

Comments

Comments on “A New Family of Cayley Graph Interconnection Networks of Constant Degree Four”

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Abstract—Vadapalli and Srimani [2] have proposed a new family of Cayley graph interconnection networks of constant degree four. Our comments show that their proposed graph is not new but is the same as the wrap-around butterfly graph. The structural kinship of the proposed graph with the de Bruijn graph is also discussed.

Index Terms—Interconnection network, Cayley graph, generator, de Bruijn graph, butterfly graph, isomorphism.

1 DEFINITION OF GRAPH $\mathcal{G}(n)$

WE first give the definition of the graph $\mathcal{G}(n)$ proposed by Vadapalli and Srimani [2].

Each node of $\mathcal{G}(n)$ is represented as a circular permutation of n different symbols in lexicographic order, where the n symbols are presented in either uncomplemented or complemented form. Let t_k , $0 \leq k \leq n-1$, denote the k th symbol in the set of n symbols. We use the English alphabet for the symbols: thus, for $n=4$, $t_0 = a$, $t_1 = b$, $t_2 = c$, and $t_3 = d$. We use t_k^* to denote either t_k or \bar{t}_k . Therefore, for n distinct symbols, there are exactly n different cyclic permutations of the symbols in lexicographic order, and, since each symbol can be present in either uncomplemented or complemented form, the node set of $\mathcal{G}(n)$ has a cardinality of $n \times 2^n$. Since each node is some cyclic permutation of the n symbols in lexicographic order, then, if $a_0 a_1 \dots a_{n-1}$ denotes the label of an arbitrary node and $a_0 = t_k^*$ for some integer k , then, for all i , $1 \leq i \leq n-1$, we have $a_i = t_{(k+i) \pmod n}^*$. Thus, the definition of $\mathcal{G}(n)$ is given as follows.

DEFINITION 1. *The graph $\mathcal{G}(n)$ is a Cayley graph whose nodes comprise the $n \times 2^n$ cyclic permutations of n distinct symbols in lexicographic order. Each symbol is presented in either uncomplemented or complemented form. Given a node represented as a string $a_0 a_1 \dots a_{n-1}$, its edges are defined by the following generators:*

$$g(a_0 a_1 \dots a_{n-1}) = a_1 a_2 \dots a_{n-1} a_0$$

$$f(a_0 a_1 \dots a_{n-1}) = a_1 a_2 \dots a_{n-1} \bar{a}_0$$

$$g^{-1}(a_0 a_1 \dots a_{n-1}) = a_{n-1} a_0 \dots a_{n-2}$$

$$f^{-1}(a_0 a_1 \dots a_{n-1}) = \bar{a}_{n-1} a_0 \dots a_{n-2}$$

If the identity permutation is $t_0 t_1 \dots t_{n-1}$, then the generator set $\Omega = \{f, g, f^{-1}, g^{-1}\}$ is given as:

$$g = t_1 t_2 \dots t_{n-1} t_0$$

$$f = t_1 t_2 \dots t_{n-1} \bar{t}_0$$

$$g^{-1} = t_{n-1} t_0 \dots t_{n-2}$$

$$f^{-1} = \bar{t}_{n-1} t_0 \dots t_{n-2}$$

Fig. 1a shows $\mathcal{G}(3)$ drawn in a “regular” fashion, which is different from that in [2]. The identity permutation of $\mathcal{G}(3)$ is abc , and the generator set is $\{bca, bc\bar{a}, cab, \bar{c}ab\}$. The nodes of $\mathcal{G}(n)$ are grouped into different columns according to the position of the first symbol t_0^* in their labels. In Fig. 1a, nodes with the symbol a in the leftmost position of their labels form the first column, nodes with the symbol a in the rightmost position form the second column, and nodes with the symbol a in the middle position form the third column. The first column is duplicated in order to give a clearer view of the connections. We use solid lines to denote the g -edges, i.e., the edges defined by the permutation g or g^{-1} , and dotted lines to denote the f -edges.

2 ISOMORPHISM TO THE WRAP-AROUND BUTTERFLY GRAPH

In this section, we prove that the graph $\mathcal{G}(n)$ is isomorphic to the wrap-around butterfly graph $\mathcal{B}(n)$.

DEFINITION 2. *The wrap-around butterfly graph $\mathcal{B}(n)$ has node-set $Z_n \times Z_2^n$. Each node is represented as a pair $\langle c, r \rangle$, where $c \in Z_n$ is the column of the node and $r \in Z_2^n$ is the row of the node. The edges of $\mathcal{B}(n)$ form butterflies (i.e., copies of the complete bipartite graph $\mathcal{K}_{2,2}$) between consecutive columns of nodes. Each node $\langle c, r \rangle$ is connected to the node $\langle c', r \rangle$ and the node $\langle c', r' \rangle$, where $c' = c + 1 \pmod n$ and r' and r differ in precisely the c th bit; the first edge is a straight edge and the second edge is a cross edge.*

Fig. 1c shows $\mathcal{B}(3)$.

An isomorphical mapping between $\mathcal{G}(n)$ and $\mathcal{B}(n)$ is as follows: Given an arbitrary node $a_0 a_1 \dots a_{n-1}$ in $\mathcal{G}(n)$ and $a_k = t_0^*$ for some k , the node a becomes $a' = a_k a_{k+1} \dots a_{n-1} a_0 \dots a_{k-1}$ after $(n-k) \pmod n$ g^{-1} operations. If we substitute a 0 for every uncomplemented symbol and a 1 for every complemented symbol in a' , and let the resulting binary string be r , then node $a_0 a_1 \dots a_{n-1}$ in $\mathcal{G}(n)$ corresponds to node $\langle n-k \pmod n, r \rangle$ in $\mathcal{B}(n)$. It is not difficult to see that this mapping is a bijection. Furthermore, the g -edges in $\mathcal{G}(n)$ correspond to the direct edges in $\mathcal{B}(n)$, while the f -edges in $\mathcal{G}(n)$ correspond to the cross edges of $\mathcal{B}(n)$. To see the latter, consider nodes $a = a_0 a_1 \dots a_{n-1}$ and $b = a_1 \dots a_{n-1} a_0$ in $\mathcal{G}(n)$. a and b are connected by a g -edge. According to the above mapping, a corresponds to the node $\langle n-k \pmod n, r \rangle$ in $\mathcal{B}(n)$, where $n-k$ and r are computed as in the above; since $b = g(a)$, $t_k^* = a_k$ is at position $k-1 \pmod n$ in b , and, so, b corresponds to the node $\langle n-k+1 \pmod n, r \rangle$; clearly, these two nodes in $\mathcal{B}(n)$ are connected by a direct edge, by the definition of $\mathcal{B}(n)$. A similar analysis can be applied to the mapping between an f -edge in $\mathcal{G}(n)$ and a cross edge in $\mathcal{B}(n)$.

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Manuscript received 7 Dec. 1996; revised 8 Aug. 1997.

For information on obtaining reprints of this article, please send e-mail to: tpd@computer.org, and reference IEEECS Log Number 100356.

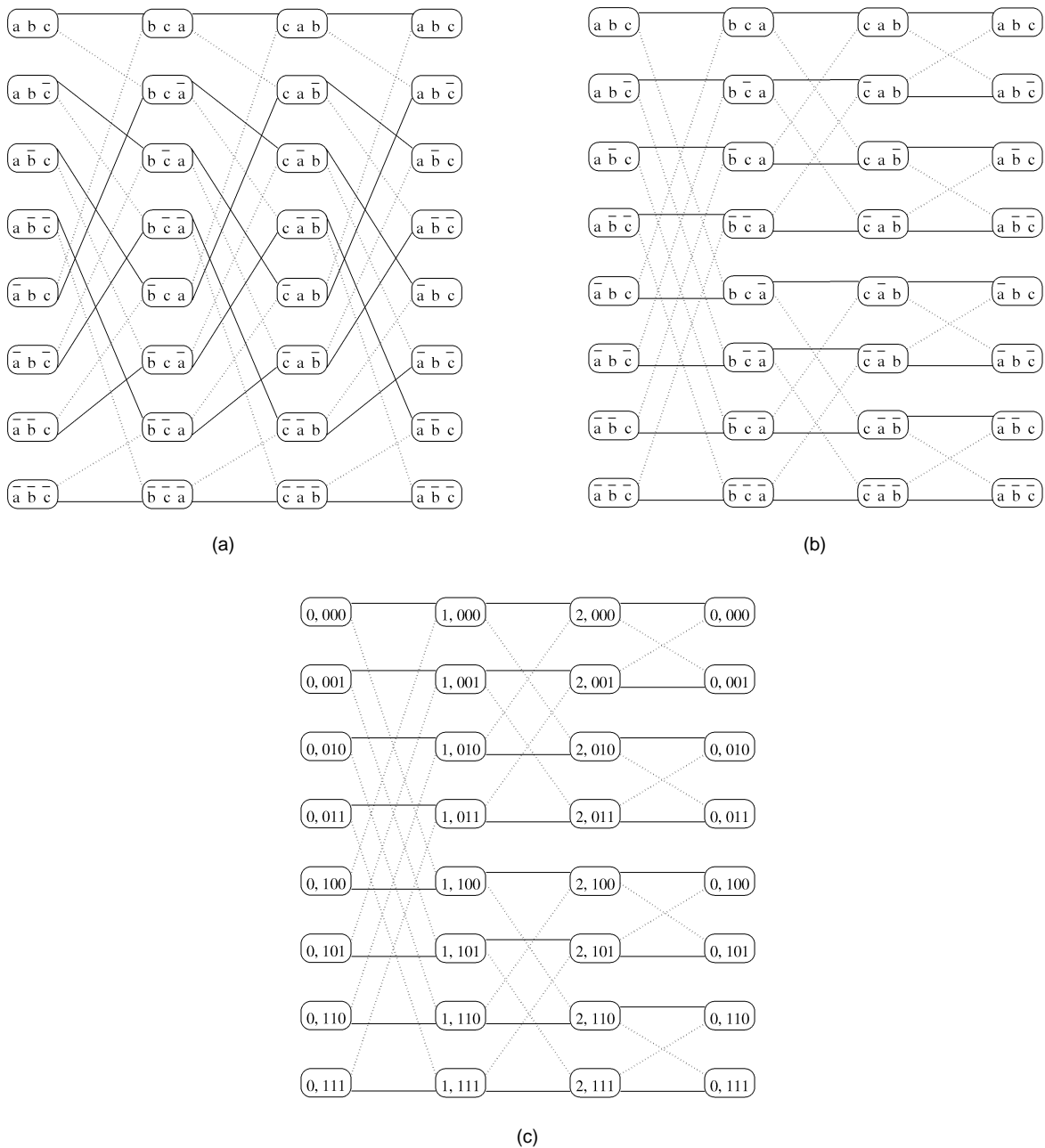


Fig. 1. (a) The proposed network, (b) after “straightening” the g -edges, (c) a wrap-around butterfly network. The solid lines are the g -edges in (a), (b), or the straight edges in (c); the dotted lines are the f -edges in (a), (b), or the cross edges in (c).

Refer again to Fig. 1 for an example. Based on the fact that a g -edge in $G(n)$ corresponds to a direct edge in $B(n)$, we “straighten” all the g -edges in $G(3)$ (Fig. 1a) (thus reordering the nodes in each column), and the result is the $G(3)$, as shown in Fig. 1b. Clearly, the latter is the same as the $B(n)$ in Fig. 1c.

3 FURTHER DISCUSSION

We have shown that the graph $G(n)$ proposed by Vadapalli and Srimani is not a new graph, but a new representation of the wrap-around butterfly graph. Indeed, $G(n) = B(n)$.

The group-theoretic relations between $B(n)$ (or $G(n)$) and the de Bruijn graph are well studied in [1], where $B(n)$ is proved to be a Cayley graph derived from the de Bruijn graph acting as a group action graph, and, inversely, the de Bruijn graph is proved to be some coset graph of $B(n)$.

The new representation in [2] shows another simple structural kinship between $G(n)$ (or $B(n)$) and the de Bruijn graph. In particular, if n distinct symbols in $G(n)$ are the same, i.e., each bit of the node address of $G(n)$ is either 0 or 1, $G(n)$ specializes to become the de Bruijn graph.

The new representation in [2] may bring about some convenience in studying the topological properties of $G(n)$ (or $B(n)$), such as optimal routing algorithms and fault tolerance.

REFERENCES

- [1] F. Annexstein, M. Baumslag and A.L. Rosenberg, “Group Action Graphs and Parallel Architectures,” *SIAM J. Computing*, vol. 19, no. 3, pp. 544–569, June 1990.
- [2] P. Vadapalli and P.K. Srimani, “A New Family of Cayley Graph Interconnection Networks of Constant Degree Four,” *IEEE Trans. Parallel and Distributed Systems*, vol. 7, no. 1, pp. 26–32, Jan. 1996.