A Geometric-Process Maintenance Model for a Deteriorating System Under a Random Environment

Yeh Lam and Yuan Lin Zhang

***Abstract*—This paper studies a geometric-process maintenance-model for a deteriorating system under a random environment. Assume that the number of random shocks, up to time t, produced by the random environment forms a counting process. Whenever a random shock arrives, the system operating time is reduced. The successive reductions in the system operating time are statistically independent and identically distributed random variables. Assume that the consecutive repair times of the system after failures, form an increasing geometric process; under the condition that the system suffers no random shock, the successive operating times of the system after repairs constitute a decreasing geometric process. A replacement policy N, by which the system is replaced at the time of the failure N, is adopted. An explicit expression for the average cost rate (long-run average cost per unit time) is derived. Then, an optimal replacement policy is determined analytically. As a particular case, a compound Poisson process model is also studied.**

***Index Terms*—Compound Poisson process, geometric process, random shocks, renewal process, replacement policy.**

ACRONYMS[1]

ACR	average cost rate: long-run average cost per unit time
Cdf	cumulative distribution function
CPPM	compound Poisson process model
GP	geometric process
GPMM	GP maintenance model
iid	s-independent and identically distributed
pdf	probability density function
RP	renewal process
RS	random shock(s)
rv	random variable
s-	implies: statistical(ly).

NOTATION

a	constant: $a \geq 1$, ratio of a decreasing GP
b	constant: $0 < b \leq 1$, ratio of an increasing GP
c	system repair-cost rate
c_R	system replacement-cost
$C(N)$	ACR with RS under replacement policy N
f_n	pdf$[X_n]$
F_n	Cdf$[X_n]$
g_n	pdf$[Y_n]$
G_n	Cdf$[Y_n]$
h	pdf$[W_n]$
H	Cdf$[W_n]$
N	number of system-failures
N^*	an optimal N for minimizing $C(N)$
r	system reward rate
W_n	reduction in the system operating time after RS #n
X_n	system operating time after repair #$(n-1)$, assuming that there is no random shock
X'_n	the real system operating time after repair #$(n-1)$
Y_n	system repair time after failure #n
Z	system replacement time
λ	E$[X_1]$
μ	E$[Y_1]$
τ	E$[Z]$.

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Y. Lam is with the Department of Statistics and Actuarial Science, The University of Hong Kong, Hong Kong, and The Northeastern University at Qinhuangdao, 066004, China (e-mail: YLam@hkustasc.hku.hk).

Y. L. Zhang is with the Institute of Applied Probability, Sanjiang University, Nanjing 210012, China and the Department of Applied Mathematics, Southeast University, Nanjing 210018, China (e-mail: YLZ@seu.edu.cn).

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[1]The singular and plural of an acronym are always spelled the same.

I. INTRODUCTION

AT THE initial stage of research in maintenance problems of a repairable system, a common assumption is "repair is perfect:" a repairable system after repair is "as good as new." Obviously, this assumption is not always true. In practice, most repairable systems are deteriorating because of the aging effect and accumulated wear. Thus, [4] introduced a minimal-repair model in which a system after repair has the same failure rate and the same effective age as at the time of failure. Reference [6] suggests an imperfect repair model, in which a repair is perfect with probability p, and a minimal repair with probability $1-p$.

An alternate approach is to introduce a monotone process model. For a deteriorating system, it is reasonable to assume that the successive operating times of the system after repairs are stochastically decreasing and the consecutive repair times of the system after failures are stochastically increasing. According to this idea, [12], [13] introduced GP model. This is a simple monotone process model but a good approximation to a more general monotone process model.

Definition 1: Given 2 r.v. X and Y, X is stochastically greater than Y (Y is stochastically less than X) if

$$\Pr\{X > z\} \geq \Pr\{Y > z\} \text{ for all real } z\text{:}$$
$$X \geq_{\rm st} Y \text{ or } Y \leq_{\rm st} X.$$

A stochastic process $\{X_n,\ n = 1, 2, \ldots\}$ is stochastically increasing (decreasing) if $X_n \leq_{\rm st} (\geq_{\rm st}) X_{n+1}$ for all $n = 1, 2, \ldots$ [29].

As a special monotone stochastic process, the GP was first introduced in [12], [13].

Definition 2: A stochastic process $\{\xi_n, n = 1, 2, \ldots\}$ is a GP, if there exists $\alpha > 0$, such that $\{\alpha^{n-1}\xi_n, n = 1, 2, \ldots\}$ forms a RP; α is the ratio of GP.

For a GP, let the Cdf$[\xi_1] = F$; then Cdf$[\xi_n] = F_n$ with $F_n(t) \equiv F(\alpha^{n-1} \cdot t),\ n = 1, 2, \ldots$; and

$$\mathrm{E}[\xi_n] = \frac{\mathrm{E}[\xi_1]}{\alpha^{n-1}}, \quad \mathrm{Var}[\xi_n] = \frac{\mathrm{Var}[\xi_1]}{\alpha^{2(n-1)}}.$$

Therefore, α, E$[\xi_1]$, and Var$[\xi_1]$ are 3 important parameters for the GP.

On the other hand, if $\alpha \geq 1$, then $\{\xi_n, n = 1, 2, \ldots\}$ is stochastically decreasing:

$$\xi_n \geq_{\mathrm{st}} \xi_{n+1}, \qquad n = 1, 2, \ldots$$

If $0 < \alpha \leq 1$, then $\{\xi_n, n = 1, 2, \ldots\}$ is stochastically increasing:

$$\xi_n \leq_{\mathrm{st}} \xi_{n+1}, \qquad n = 1, 2, \ldots$$

If $\alpha = 1$, then the GP is a RP.

Although, in many cases, the deterioration of a system is due to an internal cause such as aging and accumulated wear of the system, an external cause such as an environmental factor might be another reason for system deterioration. In practice, a precision instrument and meter installed in a power workshop might be affected by some RS due to the operation of other instruments, such as lathes or electrical machines: the operating time of the instrument and meter might be shorter. On the other hand, if an instrument and meter system are installed in a naval vessel, then the high temperature and humidity of the operating environment might reduce the operating time of the system. If a computer is invaded by some virus or attacked with a raider, the operating time of the computer is diminished, or the computer can break down. These examples show that the system is deteriorating due to an external cause. The effect of an internal cause on the system operating time can be a continuous process; while the effect of an external cause (such as a RS) might form a jump process. Therefore, in studying a maintenance problem for a repairable system, one should not only consider the internal cause but consider the effect of an RS (produced by the environment) against the system. As a result, one should study a maintenance model with RS that is also an important model in reliability theory. References [5], [7], [8] study the Poisson shock model. Later, [28] presented a generalized Poisson shock model; and [30] extended Poisson shock model to a general shock model. For the case where RS forms a semi-Markov process, [9], [33] determine the optimal replacement policy. For more references, see [1], [10], [11], [25].

This paper studies a GPMM for a system under a random environment by considering the effect of RS on the system. The replacement policy N is adopted: a failed system is replaced if the number of failures since the installation or the last replacement has reached N, otherwise it is repaired. Section II introduces the model. Section III evaluates the ACR. Section IV analytically determines an optimal replacement policy, N^*. Section V discusses particular case, CPPM.

II. Model

GPMM is studied with 6 assumptions:

Assumptions

1) A new system is installed at the beginning. It is replaced by a new and s-identical one sometime later.

2) Given that there is no random shock, then $\{X_n, n = 1, 2, \ldots\}$ form a GP with ratio $a \geq 1$ and E$[X_1] = \lambda > 0$. However, no matter whether there is an RS or not, $\{Y_n, n = 1, 2, \ldots\}$ constitutes a GP with ratio $b \leq 1$ and E$[Y_1] = \mu > 0$. Let the Cdf of X_n and Y_n be F_n and G_n, respectively; and the pdf be f_n and g_n, respectively

$$F_n(x) = F(a^{n-1} \cdot x), \quad f_n(x) = a^{n-1} \cdot f(a^{n-1} \cdot x),$$
$$G_n(y) = G(b^{n-1} \cdot y), \quad g_n(y) = b^{n-1} \cdot g(b^{n-1} \cdot y).$$

3) $N(t)$ is the number of RS up to time t produced by the random environment. $\{N(t), t \geq 0\}$ forms a counting process having stationary and s-independent increment. Whenever a shock arrives, the system operating time is reduced. $\{W_n, n = 1, 2, \ldots\}$ are iid rv; W_n is the reduction in the system operating time after RS #n. The successive reductions in the system operating time are additive.

If a system fails, it is closed so that the random environment has no effect on a failed system.

4) The processes $\{X_n, n = 1, 2, \ldots\}$, $\{Y_n,\ n = 1, 2, \ldots\}$, and rv Z are s-independent. The processes $\{X_n, n = 1, 2, \ldots\}$, $\{N(t), t \geq 0\}$, and $\{W_n, n = 1, 2, \ldots\}$ are also s-independent.

5) The replacement policy N is applied.

6) The repair-cost rate of the system is c, the replacement cost is c_R, and the reward rate of the system is c.

The completion time of repair #$(n-1)$ is denoted by t_{n-1}; the number of RS in $(t_{n-1}, t_{n-1}+t]$ produced by the environment is

$$N(t_{n-1}, t_{n-1}+t] = N(t_{n-1}+t) - N(t_{n-1});$$

$N(t_{n-1})$ and $N(t_{n-1}+t)$ are, respectively, the number of RS produced in $(0, t_{n-1}]$ and $(0, t_{n-1}+t]$; the total reduction in the operating time in $(t_{n-1}, t_{n-1}+t]$ is

$$\Delta X_{(t_{n-1}, t_{n-1}+t]} = \sum_{i=1}^{N(t_{n-1}, t_{n-1}+t]} W_i. \tag{1}$$

Consequently, under the random environment, the residual time at $t_{n-1}+t$ is

$$S_n(t) = X_n - t - \Delta X_{(t_{n-1}, t_{n-1}+t]}, \tag{2}$$

subject to $S_n(t) \geq 0$. Therefore,

$$X'_n = \inf_{t \geq 0}\{t | S_n(t) \leq 0\}. \tag{3}$$

Lemma 1 is useful for later study; the proof is trivial.

Lemma 1:

$$\begin{aligned}\mathrm{P}\{X_n - t - \Delta X_{(t_{n-1}, t_{n-1}+t]} > 0, \quad \forall t \in [0, t']\} \\ = \Pr\{X_n - t' - \Delta X_{(t_{n-1}, t_{n-1}+t']} > 0\}.\end{aligned} \tag{4}$$

Now, consider the assumptions of GPMM.

Assumption 2) shows that the system is deteriorating so that the consecutive repair times constitute an increasing GP; if there is no RS, the successive operating times form a decreasing GP. This is based on general knowledge and on the results in real-data analysis. References [16], [20] apply GP model to fit 3 real data-sets by using nonparametric and parametric methods respectively. The first data-set is the coal-mining disasters data of the intervals in days between successive coal-mining disasters in Great Britain [2]; the second data-set is the data of arrival times to unscheduled maintenance actions for the USS Halfbeak no. 3 main propulsion diesel engine [3]; the third data-set is the data of arrival times to unscheduled maintenance actions for the USS Grampus no. 4 main propulsion diesel engine [3]. The last two data-sets are sequences of successive operating times after repairs of the propulsion diesel engine. The numerical results in [16], [20] show that all 3 data-sets can be well fitted by the GP model. More real data-sets were analyzed later. By comparing the GP model with 2 inhomogeneous Poisson process models, the Weibull process model, and Cox–Lewis model, [23] shows that, on average, the GP model can fit these real data-sets better than the others. Therefore, it is reasonable to apply a decreasing GP model for the successive operating times of a system after repairs and an increasing GP model to formulate the consecutive repair-times of the system after failures. Based on this understanding, [12], [13] applied the GP model to the maintenance problem for a 1-component system. The GP model has also been applied to reliability analysis for 2-component series and parallel system [18], [19] and [21], [22]. For further reference see [26], [32].

Assumption 3) means that the effect of a random environment on the system is through a sequence of RS which shorten the operating time. In practice, many examples show that the effect of an RS is a reduction rather than a percentage-reduction in residual operating time. In other words, assume that W_n acts additively rather than multiplicatively. For example, a person suffering from second hand smoking is very serious, the effect is measured by a reduction in the lifetime. Similarly, a car damaged by traffic accidents reduces its operating time additively.

Equation (3) shows that whenever the total reduction $\Delta X_{(t_{n-1},\, t_{n-1}+t]}$ in system operating time in $(t_{n-1},\, t_{n-1}+t]$ is greater than the residual operating time $X_n - t$, then the system fails: the chance that a shock produces an immediate failure depends on the comparative distributions of $X_n - t$ and $\Delta X_{(t_{n-1},\, t_{n-1}+t]}$. To see the reasonableness of this point, consider the following examples.

In a traffic accident, all the passengers in the bus suffer the same shock, so that the reductions in their lifetimes are more or less the same, but the effects on different passengers might be quite different. An older passenger is more fragile because of having less residual lifetime than a younger passenger has; thus the older passenger can be injured more seriously than a younger passenger. The older passenger might even die, but the younger passenger might only suffer a light-injury. This situation also happens in engineering. Suppose many machines are installed in 1 workshop, all of them suffer the same shock produced by a random environment, but the effects might be different: an old machine could be destroyed whereas a new machine might be slightly damaged. This means that the effect of an RS depends on the residual lifetime of a system, if the reduction in the residual lifetime is greater than the residual time, then the system fails. Therefore (3) is realistic. These 2 examples also show why W_n acts additively, and if W_n acts multiplicatively, then system could not fail after suffering a RS.

The reasons why N is adopted are explained. Using N has a long history [24], [27]. However in a maintenance problem, besides N, policy T is also applied, wherein the affected system is replaced by a new and s-identical one at a stopping time T. For the long-run average cost, [14] and [31] show that under some mild conditions, an optimal N^* is at least as good as an optimal T^*. The same result for the total s-expected discounted cost case was proved [15], [17]. Therefore, without loss of generality, the policy N can be studied. Implementing policy N is more convenient than implementing policy T. This is an additional advantage of using policy N.

III. Average Cost Rate

In this model, a cycle is completed if a replacement is completed. Because a cycle is actually a time interval between two successive replacements, then the successive cycles form a RP. The successive cycles together with the costs incurred in each cycle make a renewal reward process. The standard result in renewal reward process shows that the ACR is [29]

$$\frac{s\text{-Expected cost incurred in a cycle}}{s\text{-Expected length of a cycle}}. \tag{5}$$

To begin, study the distribution of X_n'. For this purpose, let $N(t_{n-1},\, t_{n-1}+t]$ of RS which occur in $(t_{n-1},\, t_{n-1}+t]$ be k. Then for $t' > 0$, study the conditional probability:

$$\begin{aligned}
&\Pr\{X_n' > t' | N(t_{n-1}, t_{n-1}+t'] = k\} \\
&\quad = \Pr\left\{ X_n' = \inf_{t\geq 0}\{t | S_n(t) \leq 0\} > t' \right| \\
&\qquad\qquad \left. N(t_{n-1}, t_{n-1}+t'] = k \right\} \\
&\quad = \Pr\{S_n(t) > 0, \forall\, t \in [0,t'] | N(t_{n-1}, t_{n-1}+t'] = k\} \\
&\quad = \Pr\left\{ X_n - t - \Delta\, X_{(t_{n-1}, t_{n-1}+t]} > 0,\ \forall\, t \in [0,t'] \right| \\
&\qquad\qquad \left. N(t_{n-1}, t_{n-1}+t'] = k \right\} \\
&\quad = \Pr\left\{ X_n - t' - \Delta\, X_{(t_{n-1}, t_{n-1}+t']} > 0 \right| \\
&\qquad\qquad \left. N(t_{n-1}, t_{n-1}+t'] = k \right\} \\
&\quad = \Pr\left\{ X_n - \Delta\, X_{(t_{n-1}, t_{n-1}+t']} > t' \right| \\
&\qquad\qquad \left. N(t_{n-1}, t_{n-1}+t'] = k \right\} \\
&\quad = \Pr\left\{ X_n - \sum_{i=1}^{k} W_i > t' \right\} \\
&\quad = \int_D \int f_n(x) \cdot h_k(w)\, dx\, dw. \qquad (6)
\end{aligned}$$

$$D \equiv \{(x,w) | x > 0, w > 0, x - w > t'\},$$

$$h_k = \text{pdf}\left[\sum_{i=1}^{k} W_i \right], \tag{7}$$

and is the k-fold convolution of h with itself.

$h = \text{pdf}[W_i]$, and $H = \text{Cdf}[W_i]$. Equation (6) is due to lemma 1. Therefore, it follows from (7) that

$$\begin{aligned}\Pr\{X'_n > t' | N(t_{n-1}, t_{n-1}+t'] = k\} &= \int_0^\infty \left[\int_{t'+w}^\infty f_n(x)\,dx\right] h_k(w)\,dw \\ &= \int_0^\infty [1 - F_n(t'+w)]\,dH_k(w) \\ &= 1 - \int_0^\infty F_n(t'+w)\,dH_k(w). \end{aligned} \tag{8}$$

$H_k = \text{Cdf}[\sum_{i=1}^k W_i]$. Thus

$$\begin{aligned}\Pr\{X'_n > t') &= \sum_{k=0}^\infty [\Pr\{X'_n > t' | N(t_{n-1}, t_{n-1}+t'] = k\} \\ &\qquad \cdot \Pr\{N(t_{n-1}, t_{n-1}+t'] = k\}] \\ &= \sum_{k=0}^\infty \left[\left(1 - \int_0^\infty F_n(t'+w)\,dH_k(w)\right) \right. \\ &\qquad \left. \cdot \Pr\{N(t_{n-1}, t_{n-1}+t'] = k\}\right] \\ &= 1 - \sum_{k=0}^\infty \left[\left(\int_0^\infty F_n(t'+w)\,dH_k(w)\right) \right. \\ &\qquad \left. \cdot \Pr\{N(t') = k\}\right]. \end{aligned} \tag{9}$$

Equation (9) is due to the fact that $\{N(t), t \geq 0\}$ has a stationary increment property. Therefore, by noting that $F_n(x) = F(a^{n-1} \cdot x)$, the Cdf, I_n, of X'_n is

$$\begin{aligned} I_n(x) &= \Pr\{X'_n \leq x\} \\ &= \sum_{k=0}^\infty \left[\left(\int_0^\infty F\left(a^{n-1}\cdot(x+w)\right)\,dH_k(w)\right) \right. \\ &\qquad \left. \cdot \Pr\{N(x) = k\}\right]. \end{aligned} \tag{10}$$

By using replacement policy N, it follows from (5) that the ACR is

$$\begin{aligned} C(N) &= \frac{\mathrm{E}\left[c \cdot \sum_{n=1}^{N-1} Y_n - r \cdot \sum_{n=1}^{N} X'_n + c_R\right]}{\mathrm{E}\left[\sum_{n=1}^{N} X'_n + \sum_{n=1}^{N-1} Y_n + Z\right]} \\ &= \frac{c \cdot \sum_{n=1}^{N-1} \mathrm{E}[Y_n] - r \cdot \sum_{n=1}^{N} \mathrm{E}[X'_n] + c_R}{\sum_{n=1}^{N} \mathrm{E}[X'_n] + \sum_{n=1}^{N-1} \mathrm{E}[Y_n] + \mathrm{E}[Z]} \\ &= \frac{c \cdot \mu \cdot \sum_{n=1}^{N-1} (1/b^{n-1}) - r \cdot \sum_{n=1}^{N} \lambda'_n + c_R}{\sum_{n=1}^{N} \lambda'_n + \mu \cdot \sum_{n=1}^{N-1} (1/b^{n-1}) + \tau}, \end{aligned} \tag{11}$$

$$\lambda'_n = \mathrm{E}[X'_n] = \int_0^\infty x\,dI_n(x) \tag{12}$$

is the s-expected real operating time after repair #$(n-1)$.

Thus, the objective is to determine an optimal replacement policy N^* for minimizing the ACR: $C(N)$.

IV. OPTIMAL REPLACEMENT POLICY N^*

This section determines the optimal N^* explicitly.

First, a simple but important lemma is derived.

Lemma 2: λ'_n is nonincreasing in n.

$\{X_n, n = 1, 2, \ldots\}$ forms a decreasing GP, and $F_n = \text{Cdf}[X_n]$; thus from (9) for all real t',

$$\Pr\{X'_n > t'\} \geq \Pr\{X'_{n+1} > t'\}.$$

Thus Lemma 2 follows.

Second, rewrite (11) as:

$$C(N) = \frac{(c+r)\mu \sum_{n=1}^{N-1} (1/b^{n-1}) + \delta}{\sum_{n=1}^{N} \lambda'_n + \mu \sum_{n=1}^{N-1} (1/b^{n-1}) + \tau} - r;$$
$$\delta \equiv r \cdot \tau + c_R. \tag{13}$$

Third, introduce the auxiliary function:

$$D(N) = \frac{(c+r) \cdot \mu \cdot \left[\sum_{n=1}^{N} \lambda'_n - \lambda'_{N+1} \cdot \sum_{n=1}^{N-1} b^{N-n} + \tau\right]}{\delta \cdot \left(\lambda'_{N+1} \cdot b^{N-1} + \mu\right)}. \tag{14}$$

Lemma 3 is shown by a direct comparison of $C(N+1)$ and $C(N)$.

Lemma 3:

$$C(N+1) \gtreqqless C(N) \Longleftrightarrow D(N) \gtreqqless 1.$$

Furthermore, it is obvious that:

$$\begin{aligned} D(N+1) - D(N) &= \frac{(c + r \cdot \mu) \cdot (\lambda'_{N+1} - b \cdot \lambda'_{N+2})}{\delta \cdot b^N \cdot (\lambda'_{N+1} + \mu/b^{N-1}) \cdot (\lambda'_{N+2} + \mu/b^N)} \\ &\quad \cdot \left[\sum_{n=1}^{N+1} \lambda'_n + \mu \cdot \sum_{n=1}^{N} (1/b^{n-1}) + \tau\right] \geq 0. \end{aligned}$$

This implies:

Lemma 4: $D(N)$ is nondecreasing in N.

The combination of Lemmas 2–4 gives theorem 1:

Theorem 1: The optimal maintenance policy N^* is determined by

$$N^* = \min[N | D(N) \geq 1]. \tag{15}$$

It follows from theorem 1 that:

#1. If

$$D(1) = \frac{(c+r) \cdot \mu \cdot (\lambda'_1 + \tau)}{\delta \cdot (\lambda'_2 + \mu)} \geq 1,$$

then $N^* = 1$.

#2. If $a > 1$, $0 < b < 1$, and

$$D(\infty) = \frac{(c+r)\cdot\mu\cdot\left(\sum_{n=1}^{\infty}\lambda'_n + \tau\right)}{\delta\cdot\mu} \le 1,$$

then $N^* = \infty$.

The result of #1 is trivial. To prove #2, note that if $a > 1$, let $E[X_n] = \lambda_n$, then

$$\sum_{n=1}^{\infty}\lambda_n < \infty.$$

Hence by noting that $\lambda'_n \le \lambda_n$, then

$$\sum_{n=1}^{\infty}\lambda'_n < \infty.$$

Thus #2 is true.

Therefore,

- if $D(1) \ge 1$, the optimal policy is to replace the system whenever it fails;
- if $D(\infty) \le 1$, the optimal policy is to repair the system forever.[2]

V. A Compound Poisson-Process Model

To demonstrate the model and methodology developed in this paper, consider the special case: $\{N(t),\, t \ge 0\}$ is a Poisson process with parameter γ, i.e., the RS arrives according to a Poisson process with rate γ. Then,

$$\Pr\{N(t) = k\} = \frac{(\gamma\cdot t)^k}{k!}\cdot\exp(-\gamma\cdot t), \qquad k = 0, 1, \ldots. \tag{16}$$

Let the successive reductions in the system operating-time, by the RS, be $W_1, W_2, \ldots, W_k, \ldots$. They are iid, each having the Gamma distribution, $\Gamma(\alpha, \beta)$, with pdf h:

$$h(w) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)}\cdot w^{\alpha-1}\cdot\exp(-\beta\cdot w) & w > 0, \\ 0 & \text{elsewhere.}\end{cases} \tag{17}$$

Therefore $\sum_{i=1}^{k} W_i$ is a Gamma rv with distribution $\Gamma(k\cdot\alpha, \beta)$.

Also, let X_1 have an exponential distribution with pdf:

$$f(x) = \begin{cases} \dfrac{1}{\lambda}\cdot\exp\left(-\dfrac{x}{\lambda}\right) & x > 0, \\ 0 & \text{elsewhere.}\end{cases} \tag{18}$$

The Cdf is:

$$\begin{aligned} F_n(x) &= F(a^{n-1}\cdot x) \\ &= \begin{cases} 1 - \exp(-a^{n-1}\cdot x/\lambda) & x > 0, \\ 0 & \text{elsewhere.}\end{cases} \end{aligned} \tag{19}$$

Because $\{N(t),\, t \ge 0\}$ is a Poisson process, and $\{W_i,\, i = 1, 2, \ldots\}$ are iid, then

$$\Delta X_{(0,\,t]} = \sum_{i=1}^{N(t)} W_i$$

forms a compound Poisson process. It is a very popular and important process in application. For example, it is reasonable to assume that the number of customers arriving a supermarket by time t form a Poisson process and the amounts of money spent by the customers are iid, then the total amount of money spent by the customers forms a compound Poisson process [29].

From (10),

$$\begin{aligned} I_n(x) &= \Pr\{X'_n \le x\} \\ &= \sum_{k=0}^{\infty}\int_0^{\infty}\left[1-\exp\left(-\frac{a^{n-1}(x+w)}{\lambda}\right)\right]\cdot\frac{\beta^{k\cdot\alpha}}{\Gamma(k\cdot\alpha)} \\ &\quad \cdot w^{k\cdot\alpha-1}\cdot\exp(-\beta\cdot w)\,dw\,\frac{(\gamma\cdot x)^k}{k!}\cdot\exp(-\gamma\cdot x) \\ &= 1-\sum_{k=0}^{\infty}\left(\exp\left[-\left(\gamma+\frac{a^{n-1}}{\lambda}\right)\cdot x\right]\cdot\frac{(\gamma\cdot x)^k\cdot\beta^{k\cdot\alpha}}{k!\cdot\Gamma(k\cdot\alpha)}\right. \\ &\quad \left.\cdot\int_0^{\infty}\left[w^{k\cdot\alpha-1}\cdot\exp\left[-\left(\beta+\frac{a^{n-1}}{\lambda}\right)\cdot w\right]dw\right]\right) \\ &= 1-\sum_{k=0}^{\infty}\exp\left[-\left(\gamma+\frac{a^{n-1}}{\lambda}\right)\cdot x\right]\cdot\frac{1}{k!}\cdot\left[\frac{\gamma\cdot x\cdot\beta^\alpha}{\left(\beta+\frac{a^{n-1}}{\lambda}\right)^\alpha}\right]^k \\ &= 1-\exp\left(-\left[\gamma\cdot\left(1-\left(\frac{\beta}{\beta+\frac{a^{n-1}}{\lambda}}\right)^\alpha\right)+\frac{a^{n-1}}{\lambda}\right]\cdot x\right). \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{E}[X'_n] = \lambda'_n &= \int_0^{\infty} x\,dI_n(x) \\ &= \left(\gamma\cdot\left[1-\left(\frac{\beta}{\beta+\frac{a^{n-1}}{\lambda}}\right)^\alpha\right]+\frac{a^{n-1}}{\lambda}\right)^{-1}. \end{aligned} \tag{20}$$

If $\gamma = 0$, then the system suffers no RS, and the model reduces to the Lam model [12], [13]; if $\alpha = 1$, then $W_1, W_2, \ldots, W_k, \ldots$ are iid, each having an $\exp(\beta)$ distribution. Then (20) becomes

$$\mathrm{E}[X'_n] = \lambda'_n = \frac{\lambda}{a^{n-1}\cdot\left[1+\frac{\gamma}{\beta+\frac{a^{n-1}}{\lambda}}\right]}. \tag{21}$$

Now, substitute (20) or (21) into (14) for an explicit expression of $D(N)$. Then an optimal replacement policy can be determined by using (15) directly.

This numerical example explains how to determine N^*. Let $c = 4$, $r = 18$, $c_R = 4000$, $\lambda = 100$, $\mu = 5$, $\tau = 48$, $a = 1.01$, $b = 0.98$, $\alpha = 2$, $\beta = 3$, $\gamma = 4$.

Using (20),

$$\lambda'_n = \frac{90\,000 + 600a^{n-1} + a^{2n-2}}{a^{n-1}\cdot(3300 + 10a^{n-1} + 0.01a^{2n-2})}.$$

Substitute the given values into (11), then the results in Table I and Fig. 1 are obtained.

$C(45) = -8.9955$ is the minimum of the long-run average cost per unit time; i.e., $N^* = 45$: replace the system immediately following failure #45.

[2]The system is always repaired whenever it fails.

TABLE I
RESULTS OBTAINED FROM (11)

N	$C(N)$	N	$C(N)$	N	$C(N)$	N	$C(N)$	N	$C(N)$		
1	46.5334	11	-2.4149	21	-7.0850	31	-8.5152	41	-8.9597	60	-8.7095
2	28.2811	12	-3.2319	22	-7.3063	32	-8.5903	42	-8.9745	80	-7.7039
3	18.4841	13	-3.9280	23	-7.5047	33	-8.6571	43	-8.9853	100	-6.3007
4	12.3809	14	-4.5267	24	-7.6828	34	-8.7163	44	-8.9923	150	-2.2814
5	8.2204	15	-5.0459	25	-7.8429	35	-8.7685	45	-8.9955	200	0.9125
6	5.2074	16	-5.4992	26	-7.9868	36	-8.8142	46	-8.9954	300	3.4840
7	2.9287	17	-5.8974	27	-8.1163	37	-8.8539	47	-8.9919	400	3.9274
8	1.1482	18	-6.2488	28	-8.2327	38	-8.8879	48	-8.9854	500	3.9902
9	-0.2786	19	-6.5604	29	-8.3373	39	-8.9166	49	-8.9759	800	4.0000
10	-1.4453	20	-6.8376	30	-8.4312	40	-8.9404	50	-8.9636	1000	4.0000

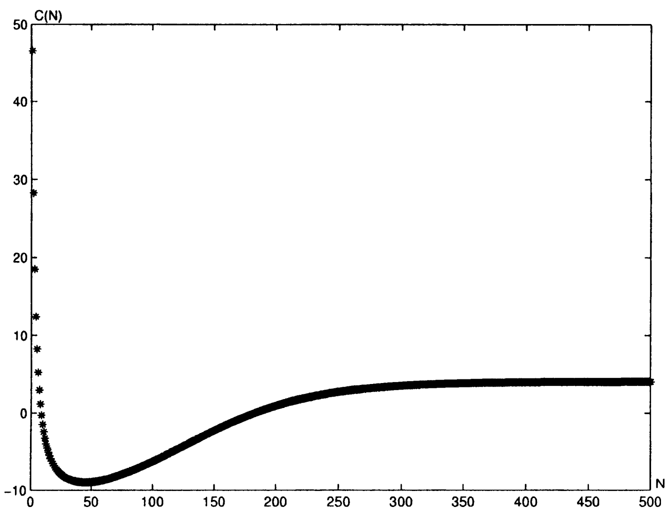

Fig. 1. Average cost rate, $C(N)$ versus N.

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REFERENCES

[1] M. Abdel-Hameed, "Optimal replacement of a system subject to shocks," *J. Appl. Prob.*, vol. 23, pp. 107–114, 1986.
[2] D. F. Andrews and A. M. Herzberg, *Data*: Springer, 1985.
[3] H. Ascher and H. Feingold, *Repairable Systems Reliability*: Marcel Dekker, 1984.
[4] R. E. Barlow and L. C. Hunter, "Optimum preventive maintenance policy," *Operations Research*, vol. 8, pp. 90–100, 1960.
[5] R. E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing*: Holt, Rinehart, Winston, 1975.
[6] M. Brown and F. Proschan, "Imperfect repair," *J. Appl. Prob.*, vol. 20, pp. 851–859, 1983.
[7] J. D. Esary, "A stochastic theory of accident survival and fatality," Ph.D. dissertation, Univ. of California, Berkely, 1957.
[8] J. D. Esary, A. W. Marshall, and F. Proschan, "Shock models and wear processes," *Ann. Prob.*, vol. 1, pp. 627–650, 1973.
[9] R. M. Feldman, "Optimal replacement with semi-Markov shock model," *J. Appl. Prob.*, vol. 13, pp. 107–114, 1976.
[10] G. Gottlieb, "Optimal replacement for shock models with general failure rate," *Oper. Res.*, vol. 30, pp. 82–92, 1982.
[11] M. Kijima, H. Morimura, and Y. Suzuki, "Periodical replacement problem without assuming minimal repair," *Eur. J. Oper. Res.*, vol. 37, pp. 194–203, 1988.
[12] Y. Lam, "A note on the optimal replacement problem," *Adv. Appl. Prob.*, vol. 20, pp. 479–482, 1988.
[13] ——, "Geometric processes and replacement problem," *Acta Mathematicae Applicatae Sinica*, vol. 4, pp. 366–377, 1988.
[14] ——, "Optimal policy for a general repair replacement model: Average reward case," *IMA J. Mathematics Applied in Business and Industry*, vol. 3, pp. 117–129, 1991.
[15] ——, "An optimal repairable replacement model for deteriorating systems," *J. Appl. Prob.*, vol. 28, pp. 843–851, 1991.
[16] ——, "Nonparametric inference for geometric processes," *Communications in Statistics—Theory and Methods*, vol. 21, pp. 2083–2105, 1992.
[17] ——, "Optimal policy for a general repair replacement model: Discounted reward case," *Communication in Statistical-Stochastic Models*, vol. 8, pp. 245–267, 1992.
[18] ——, "Calculating the rate of occurrence of failures for continuous-time Markov chains with application to a two-component parallel system," *J. Oper. Res. Soc.*, vol. 46, pp. 528–536, 1995.
[19] ——, "The rate of occurrence of failures," *J. Appl. Prob.*, vol. 34, pp. 234–247, 1997.
[20] Y. Lam and S. K. Chan, "Statistical inference for geometric processes with lognormal distribution," *Comput. Statist. Data Anal.*, vol. 27, pp. 99–112, 1998.
[21] Y. Lam and Y. L. Zhang, "Analysis of a two-component series system with a geometric process model," *Nav. Res. Logist.*, vol. 43, pp. 491–502, 1996.
[22] ——, "Analysis of a parallel system with two different units," *Acta Mathematicae Applicatae Sinica*, vol. 12, pp. 408–417, 1996.
[23] Y. Lam and L. X. Zhu, "Analysis of the data analysis from a series of events by the geometric process model," *Acta Mathematicae Applicatae Sinica*, 2000, submitted for publication.
[24] H. Morimura, "On some preventive maintenance policies for IFR," *J. Oper. Res. Soc. Jpn.*, vol. 12, pp. 94–124, 1970.
[25] T. Nakagawa, "A summary of replacement models with changing failure distributions," *R.A.I.R.O. Recherche Operationelle/Operations Research*, pp. 343–353, 1989.
[26] M. F. Neuts, R. Péréz-Ocon, and I. Torres-castro, "Repairable models with operating and repair times governed by phase type distributions," *Adv. Appl. Prob.*, vol. 32, pp. 468–479, 2000.
[27] K. S. Park, "Optimal number of minimal repairs before replacement," *IEEE Trans. Rel.*, vol. R-28, pp. 137–140, 1979.
[28] S. M. Ross, "Generalized Poisson shock models," *Ann. Prob.*, vol. 9, pp. 896–898, 1981.
[29] ——, *Stochastic Processes*, 2nd ed: Wiley, 1996.
[30] J. G. Shanthikumar and U. Sumita, "General shock models associated with correlated renewal sequences," *J. Appl. Prob.*, vol. 20, pp. 600–619, 1983.
[31] W. Stadje and D. Zuckerman, "Optimal strategies for some repair replacement models," *Adv. Appl. Prob.*, vol. 22, pp. 641–656, 1990.
[32] Y. L. Zhang, "A bivariate optimal replacement policy for a repairable system," *J. Appl. Prob.*, vol. 31, pp. 1123–1127, 1994.
[33] D. Zuckerman, "Optimal stopping in a semi-Markov shock model," *J. Appl. Prob.*, vol. 15, pp. 629–634, 1978.

Yeh Lam received the B.Sc. in Mathematics from Peking University, the M.Sc. in Statistics from Imperial College of Science and Technology, and the Ph.D. in Operations Research from University of Manchester. His research spectrum covers statistics, applied probability, and operations research. Professor Lam has published more than 50 articles in well-known journals including *Journal of the American Statistical Association*, *Biometrika*, *The Annals of Statistics*, *Journal of Applied Probability*, *Advances in Applied Probability*, *European Journal of Operational Research*, and *Journal of the Operational Research Society*. He is a Member of the International Statistical Institute and of the International Chinese Statistical Association.

Yuan Lin Zhang is a Professor in the Department of Applied Mathematics at Southeast University (SEU), and Vice Chairman of Applied Probability and Statistics Institute, SEU. His research interests include applied probability, stochastic operations research, and insurance mathematics and risk theory. Professor Zhang has published 3 books and more than 60 articles in *Journal of Applied Probability*, *Naval Research Logistics*, *Microelectronics and Reliability*, *Reliability Engineering and System Safety*, *International Journal of Systems Science*, and others. He is a Director of The 5th Operations Research Society of China (ORSC), Reliability Subsociety of ORSC, and Committee on Queueing Theory of ORSC. He is a Fellow of the Royal Statistical Society of UK, and a Member of the Institute of Mathematical Statistics of USA.