Control for Stability and Positivity: Equivalent Conditions and Computation

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***Abstract*—This paper investigates the stabilizability of linear systems with closed-loop positivity. A necessary and sufficient condition for the existence of desired state-feedback controllers guaranteeing the resultant closed-loop system to be asymptotically stable and positive is obtained. Both continuous- and discrete-time cases are considered, and all of the conditions are expressed as linear matrix inequalities which can be easily verified by using standard numerical software. Numerical examples are provided to illustrate the proposed conditions.**

***Index Terms*—Linear matrix inequality, Metzler matrix, nonnegative matrix, positive systems, stabilization.**

I. INTRODUCTION

IN MANY practical systems, variables are constrained to be nonnegative. Such constraints abound in physical systems where variables are used to represent levels of heat, population, and storage. For instance, age-structured populations described by certain Leslie models [6], compartmental models used in hydrology and biology applications, can be described by positive systems [13], [18], whose states and outputs are nonnegative whenever the initial condition and input signal are nonnegative. Since positive systems are defined on cones, not on linear spaces, many well-established results of general linear systems cannot be simply applied to positive systems. Therefore, in recent years, many researchers have shown their interests in positive systems and many fundamental results have been reported (see, for instance, [1]–[3], [7], [11], [12], [16], [17], [19], and [20] and the references therein).

Among the great number of research results obtained for positive systems, much attention has been devoted to the behavioral analysis of such systems (readers are referred to [8] and [15] for a detailed account of the recent developments in positive systems). Meanwhile, the synthesis problems under the positivity constraint seem to have received relatively less attention. More specifically, the results about how to design controllers to obtain a closed-loop system which is stable and positive are still very limited [10], [21]. That is, given a possibly unstable linear system, does there exist a controller such that the resultant closed-loop system is asymptotically stable and positive? Moreover, if the answer is yes, how can we find one?

Recently, Kaczorek [14] investigated the problem mentioned above. Using Gersgorin's theorem, existence conditions for state-feedback controllers were proposed for positive systems. It is worth mentioning that these conditions are only *sufficient*, and are only suitable for single-input systems.

In the present work, we further investigate the stabilization problem for both continuous- and discrete-time multiple-input–multiple-output (MIMO) systems under the condition that the closed-loop system is positive. Instead of using algebraic techniques which have been widely employed for the analysis of positive systems, our development is based on matrix inequalities. Based on the well-established results of Lyapunov stability theory and nonnegative matrix, equivalent conditions in terms of linear matrix inequalities (LMIs) are obtained for the existence of stabilizing state-feedback controllers. A remarkable advantage of these conditions lies in the fact that they are not only necessary and sufficient, but also can be easily verifiable by using some standard numerical software. Moreover, these conditions readily construct a desired controller if it exists. To the authors' knowledge, this work represents the first LMI treatment on control synthesis for guaranteeing asymptotic stability and positivity.

The remainder of this paper is organized as follows. Sections II and III present a necessary and sufficient condition for stabilization with positivity constraint, both for continuous- and discrete-time linear systems. Numerical examples are given in Section IV to illustrate the proposed method, and we conclude this paper in Section V.

Notations: The notations used throughout the paper are fairly standard. The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$; $\mathbb{R}_+^{m\times n}$ is the set of all $m \times n$ real matrices with nonnegative entries and $\mathbb{R}_+^n \triangleq \mathbb{R}_+^{1\times n}$; the notation $P > 0$ means that P is real symmetric and positive definite; I and 0 represent identity matrix and zero matrix, respectively; $\mathrm{diag}\{\dots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. CONTINUOUS-TIME CASE

Consider the following MIMO continuous-time system $\mathcal{C}$:

$$\mathcal{C}: \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

Manuscript received May 11, 2004; revised November 1, 2004. This work was supported in part by HKU RGC Grant 7127/02P and the National Natural Science Foundation of P.R. China under Grant 60304001. This paper was recommended by Associate Editor P. Bauer.

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Digital Object Identifier 10.1109/TCSII.2005.850525

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^l$ is the control input vector, $y(t) \in \mathbb{R}^m$ is the output vector, and $A \triangleq [a_{ij}] \in \mathbb{R}^{n\times n}$, $B \triangleq [b_{ij}] \in \mathbb{R}^{n\times l}$, $C \triangleq [c_{ij}] \in \mathbb{R}^{m\times n}$, $D \triangleq [d_{ij}] \in \mathbb{R}^{m\times l}$ are system matrices.

We first introduce the following definitions [4], [8], [15].

Definition 1: System $\mathcal{C}$ in (1) is called positive if, for any $x_0 \in \mathbb{R}_+^n$ and any $u(t) \in \mathbb{R}_+^l$ for every $t \geq 0$, we have $y(t) \in \mathbb{R}_+^m$ for $t \geq 0$.

Definition 2: A matrix M is called a Metzler matrix if all of its off-diagonal entries are nonnegative.

Assume that the state variable $x(t)$ can be directly measured, and our purpose in this section is to design a state-feedback control law of the following form:

$$\mathcal{G}: \quad u(t) = Kx(t) + v(t) \tag{2}$$

such that the closed-loop system given by

$$\begin{aligned} \mathcal{C}_c: \quad \dot{x}(t) &= \bar{A}x(t) + Bv(t), \quad x(0) = x_0 \\ y(t) &= \bar{C}x(t) + Dv(t) \end{aligned} \tag{3}$$

is asymptotically stable and positive, where $K \triangleq [k_{ij}] \in \mathbb{R}^{l\times n}$ is the controller gain to be determined and

$$\bar{A} \triangleq [\bar{a}_{ij}] = A + BK, \quad \bar{C} \triangleq [\bar{c}_{ij}] = C + DK. \tag{4}$$

The following lemmas will be essential for our derivation [8], [15].

Lemma 1: Given the system $\mathcal{C}$ in (1) and the controller $\mathcal{G}$ in (2), the closed-loop system $\mathcal{C}_c$ in (3) is positive if and only if $\bar{A}$ is a Metzler matrix and $B \in \mathbb{R}_+^{n\times l}$, $\bar{C} \in \mathbb{R}_+^{m\times n}$, and $D \in \mathbb{R}_+^{m\times l}$.

Remark 1: As the matrices B and D are invariant under state-feedback law $\mathcal{G}$ in (2), their positivity is necessary for $\mathcal{C}_c$ to be positive. However, no such condition is imposed on A and C for system $\mathcal{C}$, which means that the original system $\mathcal{C}$ in (1) is not necessarily positive. Therefore, the controller is designed not only to stabilize the system, but also to render the closed-loop system positive.

Lemma 2: Given the system $\mathcal{C}$ in (1) and the controller $\mathcal{G}$ in (2), assume the closed-loop system $\mathcal{C}_c$ in (3) is positive, then it is asymptotically stable if and only if there exists a positive diagonal matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$\bar{A}^T P + P\bar{A} < 0. \tag{5}$$

We are in a position to present our main result.

Theorem 1: Given the system $\mathcal{C}$ in (1) with $B \in \mathbb{R}_+^{n\times l}$ and $D \in \mathbb{R}_+^{m\times l}$. A controller of the form $\mathcal{G}$ in (2) such that the closed-loop system $\mathcal{C}_c$ in (3) is asymptotically stable and positive exists if and only if there exist a positive diagonal matrix $Q \triangleq \operatorname{diag}\{q_1, q_2, \ldots, q_n\}$ and a matrix $\bar{K} \triangleq [\bar{k}_{ij}] \in \mathbb{R}^{l\times n}$ satisfying

$$QA^T + \bar{K}^T B^T + AQ + B\bar{K} < 0 \tag{6}$$

$$a_{ij}q_j + \sum_{z=1}^{l} b_{iz}\bar{k}_{zj} \geq 0, \quad 1 \leq i \neq j \leq n \tag{7}$$

$$c_{ij}q_j + \sum_{z=1}^{l} d_{iz}\bar{k}_{zj} \geq 0, \quad 1 \leq i,\ j \leq n. \tag{8}$$

Under the above conditions, the matrix gain of a desired controller $\mathcal{G}$ in (2) is given by

$$K = \bar{K}Q^{-1}. \tag{9}$$

Proof: (*Sufficiency*) First, from (9), we have $k_{zj} = \bar{k}_{zj}q_j^{-1}$. By noticing $q_j > 0$, (7) and (8) trivially ensure that $\bar{A}$ is a Metzler matrix and $\bar{C} \in \mathbb{R}_+^{m\times n}$. Then, by the positivity of $B \in \mathbb{R}_+^{n\times l}$ and $D \in \mathbb{R}_+^{m\times l}$, from Lemma 1 we know that the closed-loop system is positive.

Second, from (9), we have

$$\bar{K} = KQ. \tag{10}$$

By substituting (10) into (6), we obtain

$$QA^T + QK^TB^T + AQ + BKQ < 0. \tag{11}$$

By applying to (11) the congruence transformation defined by Q^{-1} and keeping in mind (4), one gets

$$\bar{A}^T Q^{-1} + Q^{-1}\bar{A} < 0.$$

By defining $P \triangleq Q^{-1}$, we readily obtain (5). Then, from Lemma 2, we know that the closed-loop system is asymptotically stable.

(*Necessity*) Suppose there exists a controller of the form $\mathcal{G}$ in (2) such that the closed-loop system $\mathcal{C}_c$ in (3) is asymptotically stable and positive. Then, from Lemmas 1 and 2, we know that $\bar{A}$ is a Metzler matrix, $\bar{C} \in \mathbb{R}_+^{m\times n}$, and there exists a positive diagonal matrix $P \triangleq \operatorname{diag}\{p_1, p_2, \ldots, p_n\} \in \mathbb{R}^{n\times n}$ satisfying (5).

First, by applying to (5) the congruence transformation defined by P^{-1} and keeping in mind (4), one obtains

$$P^{-1}A^T + P^{-1}K^TB^T + AP^{-1} + BKP^{-1} < 0.$$

By defining

$$Q \triangleq P^{-1}, \quad \bar{K} \triangleq KQ \tag{12}$$

we readily obtain (6).

Second, $\bar{A}$ is a Metzler matrix and $\bar{C} \in \mathbb{R}_+^{m\times n}$ implies

$$a_{ij} + \sum_{z=1}^{l} b_{iz}k_{zj} \geq 0, \qquad 1 \leq i \neq j \leq n$$

$$c_{ij} + \sum_{z=1}^{l} d_{iz}k_{zj} \geq 0, \qquad 1 \leq i,\ j \leq n$$

which are trivially equivalent to (7) and (8), respectively, by noticing (12). □

Remark 2: Theorem 1 presents a necessary and sufficient condition for the existence of desired controllers. Conditions (6)–(8) are all LMIs, that is, they are convex in the matrix variables Q and $\bar{K}$; therefore, these conditions can be readily checked by using standard numerical software (such as LMI toolbox in Matlab [9]).

Remark 3: It is noted that the problem addressed in this section is similar to the stabilizability-holdability problem considered in [3, Ch. 7]. In [3], a necessary and sufficient condition is proposed for the existence of desired controllers guaranteeing the closed-loop system to be asymptotically stable and positive. The condition obtained in [3, p. 133], consisting of a set of inequalities, is in general nonlinear and thus not easy to solve. So far, no efficient algorithm has been proposed to solve that condition except for the very particular case of scalar input. In this section, we develop another necessary and sufficient condition for the existence of desired controllers guaranteeing the closed-loop system to be asymptotically stable and positive, which is numerically tractable by using standard numerical software. An obvious advantage is that the condition developed here is not only suitable for the case of scalar input, but also applicable to the case of vector input.

III. Discrete-Time Case

The results obtained in the above section can also be developed for the discrete-time case. For simplicity, unless otherwise defined, we associate the same meanings to the notations used in Section II.

Now consider the following MIMO discrete-time system $\mathcal{D}$:

$$\mathcal{D}: \quad \begin{aligned} x(t+1) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{13}$$

where $x(t)$, $u(t)$, $y(t)$, A, B, C, and D have the same dimensions and meanings as in the continuous-time case.

We introduce the following definition and assumptions [8], [15].

Definition 3: System $\mathcal{D}$ in (13) is called positive if for any $x_0 \in \mathbb{R}_+^n$ and any $u(t) \in \mathbb{R}_+^l$ for every $t \geq 0$, we have $y(t) \in \mathbb{R}_+^m$ for $t \geq 0$.

Assume the state variable $x(t)$ can be directly measured, and our purpose in this section is to design a state-feedback control law of the form (2) such that the closed-loop system given by

$$\mathcal{D}_c: \quad \begin{aligned} x(t+1) &= \bar{A}x(t) + Bv(t), \qquad x(0) = x_0 \\ y(t) &= \bar{C}x(t) + Dv(t) \end{aligned} \tag{14}$$

is asymptotically stable and positive, where $\bar{A}$ and $\bar{C}$ are defined in (4).

The following lemmas will be useful in the subsequent development [8], [15].

Lemma 3: Given the system $\mathcal{D}$ in (13) and the controller $\mathcal{G}$ in (2), the closed-loop system $\mathcal{D}_c$ in (14) is positive if and only if $\bar{A} \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times l}$, $\bar{C} \in \mathbb{R}_+^{m\times n}$, and $D \in \mathbb{R}_+^{m\times l}$.

Lemma 4: Given the system $\mathcal{D}$ in (13) and the controller $\mathcal{G}$ in (2), assume that the closed-loop system $\mathcal{D}_c$ in (14) is positive. Then it is asymptotically stable if and only if there exists a positive diagonal matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$\bar{A}^T P \bar{A} - P < 0. \tag{15}$$

Then our main result for discrete-time systems is expressed as the following theorem.

Theorem 2: Given the system $\mathcal{D}$ in (13) with $B \in \mathbb{R}_+^{n\times l}$ and $D \in \mathbb{R}_+^{m\times l}$, a controller of the form $\mathcal{G}$ in (2) such that the closed-loop system $\mathcal{D}_c$ in (14) is asymptotically stable and positive exists if and only if there exist a positive diagonal matrix $Q \triangleq \operatorname{diag}\{q_1, q_2, \ldots, q_n\}$ and a matrix $\bar{K} \triangleq [\bar{k}_{ij}] \in \mathbb{R}^{l\times n}$ satisfying

$$\begin{bmatrix} -Q & AQ+B\bar{K} \\ QA^T+\bar{K}^T B^T & -Q \end{bmatrix} < 0 \tag{16}$$

$$a_{ij}q_j + \sum_{z=1}^{l} b_{iz}\bar{k}_{zj} \geq 0, \qquad 1 \leq i,\ j \leq n \tag{17}$$

$$c_{ij}q_j + \sum_{z=1}^{l} d_{iz}\bar{k}_{zj} \geq 0, \qquad 1 \leq i,\ j \leq n. \tag{18}$$

Under the above conditions, the matrix gain of a desired controller $\mathcal{G}$ in (2) is given by (9).

Proof: (*Sufficiency*) First, (17) and (18) trivially ensure that $\bar{A} \in \mathbb{R}_+^{n\times n}$ and $\bar{C} \in \mathbb{R}_+^{m\times n}$. By the positivity of $B \in \mathbb{R}_+^{n\times l}$ and $D \in \mathbb{R}_+^{m\times l}$, from Lemma 3 we know that the closed-loop system is positive.

Next, (9) is equivalent to (10). By substituting (10) into (16), we obtain

$$\begin{bmatrix} -Q & AQ + BKQ \\ QA^T + QK^T B^T & -Q \end{bmatrix} < 0. \tag{19}$$

By applying to (19) the congruence transformation defined by $\operatorname{diag}\{Q^{-1}, Q^{-1}\}$ and keeping in mind (4), one obtains

$$\begin{bmatrix} -Q^{-1} & Q^{-1}\bar{A} \\ \bar{A}^T Q^{-1} & -Q^{-1} \end{bmatrix} < 0.$$

By defining $P \triangleq Q^{-1}$, we readily obtain (15) via Schur complement equivalence [5]. Then, from Lemma 4, we know that the closed-loop system is asymptotically stable.

(*Necessity*) Suppose there exists a controller of the form $\mathcal{G}$ in (2) such that the closed-loop system $\mathcal{D}_c$ in (14) is asymptotically stable and positive. Then, from Lemmas 3 and 4, we know that $\bar{A} \in \mathbb{R}_+^{n\times n}$, $\bar{C} \in \mathbb{R}_+^{m\times n}$, and there exists a positive diagonal matrix $P \triangleq \operatorname{diag}\{p_1, p_2, \ldots, p_n\} \in \mathbb{R}^{n\times n}$ satisfying (15).

First, by Schur complement, (15) is equivalent to

$$\begin{bmatrix} -P & P\bar{A} \\ \bar{A}^T P & -P \end{bmatrix} < 0. \tag{20}$$

By applying to (20) the congruence transformation defined by $\operatorname{diag}\{P^{-1}, P^{-1}\}$ and keeping in mind (4), one obtains

$$\begin{bmatrix} -P^{-1} & AP^{-1} + BKP^{-1} \\ P^{-1}A^T + P^{-1}K^T B^T & -P^{-1} \end{bmatrix} < 0.$$

By defining Q and $\bar{K}$ as in (12), we readily obtain (16).

Second, $\bar{A} \in \mathbb{R}_+^{n\times n}$ and $\bar{C} \in \mathbb{R}_+^{m\times n}$ trivially imply (17) and (18), respectively, by noticing (12). □

IV. Illustrative Examples

In this section, we provide several examples to illustrate the developed theories.

Example 1: Consider the following continuous-time system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 & 0.5 \\ 0.2 & -1 & 1 \\ 0.3 & 1.3 & 0.2 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 \\ 0.5 \\ 1 \end{bmatrix} u(t) \\ y(t) &= [1 \quad 2 \quad 1]x(t) + u(t). \end{aligned} \tag{21}$$

It is easy to see that the above system is positive but not asymptotically stable. Our purpose is to design a state-feedback controller of the form $\mathcal{G}$ in (2) such that the closed-loop system is asymptotically stable and positive. By applying Theorem 1, we obtain the following matrix variables:

$$Q = \text{diag}\{0.9063,\ 0.7512,\ 1.1186\}$$
$$\bar{K} = [0.0922 \quad -0.6207 \quad -0.9505].$$

Then, according to (9), the feedback gain matrix K of the controller $\mathcal{G}$ in (2) is given by

$$K = [0.1017 \quad -0.8263 \quad -0.8497]. \tag{22}$$

By (21) and (22), the matrices for the closed-loop system $\mathcal{C}_c$ in (3) are given by

$$\bar{A} = \begin{bmatrix} -0.9898 & 0.0174 & 0.4150 \\ 0.2508 & -1.4131 & 0.5751 \\ 0.4017 & 0.4737 & -0.6497 \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} 0.1 \\ 0.5 \\ 1 \end{bmatrix}$$
$$\bar{C} = [1.1017 \quad 1.1737 \quad 0.1503]$$
$$\bar{D} = 1.$$

It can be seen that $\bar{A}$ is a Metzler matrix and $\bar{B} \in \mathbb{R}_+^{n\times l}$, $\bar{C} \in \mathbb{R}_+^{m\times n}$, and $\bar{D} \in \mathbb{R}_+^{m\times l}$. In addition, $\bar{A}$ is a stable matrix.

Example 2: Consider the following example modified from Example 1:

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0.5 \\ -0.2 & -1 & 1 \\ -0.3 & 1.3 & 0.2 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 \\ 0.5 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = [1 \quad -0.1 \quad 1]x(t) + u(t) \tag{23}$$

It is easy to see that system (23) is neither stable nor positive. By applying Theorem 1, we obtain the following matrix variables:

$$Q = \text{diag}\{2.2462 \times 10^3,\ 302.2611,\ 839.8516\}$$
$$\bar{K} = [908.4768 \quad 31.2910 \quad -838.3139].$$

Then, according to (9), the feedback gain matrix K of the controller $\mathcal{G}$ in (2) is given by

$$K = [0.4045 \quad 0.1035 \quad -0.9982]. \tag{24}$$

By (23) and (24), the matrices for the closed-loop system $\mathcal{C}_c$ in (3) are given by

$$\bar{A} = \begin{bmatrix} -0.9596 & 0.1104 & 0.4002 \\ 0.0022 & -0.9482 & 0.5009 \\ 0.1045 & 1.4035 & -0.7982 \end{bmatrix}$$
$$\bar{B} = \begin{bmatrix} 0.1 \\ 0.5 \\ 1 \end{bmatrix}$$
$$\bar{C} = [1.4045 \quad 0.0035 \quad 0.0018]$$
$$\bar{D} = 1.$$

It can be seen that $\bar{A}$ is a Metzler matrix and $\bar{B} \in \mathbb{R}_+^{n\times l}$, $\bar{C} \in \mathbb{R}_+^{m\times n}$, and $\bar{D} \in \mathbb{R}_+^{m\times l}$. In addition, $\bar{A}$ is stable.

To further illustrate the effectiveness of the theories developed in this paper, let us consider a simple practical example.

Example 3: Suppose there are two stores represented by A and B. The numbers of goods in these two stores are represented by a and b, respectively. It is assumed that a and b evolving with time have the following relationship:

$$a(t+1) = a(t) + 0.3\left[b(t) - \bar{b}\right] + u_1(t)$$
$$b(t+1) = -0.2\left[a(t) - \bar{a}\right] + b(t) + u_2(t) \tag{25}$$

where $\bar{a}$, $\bar{b}$ are constant numbers. Our purpose is to keep the goods in these two stores satisfying the following two requirements:

$$1) \quad a(t) \geq \bar{a} \qquad b(t) \geq \bar{b} \text{ for all time } t \tag{26}$$
$$2) \quad a(t) \to \bar{a} \qquad b(t) \to \bar{b} \text{ as } t \to \infty. \tag{27}$$

If we select

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a(t) - \bar{a} \\ b(t) - \bar{b} \end{bmatrix} \qquad u(t) \triangleq \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}. \tag{28}$$

Then, (25) can be represented by the following state-space equation:

$$x(t+1) = Ax(t) + Bu(t) \tag{29}$$

where

$$A = \begin{bmatrix} 1 & 0.3 \\ -0.2 & 1 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Obviously, the unforced system of (29) is neither positive nor asymptotically stable. To meet the requirements in (26) and (27), we will design a state-feedback controller of the following form:

$$u(t) = Kx(t) \tag{30}$$

such that the closed-loop system

$$x(t+1) = (A + BK)x(t) \tag{31}$$

is both positive and asymptotically stable. By positivity, we have $x(t) \geq 0$ for all time t, which implies (26) by considering (28). By asymptotic stability, we have $x(t) \to 0$ as $t \to \infty$, which implies (27) by considering (28).

By employing Theorem 2, we obtain the following matrix variables:

$$Q = \text{diag}\{25.3650,\ 25.3650\}$$
$$\bar{K} = \begin{bmatrix} -16.9100 & 0.8455 \\ 13.5280 & -16.9100 \end{bmatrix}.$$

Then, according to (9), a feedback gain K for the controller (30) is given by

$$K = \begin{bmatrix} -0.6667 & 0.0333 \\ 0.5333 & -0.6667 \end{bmatrix}. \tag{32}$$

With (29) and (32), the matrix for the closed-loop system (31) is given by

$$A + BK = \begin{bmatrix} 0.3333 & 0.3333 \\ 0.3333 & 0.3333 \end{bmatrix}.$$

It can be seen that the closed-loop system is both positive and asymptotically stable, showing that the requirements in (26) and (27) are satisfied.

V. Concluding Remarks

The control problem for stability and positivity is treated in this paper for both continuous- and discrete-time linear systems. Necessary and sufficient conditions are derived in terms of LMIs for the existence of desired controllers guaranteeing the closed-loop system to be asymptotically stable and positive. Numerical examples are provided to illustrate the proposed results.

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