Theorem 1: Under Assumption 1, the dynamic output-feedback control law consisting of (8), (32), and (45) forces the aircraft (1) to globally asymptotically track the reference model (2) if the design constants k_i, $1 \le i \le 5$ are chosen such that (35) and (37) hold.

IV. SIMULATIONS

In this section, we perform a numerical simulation to illustrate the effectiveness of the proposed controller with $\varepsilon = 0.8$. The observer and control gains are chosen as: $k_{11} = k_{21} = k_{31} = 1$, $k_{12} = k_{22} = k_{32} = 2$, $k_1 = k_2 = 0.15$, $k_3 = 5$, $k_4 = k_5 = 0.4$, $k_6 = k_7 = 5$, $\delta = 0.1$, and initial conditions are $x_1(0) = y_1(0) = 5$, $\theta(0) = 0.2$, $x_2(0) = y_2(0) = \omega(0) = 0.2$, $\hat{x}_1(0) = \hat{y}_1(0) = 3$, $\hat{\theta}(0) = 0.1$, $x_{1r}(0) = y_{1r}(0) = \theta_r(0) = x_{2r}(0) = y_{2r}(0) = \omega_r(0) = 0$, $\hat{x}_2(0) = \hat{y}_2(0) = \hat{\omega}(0) = 0$. The goal is of forcing the aircraft to track a sinusoid signal of $5(\sin(0.1t) + 1.2)$ in the vertical plane generated by (2). It is shown that there exists u_{1r} for this case with $u_{1r}^* > 2$ (see Assumption 1) and that conditions (35) and (37) hold. The results including norm of observer errors are plotted in Fig. 1. It is seen that the tracking errors asymptotically converge to zero.

V. CONCLUSION

We have presented a methodology to develop an output-feedback tracking controller for an underactuated nonminimum phase VTOL aircraft. The control development was based on several changes of coordinates, Lyapunov's direct method, and an extension of applying the backstepping technique, which was presented in Sub-Step 1.1, Section III-C. Current work is underway to extend the proposed methodology to the more realistic case when the system parameters are not perfectly known and the aircraft lands arbitrarily fast.

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H_∞ Filtering for Singular Systems

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***Abstract*—This note considers the H_∞ filtering problem for linear continuous singular systems. The purpose is the design of a linear filter such that the resulting error system is regular, impulse-free and stable while the closed-loop transfer function from the disturbance to the filtering error output satisfies a prescribed H_∞-norm bound constraint. Without decomposing the original system matrices, a necessary and sufficient condition for the solvability of this problem is obtained in terms of a set of linear matrix inequalities (LMIs). When these LMIs are feasible, an explicit expression of a desired filter is given. Finally, an illustrative example is presented to demonstrate the applicability of the proposed approach.**

***Index Terms*—Continuous systems, H_∞ filtering, linear matrix inequality, singular systems.**

I. INTRODUCTION

The filtering problem, which is concerned with estimating the state variables of a dynamic system based on the corrupted system output measurement data, has been widely studied, and has found many practical applications [1], [17]. One of the most popular ways to deal with this problem is the celebrated Kalman filtering approach, which generally provides an optimal estimation of the state variables in the sense that the covariance of the estimation error is minimized [1]. It is noted that the Kalman filtering approach is based on the assumptions that the system under consideration has exactly known dynamics described by certain well-posed model, and its disturbances are stationary Gaussian noises with known statistics [1]. In some applications, however, the noise sources may not be exactly known. Also, it has been shown that the standard Kalman filtering algorithm cannot guarantee satisfactory performance when there are parameter uncertainties in a system model [5]; these limit the scope of applications of the Kalman filtering technique. In view of these, an alternative estimation method based on H_∞ filtering technique has been proposed recently. The objective is to find a filter such that the resulting filtering error system is asymptotically stable and the $\mathcal{L}_2$-induced norm (for continuous systems) or l_2-induced norm (for discrete systems) from the input disturbances to the filtering error output satisfies a prescribed H_∞ performance level. In contrast to the traditional Kalman filtering, the H_∞ filtering approach does not require knowledge of the statistical properties of the external noise. Furthermore, it has been shown that the H_∞ filtering technique provides both a guaranteed noise attenuation level and robustness against unmodeled dynamics [15]. These features render the H_∞ filtering technique useful in practical applications. Various approaches, such as the convex optimization approach or the linear matrix inequality (LMI) approach [3], [18], interpolation approach [10], polynomial equation approach [11], algebraic Riccati equation approach [15], [20], frequency domain approach [19], to name just a few, have been successfully proposed to solve the H_∞ filtering problem. It is worth

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pointing out that when parameter uncertainties appear in a system model, the robust H_∞ filtering problem has been studied recently, and a great number of results on this topic have been reported in the literature; see, e.g., [8], [9], [12], [24], and the references therein.

On the other hand, a great deal of attention has been devoted to the study of singular systems over the past decades. Singular systems are also referred to as descriptor systems, implicit systems, generalized state-space systems, differential-algebraic systems or semi-state systems. Applications of this class of systems can be found in modeling and control of electrical circuits, power systems, economics and other areas [7], [13]. A number of estimation and control issues related to singular systems have been studied and many results have been reported [6], [7], [21]–[23], [25]. The filtering problem for singular systems has also been investigated by many researchers. For example, by changing an estimable discrete-time linear singular system to an equivalent standard system via orthogonal transformations, a simple optimal filter was obtained in [2], where necessary and sufficient conditions for the convergence and stability of the derived optimal filter were provided. The Kalman filtering problem for singular systems was dealt with in [16], in which a generalization of the shuffle algorithm was used to solve this problem and some unnecessary assumptions in some earlier works were removed. For the problem of H_∞ filtering for singular systems, however, there are still no results available in the literature. This motivates the present study.

In this note, we deal with the H_∞ filtering problem for singular systems. Attention is focused on the design of a linear filter such that the resulting error system is regular, impulse-free and stable while the closed-loop transfer function from the disturbance to the filtering error output satisfies a prescribed H_∞-norm bound constraint. A necessary and sufficient condition for the solvability of this problem is obtained in terms of LMIs. The desired H_∞ filter can be constructed by solving several certain LMIs, for which the so called interior point algorithms can be resorted to [4]. It is shown that such an obtained filter is proper with a McMillan degree no more than the number of the exponential modes of the plant. It is worth pointing out that all these results are obtained without decomposing the original system matrices. An example is provided to demonstrate the effectiveness of the proposed approach.

Notation

Throughout this note, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive–semidefinite (respectively, positive definite). I is the identity matrix with appropriate dimension. The superscript "T" and "$+$" represent the transpose and the Moore–Penrose inverse, respectively. The notation $\mathcal{L}_2[0,\ \infty)$ represents the space of square-integrable vector functions over [0, ∞). For a given stable continuous-time transfer function matrix $G(s)$, its H_∞ norm is defined by $\|G\|_\infty = \sup_{\omega\in[0,\infty)} \sigma_{\max}(G(j\omega))$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

II. Problem Formulation and Definitions

Consider the following class of linear singular systems:

$$(\Sigma): \qquad E\dot{x}(t) = Ax(t) + B\omega(t) \tag{1}$$

$$y(t) = Cx(t) + D\omega(t) \tag{2}$$

$$z(t) = Lx(t) \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state; $y(t) \in \mathbb{R}^m$ is the measurement; $z(t) \in \mathbb{R}^q$ is the signal to be estimated; $\omega(t) \in \mathbb{R}^p$ is the disturbance input which belongs to $\mathcal{L}_2[0,\ \infty)$. The matrix $E \in \mathbb{R}^{n\times n}$ may be singular; we shall assume that rank $E = r \leq n$. A, B, C, D and L are known real constant matrices with appropriate dimensions.

The unforced singular system of (1) with $\omega(t) = 0$ is as follows:

$$(\Sigma_u): \qquad E\dot{x}(t) = Ax(t). \tag{4}$$

For the singular system (4), we adopt the following definition throughout this note.

Definition 1: [7], [13]

I) (Σ_u) is said to be regular if $\det(sE - A)$ is not identically zero.

II) (Σ_u) is said to be impulse-free if $\deg(\det(sE - A)) = \operatorname{rank} E$.

III) (Σ_u) is said to be stable if all the roots of $\det(sE - A) = 0$ have negative real parts.

IV) (Σ_u) is said to be admissible if it is regular, impulse-free and stable.

Now, we consider the following filter for the estimate of $z(t)$:

$$(\Sigma_f): \qquad E_f\dot{\hat{x}}(t) = A_f\hat{x}(t) + B_f y(t) \tag{5}$$

$$\hat{z}(t) = C_f\hat{x}(t) \tag{6}$$

where $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$ and $\hat{z}(t) \in \mathbb{R}^q$ are the state and the output of the filter, respectively. The matrices $E_f \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $A_f \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $B_f \in \mathbb{R}^{\hat{n}\times p}$ and $C_f \in \mathbb{R}^{q\times\hat{n}}$ are to be determined. Let

$$e(t) = [\, x(t)^T \quad \hat{x}(t)^T \,]^T \qquad \tilde{z}(t) = z(t) - \hat{z}(t). \tag{7}$$

Then, the filtering error dynamics from the systems (Σ) and (Σ_f) can be written as

$$(\tilde{\Sigma}): \qquad E_c\dot{e}(t) = A_c e(t) + B_c\omega(t) \tag{8}$$

$$\tilde{z}(t) = L_c e(t) \tag{9}$$

where

$$E_c = \begin{bmatrix} E & 0 \\ 0 & E_f \end{bmatrix} \quad A_c = \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix} \tag{10}$$

$$B_c = \begin{bmatrix} B \\ B_f D \end{bmatrix} \quad L_c = [\, L \quad -C_f \,]. \tag{11}$$

The H_∞ filter problem to be addressed in this note is formulated as follows: given the singular system (Σ) and a prescribed H_∞ bound $\gamma > 0$, determine a filter (Σ_f) in the form of (5) and (6) such that the filtering error system $(\tilde{\Sigma})$ is admissible and the transfer function from $\omega(t)$ to $\tilde{z}(t)$ given as

$$G_c(s) = L_c(sE_c - A_c)^{-1}B_c \tag{12}$$

satisfies

$$\|G_c\|_\infty < \gamma. \tag{13}$$

Remark 1: The requirement on the admissibility of the error dynamic system $(\tilde{\Sigma})$ implies that the singular system (Σ) is admissible. Therefore, the following development is based on this assumption.

III. Main Results

In this section, an LMI approach will be developed to solve the H_∞ filtering problem formulated in the previous section. First, we present the following lemmas which will be used in the proof of our main results.

Lemma 1: [14] Consider the singular system (Σ). Then, the following statements S1) and S2) are equivalent.

S1) The following conditions i) and ii) hold simultaneously.

i) The singular system (Σ) with $\omega(t) \equiv 0$ is admissible.

ii) The transfer function given as $G(s) = L(sE - A)^{-1}B$, satisfies $\|G\|_\infty < \gamma$.

S2) There exists a matrix P satisfying the following LMIs:

$$E^TP = P^TE \geq 0 \tag{14}$$

$$\begin{bmatrix} A^TP + P^TA & P^TB & L^T \\ B^TP & -\gamma^2 I & 0 \\ L & 0 & -I \end{bmatrix} < 0. \tag{15}$$

Lemma 2: [4] The matrix inequality

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_2^T & Z_3 \end{bmatrix} \geq 0$$

holds if and only if

$$Z_3 \geq 0 \quad Z_1 - Z_2 Z_3^+ Z_2^T \geq 0 \quad Z_2\left(I - Z_3 Z_3^+\right) = 0.$$

Now, we are in a position to present our main result on the H_∞ filtering problem for singular systems.

Theorem 1: Consider the linear singular system (Σ). Then, there exists a filter (Σ_f) in the form of (5) and (6) such that the H_∞ filtering problem is solvable if and only if there exist matrices X, Y, Φ, Ψ and Υ such that the LMIs shown in (16)–(19) at the bottom of the page hold. In this case, there exist nonsingular matrices $S, \tilde{S}, W$ and $\tilde{W}$ such that

$$E^T\tilde{S} = S^TE \tag{20}$$

$$EW = \tilde{W}^TE^T \tag{21}$$

$$XY^{-1} = I - \tilde{S}W \tag{22}$$

$$Y^{-1}X = I - \tilde{W}S. \tag{23}$$

Then, the parameters of a desired H_∞ filter given in the form of (14) and (15) can be chosen as

$$E_f = E \quad A_f = S^{-T}\Phi Y^{-1}W^{-1} \tag{24}$$

$$B_f = S^{-T}\Psi \quad C_f = \Upsilon Y^{-1}W^{-1}. \tag{25}$$

Proof:

Sufficiency: Suppose (16)–(19) hold. Under these conditions, we first show that there always exist nonsingular matrices S, $\tilde{S}$, W and $\tilde{W}$ such that (20)–(23) hold. To this end, we claim that the matrix Y satisfying (17) and (19) is nonsingular. If not, then there exists a vector $\eta \neq 0$ such that $Y\eta = 0$. Therefore, $\eta^T\left(A^TY + Y^TA\right)\eta = 0$. This is a contradiction since (19) implies $A^TY + Y^TA < 0$. Furthermore, we assume that, without loss of generality, $Y - X$ is nonsingular. If not, we choose $\hat{Y} = (1-\alpha)Y$, where $\alpha > 0$ is a sufficiently small scalar which is not an eigenvalue of $I - XY^{-1}$ and satisfies that (19) holds for $\hat{Y}$. Then, it can be seen that (17) and (18) are satisfied for this $\hat{Y}$. Furthermore, $\hat{Y} - X$ is nonsingular. Therefore, we can replace Y by this $\hat{Y}$ to satisfy the above requirements without violating (17)–(19). From this, it is easy to show that both $I - XY^{-1}$ and $I - Y^{-1}X$ are nonsingular. Now, we choose two nonsingular matrices M and N such that [7]

$$E = M\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}N.$$

Considering this and (16) and (17), we can deduce that the matrices X and Y can be written as

$$X = M^{-T}\begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix}N \quad Y = M^{-T}\begin{bmatrix} Y_1 & 0 \\ Y_2 & Y_3 \end{bmatrix}N \tag{26}$$

where $X_1^T = X_1 \geq 0, Y_1^T = Y_1 > 0$. Then

$$Y^{-1} = N^{-1}\begin{bmatrix} \hat{Y}_1 & 0 \\ \hat{Y}_2 & \hat{Y}_3 \end{bmatrix}M^T$$

where $\hat{Y}_1^T = \hat{Y}_1 = Y_1^{-1} > 0$. Set

$$S = M^{-T}\begin{bmatrix} S_1 & 0 \\ S_2 & S_3 \end{bmatrix}N, \ \tilde{S} = M^{-T}\begin{bmatrix} \tilde{S}_1 & 0 \\ \tilde{S}_2 & \tilde{S}_3 \end{bmatrix}N \tag{27}$$

$$W = N^{-1}\begin{bmatrix} W_1 & 0 \\ W_2 & W_3 \end{bmatrix}M^T, \tilde{W} = N^{-1}\begin{bmatrix} \tilde{W}_1 & 0 \\ \tilde{W}_2 & \tilde{W}_3 \end{bmatrix}M^T \tag{28}$$

where the matrices $S_i, \tilde{S}_i, W_i, \tilde{W}_i, i = 1, 2, 3$, are selected to satisfy

$$S_1^T = \tilde{S}_1 \quad W_1 = \tilde{W}_1^T \tag{29}$$

$$\begin{bmatrix} \tilde{S}_1W_1 & 0 \\ \tilde{S}_2W_1 + \tilde{S}_3W_2 & \tilde{S}_3W_3 \end{bmatrix} = \begin{bmatrix} I - X_1\hat{Y}_1 & 0 \\ -X_2\hat{Y}_1 - X_3\hat{Y}_2 & I - X_3\hat{Y}_3 \end{bmatrix} \tag{30}$$

$$\begin{bmatrix} \tilde{W}_1S_1 & 0 \\ \tilde{W}_2S_1 + \tilde{W}_3S_2 & \tilde{W}_3S_3 \end{bmatrix} = \begin{bmatrix} I - \hat{Y}_1X_1 & 0 \\ -\hat{Y}_2X_1 - \hat{Y}_3X_3 & I - \hat{Y}_3X_3 \end{bmatrix}. \tag{31}$$

By (26)–(31), it can be verified that the matrices $S, \tilde{S}, W$ and $\tilde{W}$ given in (27) and (28) satisfy (20)–(23). Furthermore, the nonsingularity of $I - XY^{-1}$ and $I - Y^{-1}X$ implies that the matrices $S, \tilde{S}, W$ and $\tilde{W}$ are nonsingular too.

Next, we will show that the error system ($\tilde{\Sigma}$) in the form of (8) and (9), derived from the singular system (Σ) and the filter (Σ_f) whose parameters are given in (24) and (25), is admissible and its transfer function satisfies (13). For this purpose, we define

$$\Pi_1 = \begin{bmatrix} \bar{Y} & I \\ W & 0 \end{bmatrix} \quad \Pi_2 = \begin{bmatrix} I & X \\ 0 & S \end{bmatrix} \tag{32}$$

where $\bar{Y} = Y^{-1}$. It is noted that both Π_1 and Π_2 are nonsingular. Set

$$\hat{P} = \Pi_2\Pi_1^{-1}. \tag{33}$$

Then, it can be verified that $\hat{P}$ is nonsingular, and

$$\hat{P} = \begin{bmatrix} X & \tilde{S} \\ S & -\Gamma \end{bmatrix} \tag{34}$$

where

$$\Gamma = S\bar{Y}W^{-1}. \tag{35}$$

It follows from (20)–(23) that

$$\begin{aligned} &E^T\Gamma \\ &= E^TS\bar{Y}W^{-1} = W^{-T}\left(W^TE^TS\bar{Y}\right)W^{-1} = W^{-T}E\tilde{W}S\bar{Y}W^{-1} \\ &= W^{-T}E\left(I - \bar{Y}X\right)\bar{Y}W^{-1} = W^{-T}E\left(\bar{Y} - \bar{Y}X\bar{Y}\right)W^{-1} \\ &= W^{-T}E\bar{Y}\left(Y - X\right)\bar{Y}W^{-1} = W^{-T}\bar{Y}^T\left[E^T\left(Y - X\right)\right]\bar{Y}W^{-1}. \end{aligned}$$

$$E^TX = X^TE \geq 0 \tag{16}$$

$$E^TY = Y^TE \geq 0 \tag{17}$$

$$E^T(X - Y) \geq 0 \tag{18}$$

$$\begin{bmatrix} A^TY + Y^TA & A^TX + Y^TA + C^T\Psi^T + \Phi^T & Y^TB & L^T - \Upsilon^T \\ X^TA + A^TY + \Psi C + \Phi & X^TA + A^TX + \Psi C + C^T\Psi^T & X^TB + \Psi D & L^T \\ B^TY & B^TX + D^T\Psi^T & -\gamma^2 I & 0 \\ L - \Upsilon & L & 0 & -I \end{bmatrix} < 0. \tag{19}$$

By (16)–(18), we have

$$E^T\Gamma = \Gamma^T E \le 0. \tag{36}$$

Therefore

$$\hat{E}^T\hat{P} = \hat{P}^T\hat{E} \tag{37}$$

where $\hat{E} = \mathrm{diag}(E, E)$. Noting the nonsingularity of Γ and using (36), (20), and (22), we have

$$\begin{aligned}
&E^TX + E^T\tilde{S}\Gamma^{-1}\left(\Gamma^{-T}E^T\right)^+\Gamma^{-T}E^TS\\
&=E^TX + S^TE\Gamma^{-1}\left(\Gamma^{-T}E^T\right)^+\Gamma^{-T}E^TS\\
&=E^TX + S^T\Gamma^{-T}E^T\left(\Gamma^{-T}E^T\right)^+\Gamma^{-T}E^TS\\
&=E^TX + S^T\Gamma^{-T}E^TS = E^TX + S^TE\Gamma^{-1}S\\
&=E^TX + E^T\tilde{S}\Gamma^{-1}S = E^T\left(X + \tilde{S}\Gamma^{-1}S\right)\\
&=E^T\left(X+\tilde{S}WY\right)=E^T\left[X+\left(I-X\bar{Y}\right)Y\right]=E^TY \ge 0.
\end{aligned} \tag{38}$$

Furthermore, considering that $E\Gamma^{-1}$ is symmetric, we obtain

$$\begin{aligned}
&E^T\tilde{S}\Gamma^{-1}\left[I - \left(-\Gamma^{-T}E^T\right)\left(-\Gamma^{-T}E^T\right)^+\right]\\
&=S^TE\Gamma^{-1}\left[I - (E\Gamma^{-1})^T\left(\left(E\Gamma^{-1}\right)^+\right)^T\right]\\
&=S^TE\Gamma^{-1}\left[I - \left(\left(E\Gamma^{-1}\right)^+(E\Gamma^{-1})\right)^T\right]\\
&=S^TE\Gamma^{-1}\left[I - \left(E\Gamma^{-1}\right)^+(E\Gamma^{-1})\right]\\
&=S^T\left[E\Gamma^{-1} - (E\Gamma^{-1})\left(E\Gamma^{-1}\right)^+(E\Gamma^{-1})\right] = 0.
\end{aligned} \tag{39}$$

Taking into account (38) and (39) and using Lemma 2, we can deduce

$$\begin{bmatrix} E^TX & E^T\tilde{S}\Gamma^{-1} \\ \Gamma^{-T}E^TS & -\Gamma^{-T}E^T \end{bmatrix} \ge 0. \tag{40}$$

Premultiplying (40) by $\mathrm{diag}(I,\ \Gamma^T)$ and postmultiplying (40) by $\mathrm{diag}(I,\ \Gamma)$ give

$$\begin{bmatrix} E^TX & E^T\tilde{S} \\ E^TS & -E^T\Gamma \end{bmatrix} \ge 0. \tag{41}$$

By noting (37), we can rewrite (41) as

$$\hat{E}^T\hat{P} = \hat{P}^T\hat{E} \ge 0. \tag{42}$$

On the other hand, premultiplying (19) by $\mathrm{diag}(\bar{Y}^T,\ I,\ I,\ I)$ and post-multiplying (19) by $\mathrm{diag}(\bar{Y}, I, I, I)$, we have the equation shown at the bottom of the page. This can be rewritten as

$$\begin{bmatrix} \Pi_1^TA_c^T\hat{P}\Pi_1 + \Pi_1^T\hat{P}^TA_c\Pi_1 & \Pi_1^T\hat{P}^TB_c & \Pi_1^TL_c^T \\ B_c^T\hat{P}\Pi_1 & -\gamma^2I & 0 \\ L_c\Pi_1 & 0 & -I \end{bmatrix} < 0 \tag{43}$$

where the matrices A_c, B_c and L_c are given in (10) and (11) with the parameters A_f, B_f and C_f given in (24) and (25). Then, premultiplying (43) by $\mathrm{diag}(\Pi_1^{-T}, I, I)$ and postmultiplying (43) by $\mathrm{diag}(\Pi_1^{-1}, I, I)$, we obtain

$$\begin{bmatrix} A_c^T\hat{P} + \hat{P}^TA_c & \hat{P}^TB_c & L_c^T \\ B_c^T\hat{P} & -\gamma^2I & 0 \\ L_c & 0 & -I \end{bmatrix} < 0. \tag{44}$$

Noting this and (42) and using Lemma 1, we have that the error system ($\tilde{\Sigma}$) resulting from the singular system (Σ) and the filter (Σ_f) with the parameters given in (24) and (25) is admissible and its transfer function satisfies (13). This completes the proof of sufficiency.

Necessity: Suppose that the H_∞ filtering problem is solvable; that is, there exists a filter (Σ_f) such that the error system ($\tilde{\Sigma}$) in the form of (8) and (9) is admissible and its transfer function satisfies (13). Then, by Lemma 1, there exists a matrix P_c such that

$$E_c^TP_c = P_c^TE_c \ge 0 \tag{45}$$

$$\begin{bmatrix} A_c^TP_c + P_c^TA_c & P_c^TB_c & L_c^T \\ B_c^TP_c & -\gamma^2I & 0 \\ L_c & 0 & -I \end{bmatrix} < 0. \tag{46}$$

Without loss of generality, we assume

$$\hat{n} \ge n + \mathrm{rank}\ E_f. \tag{47}$$

If not, we can choose $\tilde{E}_f \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{A}_f \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{B}_f \in \mathbb{R}^{\tilde{n}\times p}$ and $\tilde{C}_f \in \mathbb{R}^{q\times\tilde{n}}$ to replace E_f, A_f, B_f and C_f, respectively, such that $\tilde{n} \ge n + \mathrm{rank}\ E_f$ and (45) and (46) still hold for some $\tilde{P}_c$ as follows. First, set

$$\begin{aligned}
&\tilde{E}_f = \begin{bmatrix} E_f & 0 \\ 0 & 0 \end{bmatrix}\ \tilde{A}_f = \begin{bmatrix} A_f & 0 \\ 0 & -I \end{bmatrix}\ \tilde{B}_f = \begin{bmatrix} B_f \\ 0 \end{bmatrix}\\
&\tilde{C}_f = [C_f \quad 0]
\end{aligned} \tag{48}$$

with $\tilde{n} \ge n + \mathrm{rank}\ E_f$. Then, choose

$$\tilde{P}_c = \begin{bmatrix} P_c & 0 \\ 0 & I \end{bmatrix}. \tag{49}$$

It can be verified that $\tilde{E}_f$, $\tilde{A}_f$, $\tilde{B}_f$ and $\tilde{C}_f$ in (48) together with $\tilde{P}_c$ in (49) satisfy (45) and (46). Therefore, the required assumption is satisfied. Now, it is easy to see that the LMI in (46) implies

$$A_c^TP_c + P_c^TA_c < 0. \tag{50}$$

This implies that the matrix P_c is nonsingular. Write P_c and P_c^{-1} as

$$P_c = \begin{bmatrix} P_{c1} & P_{c2} \\ P_{c3} & P_{c4} \end{bmatrix} \qquad P_c^{-1} = \begin{bmatrix} Q_{c1} & Q_{c2} \\ Q_{c3} & Q_{c4} \end{bmatrix} \tag{51}$$

where the partition is compatible with that of A_c. Next, we will show that P_{c4} and Q_{c1} are nonsingular. To this end, we note that from (45) it can be seen that

$$E^TP_{c1} = P_{c1}^TE \ge 0 \tag{52}$$

$$E_f^TP_{c4} = P_{c4}^TE_f \ge 0. \tag{53}$$

On the other hand, it is easy to see that the 2–2 block of (50) gives

$$P_{c4}^TA_f + A_f^TP_{c4} < 0 \tag{54}$$

which implies that P_{c4} is nonsingular. Now, premultiplying (45) by P_c^{-T} and postmultiplying (45) by P_c^{-1} result in

$$\begin{bmatrix} \bar{Y}^TA^T + A\bar{Y} & \bar{Y}^TA^TX + A + \bar{Y}^TC^T\Psi^T + \bar{Y}^T\Phi^T & B & \bar{Y}^TL^T - \bar{Y}^T\Upsilon^T \\ X^TA\bar{Y} + A^T + \Psi C\bar{Y} + \Phi\bar{Y} & X^TA + A^TX + \Psi C + C^T\Psi^T & X^TB + \Psi D & L^T \\ B^T & B^TX + D^T\Psi^T & -\gamma^2I & 0 \\ L\bar{Y} - \Upsilon\bar{Y} & L & 0 & -I \end{bmatrix} < 0.$$

$P_c^{-T} E_c^T = E_c P_c^{-1} \geq 0$. The 1–1 block of this inequality gives

$$Q_{c1}^T E^T = E Q_{c1} \geq 0. \tag{55}$$

Premultiplying (50) by P_c^{-T} and postmultiplying (50) by P_c^{-1}, we have $P_c^{-T} A_c^T + A_c P_c^{-1} < 0$. Then, the 1–1 block of this inequality gives $AQ_{c1} + Q_{c1}^T A < 0$. This implies that Q_{c1} is nonsingular.

In the following, without loss of generality, we assume that the matrix P_{c3} is of full column rank. If not, we can choose another $\hat{P}_c$ to replace P_c such that $\hat{P}_c$ satisfies the assumption and the inequalities (45) and (46) simultaneously as follows. First, considering (47), we choose a matrix $\Omega \in \mathbb{R}^{\hat{n} \times n}$ with full-column rank satisfying $E_f^T \Omega = 0$. Then, we choose a nonsingular matrix $\mathcal{Q}$ such that

$$\mathcal{Q}\Omega = \begin{bmatrix} \Omega_1 \\ 0 \end{bmatrix}$$

where $\Omega_1 \in \mathbb{R}^{n \times n}$ is nonsingular. Partition $\mathcal{Q}P_{c3}$ as

$$\mathcal{Q}P_{c3} = \begin{bmatrix} P_{c31} \\ P_{c32} \end{bmatrix}$$

where $P_{c31} \in \mathbb{R}^{n \times n}$. Now, we choose a sufficiently small scalar $\alpha > 0$ such that α is not an eigenvalue of $-\Omega_1^{-1} P_{c31}$ and the matrix $\hat{P}_c$ defined as

$$\hat{P}_c = P_c + \alpha \begin{bmatrix} 0 & 0 \\ \Omega & 0 \end{bmatrix} \tag{56}$$

satisfies (45) and (46). Then, the 2–1 block of $\hat{P}_c$ in (56) is

$$P_{c3} + \alpha\Omega = \mathcal{Q}^{-1}\left(\mathcal{Q}P_{c3} + \alpha\mathcal{Q}\Omega\right) = \mathcal{Q}^{-1} \begin{bmatrix} \alpha\Omega_1 + P_{c31} \\ P_{c32} \end{bmatrix}$$

which is of full-column rank. Note that $E_c^T \hat{P}_c = \hat{P}_c^T E_c = E_c^T P_c \geq 0$. Thus, $\hat{P}_c$ given in (56) satisfies the assumption. Now, by some calculation, it can be verified that $Q_{c3} = -P_{c4}^{-1} P_{c3} (P_{c1} - P_{c2} P_{c4}^{-1} P_{c3})^{-1}$. Since P_{c3} can be chosen to be of full-column rank, we have that Q_{c3} can be chosen to be of full-column rank too. Set

$$\tilde{\Pi}_1 = \begin{bmatrix} Q_{c1} & I \\ Q_{c3} & 0 \end{bmatrix} \qquad \tilde{\Pi}_2 = \begin{bmatrix} I & P_{c1} \\ 0 & P_{c3} \end{bmatrix}.$$

Then, it can be seen that both $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ are of full-column rank. Furthermore, it is easy to see that

$$P_c \tilde{\Pi}_1 = \tilde{\Pi}_2. \tag{57}$$

Now, premultiplying (46) by $\mathrm{diag}(\tilde{\Pi}_1^T, I, I)$ and postmultiplying (46) by $\mathrm{diag}(\tilde{\Pi}_1, I, I)$, we have (58), as shown at the bottom of the page, where the relationship in (57) is used. Premultiplying (58) by $\mathrm{diag}(Q_{c1}^{-T}, I, I, I)$ and postmultiplying (58) by $\mathrm{diag}(Q_{c1}^{-1}, I, I, I)$ give (59), as shown at the bottom of the page. Set

$$X = P_{c1} \qquad Y = Q_{c1}^{-1}, \qquad \Phi = P_{c3}^T A_f Q_{c3} Q_{c1}^{-1}$$
$$\Psi = P_{c3}^T B_f, \qquad \Upsilon = C_f Q_{c3} Q_{c1}^{-1}.$$

Then, the inequalities in (52) and (59) provide (16) and (19), respectively. Also, (55) implies $E^T Q_{c1}^{-1} = Q_{c1}^{-T} E \geq 0$, which gives (17). Noting $Q_{c1} = \left(P_{c1} - P_{c2} P_{c4}^{-1} P_{c3}\right)^{-1}$, and using (53), we have

$$E^T (P_{c1} - Q_{c1}^{-1}) = E^T P_{c2} P_{c4}^{-1} P_{c3} = P_{c3}^T E_f P_{c4}^{-1} P_{c3} \geq 0$$

which provides (18). This completes the proof. □

Remark 2: Theorem 1 provides a necessary and sufficient condition for the design of filters such that the resulting error system is admissible and the transfer function satisfies a prescribed H_∞-norm bound constraint. It is noted that a desired filter can be obtained by solving LMIs in (16)–(19), which can be implemented easily by resorting to the so-called interior point algorithms and no tuning of parameters is required [4]. It is also worth mentioning that the optimal filter in the sense that the minimum H_∞ norm γ is approached can be obtained by solving the following optimization problem:

$$\min_{X,Y,\Phi,\Psi,\Upsilon} \gamma$$

subject to $\gamma > 0$ and the LMIs in (16)–(19).

Remark 3: From the proof of Theorem 1, it can be seen that the designed filter (Σ_f) with parameters given by (24) and (25) is admissible and the number of the finite modes of this filter equals to rank E; in other words, this filter is proper with a McMillan degree no more than the number of the exponential modes of the given plant.

Remark 4: In the case when the matrix $E = I$, that is, the singular system (Σ) reduces to a usual state-space system, it can be shown that the result presented in Theorem 1 coincides with that in [12] if there are no parameter uncertainties in a state-space model. Therefore, Theorem 1 can be regarded as an extension of the existing H_∞ results for state-space systems to singular systems.

Remark 5: In the case when a filter to be designed is assumed to be with the same dimension of the descriptor variable vector as the plant, a Luenberger-type filter can be adopted. Under the assumption of the detectability, the filter gain can be determined through the method given in [7]. It is noted that Luenberger-type filters are different from those considered in the present note. Therefore, different assumptions will be required for singular systems.

IV. An Illustrative Example

In this section, an illustrative example is provided to demonstrate the applicability of the proposed approach.

$$\begin{bmatrix} AQ_{c1}+Q_{c1}^T A^T & A+Q_{c1}^T A^T P_{c1}+Q_{c1}^T C^T B_f^T P_{c3}+Q_{c3}^T A_f^T P_{c3} & B & Q_{c1}^T L^T - Q_{c3}^T C_f^T \\ A^T+P_{c1}^T AQ_{c1}+P_{c3}^T B_f CQ_{c1}+P_{c3}^T A_f Q_{c3} & P_{c1}^T A+A^T P_{c1}+P_{c3}^T B_f C+C^T B_f^T P_{c3} & P_{c1}^T B+P_{c3}^T B_f D & L^T \\ B^T & B^T P_{c1}+D^T B_f^T P_{c3} & -\gamma^2 I & 0 \\ LQ_{c1} - C_f Q_{c3} & L & 0 & -I \end{bmatrix} < 0 \tag{58}$$

$$\begin{bmatrix} Q_{c1}^{-T} A+A^T Q_{c1}^{-1} & Q_{c1}^{-T} A+A^T P_{c1}+C^T B_f^T P_{c3}+Q_{c1}^{-T} Q_{c3}^T A_f^T P_{c3} & Q_{c1}^{-T} B & L^T - Q_{c1}^{-T} Q_{c3}^T C_f^T \\ A^T Q_{c1}^{-1}+P_{c1}^T A+P_{c3}^T B_f C+P_{c3}^T A_f Q_{c3} Q_{c1}^{-1} & P_{c1}^T A+A^T P_{c1}+P_{c3}^T B_f C+C^T B_f^T P_{c3} & P_{c1}^T B+P_{c3}^T B_f D & L^T \\ B^T Q_{c1}^{-1} & B^T P_{c1}+D^T B_f^T P_{c3} & -\gamma^2 I & 0 \\ L - C_f Q_{c3} Q_{c1}^{-1} & L & 0 & -I \end{bmatrix} < 0. \tag{59}$$

Consider the linear singular system (Σ) with parameters as follows:

$$E = \begin{bmatrix} -2 & -2 & 0 \\ -2 & 4 & -6 \\ 1 & 4 & -3 \end{bmatrix},\ A = \begin{bmatrix} 1.5 & 0.4 & -0.9 \\ -0.6 & -4.4 & 1.8 \\ 0.7 & 2.4 & 4.3 \end{bmatrix}$$
$$B = \begin{bmatrix} -0.6 & -1.1 \\ 4.2 & 5.2 \\ 5.2 & 3.7 \end{bmatrix}\ C = \begin{bmatrix} -0.4 & 1.2 & -0.4 \\ 1.7 & 1.4 & 0.7 \end{bmatrix}$$
$$D = \begin{bmatrix} 1 & 0 \\ -1 & 0.5 \end{bmatrix},\ L = \begin{bmatrix} 0 & 2 & 0 \\ -0.5 & -1 & 1.5 \end{bmatrix}.$$

The purpose of this example is to design a filter in the form of (5) and (6) such that the resulting error system is admissible and the transfer function satisfies a prespecified H_∞-norm bound. In this example, we suppose that the H_∞-norm bound γ is specified to be 0.8. To solve the problem, we resort to the Matlab LMI Control Toolbox and obtain the solutions to the LMIs in (16)–(19) as follows:

$$X = \begin{bmatrix} -0.9773 & -1.6997 & -0.3836 \\ 0.0247 & 0.6069 & -0.0293 \\ -0.2138 & -0.9876 & -0.3322 \end{bmatrix}$$
$$Y = \begin{bmatrix} -0.5301 & -0.6252 & -0.1239 \\ 0.0701 & 0.1925 & -0.0129 \\ -0.2012 & -0.3293 & -0.0908 \end{bmatrix}$$
$$\Psi = \begin{bmatrix} 0.6831 & 0.1768 \\ 1.1537 & -0.5911 \\ 2.6919 & 1.1420 \end{bmatrix}$$
$$\Phi = \begin{bmatrix} 2.4627 & 0.8777 & 0.7006 \\ 5.8168 & 6.2856 & 3.3932 \\ 0.2743 & -3.2426 & 1.4838 \end{bmatrix}$$
$$\Upsilon = \begin{bmatrix} 0 & 2 & 0 \\ -0.5 & -1 & 1.5 \end{bmatrix}.$$

Therefore, it follows from Theorem 1 that the H_∞ filtering problem is solvable. To construct a desired filter, we choose $S = \tilde{S} = X$, and

$$W = \tilde{W} = \begin{bmatrix} 4.0229 & 2.8495 & -6.2607 \\ -2.1620 & -4.3194 & 3.6649 \\ -3.2722 & 8.5221 & 11.1959 \end{bmatrix}.$$

Then, it can be shown that the matrices $S, \tilde{S}, W$ and $\tilde{W}$ chosen before satisfy (20)–(23). Thus, by Theorem 1 again, a desired filter can be constructed as

$$(\Sigma_f):\quad \begin{bmatrix} -2 & -2 & 0 \\ -2 & 4 & -6 \\ 1 & 4 & -3 \end{bmatrix} \dot{\hat{x}}(t)$$
$$= \begin{bmatrix} 4.9440 & 4.8051 & -0.2140 \\ -12.1272 & -31.7857 & 1.5735 \\ -4.9056 & -17.0092 & 6.4728 \end{bmatrix} \hat{x}(t)$$
$$+ \begin{bmatrix} 0.8920 & 0.4435 \\ -9.1515 & -5.3872 \\ -8.3272 & -3.4752 \end{bmatrix} y(t)$$
$$\hat{z}(t) = \begin{bmatrix} -0.4670 & -2.9770 & 0.0162 \\ 0.7433 & 1.5629 & -2.0665 \end{bmatrix} \hat{x}(t).$$

It is easy to verify that this filter is admissible, and the resulting error system satisfies the design requirements. Furthermore, the optimal H_∞ norm can be calculated to be 0.1895.

V. Conclusion

In this note, we have studied the H_∞ filtering problem for continuous singular systems. In terms of a set of LMIs, a necessary and sufficient condition for the existence of a linear filter has been obtained, which guarantees that the resulting error system is regular, impulse-free and stable while the closed-loop transfer function from the disturbance to the filtering error output satisfies a prescribed H_∞-norm bound constraint. An example has been provided to illustrate the effectiveness of the proposed approach.

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