Positive Real Control for Uncertain Two-Dimensional Systems

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***Abstract*—This brief deals with the problem of positive real control for uncertain two-dimensional (2-D) discrete systems described by the Fornasini-Marchesini local state-space model. The parameter uncertainty is time-invariant and norm-bounded. The problem we address is the design of a state feedback controller that robustly stabilizes the uncertain system and achieves the extended strictly positive realness of the resulting closed-loop system for all admissible uncertainties. A version of positive realness for 2-D discrete systems is established. Based on this, a condition for the solvability of the positive real control problem is derived in terms of a linear matrix inequality. Furthermore, the solution of a desired state feedback controller is also given. Finally, we provide a numerical example to demonstrate the applicability of the proposed approach.**

***Index Terms*—Fornasini-Marchesini local state-space (FMLSS) model, linear matrix inequality (LMI), positive realness, state feedback, two-dimensional (2-D) systems.**

I. Introduction

In the past decades, there has been a growing interest in the system theoretic problems of two-dimensional (2-D) discrete systems due to the rapid increase of the applicability of 2-D discrete systems theory in many areas such as image processing, seismographic data processing, thermal processes, water stream heating, and so on [11]. A great number of fundamental notions and results based on one-dimensional (1-D) discrete systems have been extended to 2-D discrete systems [5], [9], [11], [16].

On the other hand, the notion of positive realness has played an important role in control and system theory [2], [6], [18]. Applications of positive realness have been found in many areas such as the analysis of the properties of immittance or hybrid matrices of various classes of networks, the inverse problem of linear optimal control, the stability analysis for linear systems, and so on [2], [6], [8], [19]. In [1], it is also reported that positive realness has played an important role in the stability analysis for 2-D discrete systems. Recently, increasing attention has been devoted to the positive real control problem. The study of this problem is motivated by robust and nonlinear control in which a well-known fact is that the positive realness of a certain loop transfer function will guarantee the overall stability of a feedback system if uncertainty or nonlinearity can be characterized by a positive real system [18]. Furthermore, it has been shown that if the system uncertainty can be cast as a positive real transfer function and the system is strictly positive real, then the positivity theorem implies robust stability [7]. The objective of positive real control is to design controllers such that the resulting closed-loop system is stable and the closed-loop transfer function is positive real. Now, it is known that a solution to this problem for a known linear time-invariant system involves solving a pair of Riccati inequalities [17]. When parameter uncertainty appears, the results in [17] were extended in [14], where observer-based controllers were designed and an linear matrix inequality (LMI) design method was developed. The corresponding results for discrete time systems can be found in [8] and [15]. It should be pointed out that all these results were derived in the context of 1-D systems. Up to date, however, no results on positive real control problem for 2-D discrete systems is available in the literature, this problem is still open and remains challenging.

In this brief, we are concerned with the problem of positive real control for uncertain 2-D discrete systems described by the Fornasini-Marchesini local state-space (FMLSS) model. The parameter uncertainty is assumed to be time-invariant and unknown but norm-bounded. The problem to be addressed is the design of a state feedback controller such that the resulting closed-loop system is asymptotically stable and the closed-loop transfer function from the disturbance to the controlled output is *extended strictly positive real* (ESPR) for all admissible uncertainties. To solve this problem, we first establish a version of positive realness for 2-D discrete systems in terms of an LMI. It is shown that this result is an extension of the existing results of positive realness for 1-D discrete systems. Then, we obtain the results on positive realness for uncertain 2-D systems via the notion of "strong robust stability with ESPR". Based on this, a condition for the solvability of the positive real control problem is derived and the explicit formula of a desired state feedback controller is given. Finally, an example is presented to demonstrate the validity and applicability of the proposed approach.

Notation. Throughout this brief, for Hermitian matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with appropriate dimension. The superscripts "T," "$-T$,"and "$*$" represent the transpose, inverse transpose, and the complex conjugate transpose. $\mathbb{Z}^+$ denotes the set of nonnegative integers. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

II. Positive Realness Analysis

Consider an uncertain 2-D discrete-time system (Σ_Δ) described by the following FMLSS model [5], [12]:

$$\begin{aligned}(\Sigma_\Delta)\colon x(i+1,j+1) =& A_{1\Delta}x(i+1,j) + A_{2\Delta}x(i,j+1)\\ &+ B_{1\Delta}w(i+1,j)\\ &+ B_{2\Delta}w(i,j+1)\\ z(i,j) =& Cx(i,j) + Dw(i,j)\end{aligned}$$

where $x(i,j) \in \mathbb{R}^n$ is the local state vector, $w(i,j) \in \mathbb{R}^q$ is the exogenous input, $z(i,j) \in \mathbb{R}^q$ is the controlled output, $i, j \in \mathbb{Z}^+$

$$\begin{aligned}A_{1\Delta} =& A_1 + \Delta A_1 \quad A_{2\Delta} = A_2 + \Delta A_2\\ B_{1\Delta} =& B_1 + \Delta B_1 \quad B_{2\Delta} = B_2 + \Delta B_2\end{aligned}$$

where A_1, A_2, B_1, B_2, C and D are known real constant matrices with appropriate dimensions. $\Delta A_1, \Delta A_2, \Delta B_1$ and ΔB_2 are time-invariant matrices representing norm-bounded parameter uncertainties, and are assumed to be of the form [20], [21]

$$[\,\Delta A_1 \quad \Delta A_2 \quad \Delta B_1 \quad \Delta B_2\,] = MF\,[\,N_{A1} \quad N_{A2} \quad N_{B1} \quad N_{B2}\,] \tag{1}$$

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where $F \in \mathbb{R}^{g\times l}$ is an unknown real matrix satisfying

$$F^T F \leq I \tag{2}$$

and M, N_{A1}, N_{A2}, N_{B1} and N_{B2} are known real constant matrices with appropriate dimensions.

The nominal 2-D discrete-time system of (Σ_Δ) can be written as

$$\begin{aligned}(\Sigma)\colon x(i+1,j+1) =& A_1 x(i+1,j) + A_2 x(i,j+1)\\ &+ B_1 w(i+1,j) + B_2 w(i,j+1)\\ z(i,j) =& Cx(i,j) + Dw(i,j).\end{aligned}$$

Then, the square transfer function of the 2-D discrete-time system (Σ) can be written as

$$G(z_1,z_2) = C\left(z_1 z_2 I - z_1 A_1 - z_2 A_2\right)^{-1}\left(z_1 B_1 + z_2 B_2\right) + D. \tag{3}$$

We first introduce the notion of asymptotic stability of 2-D discrete-time systems.

Definition 1 [11]: The 2-D linear discrete-time system (Σ) is said to be asymptotically stable if

$$\lim_{k\to\infty} \|\chi(k)\|_E = 0$$

under the zero input $w(i,j) \equiv 0$ and $\|\chi(0)\|_E < \infty$, where

$$\|\chi(k)\|_E = \sup_{x\in\chi(k)} \|x\|, \quad \chi(k) = \{x(i,j)\colon i+j=k\}$$

and $\|x(\cdot,\cdot)\|$ is the Euclidean norm of the local state.

The following lemma gives a sufficient condition for the asymptotic stability of 2-D linear discrete-time system (Σ) in terms of an LMI.

Lemma 1 [10]: The 2-D linear discrete-time system (Σ) is asymptotically stable if there exist matrices $P>0$ and $Q>0$ such that the following LMI holds:

$$\begin{bmatrix} A_1^T P A_1 + Q - P & A_1^T P A_2 \\ A_2^T P A_1 & A_2^T P A_2 - Q\end{bmatrix} < 0. \tag{4}$$

Motivated by the notion of positive realness for 1-D discrete systems [2], we define the concept of positive realness for 2-D systems in the following.

Definition 2:

1) The 2-D discrete-time system (Σ) is said to be positive real (PR) if its transfer function $G(z_1,z_2)$ is analytic in $|z_1|>1$, $|z_2|>1$ and satisfies $G(z_1,z_2)+G^*(z_1,z_2)\geq 0$ for $|z_1|>1$, $|z_2|>1$.
2) The 2-D discrete-time system (Σ) is said to be strictly positive real (SPR) if its transfer function $G(z_1,z_2)$ is analytic in $|z_1|\geq 1$, $|z_2|\geq 1$ and satisfies $G(e^{j\theta_1},e^{j\theta_2}) + G^*(e^{j\theta_1},e^{j\theta_2}) > 0$ for $\theta_1, \theta_2 \in [0, 2\pi)$.
3) The 2-D discrete-time system (Σ) is said to be ESPR if it is SPR and $G(\infty,\infty) + G(\infty,\infty)^T > 0$.

We present a result on positive realness for 2-D discrete-time system (Σ) in the following theorem.

Theorem 1: The 2-D discrete-time system (Σ) is asymptotically stable with ESPR if there exist matrices $P>0$, $Q>0$ and $W>0$ such that the LMI shown in (5), at the bottom of the page, holds.

Proof: From (5) it is easy to see that

$$\begin{bmatrix} A_1^T P A_1 + Q - P & A_1^T P A_2 \\ A_2^T P A_1 & A_2^T P A_2 - Q\end{bmatrix} < 0. \tag{6}$$

By Lemma 1, it follows from (6) that the 2-D discrete-time system (Σ) is asymptotically stable. Therefore, $G(z_1,z_2)$ is analytic in $|z_1|\geq 1$, $|z_2|\geq 1$. Next, we shall show

$$U(e^{j\theta_1}, e^{j\theta 2}) := G(e^{j\theta_1},e^{j\theta_2}) + G^*(e^{j\theta_1},e^{j\theta_2}) > 0$$

for all $\theta_1, \theta_2 \in [0,2\pi)$. To this end, we note that (5) implies that there exist a sufficiently small scalar $\epsilon > 0$ and a matrix $Q_1 > 0$ such that (7), shown at the bottom of the page, holds. Denote

$$\begin{aligned}\tilde{A}_1 &= \begin{bmatrix} A_1 & A_2 & -B_1\end{bmatrix}\\ H &= \begin{bmatrix} Q-P+Q_1+\epsilon I & 0 & C^T \\ 0 & -Q+\epsilon I & 0 \\ C & 0 & -(D+D^T-W)\end{bmatrix}.\end{aligned} \tag{8}$$

By Schur complements [4], it follows from (7) that

$$W - B_2^T P B_2 > 0 \tag{9}$$

and

$$\tilde{A}_1^T P \tilde{A}_1 + \tilde{A}_1^T P B_2 \left(W - B_2^T P B_2\right)^{-1} B_2^T P \tilde{A}_1 + H < 0. \tag{10}$$

Let

$$\tilde{B}_2^T = \begin{bmatrix} 0 \\ 0 \\ -B_2^T\end{bmatrix}, \quad \tilde{W} = \begin{bmatrix} \epsilon I & 0 & 0 \\ 0 & \epsilon I & 0 \\ 0 & 0 & W\end{bmatrix},$$

$$\tilde{H} = \begin{bmatrix} Q-P+Q_1 & 0 & C^T \\ 0 & -Q & 0 \\ C & 0 & -(D+D^T)\end{bmatrix}.$$

Then, (10) can be rewritten as

$$\tilde{A}_1^T P \tilde{A}_1 + \tilde{A}_1^T P \tilde{B}_2 \left(\tilde{W} - \tilde{B}_2^T P \tilde{B}_2\right)^{-1} \tilde{B}_2^T P \tilde{A}_1 + \tilde{W} + \tilde{H} < 0. \tag{11}$$

$$\begin{bmatrix} A_1^T P A_1 + Q - P & A_1^T P A_2 & C^T - A_1^T P B_1 & -A_1^T P B_2 \\ A_2^T P A_1 & A_2^T P A_2 - Q & -A_2^T P B_1 & -A_2^T P B_2 \\ C - B_1^T P A_1 & -B_1^T P A_2 & -\left(D + D^T - B_1^T P B_1 - W\right) & B_1^T P B_2 \\ -B_2^T P A_1 & -B_2^T P A_2 & B_2^T P B_1 & B_2^T P B_2 - W\end{bmatrix} < 0 \tag{5}$$

$$\begin{bmatrix} A_1^T P A_1 + Q - P + Q_1 + \epsilon I & A_1^T P A_2 & C^T - A_1^T P B_1 & -A_1^T P B_2 \\ A_2^T P A_1 & A_2^T P A_2 - Q + \epsilon I & -A_2^T P B_1 & -A_2^T P B_2 \\ C - B_1^T P A_1 & -B_1^T P A_2 & -\left(D + D^T - B_1^T P B_1 - W\right) & B_1^T P B_2 \\ -B_2^T P A_1 & -B_2^T P A_2 & B_2^T P B_1 & B_2^T P B_2 - W\end{bmatrix} < 0 \tag{7}$$

Noting

$$\tilde{W} - \tilde{B}_2^T P \tilde{B}_2 > 0$$

and recalling that for any matrices X, Y and Z of appropriate dimensions with $X > 0$,

$$Y^* Z + Z^* Y \le Z^* X Z + Y^* X^{-1} Y$$

we have that for all $\theta_1, \theta_2 \in [0, 2\pi)$,

$$\begin{aligned} e^{j(\theta_2-\theta_1)} \tilde{A}_1^T P \tilde{B}_2 + e^{-j(\theta_2-\theta_1)} \tilde{B}_2^T P \tilde{A}_1 \le \tilde{A}_1^T P \tilde{B}_2 \\ \times \left(\tilde{W} - \tilde{B}_2^T P \tilde{B}_2\right)^{-1} \tilde{B}_2^T P \tilde{A}_1 + \tilde{W} - \tilde{B}_2^T P \tilde{B}_2. \end{aligned}$$

This together with (11) implies that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$\begin{aligned} &\left(\tilde{A}_1^T + e^{-j(\theta_2-\theta_1)} \tilde{B}_2^T\right) P \left(\tilde{A}_1 + e^{j(\theta_2-\theta_1)} \tilde{B}_2\right) + \tilde{H} \\ &\le \tilde{A}_1^T P \tilde{A}_1 + \tilde{A}_1^T P \tilde{B}_2 \left(\tilde{W} - \tilde{B}_2^T P \tilde{B}_2\right)^{-1} \tilde{B}_2^T P \tilde{A}_1 \\ &\quad + \tilde{W} + \tilde{H} < 0. \end{aligned}$$

That is

$$\begin{aligned} &\begin{bmatrix} A_1^T \\ A_2^T \\ -B_1^T - e^{-j(\theta_2-\theta_1)} B_2^T \end{bmatrix} P \\ &\quad \times \left[A_1 \quad A_2 \quad -B_1 - e^{j(\theta_2-\theta_1)} B_2 \right] \\ &\quad + \begin{bmatrix} Q - P + Q_1 & 0 & C^T \\ 0 & -Q & 0 \\ C & 0 & -(D + D^T) \end{bmatrix} < 0. \end{aligned} \tag{12}$$

Pre-multiplying and post-multiplying (12) by

$$\begin{bmatrix} I & e^{-j(\theta_2-\theta_1)} I & 0 \\ 0 & 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} I & 0 \\ e^{j(\theta_2-\theta_1)} I & 0 \\ 0 & I \end{bmatrix}$$

respectively, we obtain that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$\begin{aligned} &\begin{bmatrix} A(-j\theta_1, -j\theta_2)^T \\ -B(-j\theta_1, -j\theta_2)^T \end{bmatrix} P \left[A(j\theta_1, j\theta_2) \quad -B(j\theta_1, j\theta_2) \right] \\ &\qquad + \begin{bmatrix} -P + Q_1 & C^T \\ C & -(D + D^T) \end{bmatrix} < 0 \end{aligned} \tag{13}$$

where

$$\begin{aligned} A(j\theta_1, j\theta_2) &= A_1 + e^{j(\theta_2-\theta_1)} A_2 \\ B(j\theta_1, j\theta_2) &= B_1 + e^{j(\theta_2-\theta_1)} B_2. \end{aligned}$$

Using the Schur complement, it follows from (13) that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$\begin{aligned} &Q_1 - P + A(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2) \\ &\quad + \left[C^T - A(-j\theta_1, -j\theta_2)^T P B(j\theta_1, j\theta_2)\right] S(j\theta_1, j\theta_2)^{-1} \\ &\quad \times \left[C - B(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2)\right] < 0 \end{aligned} \tag{14}$$

and

$$S(j\theta_1, j\theta_2) = D + D^T - B(-j\theta_1, -j\theta_2)^T P B(j\theta_1, j\theta_2) > 0.$$

Setting

$$\Psi(j\theta_1, j\theta_2) = e^{j\theta_2} I - A(j\theta_1, j\theta_2)$$

it is easy to show that the asymptotic stability of the system (Σ) implies that $\Psi(j\theta_1, j\theta_2)$ is invertible for all $\theta_1, \theta_2 \in [0, 2\pi)$. Now, pre-multiplying and post-multiplying (14) by $B(-j\theta_1, -j\theta_2)^T \Psi(-j\theta_1, -j\theta_2)^{-T}$ and $\Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2)$ respectively, we have that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$\begin{aligned} &B(-j\theta_1, -j\theta_2)^T \Psi(-j\theta_1, -j\theta_2)^{-T} \\ &\quad \times \left[A(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2) - P\right] \Psi(j\theta_1, j\theta_2)^{-1} \\ &\quad \times B(j\theta_1, j\theta_2) + B(-j\theta_1, -j\theta_2)^T \Psi(-j\theta_1, -j\theta_2)^{-T} \\ &\quad \times \Phi(j\theta_1, j\theta_2) \Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2) \le 0 \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Phi(j\theta_1, j\theta_2) = Q_1 + &\left[C^T - A(-j\theta_1, -j\theta_2)^T P B(j\theta_1, j\theta_2)\right] \\ &\times S(j\theta_1, j\theta_2)^{-1} \left[C - B(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2)\right]. \end{aligned}$$

On the other hand, by some algebraic manipulations we can verify that the following equality holds for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$\begin{aligned} &A(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2) + \Psi(-j\theta_1, -j\theta_2)^T \\ &\quad \times P \Psi(j\theta_1, j\theta_2) + \Psi(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2) \\ &\quad + A(-j\theta_1, -j\theta_2)^T P \Psi(j\theta_1, j\theta_2) - P = 0. \end{aligned} \tag{16}$$

Pre-multiplying and post-multiplying (16) by $B(-j\theta_1, -j\theta_2)^T \Psi(-j\theta_1, -j\theta_2)^{-T}$ and $\Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2)$ respectively, and re-arranging we obtain

$$\begin{aligned} &- B(-j\theta_1, -j\theta_2)^T \Psi(-j\theta_1, -j\theta_2)^{-T} \\ &\quad \times \left[A(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2) - P\right] \\ &\quad \times \Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2) \\ &\quad = B(-j\theta_1, -j\theta_2)^T P B(j\theta_1, j\theta_2) + B(-j\theta_1, -j\theta_2)^T \\ &\qquad \times P A(j\theta_1, j\theta_2) \Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2) \\ &\qquad + B(-j\theta_1, -j\theta_2)^T \Psi(-j\theta_1, -j\theta_2)^{-T} \\ &\qquad \times A(-j\theta_1, -j\theta_2)^T P B(j\theta_1, j\theta_2) \end{aligned}$$

for all $\theta_1, \theta_2 \in [0, 2\pi)$. Considering this and (15), we have that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$\begin{aligned} &B(-j\theta_1, -j\theta_2)^T \Psi(-j\theta_1, -j\theta_2)^{-T} \Phi(j\theta_1, j\theta_2) \\ &\quad \times \Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2) - B(-j\theta_1, -j\theta_2)^T \\ &\quad \times P B(j\theta_1, j\theta_2) - B(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2) \\ &\quad \times \Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2) - B(-j\theta_1, -j\theta_2)^T \\ &\quad \times \Psi(-j\theta_1, -j\theta_2)^{-T} A(-j\theta_1, -j\theta_2)^T \\ &\quad \times P B(j\theta_1, j\theta_2) \le 0. \end{aligned} \tag{17}$$

Therefore, it follows from (17) that for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$\begin{aligned} U(e^{j\theta_1}, e^{j\theta_2}) &= D + D^T + C \Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2) \\ &\quad + B(-j\theta_1, -j\theta_2)^T \Psi(j\theta_1, j\theta_2)^{-T} C^T \\ &= S(j\theta_1, j\theta_2) + C \Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2) \\ &\quad + B(-j\theta_1, -j\theta_2)^T \Psi(j\theta_1, j\theta_2)^{-T} C^T \\ &\quad + B(-j\theta_1, -j\theta_2)^T P B(j\theta_1, j\theta_2) \\ &\ge S(j\theta_1, j\theta_2) \\ &\quad + \left[C - B(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2)\right] \\ &\quad \times \Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2) \\ &\quad + B(-j\theta_1, -j\theta_2)^T \Psi(j\theta_1, j\theta_2)^{-T} \\ &\quad \times \left[C^T - A(-j\theta_1, -j\theta_2)^T P B(j\theta_1, j\theta_2)\right] \\ &\quad + B(-j\theta_1, -j\theta_2)^T \Psi(-j\theta_1, -j\theta_2)^{-T} \\ &\quad \times \Phi(j\theta_1, j\theta_2) \Psi(j\theta_1, j\theta_2)^{-1} B(j\theta_1, j\theta_2) \\ &\ge - \left[C - B(-j\theta_1, -j\theta_2)^T P A(j\theta_1, j\theta_2)\right] \\ &\quad \times \Phi(j\theta_1, j\theta_2)^{-1} \\ &\quad \times \left[C^T - A(-j\theta_1, -j\theta_2)^T P B(j\theta_1, j\theta_2)\right] \\ &\quad + S(j\theta_1, j\theta_2). \end{aligned} \tag{18}$$

Note that

$$\Phi(j\theta_1, j\theta_2) - \left[C^T - A(-j\theta_1, -j\theta_2)^T PB(j\theta_1, j\theta_2)\right] \times S(j\theta_1, j\theta_2)^{-1}\left[C - B(-j\theta_1, -j\theta_2)^T PA(j\theta_1, j\theta_2)\right] = Q_1 > 0.$$

Therefore, for all $\theta_1, \theta_2 \in [0, 2\pi)$

$$S(j\theta_1, j\theta_2) - \left[C - B(-j\theta_1, -j\theta_2)^T PA(j\theta_1, j\theta_2)\right] \times \Phi(j\theta_1, j\theta_2)^{-1}\left[C^T - A(-j\theta_1, -j\theta_2)^T PB(j\theta_1, j\theta_2)\right] > 0.$$

From this and (18), we have that $U(e^{j\theta_1}, e^{j\theta_2}) > 0$ for all $\theta_1, \theta_2 \in [0, 2\pi)$. Thus, the 2-D discrete-time system (Σ) is ESPR. This completes the proof. □

Remark 1: Theorem 1 provides an LMI condition for the 2-D discrete-time system (Σ) to be asymptotically stable and ESPR. In the case when system (Σ) reduces to a 1-D discrete system, it is easy to show that Theorem 1 coincides with Lemma 4.2 in [8]. Therefore, Theorem 1 can be viewed as an extension of existing results on positive realness to 2-D discrete-time systems.

The positive realness result for 1-D systems has played an important role in robust positive realness analysis and synthesis for uncertain 1-D systems in both discrete and continuous contexts [15], [22]. Taking into account this and Theorem 1, we introduce the concept of strong robust stability with ESPR for the uncertain 2-D discrete-time system (Σ_Δ), which will be shown to be useful in establishing the property of robust stability with ESPR for system (Σ_Δ).

Definition 3: The uncertain 2-D discrete-time system (Σ_Δ) is said to be strongly robustly stable with ESPR if there exist matrices $P > 0$, $Q > 0$ and $W > 0$ such that the LMI shown in (19), at the bottom of the page, holds for all admissible uncertainties ΔA_1, ΔA_2, ΔB_1 and ΔB_2 satisfying (1).

Remark 2: It is worth pointing out that Definition 3 extends the notion of strong robust stability for uncertain 1-D discrete systems in [15] to the case of uncertain 2-D discrete systems.

The following theorem presents a necessary and sufficient condition for system (Σ_Δ) to be strongly robustly stable with ESPR.

Theorem 2: Consider the uncertain 2-D discrete-time system (Σ_Δ). This system is strongly robustly stable with ESPR for all admissible uncertainties if and only if there exist a scalar $\epsilon > 0$ and matrices $X > 0, Y > 0$ and $W > 0$ such that the LMI shown in (20), at the bottom of the page, holds.

Before proceeding to prove Theorem 2, we introduce the following lemmas.

Lemma 2 [13]: Let A, L, E, F and P be real matrices of appropriate dimensions with $P > 0$ and F satisfying $F^TF \le I$. Then, for any scalar $\epsilon > 0$ such that $P - \epsilon LL^T > 0$, we have

$$(A + LFE)^T P^{-1}(A + LFE) \le A^T(P - \epsilon LL^T)^{-1}A + \epsilon^{-1}E^TE.$$

Lemma 3 [21]: Let L, E, F and Q be real matrices of appropriate dimensions with Q satisfying $Q = Q^T$, then

$$Q + LFE + (LFE)^T < 0$$

for all F satisfying $F^TF \le I$, if and only if there exists a scalar $\epsilon > 0$ such that

$$Q + \epsilon LL^T + \epsilon^{-1}E^TE < 0.$$

Proof of Theorem 2: (*Sufficiency*): Suppose that there exist a scalar $\epsilon > 0$ and matrices $X > 0, Y > 0$ and $W > 0$ such that (20) is satisfied. Then, from (20) it is easy to see that

$$X - \epsilon MM^T > 0. \tag{21}$$

By Lemma 2, it can be shown that

$$\begin{aligned}&\left(\begin{bmatrix}A_1^T\\A_2^T\\-B_1^T\\-B_2^T\end{bmatrix} + \begin{bmatrix}N_{A1}^T\\N_{A2}^T\\-N_{B1}^T\\-N_{B2}^T\end{bmatrix}F^TM^T\right)X^{-1} \times \left(\begin{bmatrix}A_1^T\\A_2^T\\-B_1^T\\-B_2^T\end{bmatrix} + \begin{bmatrix}N_{A1}^T\\N_{A2}^T\\-N_{B1}^T\\-N_{B2}^T\end{bmatrix}F^TM^T\right)^T\\ &\le \begin{bmatrix}A_1^T\\A_2^T\\-B_1^T\\-B_2^T\end{bmatrix}(X - \epsilon MM^T)^{-1}\begin{bmatrix}A_1^T\\A_2^T\\-B_1^T\\-B_2^T\end{bmatrix}^T + \epsilon^{-1}\begin{bmatrix}N_{A1}^T\\N_{A2}^T\\-N_{B1}^T\\-N_{B2}^T\end{bmatrix}\begin{bmatrix}N_{A1}^T\\N_{A2}^T\\-N_{B1}^T\\-N_{B2}^T\end{bmatrix}^T.\end{aligned} \tag{22}$$

Let

$$\hat{Y} = X^{-1}YX^{-1}, \quad J = J^T = \mathrm{diag}(X, X, I, I).$$

$$\begin{bmatrix} A_{1\Delta}^TPA_{1\Delta} + Q - P & A_{1\Delta}^TPA_{2\Delta} & C^T - A_{1\Delta}^TPB_{1\Delta} & -A_{1\Delta}^TPB_{2\Delta} \\ A_{2\Delta}^TPA_{1\Delta} & A_{2\Delta}^TPA_{2\Delta} - Q & -A_{2\Delta}^TPB_{1\Delta} & -A_{2\Delta}^TPB_{2\Delta} \\ C - B_{1\Delta}^TPA_{1\Delta} & -B_{1\Delta}^TPA_{2\Delta} & -\left(D + D^T - B_{1\Delta}^TPB_{1\Delta} - W\right) & B_{1\Delta}^TPB_{2\Delta} \\ -B_{2\Delta}^TPA_{1\Delta} & -B_{2\Delta}^TPA_{2\Delta} & B_{2\Delta}^TPB_{1\Delta} & B_{2\Delta}^TPB_{2\Delta} - W \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} Y - X & 0 & XC^T & 0 & XA_1^T & XN_{A1}^T \\ 0 & -Y & 0 & 0 & XA_2^T & XN_{A2}^T \\ CX & 0 & -(D + D^T - W) & 0 & -B_1^T & -N_{B1}^T \\ 0 & 0 & 0 & -W & -B_2^T & -N_{B2}^T \\ A_1X & A_2X & -B_1 & -B_2 & \epsilon MM^T - X & 0 \\ N_{A1}X & N_{A2}X & -N_{B1} & -N_{B2} & 0 & -\epsilon I \end{bmatrix} < 0 \tag{20}$$

Then, by considering (20) and using Schur complements, we get the equation at the bottom of the page. This together with (22) implies that

$$\left(\begin{bmatrix} A_1^T \\ A_2^T \\ -B_1^T \\ -B_2^T \end{bmatrix} + \begin{bmatrix} N_{A1}^T \\ N_{A2}^T \\ -N_{B1}^T \\ -N_{B2}^T \end{bmatrix} F^T M^T \right) X^{-1} \times \left(\begin{bmatrix} A_1^T \\ A_2^T \\ -B_1^T \\ -B_2^T \end{bmatrix} + \begin{bmatrix} N_{A1}^T \\ N_{A2}^T \\ -N_{B1}^T \\ -N_{B2}^T \end{bmatrix} F^T M^T \right)^T + \begin{bmatrix} \hat{Y} - X^{-1} & 0 & C^T & 0 \\ 0 & -\hat{Y} & 0 & 0 \\ C & 0 & -(D + D^T - W) & 0 \\ 0 & 0 & 0 & -W \end{bmatrix} < 0.$$

This leads to the second equation at the bottom of the page. By Definition 3, it follows that the uncertain 2-D discrete-time system (Σ_Δ) is strongly robustly stable with ESPR for all admissible uncertainties.

(*Necessity*): Suppose the uncertain 2-D discrete-time system (Σ_Δ) is strongly robustly stable with ESPR, that is, there exist matrices $P > 0$, $Q > 0$ and $W > 0$ such that the LMI (19) holds. By Schur complements, it follows from (19) that

$$\begin{bmatrix} Q - P & 0 & C^T & 0 & A_{1\Delta}^T \\ 0 & -Q & 0 & 0 & A_{2\Delta}^T \\ C & 0 & -(D + D^T - W) & 0 & -B_{1\Delta}^T \\ 0 & 0 & 0 & -W & -B_{2\Delta}^T \\ A_{1\Delta} & A_{2\Delta} & -B_{1\Delta} & -B_{2\Delta} & -P^{-1} \end{bmatrix} < 0.$$

That is

$$\begin{bmatrix} Q - P & 0 & C^T & 0 & A_1^T \\ 0 & -Q & 0 & 0 & A_2^T \\ C & 0 & -(D + D^T - W) & 0 & -B_1^T \\ 0 & 0 & 0 & -W & -B_2^T \\ A_1 & A_2 & -B_1 & -B_2 & -P^{-1} \end{bmatrix} + \begin{bmatrix} N_{A1}^T \\ N_{A2}^T \\ -N_{B1}^T \\ -N_{B2}^T \\ 0 \end{bmatrix} F^T \begin{bmatrix} 0 & 0 & 0 & 0 & M^T \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ M \end{bmatrix} F \begin{bmatrix} N_{A1} & N_{A2} & -N_{B1} & -N_{B2} & 0 \end{bmatrix} < 0$$

for all F satisfying (2). Therefore, using Lemma 3, we have that there exists a scalar $\epsilon > 0$ such that

$$\begin{bmatrix} Q - P & 0 & C^T & 0 & A_1^T \\ 0 & -Q & 0 & 0 & A_2^T \\ C & 0 & -(D + D^T - W) & 0 & -B_1^T \\ 0 & 0 & 0 & -W & -B_2^T \\ A_1 & A_2 & -B_1 & -B_2 & -P^{-1} \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ M \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ M \end{bmatrix}^T + \epsilon^{-1} \begin{bmatrix} N_{A1}^T \\ N_{A2}^T \\ -N_{B1}^T \\ -N_{B2}^T \\ 0 \end{bmatrix} \begin{bmatrix} N_{A1}^T \\ N_{A2}^T \\ -N_{B1}^T \\ -N_{B2}^T \\ 0 \end{bmatrix}^T < 0. \quad (23)$$

$$J \left(\begin{bmatrix} A_1^T \\ A_2^T \\ -B_1^T \\ -B_2^T \end{bmatrix} (X - \epsilon M M^T)^{-1} \begin{bmatrix} A_1^T \\ A_2^T \\ -B_1^T \\ -B_2^T \end{bmatrix}^T + \epsilon^{-1} \begin{bmatrix} N_{A1}^T \\ N_{A2}^T \\ -N_{B1}^T \\ -N_{B2}^T \end{bmatrix} \begin{bmatrix} N_{A1}^T \\ N_{A2}^T \\ -N_{B1}^T \\ -N_{B2}^T \end{bmatrix}^T \right) J^T + J \begin{bmatrix} \hat{Y} - X^{-1} & 0 & C^T & 0 \\ 0 & -\hat{Y} & 0 & 0 \\ C & 0 & -(D + D^T - W) & 0 \\ 0 & 0 & 0 & -W \end{bmatrix} J^T$$

$$= \begin{bmatrix} XA_1^T \\ XA_2^T \\ -B_1^T \\ -B_2^T \end{bmatrix} (X - \epsilon M M^T)^{-1} \begin{bmatrix} XA_1^T \\ XA_2^T \\ -B_1^T \\ -B_2^T \end{bmatrix}^T + \epsilon^{-1} \begin{bmatrix} XN_{A1}^T \\ XN_{A2}^T \\ -N_{B1}^T \\ -N_{B2}^T \end{bmatrix} \begin{bmatrix} XN_{A1}^T \\ XN_{A2}^T \\ -N_{B1}^T \\ -N_{B2}^T \end{bmatrix}^T + \begin{bmatrix} Y - X & 0 & XC^T & 0 \\ 0 & -Y & 0 & 0 \\ CX & 0 & -(D + D^T - W) & 0 \\ 0 & 0 & 0 & -W \end{bmatrix} < 0$$

$$\begin{bmatrix} A_{1\Delta}^T X^{-1} A_{1\Delta} + \hat{Y} - X^{-1} & A_{1\Delta}^T X^{-1} A_{2\Delta} & C^T - A_{1\Delta}^T X^{-1} B_{1\Delta} & -A_{1\Delta}^T X^{-1} B_{2\Delta} \\ A_{2\Delta}^T X^{-1} A_{1\Delta} & A_{2\Delta}^T X^{-1} A_{2\Delta} - \hat{Y} & -A_{2\Delta}^T X^{-1} B_{1\Delta} & -A_{2\Delta}^T X^{-1} B_{2\Delta} \\ C - B_{1\Delta}^T X^{-1} A_{1\Delta} & -B_{1\Delta}^T X^{-1} A_{2\Delta} & -\left(D + D^T - B_{1\Delta}^T X^{-1} B_{1\Delta} - W\right) & B_{1\Delta}^T X^{-1} B_{2\Delta} \\ -B_{2\Delta}^T X^{-1} A_{1\Delta} & -B_{2\Delta}^T X^{-1} A_{2\Delta} & B_{2\Delta}^T X^{-1} B_{1\Delta} & B_{2\Delta}^T X^{-1} B_{2\Delta} - W \end{bmatrix} < 0$$

Pre-multiplying and post-multiplying (23) by $\mathrm{diag}(P^{-1}, P^{-1}, I, I)$, and setting $X = P^{-1}$, $Y = P^{-1}QP^{-1}$, the desired result follows immediately. □

III. Robust Positive Real Control

In this section, we consider the problem of positive real control for uncertain 2-D discrete-time systems. An LMI design approach will be developed. The uncertain 2-D discrete-time system $(\Sigma_{\Delta u})$ to be considered in this section is described by the following 2-D LSS model:

$$\begin{aligned}(\Sigma_{\Delta u})\colon x(i+1,j+1) =& A_{1\Delta}x(i+1,j)\\ &+ A_{2\Delta}x(i,j+1)\\ &+ B_{1\Delta}w(i+1,j)\\ &+ B_{2\Delta}w(i,j+1)\\ &+ B_{1\Delta u}u(i+1,j)\\ &+ B_{2\Delta u}u(i,j+1)\\ z(i,j) =& Cx(i,j) + Dw(i,j)\\ y(i,j) =& x(i,j)\end{aligned}$$

where $x(i,j) \in \mathbb{R}^n$ is the local state vector, $u(i,j) \in \mathbb{R}^m$ is the control input, $w(i,j) \in \mathbb{R}^q$ is the exogenous input, $y(i,j)$ is the measurement, $z(i,j) \in \mathbb{R}^q$ is the controlled output, $i, j \in \mathbb{Z}^+$

$$B_{1\Delta u} = B_{1u} + \Delta B_{1u} \quad B_{2\Delta u} = B_{2u} + \Delta B_{2u}$$

B_{1u} and B_{2u} are known real constant matrices with appropriate dimensions. ΔB_{1u} and ΔB_{2u} are assumed to be of the following form:

$$[\,\Delta B_{1u} \quad \Delta B_{2u}\,] = MF\,[\,N_{B1u} \quad N_{B2u}\,] \tag{24}$$

where $F \in \mathbb{R}^{g\times l}$ is an unknown real matrix satisfying (2), and N_{B1u} and N_{B2u} are known real constant matrices with appropriate dimensions. The remaining matrices are the same as in system (Σ_Δ). It is assumed that all the state variables are available for feedback.

The objective of the robust positive real control is the design of feedback controllers for system $(\Sigma_{\Delta u})$ such that the resulting closed-loop system is strongly robustly stable with ESPR for all admissible uncertainties.

The following lemma shows that if there exists a dynamic state feedback controller that achieves strong robust stability with ESPR for system $(\Sigma_{\Delta u})$, then there exists a static feedback controller that achieves the same property.

Lemma 4: Consider the uncertain 2-D discrete-time system $(\Sigma_{\Delta u})$. If there exists a proper dynamic state feedback controller for system $(\Sigma_{\Delta u})$ such that the resulting closed-loop system is strongly robustly stable with ESPR for all admissible uncertainties, then there exists a static state feedback controller that achieves the same property for system $(\Sigma_{\Delta u})$.

Proof: Suppose the following proper dynamic state feedback controller for system $(\Sigma_{\Delta u})$ achieves the property of strong robust stability with ESPR

$$\begin{aligned}(\Sigma_c)\colon \bar{x}(i+1,j+1) =& \bar{A}_1\bar{x}(i+1,j) + \bar{A}_2\bar{x}(i,j+1)\\ &+ \bar{B}_1x(i+1,j) + \bar{B}_2x(i,j+1)\\ u(i,j) =& \bar{C}\bar{x}(i,j) + \bar{D}x(i,j).\end{aligned}$$

Let

$$\tilde{x}(i,j) = [\,x(i,j)^T \quad \bar{x}(i,j)^T\,]^T.$$

Then, the resulting closed-loop system from the system $(\Sigma_{\Delta u})$ and the controller (Σ_c) can be written as

$$\begin{aligned}(\tilde{\Sigma}_c)\colon \tilde{x}(i+1,j+1) =& \tilde{A}_1\tilde{x}(i+1,j) + \tilde{A}_2\tilde{x}(i,j+1)\\ &+ \tilde{B}_1w(i+1,j) + \tilde{B}_2w(i,j+1)\\ z(i,j) =& \tilde{C}\tilde{x}(i,j) + \tilde{D}w(i,j)\end{aligned}$$

where

$$\begin{aligned}\tilde{A}_1 &= \begin{bmatrix} A_{1\Delta} + B_{1\Delta u}\bar{D} & B_{1\Delta u}\bar{C} \\ \bar{B}_1 & \bar{A}_1\end{bmatrix}\\ \tilde{A}_2 &= \begin{bmatrix} A_{2\Delta} + B_{2\Delta u}\bar{D} & B_{2\Delta u}\bar{C} \\ \bar{B}_2 & \bar{A}_2\end{bmatrix}\\ \tilde{B}_1 &= \begin{bmatrix} B_{1\Delta} \\ 0\end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} B_{2\Delta} \\ 0\end{bmatrix}\end{aligned} \tag{25}$$

$$\tilde{C} = [\,C \quad 0\,] \quad \tilde{D} = D. \tag{26}$$

From Definition 3, we have that the strong robust stability with ESPR of the closed-loop system $(\tilde{\Sigma}_c)$ implies that there exist matrices $P > 0$, $Q > 0$ and $W > 0$ such that the LMI shown in (27), at the bottom of the page, holds. By Schur complement, it is easy to show that (27) holds if and only if there exist matrices $\tilde{P} > 0$, $\tilde{Q} > 0$ and $\tilde{W} > 0$ such that

$$\begin{bmatrix} \tilde{Q} - \tilde{P} & 0 & \tilde{P}\tilde{C}^T & 0 & \tilde{P}\tilde{A}_1^T \\ 0 & -\tilde{Q} & 0 & 0 & \tilde{P}\tilde{A}_2^T \\ \tilde{C}\tilde{P} & 0 & -(\tilde{D} + \tilde{D}^T - \tilde{W}) & 0 & -\tilde{B}_1^T \\ 0 & 0 & 0 & -\tilde{W} & -\tilde{B}_2^T \\ \tilde{A}_1\tilde{P} & \tilde{A}_2\tilde{P} & -\tilde{B}_1 & -\tilde{B}_2 & -\tilde{P}\end{bmatrix} < 0. \tag{28}$$

Set

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12}^T & \tilde{P}_{22}\end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_{22}\end{bmatrix} \tag{29}$$

where the partition of $\tilde{P}$ and $\tilde{Q}$ is compatible with (25) and (26). Then, by a lengthy but routine calculation, (30), shown at the bottom of the next page, can be deduced from (28) where

$$\begin{aligned}\tilde{A}_{1\Delta} =& A_{1\Delta} + B_{1\Delta u}\left(\bar{D} + \bar{C}\tilde{P}_{12}^T\tilde{P}_{11}^{-1}\right)\\ \tilde{A}_{2\Delta} =& A_{2\Delta} + B_{2\Delta u}\left(\bar{D} + \bar{C}\tilde{P}_{12}^T\tilde{P}_{11}^{-1}\right).\end{aligned}$$

Defining

$$\begin{aligned}K =& \bar{D} + \bar{C}\tilde{P}_{12}^T\tilde{P}_{11}^{-1}, \quad \hat{P} = \tilde{P}_{11}^{-1},\\ \hat{Q} =& \tilde{P}_{11}^{-1}\tilde{Q}_{11}\tilde{P}_{11}^{-1}, \quad \hat{W} = \tilde{W}\end{aligned}$$

and applying Schur complement to (30), we have (31), shown at the bottom of the next page where

$$A_{1\Delta c} = A_{1\Delta} + B_{1\Delta u}K, \quad A_{2\Delta c} = A_{2\Delta} + B_{2\Delta u}K.$$

Now, applying the state feedback controller

$$u(i,j) = Kx(i,j)$$

to the system $(\Sigma_{\Delta u})$, we obtain the closed-loop system as

$$\begin{aligned}(\hat{\Sigma}_c)\colon x(i+1,j+1) =& A_{1\Delta c}x(i+1,j) + A_{2\Delta c}x(i,j+1)\\ &+ B_{1\Delta}w(i+1,j) + B_{2\Delta}w(i,j+1)\\ z(i,j) =& Cx(i,j) + Dw(i,j).\end{aligned}$$

$$\begin{bmatrix} \tilde{A}_1^TP\tilde{A}_1 + Q - P & \tilde{A}_1^TP\tilde{A}_2 & \tilde{C}^T - \tilde{A}_1^TP\tilde{B}_1 & -\tilde{A}_1^TP\tilde{B}_2 \\ \tilde{A}_2^TP\tilde{A}_1 & \tilde{A}_2^TP\tilde{A}_2 - Q & -\tilde{A}_2^TP\tilde{B}_1 & -\tilde{A}_2^TP\tilde{B}_2 \\ \tilde{C} - \tilde{B}_1^TP\tilde{A}_1 & -\tilde{B}_1^TP\tilde{A}_2 & -\left(\tilde{D} + \tilde{D}^T - \tilde{B}_1^TP\tilde{B}_1 - W\right) & \tilde{B}_1^TP\tilde{B}_2 \\ -\tilde{B}_2^TP\tilde{A}_1 & -\tilde{B}_2^TP\tilde{A}_2 & \tilde{B}_2^TP\tilde{B}_1 & \tilde{B}_2^TP\tilde{B}_2 - W\end{bmatrix} < 0 \tag{27}$$

By Definition 3, it follows from (31) that the system $(\hat{\Sigma}_c)$ is strongly robustly stable with ESPR. This completes the proof. □

Remark 3: Lemma 4 shows that no advantage can be obtained from the use of a proper dynamic state feedback compared with the use of a static state feedback in the context of positive real control. Similar results for 1-D continuous systems can be found in [22]. It is worth pointing out here that some 2-D system can be stabilized by a dynamic state feedback controller, but not by a static one [3]. However, in this brief, we deal with not just the stabilization problem, but the related problem of asymptotically robust stability with ESPR. Since this latter problem is more difficult, more strict conditions are required. It is in fact interesting to see from Lemma 4 that under such more strict conditions, static and dynamic state feedback controllers play the same role for the asymptotically robust stability with ESPR problem.

In view of Lemma 4, in what follows, attention will be focused on the design of static state feedback controllers to solve the positive real control problem for uncertain 2-D system $(\Sigma_{\Delta u})$. The main result of this section is given in the following theorem.

Theorem 3: Consider the uncertain 2-D discrete-time system $(\Sigma_{\Delta u})$. There exists a static state feedback controller for system $(\Sigma_{\Delta u})$ such that the resulting closed-loop system is strongly robustly stable with ESPR for all admissible uncertainties if and only if there exist a scalar $\epsilon > 0$ and matrices $X > 0, Y > 0, W > 0$ and Z, such that the following LMI holds:

$$\begin{bmatrix} H & H_1^T & H_2^T \\ H_1 & \epsilon M M^T - X & 0 \\ H_2 & 0 & -\epsilon I \end{bmatrix} < 0 \tag{32}$$

where H, H_1, and H_2 are given at the bottom of the page. Furthermore, in this case, a suitable state feedback controller can be chosen as

$$u(i,j) = ZX^{-1}x(i,j). \tag{33}$$

Proof: The proof can be carried out by using a similar argument as in the proof of Theorem 2. □

Remark 4: Theorem 3 provides an LMI condition for designing a static state feedback controller which stabilizes the uncertain 2-D discrete-time system and achieves the ESPRness property of the closed-loop system. It is worth pointing out that the LMI (32) in Theorem 3 can be solved efficiently, and no tuning of parameters is required [4].

IV. Numerical Example

In this section, we give an example to illustrate the effectiveness of the proposed method.

Consider the 2-D discrete-time system (Σ_Δ) with parameters given by

$$A_1 = \begin{bmatrix} 0.5 & 0.1 & 0.3 \\ 0.3 & -0.5 & 0.1 \\ 0.2 & 0 & 0.3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.2 & -0.1 & 0.5 \\ -0.4 & 0.1 & 0.2 \\ 0 & 0.2 & 0.5 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.5 & 0.3 & 0.1 \\ 0 & 0.2 & 0.5 \\ -1 & 0.1 & 0.4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.2 & 0 & 0.1 \\ 0.3 & 0.1 & -0.5 \\ 0.6 & -0.1 & 0.3 \end{bmatrix}$$

$$B_{1u} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 1 & 0.5 \end{bmatrix}, \quad B_{2u} = \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \\ 1 & -0.6 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.1 & 0.3 & 0.5 \\ 0.1 & 0 & 0.3 \\ 0.2 & 0.2 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1.5 & 0.5 & 0 \\ 0.1 & 0.8 & 0.2 \\ 0.3 & 0.1 & 1.6 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix} \quad N_{A1} = [0.1 \quad 0.2 \quad 0.1]$$

$$N_{A2} = [0.1 \quad 0.1 \quad 0.2] \quad N_{B1} = [0 \quad 0.1 \quad 0.2]$$

$$N_{B2} = [0.3 \quad 0.1 \quad 0] \quad N_{B1u} = [0.1 \quad 0.2]$$

$$N_{B2u} = [0.1 \quad 0.3].$$

It is required to construct a static state feedback controller that stabilizes the given 2-D discrete system while ensuring that the resulting

$$\begin{bmatrix} \tilde{Q}_{11} - \tilde{P}_{11} & 0 & \tilde{P}_{11} C^T & 0 & \tilde{P}_{11} \tilde{A}_{1\Delta}^T \\ 0 & -\tilde{Q}_{11} & 0 & 0 & \tilde{P}_{11} \tilde{A}_{2\Delta}^T \\ C\tilde{P}_{11} & 0 & -(D + D^T - \tilde{W}) & 0 & -B_{1\Delta}^T \\ 0 & 0 & 0 & -\tilde{W} & -B_{2\Delta}^T \\ \tilde{A}_{1\Delta}\tilde{P}_{11} & \tilde{A}_{2\Delta}\tilde{P}_{11} & -B_{1\Delta} & -B_{2\Delta} & -\tilde{P}_{11} \end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix} A_{1\Delta c}^T \hat{P} A_{1\Delta c} + \hat{Q} - \hat{P} & A_{1\Delta c}^T \hat{P} A_{2\Delta c} & C^T - A_{1\Delta c}^T \hat{P} B_{1\Delta} & -A_{1\Delta c}^T \hat{P} B_{2\Delta} \\ A_{2\Delta c}^T \hat{P} A_{1\Delta c} & A_{2\Delta c}^T \hat{P} A_{2\Delta c} - \hat{Q} & -A_{2\Delta c}^T \hat{P} B_{1\Delta} & -A_{2\Delta c}^T \hat{P} B_{2\Delta} \\ C - B_{1\Delta}^T \hat{P} A_{1\Delta c} & -B_{1\Delta}^T \hat{P} A_{2\Delta c} & -\left(D + D^T - B_{1\Delta}^T \hat{P} B_{1\Delta} - \hat{W}\right) & B_{1\Delta}^T \hat{P} B_{2\Delta} \\ -B_{2\Delta}^T \hat{P} A_{1\Delta c} & -B_{2\Delta}^T \hat{P} A_{2\Delta c} & B_{2\Delta}^T \hat{P} B_{1\Delta} & B_{2\Delta}^T \hat{P} B_{2\Delta} - \hat{W} \end{bmatrix} < 0 \tag{31}$$

$$H = \begin{bmatrix} Y - X & 0 & XC^T & 0 \\ 0 & -Y & 0 & 0 \\ CX & 0 & -(D + D^T - W) & 0 \\ 0 & 0 & 0 & -W \end{bmatrix}$$

$$H_1 = [A_1 X + B_{1u} Z \quad A_2 X + B_{2u} Z \quad -B_1 \quad -B_2]$$

$$H_2 = [N_{A1} X + N_{B1u} Z \quad N_{A2} X + N_{B2u} Z \quad -N_{B1} \quad -N_{B2}].$$

closed-loop system is ESPR. Now using Matlab LMI Control Toolbox and solving the LMI (32), we obtain

$$X = \begin{bmatrix} 4.6705 & 3.6760 & -2.9312 \\ 3.6760 & 7.1592 & -3.9770 \\ -2.9312 & -3.9770 & 5.6329 \end{bmatrix}$$
$$Y = \begin{bmatrix} 1.6713 & -0.0105 & -0.8890 \\ -0.0105 & 0.4083 & 0.1350 \\ -0.8890 & 0.1350 & 0.8602 \end{bmatrix}$$
$$W = \begin{bmatrix} 1.0065 & 0.0081 & -0.1192 \\ 0.0081 & 0.0558 & -0.1015 \\ -0.1192 & -0.1015 & 0.8039 \end{bmatrix}$$
$$Z = \begin{bmatrix} 0.5721 & 1.4912 & -1.9891 \\ -0.1030 & 0.9699 & -1.9244 \end{bmatrix} \quad \epsilon = 3.4374.$$

Therefore, from Theorem 3, there exists a solution to the positive real control problem. Furthermore, a desired state feedback controller can be chosen as

$$u(i,j) = \begin{bmatrix} -0.1940 & 0.0916 & -0.3894 \\ -0.3765 & 0.0497 & -0.5025 \end{bmatrix} x(i,j).$$

V. Conclusions

This brief has addressed the problem of positive real control for uncertain 2-D discrete systems described by the FMLSS model. A version of positive realness for 2-D discrete systems has been established, which has been shown to be an extension of positive realness of 1-D discrete systems. A condition of the solvability of the above problem has been presented in terms of an LMI and the explicit formula of a desired state feedback controller has been given. The proposed control law guarantees both robust stability and extended positive realness of the closed-loop system with admissible parameter uncertainties.

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Cyclostationary Noise in Radio-Frequency Communication Systems

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***Abstract*—Because of the periodically time-varying nature of some circuit blocks of a communication system, such as the mixers, the noise which is generated and processed by the system has periodically time-varying statistics. An accurate evaluation of the system output noise is not straightforward as in the case where all the circuit blocks are linear-time-invariant and the noise that they generate is time-independent. We qualitatively examine here, conditions under which we can treat the noise at the output of every circuit block of a practical communication system as if it were time-invariant, in order to simplify the noise analysis without introducing significant inaccuracy in the noise characterization of the overall communication system.**

***Index Terms*—Cyclostationary noise, mixers, noise, time varying circuits.**

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