and LA is established in the following theorem.

Theorem 3.1. *Assume the conditions in Lemma* 2.1*. For almost all sequences* $(\boldsymbol{X}_1^{(1)}, \ldots, \boldsymbol{X}_{m_1}^{(1)}, \ldots; \boldsymbol{X}_1^{(2)}, \ldots, \boldsymbol{X}_{m_2}^{(2)}, \ldots; \ldots; \boldsymbol{X}_1^{(k)}, \ldots, \boldsymbol{X}_{m_k}^{(k)}, \ldots)$*, of independent* $d \times 1$ *random vectors having finite fourth moments with* $E(\boldsymbol{X}_j^{(i)}) = \boldsymbol{\mu}^{(i)}$ *and* $cov(\boldsymbol{X}_j^{(i)}) = \boldsymbol{\Sigma}$ *for* $i = 1, \ldots, k$*, the conditional distribution of* LM_B (LA_B) *given the finite samples* $(\boldsymbol{X}_1^{(1)}, \ldots, \boldsymbol{X}_{m_1}^{(1)}; \boldsymbol{X}_1^{(2)}, \ldots, \boldsymbol{X}_{m_2}^{(2)}, \ldots; \boldsymbol{X}_1^{(k)}, \ldots, \boldsymbol{X}_{m_k}^{(k)})$ *converges to the unconditional asymptotic distribution of* LM (LA)*.*

In view of this asymptotic equivalence, the critical value of LM (or LA) for testing H_0 can be calculated by repeated bootstrap sampling from the given sample data.

3.2. Random symmetrization

When the k samples have the same size, say m, we suggest another conditional test procedure which is much easier to implement. The motivation of the method is given below. We also give a brief justification for the exact validity of the random symmetrization test in a special case. The asymptotic validity will be stated as a theorem.

Consider the two-sample case as an illustration. Suppose that under the null hypothesis, all variables $(\boldsymbol{X}_1^{(1)}, \boldsymbol{X}_2^{(1)}, \ldots, \boldsymbol{X}_m^{(1)})$ and $(\boldsymbol{X}_1^{(2)}, \boldsymbol{X}_2^{(2)}, \ldots, \boldsymbol{X}_m^{(2)})$ are i.i.d. from a d-dimensional distribution with a given mean. Without loss of generality, assume the mean to be zero. Let Y_j be $[(\boldsymbol{X}_j^{(1)})(\boldsymbol{X}_j^{(1)})^T - (\boldsymbol{X}_j^{(2)})(\boldsymbol{X}_j^{(2)})^T]$.

By assumption, Y_j has a symmetric distribution. For a random sign e_j independent of Y_j, Y_j and e_jY_j are identical in distribution and e_j is independent of e_jY_j. The latter assertion can be seen by invoking the independence of e_j and Y_j and the symmetry of Y_j. Therefore, for any statistic $T(Y_1, \ldots, Y_m)$, its distribution is the same as that of $T(e_1Y_1, \ldots, e_mY_m)$ where e_i's are i.i.d. random signs. Consequently, generate r sets of random signs $(e_1, \ldots, e_m)$, and then obtain r values of $T(e_1Y_1, \ldots, e_mY_m)$, say $T^1, \ldots T^r$. Denote the value of the original T as T^0. We know that T^i, $i = 0, 1, \ldots, r$, are $r + 1$ i.i.d. variables. Suppose for the moment the null hypothesis will be rejected for large value of T (for two-sided tests, modifications are easily done). The p-value can be estimated by the fraction of values in $T^0, T^1, \ldots, T^r$ that are larger than or equal to T^0. If the estimated p-value is smaller than the nominal level α, the null hypothesis will be rejected. This explains the exact validity of the RAS approximation.

In practice, one cannot assume that variables in different samples are i.i.d. and the mean is known. In the following we give the detail of constructing tests and of the consistency of the random symmetrization approximation for the general case.

Since the random symmetrization is also a conditional test, we can work with the standardized data as in the bootstrap procedure:

$$\boldsymbol{Z}_j^{(i)} = \hat{\boldsymbol{\Sigma}}^{-1/2}(\boldsymbol{X}_j^{(i)} - \hat{\boldsymbol{\mu}}^{(i)}) \ , \quad j = 1, \ldots, m, \ i = 1, \ldots, k \ . \tag{3.4}$$

Let $\{e_1, \ldots, e_m\}$ be a set of random signs, the random symmetrization of $\hat{\boldsymbol{\Sigma}}^{-1/2}(\hat{\boldsymbol{\Sigma}}^{(l)} - \hat{\boldsymbol{\Sigma}}^{(i)})\hat{\boldsymbol{\Sigma}}^{-1/2}$ is, $1 \le i < l \le k$,

$$\boldsymbol{W}_{li} = \frac{1}{m}\sum_{j=1}^{m} e_j[\boldsymbol{Z}_j^{(l)}(\boldsymbol{Z}_j^{(l)})^T - \boldsymbol{Z}_j^{(i)}(\boldsymbol{Z}_j^{(i)})^T]. \tag{3.5}$$

The RAS counterparts of LM and LA are

$$LM_R = \frac{2}{k(k-1)}\sum_{i<l} \max\Big\{\text{absolute eigenvalues of } \sqrt{\frac{m^2}{N}}\boldsymbol{W}_{li}\Big\}, \tag{3.6a}$$

$$LA_R = \frac{2}{k(k-1)}\sum_{i<l} \text{average}\ \Big\{\text{absolute eigenvalues of } \sqrt{\frac{m^2}{N}}\boldsymbol{W}_{li}\Big\}. \tag{3.6b}$$

We need to verify that LM_R (LA_R) is asymptotically equivalent to LM (LA).

Theorem 3.2. *Under the assumptions of Lemma* 2.1*, the conditional distribution of LM_R (LA_R) given the data converges to the unconditional asymptotic distribution of LM (LA).*

The p-value is estimated as at the end of Section 3.1. Let $LM_R^{(1)}, \ldots, LM_R^{(r)}$ be r replications of RAS with r independent sets of random signs and let $LM_R^{(0)}$

be the value of the original test statistic LM. The estimated p-value equals the fraction of the values which are greater than or equal to $LM_R^{(0)}$. The same procedure can be applied to LA_R.

3.3. Permutation test

A drawback of the RAS is its restriction to equal sample size. The permutation test can be applied to samples of unequal sizes. It also has some advantages over the bootstrap, but is harder to implement than random symmetrization. It is easy to see that, similar to RAS, when all variables in samples are i.i.d., the exact validity of the permutation tests can be achieved. The justification is similar to that described for RAS, as follows.

Pool the standardized data

$$\hat{\boldsymbol{\Sigma}}^{-1/2}(\boldsymbol{X}_j^{(i)} - \hat{\boldsymbol{\mu}}^{(i)}) \ , \quad j = 1, \ldots, m_i, \ i = 1, \ldots, k, \tag{3.7}$$

into a sample of size N, then randomly divide it into k samples such that the ith sample has size m_i. Denote the ith sample by $\boldsymbol{Z}_j^{(i)}$, $j = 1, \ldots, m_i$, and let

$$\hat{\boldsymbol{\Sigma}}_P^{(i)} = 1/m_i \sum_{j=1}^{m_i} \boldsymbol{Z}_j^{(i)} (\boldsymbol{Z}_j^{(i)})^T \ . \tag{3.8}$$

The permutation test statistics are

$$LM_P = \frac{2}{k(k-1)} \sum_{i<l} \max \left\{ \text{absolute eigenvalues of } \sqrt{\frac{m_l m_i}{N}} (\hat{\boldsymbol{\Sigma}}_P^{(l)} - \hat{\boldsymbol{\Sigma}}_P^{(i)}) \right\}, \tag{3.9a}$$

$$LM_P = \frac{2}{k(k-1)} \sum_{i<l} \text{average} \left\{ \text{absolute eigenvalues of } \sqrt{\frac{m_l m_i}{N}} (\hat{\boldsymbol{\Sigma}}_P^{(l)} - \hat{\boldsymbol{\Sigma}}_P^{(i)}) \right\}. \tag{3.9b}$$

Analogous to random symmetrization, the exact validity of the permutation tests for the case of given means can be obtained. In fact, under the null hypothesis, the permutation counterpart has the same distribution as that of the original test statistic. Therefore, similar to that illustrated for RAS, exact validity can be expected. As with RAS it is of course restrictive, but simulation studies show that in unknown mean cases, permutation tests outperform bootstrap tests in getting closer to the nominal level. The following covers the asymptotic validity of permutation tests for the general case.

Theorem 3.3. *Under the assumptions of Lemma* 2.1*, the conditional distribution of* LM_P (LA_P) *given the data converges to the unconditional asymptotic distribution of* LM (LA)*.*

With r independent random permutations, we have r replications of (3.9), $LM_P^{(1)}, \ldots, LM_P^{(r)}$. The p-value can be estimated as in preceding procedures.

3.4. Monte Carlo simulation

This section reports the results of some Monte Carlo studies. These are carried out to compare the three procedures using three families of multivariate distributions: multinormal $N(\mathbf{0}, \boldsymbol{I}_d)$, multivariate t-distribution $MT(5; \mathbf{0}, \boldsymbol{I}_d)$, and a contaminated normal distribution $NC_2(\mathbf{0}, \boldsymbol{I}_d)$ whose components are independent, each being $N(0,1)$ with probability 0.9 and a $\chi^2_{(2)}$ with probability 0.1. We consider $k=2$ and $k=6$, and the dimension of random vector $d=2$ and $d=5$. The nominal 5% level of significance is chosen. In each procedure, the number of replications for calculating a critical value is $r=500$. Each actual proportion of rejections of H_0 is based on 1000 simulations. As expected, the tests perform better when the means are known. Here we only report results relating to the case of an unknown mean. As one can see from Table 1, most of the time the actual proportion of rejection by permutation (PERM) is closer to the nominal α than is the bootstrap(BOOT), and in this aspect random symmetrization (RAS) is comparable to the bootstrap (BOOT). For different sample sizes, where RAS is not available, the results are given in Table 2. The table shows that PERM is better than BOOT in 8 of 12 simulations. Comparing LA with LM, we found that with equal sample sizes, when $k=2$ LA is worse than LM most of the time; when $k=6$, LA is better in all cases. With different sample sizes, LA is better than LM most of the time.

Table 1. Percentage of times H_0 (2.1) was rejected.

$k=2, m_1=m_2=20$

		$N(\mathbf{0}, \boldsymbol{I}_d)$			$MT(5; \mathbf{0}, \boldsymbol{I}_d)$			$NC_2(\mathbf{0}, \boldsymbol{I}_d)$		
		RAS	BOOT	PERM	RAS	BOOT	PERM	RAS	BOOT	PERM
$d=2$	LM	0.053	0.045	0.048	0.053	0.046	0.054	0.055	0.046	0.056
	LA	0.059	0.057	0.054	0.062	0.046	0.057	0.063	0.060	0.061
$d=5$	LM	0.055	0.045	0.056	0.050	0.044	0.051	0.057	0.039	0.053
	LA	0.054	0.047	0.054	0.059	0.031	0.060	0.063	0.041	0.059

$k=6, m_i=20,\ i=1,\ldots,6$

		$N(\mathbf{0}, \boldsymbol{I}_d)$			$MT(5; \mathbf{0}, \boldsymbol{I}_d)$			$NC_2(\mathbf{0}, \boldsymbol{I}_d)$		
		RAS	BOOT	PERM	RAS	BOOT	PERM	RAS	BOOT	PERM
$d=2$	LM	0.033	0.062	0.060	0.032	0.064	0.059	0.033	0.061	0.058
	LA	0.053	0.058	0.057	0.060	0.060	0.056	0.057	0.060	0.055
$d=5$	LM	0.040	0.063	0.059	0.043	0.058	0.056	0.043	0.057	0.063
	LA	0.056	0.060	0.054	0.049	0.045	0.053	0.055	0.054	0.055

Table 2. Percentage of times H_0 (2.1) was rejected.

$k = 2$, $m_1 = 20, m_2 = 40$

		$N(\mathbf{0}, \boldsymbol{I}_d)$		$MT(5; \mathbf{0}, \boldsymbol{I}_d)$		$NC_2(\mathbf{0}, \boldsymbol{I}_d)$	
		BOOT	PERM	BOOT	PERM	BOOT	PERM
$d = 2$	LM	0.042	0.041	0.046	0.045	0.058	0.055
	LA	0.054	0.053	0.058	0.056	0.054	0.053
$d = 6$	LM	0.047	0.049	0.053	0.055	0.064	0.060
	LA	0.055	0.052	0.049	0.048	0.053	0.053

$k = 6$, $n_1 = n_2 = 20, n_3 = n_4 = 30, n_5 = n_6 = 40$

		$N(\mathbf{0}, \boldsymbol{I}_d)$		$MT(5; \mathbf{0}, \boldsymbol{I}_d)$		NC	
		BOOT	PERM	BOOT	PERM	BOOT	PERM
$d = 2$	LM	0.060	0.057	0.045	0.043	0.054	0.052
	LA	0.056	0.055	0.041	0.045	0.055	0.054
$d = 6$	LM	0.066	0.062	0.065	0.063	0.064	0.059
	LA	0.060	0.057	0.043	0.045	0.045	0.055

The bootstrap results in these Monte Carlo studies also provide some evidence for comparing the test statistics LM and LA of (3.5) with Bartlett's statistic. As shown in Zhang and Boos ((1992), p.428), the bootstrap procedure for Bartlett's homogeneity test performs worse as dimension d (they used p for dimension) increases. The bootstrap procedure of our tests is quite stable; see Table 1. When $m_1 = m_2 = 20$, $\alpha = 0.05$ and $d = 2$, our proportions of rejections for three distributions are 0.045, 0.046 and 0.046, against their 0.046, 0.045 and 0.50 respectively. But when dimension increases to $d = 5$, ours become 0.045, 0.044 and 0.039, against their 0.012, 0.023 and 0.019 respectively.

The power of the tests was also studied for $k = 2$ samples with sample size $m_1 = m_2 = 20$, for dimension $d = 2$. Multinormal and multivariate-t distributions, $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $MT(5, 0, \boldsymbol{\Sigma})$, were used to generate data. We pair the identity matrix $\boldsymbol{I}_2$ with $\boldsymbol{C}_2$ and with $\boldsymbol{V}_2$, respectively, where

$$\boldsymbol{C}_2 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \quad \boldsymbol{V}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

The results are given in Table 3. For comparison with Bartlett's test as studied by Zhang and Boos (1992), we also calculate the adjusted power. The rows marked $(a - p)$ are obtained by using as critical values the 5th percentile of the empirical distribution of p values under H_0 obtained in constructing Table 1. The table shows that the bootstrap has higher power when the distribution is normal, but lower power than the permutation test and random symmetrization in the case of multivariate t. This is also true for simulation results in the three-dimensional case, not shown here in order to save space. Furthermore, LA has better performance than LM most of the time.

Table 3. Power study for $d = 2$, $k = 2$, $m_1 = m_2 = 20$.

	$N(\mathbf{0}, \boldsymbol{C_2})$ & $N(\mathbf{0}, \boldsymbol{I_2})$			$N(\mathbf{0}, \boldsymbol{V_2})$ & $N(\mathbf{0}, \boldsymbol{I_2})$		
	RAS	BOOT	PERM	RAS	BOOT	PERM
LM	0.193	0.227	0.193	0.753	0.781	0.762
$LM(a-p)$	0.229	0.246	0.223	0.815	0.825	0.817
LA	0.256	0.276	0.265	0.784	0.770	0.770
$LA(a-p)$	0.296	0.299	0.297	0.817	0.818	0.814

	$MT(5; \mathbf{0}, \boldsymbol{C_2})$ & $MT(5; \mathbf{0}, \boldsymbol{I_2})$			$MT(5; \mathbf{0}, \boldsymbol{V_2})$ & $Mt(5; \mathbf{0}, \boldsymbol{I_2})$		
	RAS	BOOT	PERM	RAS	BOOT	PERM
LM	0.228	0.231	0.233	0.553	0.521	0.561
$LM(a-p)$	0.261	0.262	0.265	0.636	0.604	0.823
LA	0.230	0.229	0.234	0.570	0.538	0.589
$LA(a-p)$	0.274	0.270	0.275	0.640	0.617	0.637

The power studies of the bootstrap are in favor of our tests. Zhang and Boos (1992) performed power studies of the bootstrapped Bartlett's test with $\boldsymbol{C}_2$ and $\boldsymbol{V}_2$. The corresponding values (BartlettB) in Table 2 of Zhang and Boos (1992) and in our Table 3 are collated below for easier comparison ($\alpha = 0.05, k = 2, d = 2, m_1 = m_2 = 20$), where the first column is for the case of $N(0, \boldsymbol{V}_2)$ against $N(0, \boldsymbol{I}_2)$, the second column for $MT(5, 0, \boldsymbol{V}_2)$ against $MT(5, 0, \boldsymbol{I}_2)$, the third column for $N(0, \boldsymbol{C}_2)$ against $N(0, \boldsymbol{I}_2)$, and the fourth column for $MT(5, 0, \boldsymbol{C}_2)$ against $MT(5, 0, \boldsymbol{I}_2)$. The values in parentheses are the adjusted powers, assuming the population means to be unknown parameters:

BartlettB	0.642(0.657)	0.487(0.525)	0.233(0.243)	0.155(0.177)
BOOT of (2.5b)	0.770(0.818)	0.538(0.617)	0.276(0.299)	0.229(0.270)
RAS of (2.5b)	0.784(0.817)	0.570(0.640)	0.256(0.296)	0.230(0.274)
PERM of (2.5b)	0.770(0.814)	0.589(0.637)	0.265(0.297)	0.234(0.275)

In summary, we have the following recommendations: (1) the average value test outperforms the maximum value test; (2) the test of (2.5) is prefered over Bartlett's test, with or without the bootstrap; (3) if sample sizes are equal, use the random symmetrization procedure for easier implementation even though its power performance may be slightly worse; (4) if sample sizes are not equal, the permutation procedure is a good choice.

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Appendix

To simplify notation we rewrite, in the two-sample case, m_1 as m and the second sample as $(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n)$. In this way, $N = m + n$. It is clear that the convergence of the test statistics follows the convergence of the random matrices defined. Hence we deal with convergence of random matrices. Furthermore note that in the k-sample case the estimate $\hat{\boldsymbol{\Sigma}}$ of the covariance matrix based on all data converges to a constant matrix in probability and will not affect the limiting behavior of the test statistics. Hence, we simply regard it as an identity matrix when studying the limit properties of tests.

The proof of Lemma 2.1 is simple and the proof of Theorem 3.1 is a direct application of Theorem 2.4 in Giné and Zinn (1990).

Proof of Theorem 3.2. We have to first prove the asymptotic normality of $\{\sqrt{m_i m_l/N}\boldsymbol{W}_{li}, 1 \le i < l \le k\}$. We need only prove the asymptotic normality of all linear combinations of the matrices $\sqrt{m_i m_l/N}\boldsymbol{W}_{li}$ having asymptotically the same covariance structure as that of the limiting random matrices in Theorem 3.1. That is, for any constants b_{il} with at least one being nonzero, $\sum_{1\le i<l\le k} b_{il}\sqrt{m_i m_l/N}\boldsymbol{W}_{li}$ is asymptotically normal in the sense of Lemma 2.1. These can be derived by the above with some more calculation, details being omited. The proof is completed.

Proof of Theorem 3.3. We first prove the convergence of the permutation empirical process in the two-sample case. Write m_1 as m and the second sample as $\{\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n\}$, $N = m + n$. Let F_m and F_m^P be the empirical distributions based on $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m\}$ and $\{\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_m\}$ respectively, and G_n and G_n^P the empirical distributions based on $\{\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n\}$ and $\{\boldsymbol{Z}_{m+1}, \ldots, \boldsymbol{Z}_{m+n}\}$. Further, let $H_N(t) = (m/N)F_m(t) + (n/N)G_n(t))$ and $H(t) = \lambda F(t) + (1-\lambda)G(t))$. Applying Theorem 1 of Præstgaard ((1995), p.309), for almost all series $\{\boldsymbol{X}_i\}$ and $\{\boldsymbol{Y}_i\}$,

$$\begin{aligned}\{\sqrt{nm/N}(F_m^P(t) - G_n^P(t)) : t \in R^1\} &= \{\sqrt{mN/n}(F_m^P(t) - H_N(t)) : t \in R^1\} \\ &\Longrightarrow RV_H =: \{RV_H(t) : t \in R^1\}, \qquad \text{(A.1)}\end{aligned}$$

where RV_H is a P-Brownian bridge with $H = \lambda F + (1-\lambda)G$. The convergence is convergence in distribution in $l^\infty(\mathcal{F})$, consisting of bounded, real-valued functions defined on $\mathcal{F}$, the class of indicator functions of half spaces $\{\boldsymbol{a}^\tau \cdot \le t\}$. As usual, (see, e.g., Gine and Zinn, (1984)), the supremum norm on this space is considered. Note that all sample paths of RV_H are contained in $C(\mathcal{F}, H)$, a sub-collection consisting of all bounded, uniformly continuous functions under the $L^2(H)$-seminorm $d^2(f,g) = E_H(f-g)^2 - (E_H(f-g))^2$. It is known that $C(\mathcal{F}, H)$ is separable (e.g., see Pollard, (1984), p.169, ex.7). Furthermore, any point in $C(\mathcal{F}, H)$ can easily be showed to be completely regular (Pollard, (1984, p.67)). By the representation theory (e.g., Pollard, (1984), p.71), we have, under uniform norm,

$$\{\sqrt{mN/n}(F_m^P(t) - H_N(t)) : t \in R^1\} \longrightarrow \{RV_H(t) : t \in R^1\} \quad a.s. \tag{A.2}$$

We now turn to the proof for $\sqrt{mn/N}\{(\hat{\mathbf{\Sigma}}_P^{(1)} - \hat{\mathbf{\Sigma}}_P^{(2)})$. Consider the upper-left element on the diagonal of the matrix, $\sqrt{mn/N}\{1/m\sum_{i=1}^m[(Z_i^P)^2 - 1/n\sum_{j=1}^n[(Z_{i+m}^P)^2]\}$. Write it as

$$\sqrt{mn/N}\int(t^2-\sigma^2)d\,(F_m^P(t) - G_n^P(t) = \int(t^2-\sigma^2)d\,\{\sqrt{mN/n}(F_m^P(t) - H_N(t))\}.$$

From (4.2), it converges to $T^P := \int(t^2-\sigma^2)d\,RV_H(t)$ $a.s.$ This stochastic integral is distributed normally. The work remaining is to check that its variance coincides with the variance of the upper-left element of $\boldsymbol{W}_1$, $E(x^2-\sigma^2)^2$. Note that under the condition of Theorem 3.3, $H = F$. Via some elementary calculations, we have

$$\begin{aligned} E((T^P)^2) &= E(\int(t^2-\sigma^2)(t_1^2-\sigma^2)d\,RV_H(t)d\,RV_H(t_1)) \\ &= \int(t^2-\sigma^2)^2E(d\,RV_H(t))^2) \\ &= \int(t^2-\sigma^2)^2d\,F(t) = E(x^2-\sigma^2)^2. \end{aligned} \tag{A.3}$$

The third equation uses $\int(t^2-\sigma^2)^2(d\,F(t))^2 = 0$.

For the general case, we start with a lemma. We consider only the 3-sample case, more samples can be treated with more complicated calculations. Let the data be $\{\boldsymbol{X}_1^{(1)}, \ldots, \boldsymbol{X}_{m_1}^{(1)}, \boldsymbol{X}_1^{(2)}, \ldots, \boldsymbol{X}_{m_2}^{(2)}, \boldsymbol{X}_1^{(3)}, \ldots, \boldsymbol{X}_{m_3}^{(3)}\}$ and let $\{\boldsymbol{Z}_1^{(1)}, \ldots, \boldsymbol{Z}_{m_1}^{(1)}$, $\boldsymbol{Z}_1^{(2)}, \ldots, \boldsymbol{Z}_{m_2}^{(2)}$, $\boldsymbol{Z}_1^{(3)}, \ldots, \boldsymbol{Z}_{m_3}^{(3)}\}$ be the data generated by permutation. Let $N = m_1 + m_2 + m_3$. Further, for $l = 1, 2, 3$, let $F_{m_l l}$ and $F_{m_l l}^P$ be, respectively, the empirical distributions based on $\{\boldsymbol{X}_1^{(l)}, \ldots, \boldsymbol{X}_{m_l}^{(l)}\}$, and $\{\boldsymbol{Z}_1^{(l)}, \ldots, \boldsymbol{Z}_{m_l}^{(l)}\}$, and let $H_{N-m_1}(t) = (m_2/(N-m_1))F_{m_2 2}(t) + (m_3/N)F_{m_3 3}(t))$.

Lemma. *Under the conditions of Theorem* 3.3*, the conditional empirical process* $\{\sqrt{m_2(N-m_1)/m_3}(F_{m_2 2}^P(t) - H_{N-m_1}(t)) : t \in R^1\}$ *given* $\{\boldsymbol{Z}_1^{(1)}, \ldots, \boldsymbol{Z}_{m_1}^{(1)}\}$ *converges weakly to* $\{RV_F(t) : t \in R^1\}$*, where* F *is the distribution of the random variable* $\boldsymbol{X}$.

Proof. Note that when $\{\boldsymbol{Z}_1^{(1)},\ldots,\boldsymbol{Z}_{m_1}^{(1)}\}$ is given, the process $\{\sqrt{m_2(N-m_1)/m_3}\,(F_{m_2 2}^P(t)-H_{N-m_1}(t)) : t\in R^1\}$ is almost the same as that in (4.2). Following the arguments used in the proof of Theorem 1 of Præstgaard (1995), we can derive the conclusion. Details are omitted.

We now turn to the proof of the theorem and first consider the asymptotic normality of $\{(\hat{\boldsymbol{\Sigma}}_P^i - \hat{\boldsymbol{\Sigma}}_P^l),\ 1\le i<l\le 3\}$. We show, for any constants b_{il} with at least one being nonzero, $\sum_{1\le i<l\le 3} b_{il}\sqrt{m_i m_l/N}(\hat{\boldsymbol{\Sigma}}_P^{(i)} - \hat{\boldsymbol{\Sigma}}_P^{(l)})$ is asymptotically normal in the sense of Lemma 2.1. As in the two-sample case, we consider the empirical permutation process $\sum_{1\le i<l\le 3} b_{il}\sqrt{m_i m_l/N}(F_{m_i i}^P - F_{m_l l}^P)$ first, convergence of $\{(\hat{\boldsymbol{\Sigma}}_P^{(i)} - \hat{\boldsymbol{\Sigma}}_P^{(l)}),\ 1\le i<l\le 3\}$ will be a consequence. Invoking $F_{m_3 3}^P = 1/m_3(NF_N - m_1F_{m1}^P - m_2F_{m2}^P)$, it can be verified that

$$\begin{aligned}
&\sum_{1\le i<l\le 3} b_{il}\sqrt{m_i m_l/N}(F_{m_i i}^P - F_{m_l l}^P)\\
&= \sqrt{m_1}(\sqrt{m_2/N}b_{12} + (1+m_2/m_3)\sqrt{m_3/N}(b_{13} + \sqrt{m_1m_2/(m_3N}b_{23}(F_{m_1}^P - F_N)\\
&+ \sqrt{m_2}(\sqrt{m_1/N}b_{12} + (1+m_2/m_3)\sqrt{m_3/N}(b_{23} + \sqrt{m_1m_2/(m_3N}b_{13}(F_{m_2}^P - F_N)\\
&=: \sqrt{m_1}b_{n1}(F_{m_1}^P - F_N) + \sqrt{m_2}b_{n2}(F_{m_2}^P - F_N), \qquad (A.4)
\end{aligned}$$

where b_{n1} and b_{n2} go to constants. Further, noting that $F_N = m_1/N(F_{m1}^P - F_{N-m_1}^P) + F_{N-m_1}^P$,

$$\begin{aligned}
&\sqrt{m_1}b_{n1}(F_{m_1}^P - F_N) + \sqrt{m_2}b_{n2}(F_{m_2}^P - F_N)\\
&= \sqrt{m_1}(b_{n1} + b_{N2}\sqrt{m_1m_2}/N)(F_{m_1}^P - F_N) + \sqrt{m_2}b_{n2}(F_{m_2}^P - F_{N-m_1}^P). \quad (A.5)
\end{aligned}$$

Note that $(F_{m_1}^P - F_N)$ is conditionally independent of $(F_{m_2}^P - F_{N-m_1}^P)$. Together with the proof for the two-sample case and the lemma, the process then converges weakly to a Gaussian process. The work remaining is to check that the limiting covariance structure of $\sum_{1\le i<l\le 3} b_{il}\sqrt{m_i m_l/N}(F_{m_i i} - F_{m_l l})$ coincides with that of $\sum_{1\le i<l\le 3} b_{il}\sqrt{m_i m_l/N}(F_{m_i i}^P - F_{m_l l}^P)$. As above,

$$\begin{aligned}
&\sum_{1\le i<l\le 3} b_{il}\sqrt{m_i m_l/N}(F_{m_i i} - F_{m_l l})\\
&= \sqrt{m_1}(b_{n1} + b_{N2}\sqrt{m_1m_2}/N)(F_{m_1} - F_N) + \sqrt{m_2}b_{n2}(F_{m_2} - F_{N-m_1}). \quad (A.6)
\end{aligned}$$

It is enough to show that

(1) $\sqrt{m_1N/(N-m_1)}(F_{m_1}^P - F_N)$ and $\sqrt{m_1(N-m_1)/m_3}(F_{m_2}^P - F_{N-m_1}^P)$ have the same limiting covariance structures as those of $\sqrt{m_1N/(N-m_1)}(F_{m_1} - F_N)$ and $\sqrt{m_1(N-m_1)/m_3}(F_{m_2} - F_{N-m_1})$ respectively, and

(2) $F_{m_1} - F_N$ is uncorrelated with $F_{m_2} - F_{N-m_1}$.

For (1), noting that $F_{m_1} - F_N = (N-m_1)/N(F_{m_1} - F_{N-m_1})$, it is easy to see that the covariance at (t, t_1) is

$$\begin{aligned} R(t,t_1) &= \frac{m_1(N-m_1)}{N}(\frac{1}{m_1} + \frac{1}{(N-m_1)})(F(t \wedge t_1) - F(t)F_x(t_1)) \\ &= F(t \wedge t_1) - F(t)F(t_1), \end{aligned} \tag{A.7}$$

the covariance structure of a P-Brownian bridge, where "$\wedge$" denotes minimum; similarly for $(\sqrt{m_1(N-m_1)/m_3}(F_{m_2} - F_{N-m_1})$.

For (2), via elementary calculations we have, applying the independence of the variables having the common distribution F,

$$\begin{aligned} &E(F_{m_1}(t) - F_N(t))(F_{m_2}(t_1) - F_{N-m_1}(t_1)) \\ &= -\frac{m_3}{N} E(F_{N-m_1}(t) - F(t))(F_{m_2}(t_1) - F_{m_3}(t_1)) \\ &= -\frac{m_3}{(N(N-m_1))} [(F(t \wedge t_1) - F(t)F(t_1)) - (F(t \wedge t_1) - F(t)F(t_1))] = 0. \end{aligned}$$

The proof is complete.

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