Nonlinear Filtering for State Delayed Systems with Markovian Switching

Zidong Wang, James Lam and K. J. Burnham

Abstract— This paper deals with the filtering problem for a general class of nonlinear time-delay systems with Markovian jumping parameters. The nonlinear time-delay stochastic systems may switch from one to the others according to the behavior of a Markov chain. The purpose of the problem addressed is to design a nonlinear full-order filter such that the dynamics of the estimation error is guaranteed to be stochastically exponentially stable in the mean square. Both filter analysis and synthesis problems are investigated. Sufficient conditions are established for the existence of the desired exponential filters, which are expressed in terms of the solutions to a set of Linear Matrix Inequalities (LMIs). The explicit expression of the desired filters is also provided.

Keywords— Nonlinear filtering; Stochastic exponential stability; Nonlinear systems; Markovian jump systems; Time-delay systems; Linear matrix inequalities

I. Introduction

Nonlinear filtering is one of the important issues in signal processing, and has been an active research area over the past three decades. Some recent representative work on nonlinear filtering in the deterministic case can be found in [4], [10]. For the stochastic case, the nonlinear filtering problem has been extensively studied, see [6] for a survey. In particular, the nonlinear filtering problem was investigated in [17] through the concepts of observers for stochastic nonlinear systems, and an important stochastic stability approach to designing the observers with guaranteed convergence was developed.

It is now well known that the delayed state is very often the cause for instability and poor performance of systems. In the past few years, we have seen an increasing interest in the controller as well as observer designs for *linear* systems with certain types of time-delays, see [11], [15], [16] for more details. However, the nonlinear filtering problem for general time-delay stochastic systems has received very little attention. In [14], the nonlinear filtering problem was studied for uncertain time-delay stochastic systems where the nonlinearities were introduced in the form of additional nonlinear disturbances.

On the other hand, many physical systems are subject to frequent unpredictable structural changes, such as random failures, repairs of sudden environment disturbances, abrupt variation of the operating point on a nonlinear plant, etc. Markovian jump systems (MJS), which comprise an important family of models subject to abrupt variations, are very often used to describe the above class of systems. In the past decade, the optimal regulator, controllability, observability, stability and stabilization problems have been studied for *jump* linear systems (JLSs), see e.g. [2], [3], [13] and references therein. Also, the filtering problem for JLSs has recently gained initial attention, see e.g. [12].

In practice, a nonlinear system with Markovian jumping parameters may be more reasonable to account for the nonlinearities and structural changes. To the best of the authors' knowledge, so far, there have been very few papers dealing with filter design problem for general *nonlinear time-delay* systems with or without Markovian jump parameters. This situation encourage us to study the filtering problem for a class of nonlinear time-delay systems with Markovian switching.

This paper is concerned with the exponential filtering problem for nonlinear jump time-delay systems. Our aim is to design a nonlinear full-order filter such that the dynamics of the estimation error of each system mode is stochastically exponentially stable in the mean square. We show that both the filter analysis and the filter synthesis problems can be solved in terms of the solutions to a set of linear matrix inequalities (LMIs, see [1]). Therefore, in our study, the powerful Matlab LMI toolbox ([5]) can be ideally employed to facilitate the filter design problem.

Notation. We let $h > 0$ and $C([-h, 0]; \mathbb{R}^n)$ denote the family of continuous functions φ from $[-h, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-h\leq\theta\leq 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. The operator norm of a matrix A is defined by $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^TA)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C([-h, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0} \mathbb{E}|\xi(\theta)|^p < \infty$ where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P.

II. Problem formulation and assumptions

Let $\{r(t),\ t \geq 0\}$ be a right-continuous Markov process on the probability space which takes values in the finite

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space $\mathcal{S} = \{1, 2, \ldots, N\}$ with generator $\Pi = (\gamma_{ij})$ $(i, j \in \mathcal{S})$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j \end{cases}$$

where $\Delta > 0$ and $\lim_{\Delta \to 0} o(\Delta)/\Delta = 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ and $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$.

Let us consider a nonlinear state delayed jump system in a fixed complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ described by:

$$\dot{x}(t) = f(x(t), u(t), r(t)) + g(x(t-\tau), r(t)), \quad (1)$$

$$x(t) = \varphi(t), \quad r(t) = r(0), \quad t \in [-\tau, 0], \quad (2)$$

$$y(t) = h(x(t), r(t)), \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the deterministic input, $y(t) \in \mathbb{R}^p$ is the measurement output, and $f(\cdot,\cdot,\cdot) \in \mathbb{R}^n$, $g(\cdot,\cdot) \in \mathbb{R}^n$, $h(\cdot,\cdot) \in \mathbb{R}^p$ are nonlinear vector functions. τ denotes the state delay, $\varphi(t)$ is a continuous vector valued initial function.

Assumption 1: For any fixed system mode $r(t) = i \in \mathcal{S}$, the nonlinear vector functions $f(\cdot,\cdot,\cdot)$, $g(\cdot,\cdot)$, $h(\cdot,\cdot)$ are assumed to satisfy $f(0, 0, r(t)) = 0$, $g(0, r(t)) = 0$, $h(0, r(t)) = 0$ and

$$\left| f(x(t)+\sigma, u(t)+\delta, r(t)) - f(x(t), u(t), r(t)) - \begin{bmatrix} A(r(t)) & B(r(t)) \end{bmatrix} \begin{bmatrix} \sigma \\ \delta \end{bmatrix} \right| \leq a_{11}(r(t)) \left| \begin{bmatrix} \sigma \\ \delta \end{bmatrix} \right|, \quad (4)$$

$$\left| g(x(t-\tau)+\sigma, r(t)) - g(x(t-\tau), r(t)) - A_d(r(t))\sigma \right| \leq a_{22}(r(t))|\sigma|, \quad (5)$$

$$\left| h(x(t)+\sigma, r(t)) - g(x(t), r(t)) - C(r(t))\sigma \right| \leq a_{33}(r(t))|\sigma|, \quad (6)$$

where $A(r(t)) \in \mathbb{R}^{n \times n}$, $B(r(t)) \in \mathbb{R}^{n \times m}$, $A_d(r(t)) \in \mathbb{R}^{n \times n}$, $C(r(t)) \in \mathbb{R}^{p \times n}$ are known constant matrices, $\sigma \in \mathbb{R}^n$, $\delta \in \mathbb{R}^m$ are vectors, $a_{11}(r(t))$, $a_{22}(r(t))$, and $a_{33}(r(t))$ are known positive constants.

Remark 1: The system (1)-(3) is called a nonlinear time-delay system with jumping parameters, and can therefore be utilized to describe many important physical systems subject to nonlinearities, time-delay, random failures and structural changes. The nonlinear descriptions (4)-(6) (see [17]) reflect the "distance" between the originally nonlinear model (1)-(3) and the "nominal" linear model whose system parameters are $(A(r(t)), B(r(t)), A_d(r(t)), C(r(t)))$.

Throughout this paper, we shall employ the full-order nonlinear filter being of the following structure

$$\begin{aligned} \dot{\hat{x}}(t) &= f(\hat{x}(t), u(t), r(t)) + g(\hat{x}(t-\tau), r(t)) \\ &\quad + K(r(t))[y(t) - h(\hat{x}(t), r(t))] \end{aligned} \quad (7)$$

where $\hat{x}$ is the state estimate and the constant gains $K(r(t))$ are the filter parameters to be designed.

Notice that the Markov process $\{r(t),\ t \geq 0\}$ takes values in the finite space $\mathcal{S} = \{1, 2, \ldots, N\}$. For notation convenience, we write:

$$A(i) := A_i, \quad A_d(i) := A_{di}, \quad C(i) := C_i, \quad (8)$$

$$a_{11}(i) := a_{11i}, \quad a_{22}(i) := a_{22i}, \quad a_{33}(i) := a_{33i}. \quad (9)$$

Let the error state be

$$e(t) = x(t) - \hat{x}(t), \quad (10)$$

then it follows from (1)-(3) and (7) that

$$\begin{aligned} \dot{e}(t) &= f(x(t), u(t), r(t)) - f(\hat{x}(t), u(t), r(t)) \\ &\quad + g(x(t-\tau), r(t)) - g(\hat{x}(t-\tau), r(t)) \\ &\quad - K(r(t))[h(x(t), r(t)) - h(\hat{x}(t), r(t))]. \end{aligned} \quad (11)$$

Now we shall work on the system mode $r(t) = i$, $\forall i \in \mathcal{S}$. To continue, we introduce the following definitions:

$$\begin{aligned} l_i(t) :=& f(x(t), u(t), i) - f(\hat{x}(t), u(t), i) \\ &- A(i)e(t), \end{aligned} \quad (12)$$

$$\begin{aligned} m_i(t-\tau) :=& g(x(t-\tau), i) - g(\hat{x}(t-\tau), i) \\ &- A_d(i)e(t-\tau), \end{aligned} \quad (13)$$

$$n_i(t) := h(x(t), i) - h(\hat{x}(t), i) - C(i)e(t). \quad (14)$$

Then, we can obtain from (8)-(14) that

$$\begin{aligned} \dot{e}(t) &= (A_i - K_iC_i)e(t) + A_{di}e(t-\tau) + l_i(t) \\ &\quad + m_i(t-\tau) - K_in_i(t). \end{aligned} \quad (15)$$

Assumption 2: For all $\zeta \in [-\tau, 0]$, there exists a scalar $\eta > 0$ such that $|e(t+\zeta)| \leq \eta|e(t)|$.

As mentioned in [3], Assumption 2 is not restrictive since the scalar $\eta > 0$ can be chosen arbitrarily. Now, let $e(t;\xi)$ denote the state trajectory from the initial data $e(\theta) = \xi(\theta)$ on $-\tau \leq \theta \leq 0$ in $L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$. It is clear from our assumption (1) that the system (15) admits a trivial solution $e(t; 0) \equiv 0$ corresponding to the initial data $\xi = 0$.

Definition 1: For every $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, the trivial solution of (15) is asymptotically stable in the mean square if

$$\lim_{t \to \infty} \mathbb{E}|e(t;\xi)|^2 = 0; \quad (16)$$

and is exponentially stable in the mean square if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\mathbb{E}|e(t;\xi)|^2 \leq \alpha e^{-\beta t} \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2. \quad (17)$$

The primary objective of this paper is to provide a practical design procedure for an exponential filter of the nonlinear time-delay system (1)-(3). In other words, we shall design the filter parameter K_i such that the dynamics of the estimation error (i.e., the solution of the system (15)) is guaranteed to be stochastically exponentially stable.

III. Main results and proofs

Lemma 1: Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\varepsilon > 0$. Then

$$x^Ty + y^Tx \leq \varepsilon x^Tx + \varepsilon^{-1}y^Ty.$$

The following theorem, which acts as a main key for solving the addressed nonlinear filtering problem, shows that the exponential stability of a given filter for the nonlinear time-delay stochastic system (1)-(3) can be guaranteed if positive definite solutions to a set of modified algebraic Riccati-like matrix inequalities (quadratic matrix inequalities) are known to exist.

Theorem 1: Let the filter parameters K_i be given. If there exists a sequence of positive scalars $\{\varepsilon_{1i}, \varepsilon_{2i}, \varepsilon_{3i}, \varepsilon_{4i}, i \in \mathcal{S}\}$ such that the following matrix inequalities

$$\begin{aligned}(A_i - K_iC_i)^T P_i + P_i(A_i - K_iC_i) + \sum_{j=1}^{N}\gamma_{ij}P_j \\ +P_i[(\varepsilon_{1i} + \varepsilon_{2i} + \varepsilon_{3i})I + \varepsilon_{4i}K_iK_i^T]P_i \\ +(\varepsilon_{2i}^{-1}a_{11i}^2 + \varepsilon_{4i}^{-1}a_{33i}^2)I + Q_i < 0 \end{aligned} \tag{18}$$

where

$$Q_i := \varepsilon_{1i}^{-1}A_{di}^T A_{di} + \varepsilon_{3i}^{-1}a_{22i}^2 I \tag{19}$$

have positive definite solutions $P_i > 0$, then system (15) is exponentially stable in the mean square.

Proof: Fix $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ arbitrarily and write $e(t;\xi) = e(t)$. For $(e(t), t) \in \mathbb{R}^n \times \mathbb{R}_+$, we define the stochastic Lyapunov functional $V(\cdot) : \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ as

$$\begin{aligned} V(e(t), r(t) = i) :=& V(e(t), t, i) \\ =& e^T(t)P_ie(t) + \int_{t-\tau}^{t} e^T(s)Q_ie(s)ds, \end{aligned} \tag{20}$$

where P_i is the positive definite solution to the matrix inequality (18) and $Q_i > 0$ is defined in (19).

The weak infinitesimal operator $\mathcal{A}$ (see [7]) of the stochastic process $\{r(t), e(t)\}$ $(t \geq 0)$ is given by

$$\begin{aligned} &\mathcal{A}V(e(t), r(t)) \\ =& \lim_{\Delta\to 0}\frac{1}{\Delta}[\mathbb{E}\{V(x(t+\Delta), r(t+\Delta))|x(t), r(t) = i\} \\ & -V(x(t), r(t) = i)] \\ =& e^T(t)[(A_i - K_iC_i)^T P_i + P_i(A_i - K_iC_i) \\ & + \sum_{j=1}^{N}\gamma_{ij}P_j + Q_i]e(t) \\ & + e^T(t)P_iA_{di}e(t-\tau) + e^T(t-\tau)A_{di}^T P_ie(t) \\ & + e^T(t)P_i[l_i(t) + m_i(t-\tau) - K_in_i(t)] \\ & + [l_i(t) + m_i(t-\tau) - K_in_i(t)]^T P_ie(t) \\ & - e^T(t-\tau)Q_ie(t-\tau). \end{aligned} \tag{21}$$

Let ε_{1i}, ε_{2i}, ε_{3i}, ε_{4i} be positive scalars. It then follows from Lemma 1 that

$$\begin{aligned} & e^T(t)P_iA_{di}e(t-\tau) + e^T(t-\tau)A_{di}^T P_ie(t) \\ \leq & \varepsilon_{1i}e^T(t)P_i^2e(t) + \varepsilon_{1i}^{-1}e^T(t-\tau)A_{di}^T A_{di}e(t-\tau). \end{aligned} \tag{22}$$

Also, it results from the Assumption 1, the definitions (10) and (12)-(14) that

$$l_i^T(t)l_i(t) \leq a_{11i}^2|e(t)|^2 = a_{11i}^2e^T(t)e(t), \tag{23}$$

$$m_i^T(t-\tau)m_i(t-\tau) \leq a_{22i}^2e^T(t-\tau)e(t-\tau), \tag{24}$$

$$n_i^T(t)n_i(t) \leq a_{33i}^2|e(t)|^2 = a_{33i}^2e^T(t)e(t). \tag{25}$$

Considering (23)-(25), we can obtain from Lemma 1 that

$$\begin{aligned} & e^T(t)P_il_i(t) + l_i^T(t)P_ie(t) \\ & \leq e^T(t)(\varepsilon_{2i}P_i^2 + \varepsilon_{2i}^{-1}a_{11i}^2I)e(t), \end{aligned} \tag{26}$$

$$\begin{aligned} & e^T(t)P_im_i(t-\tau) + m_i^T(t-\tau)P_ie(t) \\ \leq & \varepsilon_{3i}e^T(t)P_i^2e(t) + \varepsilon_{3i}^{-1}a_{22i}^2e^T(t-\tau)e(t-\tau), \end{aligned} \tag{27}$$

$$\begin{aligned} & -e^T(t)P_iK_in_i(t) - n_i^T(t)K_i^T P_ie(t) \\ \leq & \varepsilon_{4i}e^T(t)(P_iK_iK_i^T P_i)e(t) + \varepsilon_{4i}^{-1}a_{33i}^2e^T(t)e(t). \end{aligned} \tag{28}$$

For simplicity, we denote

$$\begin{aligned} \Pi_i :=& (A_i - K_iC_i)^T P_i + P_i(A_i - K_iC_i) + \sum_{j=1}^{N}\gamma_{ij}P_j \\ & +P_i[(\varepsilon_{1i} + \varepsilon_{2i} + \varepsilon_{3i})I + \varepsilon_{4i}K_iK_i^T]P_i \\ & +(\varepsilon_{2i}^{-1}a_{11i}^2 + \varepsilon_{4i}^{-1}a_{33i}^2)I + \varepsilon_{1i}^{-1}A_{di}^T A_{di} + \varepsilon_{3i}^{-1}a_{22i}^2I, \end{aligned} \tag{29}$$

then (18) and (19) result in that $\Pi_i < 0$.

Substituting (19), (22) and (26)-(28) into (21) yields

$$\mathcal{A}V(e(t), i) \leq e^T(t)\Pi_ie(t) \leq -\lambda_{\min}(-\Pi_i)e^T(t)e(t). \tag{30}$$

It follows from Assumption 2 that

$$\begin{aligned} \frac{\mathcal{A}V(e(t),i)}{V(e(t),i)} \leq & \frac{-\lambda_{\min}(-\Pi_i)e^T(t)e(t)}{e^T(t)P_ie(t) + \int_{t-\tau}^{t} e^T(s)Q_ie(s)ds} \\ \leq & \frac{-\lambda_{\min}(-\Pi_i)|e(t)|^2}{\lambda_{\max}(P_i)|e(t)|^2 + \tau\eta^2\lambda_{\max}(Q_i)|e(t)|^2} \\ \leq & -\min_{i\in\mathcal{S}}\left\{\frac{\lambda_{\min}(-\Pi_i)}{\lambda_{\max}(P_i) + \tau\eta^2\lambda_{\max}(Q_i)}\right\} \\ := & -\kappa \end{aligned}$$

and therefore $\kappa > 0$ and $\mathcal{A}V(e(t), i) \leq -\kappa V(e(t), i)$. Then, similar to the proof of Theorem 1 in [3], by employing the Dynkin's formula and the Gronwall-Bellman lemma, we can prove that $\mathbb{E}\{V(e(t), i)\} \leq e^{-\kappa t}V(e(0), i)$. It then follows that the nonlinear jump stochastic time-delay system (15) is asymptotically stable in the mean square. To show the expected exponential stability (in the mean square) of the system (15), we need to perform some standard manipulations on the relation (30) by utilizing the technique developed in [8], [9]. The details are along the similar line of the proof of Theorem 2.1 in [9], and are thus omitted here. We just mention that, for the exponential stability of (15), the required constant $\beta > 0$ in (17) is the unique root of the equation

$$\lambda_{\min}(-\Pi_i) - \beta\lambda_{\max}(P_i) - \beta\tau\lambda_{\max}(Q_i)e^{\beta\tau} = 0, \tag{31}$$

and the required constant $\alpha > 0$ can be determined by

$$\alpha := \lambda_{\min}^{-1}(P_i)\left[\lambda_{\max}(P_i) + \tau\lambda_{\max}(Q_i)(1 + \tau e^{\beta\tau})\right].$$

This completes the proof of Theorem 1. ■

The following corollary reveals that, for the nonlinear time-delay jump system (15), the exponential stability in the mean square also implies the almost surely exponential stability. The proof can be found in [8].

Corollary 1: Under the same conditions as in Theorem 1, the nonlinear time-delay system (15) is almost surely exponentially stable in the mean square. That is,

$$\lim_{t\to\infty}\sup\frac{1}{t}\log|e(t;\xi)|\leq-\frac{\beta}{2}$$

almost surely holds for all $\xi\in L^2_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ where $\beta>0$ is the the unique root of the equation (31).

Having obtained the analysis results in Theorem 1, we are now ready to tackle the corresponding synthesis problem. That is, we need to derive the *explicit* expression of expected filter gains and propose a practical design procedure. It should be pointed out that, in most literature concerning nonlinear filtering, the solution to the nonlinear filtering problem has not been given as an explicit representation.

For presentation convenience, we further define

$$\begin{aligned}\Gamma_i &:= A_i^TP_i+P_iA_i+\sum_{j=1}^N\gamma_{ij}P_j+(\varepsilon_{1i}+\varepsilon_{2i}+\varepsilon_{3i})P_i^2\\&\quad+(\varepsilon_{2i}^{-1}a_{11i}^2+\varepsilon_{4i}^{-1}a_{33i}^2)I+Q_i, \quad (32)\\ \Xi_i &:= A_i^TP_i+P_iA_i+\sum_{j=1}^N\gamma_{ij}P_j+(\varepsilon_{1i}+\varepsilon_{2i}+\varepsilon_{3i})P_i^2\\&\quad+\varepsilon_{1i}^{-1}A_{di}^TA_{di}+(\varepsilon_{2i}^{-1}a_{11i}^2+\varepsilon_{3i}^{-1}a_{22i}^2+\varepsilon_{4i}^{-1}a_{33i}^2)I\\&\quad-\varepsilon_{4i}^{-1}C_i^TC_i, \quad (33)\\ \Theta_i &:= [P_i\ \ \mu_{1i}A_{di}^T\ \ P_i\ \ \mu_{2i}a_{11i}I\ \ P_i\ \ \mu_{3i}a_{22i}I] \quad (34)\end{aligned}$$

where Q_i is defined in (19).

In principle, our task now consists of two parts. One is to find the necessary and sufficient conditions for the existence of filter gains K_i such that there exist positive definite matrices P_i satisfying (18), and the other one is to express all expected filter gains in terms of the positive definite solutions P_i and, if any, some other free parameters. The following theorem accomplishes the above specified task.

Theorem 2: There exist a sequence of positive scalars $\{\varepsilon_{1i},\ \varepsilon_{2i},\ \varepsilon_{3i},\ \varepsilon_{4i},\ i\in\mathcal{S}\}$ and positive definite matrices P_i such that the matrix inequalities (18) (for $i\in\mathcal{S}$) have solutions K_i if and only if one of the following two assertions holds:

(1) There exist a sequence of positive scalars $\{\varepsilon_{1i},\cdots,\varepsilon_{4i},\ i\in\mathcal{S}\}$ and positive definite matrices P_i such that $\Xi_i<0$ where Ξ is defined in (33).

(2) There exist a sequence of positive scalars $\{\mu_{1i},\cdots,\mu_{4i},\ i\in\mathcal{S}\}$ and positive definite matrices P_i such that the following set of linear matrix inequalities

$$\begin{bmatrix}\Upsilon_i & \Theta_i\\ \Theta_i^T & -\mathrm{diag}\{\mu_{1i}I,\mu_{1i}I,\mu_{2i}I,\mu_{2i}I,\mu_{3i}I,\mu_{3i}I\}\end{bmatrix}<0 \quad (35)$$

hold, where

$$\Upsilon_i:=A_i^TP_i+P_iA_i+\sum_{j=1}^N\gamma_{ij}P_j+\mu_{4i}(a_{33i}^2I-C_i^TC_i). \quad (36)$$

Furthermore, if (35) is true for positive scalars $\mu_{1i},\ \mu_{2i},\ \mu_{3i},\ \mu_{4i}$ and positive definite matrices P_i, all matrices K_i meeting the matrix inequalities (18) can then be parameterized by

$$K_i=\mu_{4i}P_i^{-1}C_i^T+\mu_{4i}^{1/2}P_i^{-1}\Lambda_iU_i \quad (37)$$

where $\Lambda_i\in\mathbb{R}^{n\times p}$ is any matrix satisfying

$$\Lambda_i\Lambda_i^T<-\Xi_i \quad (38)$$

for $\varepsilon_{ki}=\mu_{ki}^{-1}$ $(k=1,2,3,4)$ and $U_i\in\mathbb{R}^{p\times p}$ is an arbitrary orthogonal matrix (i.e., $U_iU_i^T=I$).

Proof: It is straightforward to rearrange the matrix inequality (18) as

$$-C_i^TK_i^TP_i-P_iK_iC_i+\varepsilon_{4i}P_iK_iK_i^TP_i+\Gamma_i<0, \quad (39)$$

where Γ_i is defined in (32), or

$$\begin{aligned}&[\varepsilon_{4i}^{1/2}P_iK_i-\varepsilon_{4i}^{-1/2}C_i^T][\varepsilon_{4i}^{1/2}P_iK_i-\varepsilon_{4i}^{-1/2}C_i^T]^T\\&\quad<\varepsilon_{4i}^{-1}C_i^TC_i-\Gamma_i. \quad (40)\end{aligned}$$

It is apparent that there exist filter gain matrices K_i such that the inequalities (18) (or equivalently, (40) for $i\in\mathcal{S}$) hold for some positive scalars $\varepsilon_{1i},\ \varepsilon_{2i},\ \varepsilon_{3i},\ \varepsilon_{4i}$ and positive definite matrix P_i if and only if the right-hand side of (40) is positive definite. That is,

$$\begin{aligned}&A_i^TP_i+P_iA_i+\sum_{j=1}^N\gamma_{ij}P_j+(\varepsilon_{1i}+\varepsilon_{2i}+\varepsilon_{3i})P_i^2\\&\quad+\varepsilon_{1i}^{-1}A_{di}^TA_{di}+(\varepsilon_{2i}^{-1}a_{11i}^2+\varepsilon_{3i}^{-1}a_{22i}^2\\&\quad+\varepsilon_{4i}^{-1}a_{33i}^2)I-\varepsilon_{4i}^{-1}C_i^TC_i<0 \quad (41)\end{aligned}$$

or $\Xi_i<0$ holds.

Notice that (41) is neither linear on P_i nor linear on $\varepsilon_{1i},\ \varepsilon_{2i},\ \varepsilon_{3i},\ \varepsilon_{4i}$. Our next goal is to convert (41) into an LMI so that the powerful Matlab LMI Toolbox can be applied. To do this, we continue to rewrite (41) as

$$\Upsilon_i+\Omega_i\Omega_i^T<0 \quad (42)$$

where Υ_i is defined in (36) (let $\mu_{4i}:=\varepsilon_{4i}^{-1}$) and

$$\Omega_i:=[\Omega_{1i}\ \ \Omega_{2i}], \quad (43)$$

where

$$\begin{aligned}\Omega_{1i}&=[\varepsilon_{1i}^{1/2}P_i\ \ \varepsilon_{1i}^{-1/2}A_{di}^T\ \ \varepsilon_{2i}^{1/2}P_i],\\ \Omega_{2i}&=[\varepsilon_{2i}^{-1/2}a_{11i}I\ \ \varepsilon_{3i}^{1/2}P_i\ \ \varepsilon_{3i}^{-1/2}a_{22i}I].\end{aligned}$$

It follows from Schur Complement Lemma that (42) holds if and only if the following inequality holds:

$$\begin{bmatrix}\Upsilon_i & \Omega_i\\ \Omega_i^T & -I\end{bmatrix}<0. \quad (44)$$

Let

$$\mu_{ki}:=\varepsilon_{ki}^{-1},\quad k=1,2,3,4. \quad (45)$$

Pre- and post-multiplying the inequality (44) by

$$\mathrm{diag}\{I, \varepsilon_{1i}^{-1/2}I, \varepsilon_{1i}^{-1/2}I, \varepsilon_{2i}^{-1/2}I, \varepsilon_{2i}^{-1/2}I, \varepsilon_{3i}^{-1/2}I, \varepsilon_{3i}^{-1/2}I\}$$

yield (35). This proves the first part of this theorem.

Suppose now that (35) is true. Note that the dimension of the filter gain K_i is $n \times p$ and $p \leq n$. From (40) and the definition of $\Lambda_i \in \mathbb{R}^{n \times p}$ in (38), we have

$$[\varepsilon_{4i}^{1/2}P_iK_i - \varepsilon_{4i}^{-1/2}C_i^T][\varepsilon_{4i}^{1/2}P_iK_i - \varepsilon_{4i}^{-1/2}C_i^T]^T = \Lambda_i\Lambda_i^T. \quad (46)$$

It then follows from [15] that (46) holds if and only if

$$\varepsilon_{4i}^{1/2}P_iK_i - \varepsilon_{4i}^{-1/2}C_i^T = \Lambda_iU_i, \quad (47)$$

where $U_i \in \mathbb{R}^{p \times p}$ is an arbitrary orthogonal matrix. Therefore, the expression (37) follows from (47) immediately, and the proof of this theorem is complete. ∎

As a summary, we give our main results as follows that are easily derived from Theorem 1 and Theorem 2.

Theorem 3: Consider the nonlinear jump state delayed system (1)-(3) with the nonlinear filter (7). If there exist a sequence of positive scalars $\{\mu_{1i}, \mu_{2i}, \mu_{3i}, \mu_{4i}, i \in \mathcal{S}\}$ and positive definite matrices P_i $(i \in \mathcal{S})$ such that the LMIs (35) hold, then the filter (7) with its parameter given in (37) will be such that the dynamics of the estimation error (i.e., the solution of the error-state system (15)) is stochastically exponentially stable in the mean square.

Remark 2: The solution to the addressed filter design problem for nonlinear jump time-delay systems is given in Theorem 3. Note that the design procedure of the filter parameters depends solely on the feasibility of the LMIs (35) that are linear on the scalar variables $\mu_{1i} > 0$, $\mu_{2i} > 0$, $\mu_{3i} > 0$, $\mu_{4i} > 0$ and the matrix variable $P_i > 0$. Fortunately, with the recently developed Matlab LMI Toolbox [5], we can check the solvability of the LMIs (35) readily and reliably. This makes our proposed design approach very practical.

Remark 3: We can see that, if the set of desired filter gains is not empty, it must be very large. We may utilize the freedom (such as the choices of matrices Λ_i and U_i) in the filter design to improve other system properties. One of the future research topics is to exploit such remaining freedom to achieve the specified reliable constraint on the filtering process. Also, we point out that it is not difficult to obtain parallel results for the multi-delay case, and for the case where there are bounded nonlinearities and uncertain disturbances. The reason why we discuss the relatively simple system (1)-(3) associated with (4)-(6) is to make our theory more understandable and to avoid unnecessarily complicated notations.

The simulation results, which verify that our expected performance is well achieved, are omitted here due to space limitation.

IV. Conclusions

In this paper we have investigated the filter design problem for a class of nonlinear time-delay systems with Markov jumping parameters. Both the filter analysis and design issues have been discussed in detail by means of linear matrix inequalities. We have derived the existence conditions as well as the analytical parameterization of desired filters. The method relies not on the optimization theory but on Lyapunov type stochastic stability results that can guarantee a mean square exponential rate of convergence for the estimation error. It has been emphasized that, the desired exponential filters for this class of nonlinear time-delay systems, when they exist, are usually a large set, and the remaining freedom can be used to meet other expected performance requirements.

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