New Approach to Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Filtering for Polytopic Discrete-Time Systems

Huijun Gao, James Lam, *Senior Member, IEEE*, Lihua Xie, *Senior Member, IEEE*, and Changhong Wang

***Abstract*—This paper revisits the problem of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering for polytopic discrete-time systems. Differing from previous results in the quadratic framework, the filter design makes full use of the parameter-dependent stability idea: Not only is the filter dependent of the parameters (which are assumed to reside in a polytope and be measurable online), but in addition, the Lyapunov matrices are different for the entire polytope domain, as well as for different channels with respect to the mixed performances. These ideas are realized by introducing additional slack variables to the well-established performance conditions and by employing new bounding techniques, which results in a much less conservative filter design method. A numerical example is presented to illustrate the effectiveness and advantage of the developed filter design method.**

***Index Terms*—Discrete-time systems, linear matrix inequality, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering, polytopic systems.**

I. Introduction

It is well known that state estimation of dynamic systems with both process and measurement noise inputs is a very important and challenging problem in engineering applications [1], [19]. The celebrated Kalman filtering (also called $\mathcal{H}_2$ filtering) [12], [27] has found many applications in aerospace [20], economics [16] etc., which minimizes the $\mathcal{H}_2$ norm of the filtering error transfer function under the assumption that the noise processes have known power spectral densities. In many practical situations, however, we may not be able to have exactly known information on the spectral densities of the noise processes. In such cases, an alternative is to reformulate the estimation problem in an $\mathcal{H}_\infty$ filtering framework, which has been well addressed for different systems through different techniques during the past decade (see, for instance, [6], [8], [10], [15], [28], and the references therein). It is noted that although $\mathcal{H}_\infty$ filtering offers much better robustness in performance than $\mathcal{H}_2$ filtering, $\mathcal{H}_\infty$ filtering may be very conservative and may lead to a large intolerable estimation error variance when the system is driven by white noise signals. Therefore, to capture the benefits of both pure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filters, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem, which simultaneously takes into account the presence of two kinds of exogenous signals (that is, the energy-bounded disturbance input and the stochastic disturbance input with known statistics), was introduced in [3]. An important application of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering in aerospace can be found in [21].

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem consists of the minimization of an upper bound of the $\mathcal{H}_2$ norm of the filtering error system while a prescribed $\mathcal{H}_\infty$ attenuation level is guaranteed, allowing us to make a tradeoff between the performance of the $\mathcal{H}_2$ filter and that of the $\mathcal{H}_\infty$ filter. Up until now, several approaches have been proposed to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem: Bernstein and Haddad solved this problem by transforming it into an auxiliary minimization problem, on which by using the Lagrange multiplier technique, an upper bound on the $\mathcal{H}_2$ filtering error variance was given by solving a set of coupled Riccati and Lyapunov equations [3]; a time domain game theoretic approach was proposed to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem through a set of coupled Riccati equations in [4] and [24]. In [13] and [21], a convex optimization approach to obtain the solutions via affine symmetric matrix inequalities was used.

Very recently, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem has also been considered for systems with parameter uncertainties, which are inherent to physical systems and must be taken into consideration in a realistic design. For norm-bounded uncertain systems, solutions to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem have been given for both continuous- and discrete-time systems by using Riccati-like approaches [25], [26]. In addition, for polytopic uncertain systems, Palhares and Peres [17] presented a linear matrix inequality (LMI) approach to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem in the discrete-time case, where admissible filters can be found by solving a set of LMIs. It is worth emphasizing that most of the aforementioned results are based on the quadratic stability notion, reflected from the following two aspects.

1) A single Lyapunov matrix is used for different performance channels.
2) The Lyapunov matrix remains fixed for the entire uncertainty domain.

The quadratic stability has been largely used for robust analysis and synthesis in the past decades. Although being specially adequate for arbitrarily fast time-varying parameters, methods based on quadratic stability can produce conservative results since the same parameter-independent Lyapunov function must be used for the entire uncertainty domain. One well-recognized way of overcoming this conservativeness is to consider a parameter-dependent Lyapunov function. An example of a less-conservative stability condition based on parameter-dependent Lya-

Manuscript received June 15, 2004; revised October 13, 2004. This work was supported in part by RGC 7103/01P. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Zidong Wang.

H. Gao and C. Wang are with the Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin 150001, China (e-mail: hjgao@hit.edu.cn).

J. Lam is with the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong (e-mail: james.lam@hku.hk).

L. Xie is with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: ELHXIE@ntu.edu.sg).

Digital Object Identifier 10.1109/TSP.2005.851116

punov functions can be found in [18]. This idea has been subsequently applied to the filter designs in a few contexts [9], [11], [23]. The parameter-dependent stability has been proved to be not only suitable for dealing with uncertain parameters but also very useful for multiobjective synthesis problems [2].

To reduce the conservativeness mentioned above, in this paper, we revisit the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem. More specifically, we present a new approach to solve this filtering problem for polytopic discrete-time systems. Differently from previous results in the quadratic framework, the filter design makes full use of the parameter-dependent stability idea: Not only is the filter is dependent of the parameters (which are assumed to reside in a polytope and be measurable online), but in addition, the Lyapunov matrices are different for the entire polytope domain, as well as for different channels with respect to the mixed performances. These ideas are realized by introducing additional slack variables to the well-established performance conditions and by employing new bounding techniques, with the result of a much less conservative filter design method. A numerical example is presented to illustrate the effectiveness and advantage of the developed filter design method.

Notations: The notations used throughout the paper are fairly standard. The superscript "T"stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$, and the notation $P > 0$ means that P is real symmetric and positive definite. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry, and diag$\{\ldots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. $l_2[0,\infty)$ is the space of square-summable vector functions over $[0,\infty)$.

II. Problem Description and Preliminaries

A. Problem Description

Consider the following stable discrete-time linear system:

$$\begin{aligned}\mathcal{S}\colon x(t+1) &= A(\lambda)x(t) + B(\lambda)v(t) + E(\lambda)w(t)\\ y(t) &= C(\lambda)x(t) + D(\lambda)v(t) + F(\lambda)w(t)\\ z(t) &= L(\lambda)x(t)\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $y(t) \in \mathbb{R}^m$ is the measured output; $z(t) \in \mathbb{R}^p$ is the signal to be estimated; and $v(t) \in \mathbb{R}^l, w(t) \in \mathbb{R}^q$ are disturbance inputs. As in previous mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems [13], we use $v(t)$ and $w(t)$ to represent two different classes of noises:

$$\begin{cases} v(t), & \text{white noise processes}\\ w(t), & \text{energy bounded input signals.}\end{cases}$$

In addition, $A(\lambda), B(\lambda), E(\lambda), C(\lambda), D(\lambda), F(\lambda)$, and $L(\lambda)$ are appropriately dimensioned matrices. It is assumed that

$$\Omega(\lambda) \triangleq (A(\lambda), B(\lambda), E(\lambda), C(\lambda), D(\lambda), F(\lambda), L(\lambda)) \in \mathcal{R}$$

where $\mathcal{R}$ is a given convex bounded polyhedral domain described by s vertices:

$$\mathcal{R} \triangleq \left\{ \Omega(\lambda) \,\middle|\, \Omega(\lambda) = \sum_{i=1}^{s} \lambda_i \Omega_i; \sum_{i=1}^{s} \lambda_i = 1, \lambda_i \geq 0 \right\}$$

and $\Omega_i \triangleq (A_i, B_i, E_i, C_i, D_i, F_i, L_i)$ denotes the ith vertex of the polytope. It is also assumed that λ does not depend explicitly on the time variable but can be measured online. The parameter λ can vary slowly due to changes in temperature, wind, pressure, humidity, atmosphere, or operating points [14].

Here, we are interested in estimating the signal $z(t)$ by a parameter-dependent filter of general structure described by $\mathcal{F}$

$$\begin{aligned}\mathcal{F}\colon x_F(t+1) &= A_F(\lambda)x_F(t) + B_F(\lambda)y(t)\\ z_F(t) &= C_F(\lambda)x_F(t)\end{aligned} \tag{2}$$

where $x_F(t) \in \mathbb{R}^n$ is the filter state vector, and $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$ are appropriately dimensioned parameter-dependent filter matrices to be determined.

Augmenting the model of $\mathcal{S}$ to include the states of the filter, we obtain the filtering error system $\mathcal{E}$:

$$\begin{aligned}\mathcal{E}\colon \xi(t+1) &= \bar{A}(\lambda)\xi(t) + \bar{B}(\lambda)v(t) + \bar{E}(\lambda)w(t)\\ e(t) &= \bar{C}(\lambda)\xi(t)\end{aligned} \tag{3}$$

where $\xi(t) = [x^T(t)\ x_F^T(t)]^T, e(t) = z(t) - z_F(t)$, and

$$\begin{aligned}\bar{A}(\lambda) &= \begin{bmatrix} A(\lambda) & 0\\ B_F(\lambda)C(\lambda) & A_F(\lambda)\end{bmatrix}\\ \bar{B}(\lambda) &= \begin{bmatrix} B(\lambda)\\ B_F(\lambda)D(\lambda)\end{bmatrix}\\ \bar{E}(\lambda) &= \begin{bmatrix} E(\lambda)\\ B_F(\lambda)F(\lambda)\end{bmatrix}\\ \bar{C}(\lambda) &= [L(\lambda) \quad -C_F(\lambda)]\end{aligned} \tag{4}$$

Then, similarly to [17], the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem to be addressed in this paper can be expressed as follows.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Filtering Problem: Given system $\mathcal{S}$ in (1), determine the parameter-dependent matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$ of the filter $\mathcal{F}$ in (2), such that the filtering error system $\mathcal{E}$ in (3) is asymptotically stable, and $z_F(t)$ is a good estimation of $z(t)$ in the sense of the $\mathcal{H}_2$ performance with respect to the white noise $v(t)$ and the $\mathcal{H}_\infty$ performance with respect to the energy bounded noise $w(t)$. More specifically, the aim of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering is to find a stable filter $\mathcal{F}$ in the form of (2) such that $\max_\lambda \|T_{ev}(\lambda)\|_2^2 < \beta$ and $\max_\lambda \|T_{ew}(\lambda)\|_\infty < \gamma$ are assured, where $T_{ev}(\lambda)$ denotes the operator from $v(t)$ to $e(t)$, and $T_{ew}(\lambda)$ denotes the operator from $w(t)$ to $e(t)$, respectively. Filters satisfying the above conditions are called parameter-dependent mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters.

Throughout the paper, we make the following assumption.

Assumption 1: System $\mathcal{S}$ in (1) is asymptotically stable for any $\Omega(\lambda) \in \mathcal{R}$.

B. Preliminaries

To solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem formulated above, we need some preliminary results.

Lemma 1: Supposing that system $\mathcal{S}$ in (1) and filter $\mathcal{F}$ in (2) are given, the filtering error system $\mathcal{E}$ in (3) is asymptotically stable with $\|T_{ev}(\lambda)\|_2^2 < \beta$ if and only if there exists a matrix function $Q(\lambda) > 0$ satisfying

$$\mathrm{tr}(\bar{B}^T(\lambda)Q(\lambda)\bar{B}(\lambda)) < \beta \quad (5)$$

$$\bar{A}^T(\lambda)Q(\lambda)\bar{A}(\lambda) - Q(\lambda) + \bar{C}^T(\lambda)\bar{C}(\lambda) < 0. \quad (6)$$

Lemma 2: Supposing that system $\mathcal{S}$ in (1) and filter $\mathcal{F}$ in (2) are given, the filtering error system $\mathcal{E}$ in (3) is asymptotically stable with $\|T_{ew}(\lambda)\|_\infty < \gamma$ if and only if there exists a matrix function $P(\lambda) > 0$ satisfying

$$\begin{bmatrix} -P(\lambda) & P(\lambda)\bar{A}(\lambda) & P(\lambda)\bar{E}(\lambda) & 0 \\ * & -P(\lambda) & 0 & \bar{C}^T(\lambda) \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0. \quad (7)$$

The above two lemmas characterize the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances for discrete-time systems by using LMI representations. For synthesis purposes (to keep the filtering synthesis problem tractable), based on these two lemmas, and by following similar arguments as in [17], the following lemma can be used to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem.

Lemma 3: Supposing that system $\mathcal{S}$ in (1) and filter $\mathcal{F}$ in (2) are given, the filtering error system $\mathcal{E}$ in (3) is asymptotically stable with $\|T_{ew}(\lambda)\|_\infty < \gamma$ and $\|T_{ev}(\lambda)\|_2^2 < \beta$ if there exists a matrix function $Q(\lambda) > 0$ satisfying (5) and

$$\begin{bmatrix} -Q(\lambda) & Q(\lambda)\bar{A}(\lambda) & Q(\lambda)\bar{E}(\lambda) & 0 \\ * & -Q(\lambda) & 0 & \bar{C}^T(\lambda) \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0. \quad (8)$$

Remark 1: It can be easily seen that Lemma 3 actually combines Lemmas 1 and 2 by setting $Q(\lambda) = P(\lambda)$, with the result that condition (6) is embedded in (8). In such a way, the system matrices $(\bar{A}(\lambda), \bar{B}(\lambda), \bar{C}(\lambda))$ involve only one positive definite matrix variable $Q(\lambda)$, and therefore, by partitioning $Q(\lambda)$, we can readily solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem by following the linearization procedure presented in [22]. This is the main idea used in [17] for solving the robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem. However, it is worth mentioning that the filtering result developed in [17] has introduced some overdesign, which comes from the following two aspects.

1) The mixed performance condition used in [17] (i.e., Lemma 3) is conservative due to the imposition of $Q(\lambda) = P(\lambda)$.
2) In solving the filtering synthesis problem, the final result is based on the quadratic stability idea (that is, a fixed Lyapunov matrix $Q(\lambda) \equiv Q$ has been used for the entire polytope domain).

In the following, we will present a new approach to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem formulated in the above subsection. This approach reduces the conservativeness of the previous result from both of the above two aspects to some extent.

III. Filtering Results

A. New Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Performance

In this subsection, we present a new mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance. The following lemmas play an important role, which can be proved by following similar lines of arguments as in [5].

Lemma 4: Supposing that system $\mathcal{S}$ in (1) and filter $\mathcal{F}$ in (2) are given, the filtering error system $\mathcal{E}$ in (3) is asymptotically stable with $\|T_{ev}(\lambda)\|_2^2 < \beta$ if and only if there exist matrix functions $Q(\lambda) > 0, \Pi(\lambda) > 0$, and $W(\lambda)$, satisfying

$$\mathrm{tr}\Pi(\lambda) < \beta \quad (9)$$

$$\begin{bmatrix} \Gamma_1 & W^T(\lambda)\bar{B}(\lambda) \\ * & -\Pi(\lambda) \end{bmatrix} < 0 \quad (10)$$

$$\begin{bmatrix} \Gamma_1 & W^T(\lambda)\bar{A}(\lambda) & 0 \\ * & -Q(\lambda) & \bar{C}^T(\lambda) \\ * & * & -I \end{bmatrix} < 0 \quad (11)$$

where $\Gamma_1 \triangleq Q(\lambda) - W^T(\lambda) - W(\lambda)$.

Lemma 5: Supposing that system $\mathcal{S}$ in (1) and filter $\mathcal{F}$ in (2) are given, the filtering error system $\mathcal{E}$ in (3) is asymptotically stable with $\|T_{ew}(\lambda)\|_\infty < \gamma$ if and only if there exist matrix functions $P(\lambda) > 0$ and $V(\lambda)$ satisfying

$$\begin{bmatrix} \Gamma_2 & V^T(\lambda)\bar{A}(\lambda) & V^T(\lambda)\bar{E}(\lambda) & 0 \\ * & -P(\lambda) & 0 & \bar{C}^T(\lambda) \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \quad (12)$$

where $\Gamma_2 \triangleq P(\lambda) - V^T(\lambda) - V(\lambda)$.

The above two lemmas present improved versions of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances for discrete-time systems. An important feature of Lemmas 4 and 5 lies in the fact that the conditions in these lemmas do not contain product terms between the Lyapunov matrices $Q(\lambda), P(\lambda)$ and the system matrices. This feature will enable us to obtain an improved mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance as follows.

Proposition 1: Supposing that system $\mathcal{S}$ in (1) and filter $\mathcal{F}$ in (2) are given, the filtering error system $\mathcal{E}$ in (3) is asymptotically stable with $\|T_{ew}(\lambda)\|_\infty < \gamma$ and $\|T_{ev}(\lambda)\|_2^2 < \beta$ if there exist matrix functions $Q(\lambda) > 0, P(\lambda) > 0, \Pi(\lambda) > 0$, and $W(\lambda)$ satisfying (9)–(11) and

$$\begin{bmatrix} \Gamma_3 & W^T(\lambda)\bar{A}(\lambda) & W^T(\lambda)\bar{E}(\lambda) & 0 \\ * & -P(\lambda) & 0 & \bar{C}^T(\lambda) \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \quad (13)$$

where $\Gamma_3 \triangleq P(\lambda) - W^T(\lambda) - W(\lambda)$.

Remark 2: Proposition 1 presents an improved version of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance. Differently from Lemma 3, where we set $Q(\lambda) = P(\lambda)$ for different performance objectives, here, we set $W(\lambda) = V(\lambda)$ (note that $W(\lambda)$ and $V(\lambda)$ are matrices without any structural restriction). In addition, if we impose the extra condition $W^T(\lambda) = W(\lambda) = P(\lambda) = Q(\lambda)$ in Proposition 1, we readily recover Lemma 3, which means that the condition in Proposition 1 is weaker than that in Lemma 3 (in other words, Proposition 1 is potentially less conservative than Lemma 3).

The following development will be based on Proposition 1.

B. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Filtering Result

An improved version of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance has been formulated in Proposition 1. It is noted that if the filter matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$ are given, the conditions in Proposition 1 are linear matrix inequalities over the decision variables $Q(\lambda), P(\lambda), \Pi(\lambda)$, and $W(\lambda)$ for fixed λ. However, since our purpose is to determine the filter matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$, the above conditions are actually nonlinear matrix inequalities. In addition, to test the feasibility of these conditions is an infinite-dimensional problem in terms of the parameter λ. Our main objective hereafter is to transform them into finite-dimensional LMI conditions. The following proposition presents a preliminary result.

Proposition 2: Given system $\mathcal{S}$ in (1). There exist filter matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$ and matrices $Q(\lambda) > 0, P(\lambda) > 0, \Pi(\lambda) > 0$, and $W(\lambda)$ satisfying (9)–(11) and (13), if and only if there exist matrices $\Pi(\lambda) > 0, \bar{P}(\lambda) \triangleq [\begin{smallmatrix} \bar{P}_1(\lambda) & \bar{P}_2(\lambda) \\ * & \bar{P}_3(\lambda) \end{smallmatrix}] > 0, \bar{Q}(\lambda) \triangleq [\begin{smallmatrix} \bar{Q}_1(\lambda) & \bar{Q}_2(\lambda) \\ * & \bar{Q}_3(\lambda) \end{smallmatrix}] > 0, R(\lambda), S(\lambda), T(\lambda), \bar{A}_F(\lambda), \bar{B}_F(\lambda)$, and $\bar{C}_F(\lambda)$ satisfying (9) and (14)–(16), shown at the bottom of the page, where

$$
\begin{aligned}
\Gamma_4 &\triangleq \bar{Q}_1(\lambda) - R^T(\lambda) - R(\lambda)\\
\Gamma_5 &\triangleq \bar{Q}_2(\lambda) - T(\lambda) - S(\lambda)\\
\Gamma_6 &\triangleq \bar{Q}_3(\lambda) - T(\lambda) - T^T(\lambda)\\
\Gamma_7 &\triangleq \bar{P}_1(\lambda) - R^T(\lambda) - R(\lambda)\\
\Gamma_8 &\triangleq \bar{P}_2(\lambda) - T(\lambda) - S(\lambda)\\
\Gamma_9 &\triangleq \bar{P}_3(\lambda) - T(\lambda) - T^T(\lambda).
\end{aligned}
$$

Moreover, under the above conditions, the matrix functions for an admissible parameter-dependent mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter are given by

$$
\begin{bmatrix} A_F(\lambda) & B_F(\lambda) \\ C_F(\lambda) & 0 \end{bmatrix} = \begin{bmatrix} T^{-1}(\lambda) & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} \bar{A}_F(\lambda) & \bar{B}_F(\lambda) \\ \bar{C}_F(\lambda) & 0 \end{bmatrix}. \quad (17)
$$

Proof (Necessity): Suppose there exist filter matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$ and matrices $Q(\lambda) > 0, P(\lambda) > 0, \Pi(\lambda) > 0$, and $W(\lambda)$ satisfying (9)–(11) and (13). Let the matrix functions $P(\lambda), Q(\lambda)$, and $W(\lambda)$ be partitioned as

$$
\begin{aligned}
P(\lambda) &\triangleq \begin{bmatrix} P_1(\lambda) & P_2(\lambda) \\ P_2^T(\lambda) & P_3(\lambda) \end{bmatrix}\\
Q(\lambda) &\triangleq \begin{bmatrix} Q_1(\lambda) & Q_2(\lambda) \\ Q_2^T(\lambda) & Q_3(\lambda) \end{bmatrix}\\
W(\lambda) &= \begin{bmatrix} W_1(\lambda) & W_2(\lambda) \\ W_4(\lambda) & W_3(\lambda) \end{bmatrix}.
\end{aligned} \quad (18)
$$

First, (13) implies $P(\lambda) - W^T(\lambda) - W(\lambda) < 0$ and $P(\lambda) > 0$, and then, we have $W^T(\lambda) + W(\lambda) > 0$; thus, $W(\lambda)$ is nonsingular, which leads to the nonsingularity of $W_3(\lambda)$. Due to the strict nature of the LMI constraints and by invoking a small perturbation if necessary, we can assume that $W_4(\lambda)$ is nonsingular without loss of generality [2]. Define the following invertible matrix functions:

$$
\begin{aligned}
J(\lambda) &\triangleq \begin{bmatrix} I & 0 \\ 0 & W_3^{-1}(\lambda)W_4(\lambda) \end{bmatrix}\\
K_1(\lambda) &\triangleq \mathrm{diag}\{J(\lambda), I\}\\
K_2(\lambda) &\triangleq \mathrm{diag}\{J(\lambda), J(\lambda), I\}\\
K_3(\lambda) &\triangleq \mathrm{diag}\{J(\lambda), J(\lambda), I, I\}
\end{aligned} \quad (19)
$$

and define

$$
\begin{aligned}
\bar{Q}(\lambda) &\triangleq \begin{bmatrix} \bar{Q}_1(\lambda) & \bar{Q}_2(\lambda) \\ * & \bar{Q}_3(\lambda) \end{bmatrix} = J^T(\lambda)Q(\lambda)J(\lambda)\\
\bar{P}(\lambda) &\triangleq \begin{bmatrix} \bar{P}_1(\lambda) & \bar{P}_2(\lambda) \\ * & \bar{P}_3(\lambda) \end{bmatrix} = J^T(\lambda)P(\lambda)J(\lambda).
\end{aligned} \quad (20)
$$

$$
\Phi(\lambda) \triangleq \begin{bmatrix} \Gamma_4 & \Gamma_5 & R^T(\lambda)B(\lambda) + \bar{B}_F(\lambda)D(\lambda) \\ * & \Gamma_6 & S^T(\lambda)B(\lambda) + \bar{B}_F(\lambda)D(\lambda) \\ * & * & -\Pi(\lambda) \end{bmatrix} < 0 \quad (14)
$$

$$
\Psi(\lambda) \triangleq \begin{bmatrix} \Gamma_4 & \Gamma_5 & R^T(\lambda)A(\lambda) + \bar{B}_F(\lambda)C(\lambda) & \bar{A}_F(\lambda) & 0 \\ * & \Gamma_6 & S^T(\lambda)A(\lambda) + \bar{B}_F(\lambda)C(\lambda) & \bar{A}_F(\lambda) & 0 \\ * & * & -\bar{Q}_1(\lambda) & -\bar{Q}_2(\lambda) & L^T(\lambda) \\ * & * & * & -\bar{Q}_3(\lambda) & -\bar{C}_F^T(\lambda) \\ * & * & * & * & -I \end{bmatrix} < 0 \quad (15)
$$

$$
\Theta(\lambda) \triangleq \begin{bmatrix} \Gamma_7 & \Gamma_8 & R^T(\lambda)A(\lambda) + \bar{B}_F(\lambda)C(\lambda) & \bar{A}_F(\lambda) & R^T(\lambda)E(\lambda) + \bar{B}_F(\lambda)F(\lambda) & 0 \\ * & \Gamma_9 & S^T(\lambda)A(\lambda) + \bar{B}_F(\lambda)C(\lambda) & \bar{A}_F(\lambda) & S^T(\lambda)E(\lambda) + \bar{B}_F(\lambda)F(\lambda) & 0 \\ * & * & -\bar{P}_1(\lambda) & -\bar{P}_2(\lambda) & 0 & L^T(\lambda) \\ * & * & * & -\bar{P}_3(\lambda) & 0 & -\bar{C}_F^T(\lambda) \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \quad (16)
$$

Then, performing congruence transformations to (10) by $K_1(\lambda)$, to (11) by $K_2(\lambda)$, to (12) by $K_3(\lambda)$, together with the consideration of (4) yields

$$\begin{bmatrix} \bar{Q}(\lambda)-\Psi_1 & \Psi_2 \\ * & -\Pi(\lambda) \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} \bar{Q}(\lambda)-\Psi_1 & \Psi_3 & 0 \\ * & -\bar{Q}(\lambda) & \Psi_4 \\ * & * & -I \end{bmatrix} < 0 \tag{22}$$

$$\begin{bmatrix} \bar{P}(\lambda)-\Psi_1 & \Psi_3 & \Psi_5 & 0 \\ * & -\bar{P}(\lambda) & 0 & \Psi_4 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{23}$$

where we have the first equation at the bottom of the page. By defining

$$R(\lambda) \triangleq W_1(\lambda) \tag{24}$$

$$S(\lambda) \triangleq W_2(\lambda)W_3^{-1}(\lambda)W_4(\lambda) \tag{25}$$

$$T(\lambda) \triangleq W_4^T(\lambda)W_3^{-1}(\lambda)W_4(\lambda) \tag{26}$$

$$\begin{bmatrix} \bar{A}_F(\lambda) & \bar{B}_F(\lambda) \\ \bar{C}_F(\lambda) & 0 \end{bmatrix} \triangleq \begin{bmatrix} W_4^T(\lambda) & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} A_F(\lambda) & B_F(\lambda) \\ C_F(\lambda) & 0 \end{bmatrix} \times \begin{bmatrix} W_3^{-1}(\lambda)W_4(\lambda) & 0 \\ 0 & I \end{bmatrix} \tag{27}$$

(21)–(23) are equivalent to (14)–(16), respectively, and the necessity is proved.

Sufficiency: Supposing that there exist matrices $\Pi(\lambda) > 0, \bar{P}(\lambda) > 0, \bar{Q}(\lambda) > 0, R(\lambda), S(\lambda), T(\lambda), \bar{A}_F(\lambda), \bar{B}_F(\lambda)$, and $\bar{C}_F(\lambda)$ satisfying (9) and (14)–(16), we will prove that there must exist filter matrices $(A_F(\lambda), B_F(\lambda), C_F(\lambda))$, and matrices $Q(\lambda) > 0, P(\lambda) > 0, \Pi(\lambda) > 0$, and $W(\lambda)$ satisfying (9)–(11) and (13).

First (15) implies $T(\lambda) + T^T(\lambda) - \bar{Q}_3(\lambda) > 0$, and then, we know that $T(\lambda)$ is nonsingular due to $\bar{Q}_3(\lambda) > 0$. Thus, one can always find square and nonsingular matrix functions $W_3(\lambda)$ and $W_4(\lambda)$ satisfying (26). Now, introduce the matrix functions $J(\lambda), K_1(\lambda), K_2(\lambda), K_3(\lambda)$, as defined in (19) and

$$W(\lambda) \triangleq \begin{bmatrix} R(\lambda) & S(\lambda)W_4^{-1}(\lambda)W_3(\lambda) \\ W_3(\lambda) & W_4(\lambda) \end{bmatrix}$$

$$Q(\lambda) \triangleq J^{-T}(\lambda)\bar{Q}(\lambda)J^{-1}(\lambda)$$

$$P(\lambda) \triangleq J^{-T}(\lambda)\bar{P}(\lambda)J^{-1}(\lambda)$$

$$\begin{bmatrix} A_F(\lambda) & B_F(\lambda) \\ C_F(\lambda) & 0 \end{bmatrix} \triangleq \begin{bmatrix} W_4^{-T}(\lambda) & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} \bar{A}_F(\lambda) & \bar{B}_F(\lambda) \\ \bar{C}_F(\lambda) & 0 \end{bmatrix} \times \begin{bmatrix} W_4^{-1}(\lambda)W_3(\lambda) & 0 \\ 0 & I \end{bmatrix}. \tag{28}$$

Then, we have $Q(\lambda) > 0$ and $P(\lambda) > 0$. Now, by some algebraic matrix manipulations, it can be established that (14)–(16) are equivalent to (29)–(31), shown at the bottom of the page.

Now, performing congruence transformations to (29) by $K_1^{-1}(\lambda)$, to (30) by $K_2^{-1}(\lambda)$, and to (31) by $K_3^{-1}(\lambda)$ yields (10), (11), and (13), and the sufficiency proof is completed.

$$\Psi_1 \triangleq \begin{bmatrix} W_1^T(\lambda)+W_1(\lambda) & W_4^T(\lambda)W_3^{-1}(\lambda)W_4(\lambda)+W_2(\lambda)W_3^{-1}(\lambda)W_4(\lambda) \\ * & W_4^T(\lambda)W_3^{-1}(\lambda)W_4(\lambda)+W_4^T(\lambda)W_3^{-T}(\lambda)W_4(\lambda) \end{bmatrix}$$

$$\Psi_2 \triangleq \begin{bmatrix} W_1^T(\lambda)B(\lambda)+W_4^T(\lambda)B_F(\lambda)D(\lambda) \\ W_4^T(\lambda)W_3^{-T}(\lambda)W_2^T(\lambda)B(\lambda)+W_4^T(\lambda)B_F(\lambda)D(\lambda) \end{bmatrix}$$

$$\Psi_3 \triangleq \begin{bmatrix} W_1^T(\lambda)A(\lambda)+W_4^T(\lambda)B_F(\lambda)C(\lambda) & W_4^T(\lambda)A_F(\lambda)W_3^{-1}(\lambda)W_4(\lambda) \\ W_4^T(\lambda)W_3^{-T}(\lambda)W_2^T(\lambda)A(\lambda)+W_4^T(\lambda)B_F(\lambda)C(\lambda) & W_4^T(\lambda)A_F(\lambda)W_3^{-1}(\lambda)W_4(\lambda) \end{bmatrix}$$

$$\Psi_4 \triangleq \begin{bmatrix} L^T(\lambda) \\ -W_4^T(\lambda)W_3^{-T}(\lambda)C_F^T(\lambda) \end{bmatrix}$$

$$\Psi_5 \triangleq \begin{bmatrix} W_1^T(\lambda)E(\lambda)+W_4^T(\lambda)B_F(\lambda)F(\lambda) \\ W_4^T(\lambda)W_3^{-T}(\lambda)W_2^T(\lambda)E(\lambda)+W_4^T(\lambda)B_F(\lambda)F(\lambda) \end{bmatrix}$$

$$\begin{bmatrix} J^T(\lambda)\Gamma_1 J(\lambda) & J^T(\lambda)W^T(\lambda)\bar{B}(\lambda) \\ * & -\Pi(\lambda) \end{bmatrix} < 0 \tag{29}$$

$$\begin{bmatrix} J^T(\lambda)\Gamma_1 J(\lambda) & J^T(\lambda)W^T(\lambda)\bar{A}(\lambda)J(\lambda) & 0 \\ * & -J^T(\lambda)Q(\lambda)J(\lambda) & J^T(\lambda)\bar{C}^T(\lambda) \\ * & * & -I \end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix} J^T(\lambda)\Gamma_2 J(\lambda) & J^T(\lambda)W^T(\lambda)\bar{A}(\lambda)J(\lambda) & J^T(\lambda)W^T(\lambda)\bar{E}(\lambda) & 0 \\ * & -J^T(\lambda)P(\lambda)J(\lambda) & 0 & J^T(\lambda)\bar{C}^T(\lambda) \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix}. < 0 \tag{31}$$

Proof of Second Part: If the condition in Proposition 2 has a set of feasible solutions $\{\Pi(\lambda), \bar{P}(\lambda), \bar{Q}(\lambda), R(\lambda), S(\lambda), T(\lambda), \bar{A}_F(\lambda), \bar{B}_F(\lambda), \bar{C}_F(\lambda)\}$, from the above proof, we know that the filter with a state-space realization $(\bar{A}(\lambda), \bar{B}(\lambda), \bar{C}(\lambda))$ defined in (28) guarantees the filtering error system $\mathcal{E}$ to be asymptotically stable with $\|T_{ew}(z,\lambda)\|_\infty < \gamma$ and $\|T_{ev}(z,\lambda)\|_2^2 < \beta$. Now, denote the operator from $y(t)$ to $z_F(t)$ by $T_{z_F y}(\lambda) \triangleq (A_F(\lambda), B_F(\lambda), C_F(\lambda))$; then, we have that $T_{z_F y}(\lambda)$ is equivalent to $\hat{T}_{z_F y}(\lambda)$ by similarity transformation, where

$$\hat{T}_{z_F y}(\lambda) \triangleq \big(W_4^{-1}(\lambda)W_3(\lambda)A_F(\lambda)W_3^{-1}(\lambda)W_4(\lambda) \\ W_4^{-1}(\lambda)W_3(\lambda)B_F(\lambda), C_F(\lambda)W_3^{-1}(\lambda)W_4(\lambda)\big).$$

By substituting the matrices with (28) and by considering the relationship (26), we have

$$\hat{T}_{z_F y}(\lambda) = (T^{-1}(\lambda)\bar{A}_F(\lambda), T^{-1}(\lambda)\bar{B}_F(\lambda), \bar{C}_F(\lambda)).$$

Therefore, an admissible filter can be given by (17), and the proof is completed. □

Remark 3: Proposition 1 is a preliminary result for solving the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem. It casts the nonlinear matrix inequality condition in Proposition 1 into an LMI condition by using linearization procedures on which desired filters can be constructed by using the obtained matrix functions $\Pi(\lambda), \bar{P}(\lambda), \bar{Q}(\lambda), R(\lambda), S(\lambda), T(\lambda), \bar{A}_F(\lambda), \bar{B}_F(\lambda)$, and $\bar{C}_F(\lambda)$. However, these LMI conditions still cannot be implemented since they are not convex in the parameter λ. It is noted that if we set $\Pi(\lambda) \equiv \Pi, \bar{P}(\lambda) \equiv \bar{P}, \bar{Q}(\lambda) \equiv \bar{Q}, R(\lambda) \equiv R, S(\lambda) \equiv S, T(\lambda) \equiv T, \bar{A}_F(\lambda) \equiv \bar{A}_F, \bar{B}_F(\lambda) \equiv \bar{B}_F$, and $\bar{C}_F(\lambda) \equiv \bar{C}_F$, we will readily obtain a robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering result in the quadratic framework similar to that obtained in [17].

To obtain less conservative results than [17], in the following, we will introduce new techniques that help convexify the matrix inequalities in Proposition 1, leading to LMIs that depend only on the vertices of the polytope $\mathcal{R}$. Then, we have the main filtering result in the following theorem.

Theorem 1 (Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Filtering): Given system $\mathcal{S}$ in (1), an admissible parameter-dependent mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter in the form of $\mathcal{F}$ in (2) exists if there exist matrices $\Pi_i > 0, \bar{P}_i \triangleq \begin{bmatrix} \bar{P}_{1i} & \bar{P}_{2i} \\ * & \bar{P}_{3i}\end{bmatrix} > 0, \bar{Q}_i \triangleq \begin{bmatrix} \bar{Q}_{1i} & \bar{Q}_{2i} \\ * & \bar{Q}_{3i}\end{bmatrix} > 0, R_i, S_i, T_i, \bar{A}_{Fi}, \bar{B}_{Fi}, \bar{C}_{Fi}, \Lambda_{ij}, \Sigma_{ij}$, and Ξ_{ij} satisfying

$$\mathrm{tr}\Pi_i < \beta, \quad (i = 1, \ldots, s) \tag{32}$$

$$\Phi_{ij} + \Phi_{ji} - \Lambda_{ij} - \Lambda_{ij}^T \le 0, \quad (1 \le i < j \le s) \tag{33}$$

$$\Psi_{ij} + \Psi_{ji} - \Sigma_{ij} - \Sigma_{ij}^T \le 0, \quad (1 \le i < j \le s) \tag{34}$$

$$\Theta_{ij} + \Theta_{ji} - \Xi_{ij} - \Xi_{ij}^T \le 0, \quad (1 \le i < j \le s) \tag{35}$$

$$\Lambda \triangleq \begin{bmatrix} \Phi_{11} & \Lambda_{12} & \cdots & \Lambda_{1s} \\ * & \Phi_{22} & \cdots & \Lambda_{2s} \\ * & * & \ddots & \vdots \\ * & * & * & \Phi_{ss} \end{bmatrix} < 0 \tag{36}$$

$$\Sigma \triangleq \begin{bmatrix} \Psi_{11} & \Sigma_{12} & \cdots & \Sigma_{1s} \\ * & \Psi_{22} & \cdots & \Sigma_{2s} \\ * & * & \ddots & \vdots \\ * & * & * & \Psi_{ss} \end{bmatrix} < 0 \tag{37}$$

$$\Xi \triangleq \begin{bmatrix} \Theta_{11} & \Xi_{12} & \cdots & \Xi_{1s} \\ * & \Theta_{22} & \cdots & \Xi_{2s} \\ * & * & \ddots & \vdots \\ * & * & * & \Theta_{ss} \end{bmatrix} < 0 \tag{38}$$

where we have the equation at the bottom of the page.

Moreover, under the above conditions, the matrix functions for an admissible mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter in the form of (2) are given by

$$\begin{bmatrix} A_F(\lambda) & B_F(\lambda) \\ C_F(\lambda) & 0 \end{bmatrix} = \begin{bmatrix} \left(\sum_{i=1}^s \lambda_i T_i\right)^{-1} & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} \sum_{i=1}^s \lambda_i \bar{A}_{Fi} & \sum_{i=1}^s \lambda_i \bar{B}_{Fi} \\ \sum_{i=1}^s \lambda_i \bar{C}_{Fi} & 0 \end{bmatrix}. \tag{39}$$

Proof: From Propositions 1 and 2, an admissible parameter-dependent mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter in the form of $\mathcal{F}$ in (2) exists if there exist matrix functions $\Pi(\lambda) > 0, \bar{P}(\lambda) > 0, \bar{Q}(\lambda) > 0, R(\lambda), S(\lambda), T(\lambda), \bar{A}_F(\lambda), \bar{B}_F(\lambda)$, and $\bar{C}_F(\lambda)$ satisfying (9)

$$\Phi_{ij} \triangleq \begin{bmatrix} \bar{Q}_{1i} - R_i^T - R_i & \bar{Q}_{2i} - T_i - S_i & R_i^T B_j + \bar{B}_{Fi} D_j \\ * & \bar{Q}_{3i} - T_i - T_i^T & S_i^T B_j + \bar{B}_{Fi} D_j \\ * & * & -\Pi_i \end{bmatrix}$$

$$\Psi_{ij} \triangleq \begin{bmatrix} \bar{Q}_{1i} - R_i^T - R_i & \bar{Q}_{2i} - T_i - S_i & R_i^T A_j + \bar{B}_{Fi} C_j & \bar{A}_{Fi} & 0 \\ * & \bar{Q}_{3i} - T_i - T_i^T & S_i^T A_j + \bar{B}_{Fi} C_j & \bar{A}_{Fi} & 0 \\ * & * & -\bar{Q}_{1i} & -\bar{Q}_{2i} & L_j^T \\ * & * & * & -\bar{Q}_{3i} & -\bar{C}_{Fi}^T \\ * & * & * & * & -I \end{bmatrix}$$

$$\Theta_{ij} \triangleq \begin{bmatrix} \bar{P}_{1i} - R_i^T - R_i & \bar{P}_{2i} - T_i - S_i & R_i^T A_j + \bar{B}_{Fi} C_j & \bar{A}_{Fi} & R_i^T E_j + \bar{B}_{Fi} F_j & 0 \\ * & \bar{P}_{3i} - T_i - T_i^T & S_i^T A_j + \bar{B}_{Fi} C_j & \bar{A}_{Fi} & S_i^T E_j + \bar{B}_{Fi} F_j & 0 \\ * & * & -\bar{P}_{1i} & -\bar{P}_{2i} & 0 & L_j^T \\ * & * & * & -\bar{P}_{3i} & 0 & -\bar{C}_{Fi}^T \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -I \end{bmatrix}$$

and (14)–(16). Now, assume that the above matrix functions are of the following form

$$\begin{aligned}
\Pi(\lambda) &= \sum_{i=1}^{s}\lambda_i\Pi_i, \quad \bar{P}(\lambda) = \sum_{i=1}^{s}\lambda_i\bar{P}_i \\
&= \sum_{i=1}^{s}\lambda_i\begin{bmatrix}\bar{P}_{1i} & \bar{P}_{2i}\\ * & \bar{P}_{3i}\end{bmatrix}\\
\bar{Q}(\lambda) &= \sum_{i=1}^{s}\lambda_i\bar{Q}_i = \sum_{i=1}^{s}\lambda_i\begin{bmatrix}\bar{Q}_{1i} & \bar{Q}_{2i}\\ * & \bar{Q}_{3i}\end{bmatrix}\\
R(\lambda) &= \sum_{i=1}^{s}\lambda_iR_i, \quad S(\lambda) = \sum_{i=1}^{s}\lambda_iS_i\\
T(\lambda) &= \sum_{i=1}^{s}\lambda_iT_i\\
\bar{A}_F(\lambda) &= \sum_{i=1}^{s}\lambda_i\bar{A}_{Fi}, \quad \bar{B}_F(\lambda) = \sum_{i=1}^{s}\lambda_i\bar{B}_{Fi}\\
\bar{C}_F(\lambda) &= \sum_{i=1}^{s}\lambda_i\bar{C}_{Fi}.
\end{aligned} \tag{40}$$

Then, with (40), it is not difficult to rewrite $\Phi(\lambda), \Psi(\lambda)$, and $\Theta(\lambda)$ in (14)–(16) as

$$\begin{aligned}
\Phi(\lambda) &= \sum_{j=1}^{s}\sum_{i=1}^{s}\lambda_i\lambda_j\Phi_{ij}\\
&= \sum_{i=1}^{s}\lambda_i^2\Phi_{ii} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\lambda_i\lambda_j(\Phi_{ij}+\Phi_{ji})
\end{aligned} \tag{41}$$

$$\begin{aligned}
\Psi(\lambda) &= \sum_{j=1}^{s}\sum_{i=1}^{s}\lambda_i\lambda_j\Psi_{ij}\\
&= \sum_{i=1}^{s}\lambda_i^2\Psi_{ii} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\lambda_i\lambda_j(\Psi_{ij}+\Psi_{ji})
\end{aligned} \tag{42}$$

$$\begin{aligned}
\Theta(\lambda) &= \sum_{j=1}^{s}\sum_{i=1}^{s}\lambda_i\lambda_j\Theta_{ij}\\
&= \sum_{i=1}^{s}\lambda_i^2\Theta_{ii} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\lambda_i\lambda_j(\Theta_{ij}+\Theta_{ji}).
\end{aligned} \tag{43}$$

On the other hand, (33)–(35) are equivalent to

$$\Phi_{ij}+\Phi_{ji} \le \Lambda_{ij}+\Lambda_{ij}^T, \quad (1 \le i < j \le s) \tag{44}$$
$$\Psi_{ij}+\Psi_{ji} \le \Xi_{ij}+\Xi_{ij}^T, \quad (1 \le i < j \le s) \tag{45}$$
$$\Theta_{ij}+\Theta_{ji} \le \Sigma_{ij}+\Sigma_{ij}^T, \quad (1 \le i < j \le s). \tag{46}$$

Then, from (41)–(46), we have

$$\begin{aligned}
\Phi(\lambda) &\le \sum_{i=1}^{s}\lambda_i^2\Phi_{ii} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\lambda_i\lambda_j\left\{\Lambda_{ij}+\Lambda_{ij}^T\right\} = \eta^T\Lambda\eta\\
\Psi(\lambda) &\le \sum_{i=1}^{s}\lambda_i^2\Psi_{ii} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\lambda_i\lambda_j\left\{\Xi_{ij}+\Xi_{ij}^T\right\} = \eta^T\Xi\eta\\
\Theta(\lambda) &\le \sum_{i=1}^{s}\lambda_i^2\Theta_{ii} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\lambda_i\lambda_j\left\{\Sigma_{ij}+\Sigma_{ij}^T\right\} = \eta^T\Sigma\eta
\end{aligned}$$

where $\eta \triangleq [\,\lambda_1 I \;\; \lambda_2 I \;\; \cdots \;\; \lambda_s I\,]^T$. Inequalities (36)–(38) guarantee $\Phi(\lambda) < 0, \Psi(\lambda) < 0$, and $\Theta(\lambda) < 0$. In addition, (32) is equivalent to (9), and the first part of the proof is completed.

By substituting the matrices defined in (40) into (17), we readily obtain (39), and the proof is completed. □

Remark 4: The idea behind Theorem 1 is to use convex combinations of vertex matrices in the form of (40) to substitute the matrix functions in Proposition 2. With the introduction of these matrices, and by means of the bounding technique used in the Proof of Theorem 1, the infinite-dimensional nonlinear matrix inequality conditions in Proposition 2 are cast into finite-dimensional LMI conditions, which depend only on the vertex matrices of the polytope $\mathcal{R}$ and, therefore, can be readily checked by using standard numerical software [7].

Remark 5: From the Proofs of Proposition 2 and Theorem 1, it is not difficult to see that in solving the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering problem, we actually define multiple Lyapunov functions for each performance objective, that is, $P(\lambda)$ takes the form of $P(\lambda) = \sum_{i=1}^{s}\lambda_iP_i$, and $Q(\lambda)$ takes the form $Q(\lambda) = \sum_{i=1}^{s}\lambda_iQ_i$. The filter design based on parameter-dependent Lyapunov functions has been investigated in [9] and [23], where parameter-dependent idea is realized at the expense of setting an additional slack variable to be constant for each vertex of the polytope. Notably, here in Theorem 1, we do not set any matrix variable to be constant for the whole polytope domain, and therefore, Theorem 1 has the potential to yield less conservative results in applications where the parameters involved can be measured online, which will be illustrated via a numerical example in the next section.

Remark 6: Note that the conditions in Theorem 1 are LMIs not only over the matrix variables but also over the scalars γ and β. This implies that the scalars γ and β can be included as optimization variables to obtain a reduction of the attenuation level bound. As in [17], it is usually desired to design filters with minimized $\mathcal{H}_2$ performance β and prescribed $\mathcal{H}_\infty$ performance γ, which can be readily found by solving the following convex optimization problem:

$$\text{Minimize } \beta \quad \text{subject to (32)–(36) for given } \gamma.$$

IV. Illustrative Example

Consider the following numerical example borrowed from [17] with small modifications:

$$\begin{aligned}
x(t+1) &= \begin{bmatrix}0 & -0.8\\ 1.2+0.1a & -0.5\end{bmatrix}x(t)\\
&\quad + \begin{bmatrix}0\\1\end{bmatrix}v(t) + \begin{bmatrix}-0.45\\ 0.35\end{bmatrix}w(t)\\
y(t) &= [\,0.35 \quad -0.65\,]\,x(t) + 1.3v(t) + 0.4w(t)\\
z(t) &= [0.2 \quad 0]x(t)
\end{aligned} \tag{47}$$

where a satisfies $|a| \le \bar{a}$. This system can be modeled with a two-vertex polytope.

First, consider the nominal system (corresponding to $a = 0$). Fixing the $\mathcal{H}_\infty$ performance $\gamma = 0.2$, by Theorem 1, the obtained minimum $\mathcal{H}_2$ performance of admissible mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters is $\beta = 0.0331$, and the associated matrices for filter (2)

TABLE I
MINIMUM $\mathcal{H}_2$ PERFORMANCE FOR DIFFERENT CASES

Prescribed $\mathcal{H}_\infty$ performance γ	0.2			0.3		
$\bar{a}$	0.1	0.13	0.4	0.1	0.21	0.4
Minumum β by Theorem 1	0.0346	0.0350	0.0392	0.0021	0.0028	0.0041
Minumum β by [17]	0.0705	infeasible	infeasible	0.0244	infeasible	infeasible

are given by (note that for the nominal case, the filter matrices are constant)

$$A_F = \begin{bmatrix} -0.1552 & -0.5228 \\ 0.7225 & 0.3843 \end{bmatrix}, \quad B_F = \begin{bmatrix} -0.4311 \\ -1.3404 \end{bmatrix}$$
$$C_F = [-0.2000 \quad 0].$$

The actual calculated performances of the filtering error system by connecting the above filter to the original system are given by $\|T_{ev}(z)\|_2^2 = 0.0261$ and $\|T_{ew}(z)\|_\infty = 0.1986$, which are below their corresponding prescribed values, showing the effectiveness of the filter design method. In addition, by [17, Corol. 1], the obtained minimum $\mathcal{H}_2$ performance of admissible mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters is $\beta = 0.0429$ for prescribed $\mathcal{H}_\infty$ performance $\gamma = 0.2$, which is higher than that obtained by our method, showing the lesser conservativeness of the filter design method developed in this paper. As analyzed previously, the lesser conservativeness comes naturally as a result of the introduction of different Lyapunov matrices for different performance channels.

Now, consider the case $|a| \leq \bar{a}$. Assuming $\bar{a} = 0.1$ and fixing the $\mathcal{H}_\infty$ performance $\gamma = 0.2$, by Theorem 1, the obtained minimum $\mathcal{H}_2$ performance of admissible parameter-dependent mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters is $\beta = 0.0346$, and the associated matrices needed for the calculation of (39) are given by

$$T_1 = \begin{bmatrix} 0.0561 & 0.0068 \\ 0.0071 & 0.0287 \end{bmatrix}$$
$$T_2 = \begin{bmatrix} 0.0567 & 0.0071 \\ 0.0068 & 0.0308 \end{bmatrix}$$
$$\bar{A}_{F1} = \begin{bmatrix} -0.0041 & -0.0259 \\ 0.0197 & 0.0077 \end{bmatrix}$$
$$\bar{A}_{F2} = \begin{bmatrix} -0.0038 & -0.0273 \\ 0.0206 & 0.0082 \end{bmatrix}$$
$$\bar{B}_{F1} = \begin{bmatrix} -0.0345 \\ -0.0422 \end{bmatrix}, \quad \bar{B}_{F2} = \begin{bmatrix} -0.0339 \\ -0.0442 \end{bmatrix}$$
$$\bar{C}_{F1} = [-0.2000 \quad 0], \quad \bar{C}_{F2} = [-0.2001 \quad 0]. \tag{48}$$

By calculation, it is found that the maximum $\mathcal{H}_2$ norm of the filtering error system for different a by connecting the above filter to the original system is 0.0276, and the maximum $\mathcal{H}_\infty$ norm is 0.1984, which are all below their prescribed values. By [17, Corol. 1], the obtained minimum $\mathcal{H}_2$ performance of admissible robust mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters is $\beta = 0.0705$ for prescribed $\mathcal{H}_\infty$ performance $\gamma = 0.2$, which is higher than that obtained by our method, showing again the lesser conservativeness of the filter design method developed in this paper for applications where the parameter a is measurable online. The filter matrix functions with respect to the value of a can be given explicitly by

$$A_F(a) = \left(\frac{\bar{a}+a}{2\bar{a}}T_1 + \frac{\bar{a}-a}{2\bar{a}}T_2\right)^{-1} \times \left(\frac{\bar{a}+a}{2\bar{a}}\bar{A}_{F1} + \frac{\bar{a}-a}{2\bar{a}}\bar{A}_{F2}\right)$$
$$B_F(a) = \left(\frac{\bar{a}+a}{2\bar{a}}T_1 + \frac{\bar{a}-a}{2\bar{a}}T_2\right)^{-1} \times \left(\frac{\bar{a}+a}{2\bar{a}}\bar{B}_{F1} + \frac{\bar{a}-a}{2\bar{a}}\bar{B}_{F2}\right)$$
$$C_F(a) = \frac{\bar{a}+a}{2\bar{a}}\bar{C}_{F1} + \frac{\bar{a}-a}{2\bar{a}}\bar{C}_{F2}.$$

In order to provide relatively complete information, Table I presents a comparison between minimum $\mathcal{H}_2$ performance obtained by using Theorem 1 and [17, Corol. 1] for different cases. From the table, it can be seen that the filter design method presented in this paper produces much less conservative results. Notably for $\bar{a} = 0.4$, where the previous method fails to find feasible solutions, Theorem 1 is still able to provide desired filters.

V. CONCLUSION

This paper has presented a novel approach to design mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters for discrete-time systems with polytopic bounded parameters. Given a stable discrete-time linear system with parameters residing in a polytope, attention is focused on the design of parameter-dependent filters such that the filtering error system is asymptotically stable and has guaranteed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances for different performance objectives. Sufficient conditions are obtained for the existence of admissible filters in terms of a set of linear matrix inequalities, upon which the filter designs are cast into convex optimization problems. In solving this filtering problem, to reduce the conservativeness, we have made full use of the parameter-dependent stability idea: The designed filters are dependent of the slow time-varying parameters (which are assumed to be measurable online); the Lyapunov matrices for different performance objectives are enabled to be different by the introduction of additional slack variables, and multiple Lyapunov matrices have been developed for the entire polytope domain using some new bounding techniques. Based on the above three aspects that appear to be quite different from previous results in the quadratic framework, the filter design method developed in this paper is much less conservative than previous ones, as is illustrated via a numerical example. Finally, it is worth pointing out that a disadvantage of the filter design approach developed here is that the number of LMIs and matrix variables will be very large when the vertex number of the polytope increases. One possible way to reduce the number of the matrix variables in Theorem 1 is to simply set $\Lambda_{ij} = 0, \Sigma_{ij} = 0$, and $\Xi_{ij} = 0$, which will introduce some degree of conservativeness.

ACKNOWLEDGMENT

The authors are grateful to the anonymous reviewers for their valuable comments and suggestions that helped them improve the presentation of the paper. Special thanks are given to one reviewer who has shown his/her strong interest in this work by carefully examining the simulation and provided interesting discussions.

REFERENCES

[1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*. Englewood Cliffs, NJ: Prentice-Hall, 1979.
[2] P. Apkarian, H. D. Tuan, and J. Bernussou, "Continuous-time analysis, eigenstructure assignment, and H_2 synthesis with enhanced linear matrix inequalities (LMI) characterizations," *IEEE Trans. Autom. Control*, vol. 46, no. 12, pp. 1941–1946, Dec. 2001.
[3] D. S. Bernstein and W. M. Haddad, "Steady-state Kalman filtering with an error bounded," *Syst. Control Lett.*, vol. 12, pp. 9–16, 1989.
[4] X. Chen and K. Zhou, "Multiobjective filtering design," in *Proc. IEEE Canadian Conf. Elect. Comput. Eng.*, Edmonton, AB, Canada, 1999, pp. 708–713.
[5] M. C. De Oliveira, J. C. Geromel, and J. Bernussou, "Extended H_2 and H_∞ norm characterizations and controller parametrizations for discrete-time systems," *Int. J. Control*, vol. 75, pp. 666–679, 2002.
[6] A. Elsayed and M. J. Grimble, "A new approach to H_∞ design of optimal digital linear filters," *IMA J. Math. Control Inf.*, vol. 6, pp. 233–251, 1989.
[7] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, *LMI Control Toolbox User's Guide*. Natick, MA: Mathworks, 1995.
[8] H. Gao and C. Wang, "Delay-dependent robust H_∞ and L_2-L_∞ filtering for a class of uncertain nonlinear time-delay systems," *IEEE Trans. Autom. Control*, vol. 48, no. 9, pp. 1661–1666, Sep. 2003.
[9] ——, "Robust energy-to-peak filtering with improved LMI representations," *Proc. Inst. Elect. Eng. J: Vision, Image, Signal Process.*, vol. 150, no. 2, pp. 82–89, 2003.
[10] K. M. Grigoriadis and J. T. Watson, "Reduced order H_∞ and L_2-L_∞ filtering via linear matrix inequalities," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 33, no. 4, pp. 1326–1338, Oct. 1997.
[11] N. T. Hoang, H. D. Tuan, P. Apkarian, and S. Hosoe, "Gain-scheduled filtering for time-varying discrete systems," *IEEE Trans. Signal Process.*, vol. 52, no. 9, pp. 2464–2476, Sep. 2004.
[12] R. E. Kalman, "A new approach to linear filtering and prediction problems," *J. Basic Eng.*, vol. 82D, no. 1, pp. 35–45, 1960.
[13] P. P. Khargonekar, M. A. Rotea, and E. Baeyens, "Mixed H_2/H_∞ filtering," *Int. J. Robust Nonlinear Control*, vol. 6, no. 6, pp. 313–330, 1996.
[14] V. J. S. Leite, V. F. Montagner, and P. L. D. Peres, "Robust pole location by parameter dependent state feedback control," in *Proc. 41st IEEE Conf. Decision Contr.*, Las Vegas, NV, 2002, pp. 1864–1869.
[15] H. Li and M. Fu, "A linear matrix inequality approach to robust H_∞ filtering," *IEEE Trans. Signal Process.*, vol. 45, no. 9, pp. 2338–2350, Sep. 1997.
[16] X. Liu, X. Zhang, and X. Xin, "Application of Kalman filter in agricultural economic forecasting," in *Proc. IEEE Int. Conf. Syst., Man, Cybern.*, 1996, pp. 2185–2189.
[17] R. M. Palhares and P. L. D. Peres, "LMI approach to the mixed H_2/H_∞ filtering design for discrete-time systems," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 37, no. 1, pp. 292–296, Jan. 2001.
[18] D. Peaucelle, D. Arzelier, J. Bachelier, and J. Bernussou, "A new robust D-stability condition for real convex polytopic uncertainty," *Syst. Control Lett.*, vol. 40, pp. 21–30, 2000.
[19] I. R. Petersen and A. V. Savkin, *Robust Kalman Filtering for Signals and Systems With Large Uncertainties*. Boston, MA: Birkhauser, 1999.
[20] H. Qi and J. B. Moore, "Direct Kalman filtering approach for GPS/INS integration," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 38, no. 2, pp. 687–693, Apr. 2002.
[21] H. Rotstein, M. Sznaier, and M. Idan, "H_2/H_∞ filtering theory and an aerospace application," *Int. J. Robust Nonlinear Control*, vol. 6, pp. 347–366, 1996.
[22] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Trans. Autom. Control*, vol. 42, no. 7, pp. 896–911, Jul. 1997.
[23] U. Shaked, L. Xie, and Y. C. Soh, "New approaches to robust minimum variance filter design," *IEEE Trans. Signal Process.*, vol. 49, no. 11, pp. 2620–2629, Nov. 2001.
[24] Y. Theodor and U. Shaked, "A dynamic game approach to mixed H_2/H_∞ estimation," *Int. J. Robust Nonlinear Control*, vol. 6, no. 4, pp. 331–345, 1996.
[25] Z. Wang and B. Huang, "Robust H_2/H_∞filtering for linear systems with error variance constraints," *IEEE Trans. Signal Process.*, vol. 48, no. 8, pp. 2463–2467, Aug. 2000.
[26] Z. Wang and H. Unbehauen, "Robust H_2/H_∞ state estimation for systems with error variance constraints: The continuous-time case," *IEEE Trans. Autom. Control*, vol. 44, no. 5, pp. 1061–1065, May 1999.
[27] L. Xie, Y. C. Soh, and C. E. de Souza, "Robust Kalman filtering for uncertain discrete-time systems," *IEEE Trans. Autom. Control*, vol. 39, no. 6, pp. 1310–1314, Jun. 1994.
[28] S. Xu and T. Chen, "Reduced-order H_∞ filtering for stochastic systems," *IEEE Trans. Signal Process.*, vol. 50, no. 12, pp. 2998–3007, Dec. 2002.

Huijun Gao was born in Heilongjiang Province, China, in 1976. He received the M.S. degree in electrical engineering from Shenyang University of Technology, Shengyang, China, in 2001 and the Ph.D. degree in control science and engineering from Harbin Institute of Technology, Harbin, China, in 2005.

From November 2003 to September 2004, he served as a Research Associate with the Department of Mechanical Engineering, University of Hong Kong, Hong Kong. He is now a lecturer with the Space Control and Inertial Technology Research Center, Harbin Institute of Technology. His research interests include robust control/filter theory, model reduction, time-delay systems, and their applications.

James Lam (SM'99) received the first-class B.Sc. degree in mechanical engineering from the University of Manchester, Manchester, U.K., in 1983. He received the M.Phil. and Ph.D. degrees in control engineering in 1985 and 1988, respectively, from the University of Cambridge Cambridge, U.K. His postdoctoral research was carried out in the Australian National University, Canberra, Australia, between 1990 and 1992.

His research interests include model reduction, delay systems, descriptor systems, stochastic systems, multidimensional systems, robust control and filtering, fault detection, and reliable control. He has held faculty positions at the City University of Hong Kong and the University of Melbourne, Parkville, Australia. He is now an Associate Professor with the Department of Mechanical Engineering, the University of Hong Kong, and holds a Concurrent Professorship at the Northeastern University, Shenyang, China; a Guest Professorship at the Huazhong University of Science and Technology, Wuhan, China; a Consulting Professorship at the South China University of Technology, Guangzhou, China; and Guest Professorship of Shandong University, Jinan, China. He is an Associate Editor of the *Asian Journal of Control* and the *International Journal of Applied Mathematics and Computer Science*.

Dr. Lam is a Chartered Mathematician, a Fellow of the Institute of Mathematics and Its Applications (U.K.), and a Member of the Institution of Electrical Engineers (U.K.). He received the Ashbury Scholarship, the A.H. Gibson Prize, and the H. Wright Baker Prize for his academic performance. Dr. Lam is a Scholar (1984) and Fellow (1990) of the Croucher Foundation.

Lihua Xie (SM'97) received the B.E. and M.E. degrees in electrical engineering from Nanjing University of Science and Technology, Nanjing, China, in 1983 and 1986, respectively, and the Ph.D. degree in electrical engineering from the University of Newcastle, Callaghan, Australia, in 1992. engineering from the University of Newcastle, Callaghan, Australia, in 1992.

From April 1986 to January 1989, he held teaching appointments with the Department of Automatic Control, Nanjing University of Science and Technology. He has been with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, since 1992, where he is currently a professor. His current research interests include robust control, estimation theory and signal processing, and networked control systems. He has authored and co-authored one book and over 110 articles in refereed international journals.

Dr. Xie was the General Co-Chair of the 2002 International Conference on Control and Automation, Xiamen, China, the Program Chair of the Fourth International Conference on Control and Automation, Montreal, QC, Canada, June 2003, and the Program Chair of the Eighth International Conference on Control, Automation, Robotics, and Vision, Kunming, China, December 2004. He is currently an associate editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, the *International Journal of Control, Automation, and Systems,* and the *Journal of Control Theory and Applications*. He served as an associate editor of the Conference Editorial Board, IEEE Control Systems Society, from 2000 to 2004.

Changhong Wang received B.E., M.E., and Ph.D. degrees from Harbin Institute of Technology, Harbin, China, in 1983, 1986, and 1991, respectively.

From 1993 to 1994, he was a Postdoctoral Fellow with the Automation Department, Faculte Polytechnique de Mons, Mons, Belgium. In 1986, he joined the Department of Control Science and Control Engineering, Harbin Institute of Technology, where he is presently a full professor. He has served as the Director of Control Theory and Application for five years and as the Chairman of the Department of Control Science and Control Engineering for three years. Currently, he is the Deputy Dean of Academy of Industrial Technology, the Deputy Dean of School of Astronautics, and the Director of the Space Control and Inertial Technology Research Center, Harbin Institute of Technology. His research interests include intelligent control and intelligent systems, inertial technology and its testing equipment, robotics, precision servo systems, and network control. He has published over 80 refereed papers in technical journals, books, and conference proceedings.

Prof. Wang serves as the Director of China Inertial Technology Association, the Director of China Computer Simulation Association, and the Director of the Heilongjiang Automation Association.