IV. Conclusion

In this note, an algorithm is provided for testing the diagnosability of discrete-event systems. Compared to the existing testing method in [4], our algorithm does not require the construction of a diagnoser for the system. The complexity of our algorithm is of fourth order in the number of states of the system and linear in the number of failure types of the system, whereas the complexity of the testing method in [4] is exponential in the number of states of the system and doubly exponential in the number of failure types of the system.

References

[1] Y. L. Chen and G. Provan, "Modeling and diagnosis of timed discrete-event systems—A factory automation example," in *Proc. Amer. Control Conf.*, Albuquerque, NM, June 1997, pp. 31–36.

[2] L. Holloway and S. Chand, "Time templates for discrete event fault monitoring in manufacturing systems," in *Proc. 1994 Amer. Control Conf.*, 1994, pp. 701–706.

[3] F. Lin, "Diagnosability of discrete-event systems and its applications," *J. Discrete Event Dyn. Syst.: Theory Appl.*, vol. 4, no. 2, pp. 197–212, May 1994.

[4] M. Sampath, R. Sengupta, S. Lafortune, K. Sinnamohideen, and D. Teneketzis, "Diagnosability of discrete-event systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1555–1575, Sept. 1995.

[5] S. H. Zad, R. H. Kwong, and W. M. Wonham, "Fault diagnosis in timed discrete-event systems," in *Proc. 38th IEEE Conf. Decision Control*, Phoenix, AZ, Dec. 1999, pp. 1756–1761.

[6] G. Westerman, R. Kumar, C. Stroud, and J. R. Heath, "Discrete-event systems approach for delay fault analysis in digital circuits," in *Proc. 1998 Amer. Control Conf.*, Philadelphia, PA, 1998.

H_∞ and Positive-Real Control for Linear Neutral Delay Systems

Shengyuan Xu, James Lam, and Chengwu Yang

***Abstract*—This note is concerned with the H_∞ and positive-real control problems for linear neutral delay systems. The purpose of H_∞ control is the design of a memoryless state feedback controller which stabilizes the neutral delay system and reduces the H_∞ norm of the closed-loop transfer function from the disturbance to the controlled output to a prescribed level, while the purpose of positive-real control is to design a memoryless state feedback controller such that the resulting closed-loop system is stable and the closed-loop transfer function is extended strictly positive real. Sufficient conditions for the existence of the desired controllers are given in terms of a linear matrix inequality (LMI). When this LMI is feasible, the expected memoryless state feedback controllers can be easily constructed via convex optimization.**

***Index Terms*—H_∞ control, linear matrix inequality, memoryless state feedback, neutral delay systems, positive-real control.**

I. Introduction

Since the late 1980s, the H_∞ control problem has attracted much attention due to its both practical and theoretical importance. Various approaches have been developed and a great number of results for continuous systems as well as discrete systems have been reported in the literature; see, for instance, [4], [18]. Very recently, interest has been focused on H_∞ control problem for delay systems. Lee *et al.* [7] generalized the H_∞ results for continuous systems to systems with state delay, which was further extended to systems with both state and input delays in [3] and [9], respectively. In the context of discrete systems with state delay, similar results can be found in [12] and references therein.

On the other hand, since the introduction of the notion of positive realness, many researchers have considered the positive-real control problem for linear time-invariant systems [1], [15]. The objective is to design controllers such that the resulting closed-loop system is stable and the closed-loop transfer function is positive real. It has been shown in [13] that a solution to this problem involves solving a pair of Riccati inequalities. These results have been extended to uncertain linear systems with time-invariant uncertainty in [11] and [16], respectively. It is worth noting that some positive realness results have also been generalized to time-delay systems [8].

Recently, much attention has been focused on the study of the theory of neutral delay systems and some issues, such as stability and stabilization, related to such systems have been studied [5], [10], [14]. To date, however, very little attention has been drawn to the problem of H_∞ control, as well as positive-real control, for linear neutral delay systems, these are more complex and still open.

In this note, we deal with the H_∞ control and positive-real control problems for linear neutral delay systems. The size of the delays appearing in the state and derivative of the state may not be identical. The H_∞ control problem we address is to design a memoryless state feedback controller such that the resulting closed-loop system is asymptotically stable while the closed-loop transfer function from the disturbance to the controlled output meets a prescribed H_∞-norm bound constraint. In terms of a linear matrix inequality, a sufficient condition for the existence of H_∞ state feedback controllers is presented. Then, based on the relationship between bounded realness and positive realness and the results on H_∞ control, we obtain a sufficient condition for *extended strictly positive realness* (ESPR) for neutral delay systems. The condition for the solvability of positive-real control problem is also given in terms of a linear matrix inequality.

Notation: Throughout this note, for symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with appropriate dimension. The superscript "T" and "$*$" represent the transpose and the complex conjugate transpose. $\|x\|$ is the Euclidean norm of the vector x. For a given stable transfer function matrix $G(s)$, its H_∞ norm is given by $\|G(s)\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}[G(j\omega)]$, where $\sigma_{\max}$ represents the maximum singular value of a matrix. $\rho(A)$ denotes spectral radius of a matrix A. $L_2[0, \infty)$ stands for the space of square integrable functions on $[0, \infty)$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

II. Main Results

Consider the following linear neutral delay system:

$$\begin{aligned}\dot{x}(t) = Ax(t) + A_h x(t-h) + A_d\dot{x}(t-d)&\\ + Bu(t) + E\omega(t)& \qquad (1)\end{aligned}$$

$$z(t) = Cx(t) + D\omega(t) \qquad (2)$$

$$x(t_0+\theta) = \phi(\theta) \qquad \forall\,\theta\in[-l, 0] \qquad (3)$$

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where

$x(t) \in \mathbb{R}^n$	state;
$u(t) \in \mathbb{R}^m$	control input;
$\omega(t) \in \mathbb{R}^q$	disturbance input which belongs to $L_2[0, \infty)$;
$z(t) \in \mathbb{R}^s$	controlled output;
A, A_h, A_d, B, E, C and D	known real constant matrices with appropriate dimensions;
$d > 0, h > 0$	constant time-delays, d may not be equal to h;
$l = \max\{h, d\}$ and $\phi(\theta)$	continuously differentiable initial function on $[-l, 0]$.

Now consider the following memoryless linear state feedback controller:

$$u(t) = Fx(t), \qquad F \in \mathbb{R}^{m\times n}. \tag{4}$$

The resulting closed-loop system from (1)–(4) can then be written as

$$\dot{x}(t) = A_c x(t) + A_h x(t-h) + A_d \dot{x}(t-d) + E\omega(t) \tag{5}$$

$$z(t) = Cx(t) + D\omega(t) \tag{6}$$

where $A_c = A + BF$, and the closed-loop transfer function matrix $G_{z\omega}(s)$ from the disturbance $\omega(t)$ to the controlled output $z(t)$ is given by

$$G_{z\omega}(s) = C\left[s\left(I - A_d e^{-sd}\right) - \left(A_c + A_h e^{-sh}\right)\right]^{-1} E + D. \tag{7}$$

We first consider the H_∞ control problem. The purpose is to determine the state feedback controller (4) such that the following requirements are met:

R2) the closed-loop system is asymptotically stable when $\omega(t) = 0$;

R3) the H_∞ norm of the closed-loop transfer function $G_{z\omega}(s)$ satisfies the constraint

$$\|G_{z\omega}(s)\|_\infty < \gamma \tag{8}$$

where $\gamma > 0$ is a prescribed scalar.

To solve the H_∞ control problem formulated above, we first give a sufficient condition for the asymptotic stability of linear neutral delay systems.

Lemma 1: Consider the neutral delay system (1) with $u(t) \equiv 0$ and $\omega(t) \equiv 0$, that is

$$\dot{x}(t) = Ax(t) + A_h x(t-h) + A_d \dot{x}(t-d) \tag{9}$$

$$x_{t_0} = x(t_0 + \theta) = \phi(\theta) \qquad \forall\,\theta \in [-l, 0]. \tag{10}$$

If there exist matrices $P > 0$, $Q > 0$, and $S > 0$ such that

$$\begin{aligned} &PA + A^T P + Q + S + (Q + S + PA)A_d W^{-1} \\ &\quad \cdot A_d^T (Q + S + PA)^T + PA_h S^{-1} A_h^T P < 0 \end{aligned} \tag{11}$$

$$W = Q - A_d^T (Q + S) A_d > 0 \tag{12}$$

then the system (9) and (10) is asymptotically stable.

Proof: Define a difference operator $\mathbb{D}$ as

$$\mathbb{D}(\phi) = \phi(0) - A_d \phi(-d). \tag{13}$$

From (12), it is easy to show that

$$A_d^T Q A_d - Q < 0.$$

Thus, the operator $\mathbb{D}$ is stable.

Now, we introduce the following Lyapunov functional candidate for the system (9) and (10):

$$\begin{aligned} V(x_t) &= (x(t) - A_d x(t-d))^T P (x(t) - A_d x(t-d)) \\ &\quad + \int_{t-d}^{t} x(\tau)^T Q x(\tau) d\tau + \int_{t-h}^{t} x(\tau)^T S x(\tau)\, d\tau \end{aligned} \tag{14}$$

where $x_t = x(t+\theta)$, $\theta \in [-l, 0]$. It can be shown that there exist scalars $c_1 > 0$ and $c_2 > 0$ such that the following holds:

$$c_1 \|\mathbb{D}(\phi)\|^2 \le V(\phi) \le c_2 \sup_{\theta \in [-l,\, 0]} \|\phi(\theta)\|^2. \tag{15}$$

Differentiating $V(x_t)$ along the solution of (9) and (10) results in

$$\begin{aligned} \dot{V}(x_t) &= 2(x(t) - A_d x(t-d))^T P (Ax(t) + A_h x(t-h)) \\ &\quad + x(t)^T (Q+S) x(t) - x(t-d)^T Q x(t-d) \\ &\quad - x(t-h)^T S x(t-h) \\ &= (x(t) - A_d x(t-d))^T (PA + A^T P + Q + S) \\ &\quad \cdot (x(t) - A_d x(t-d)) - x(t-d)^T W x(t-d) \\ &\quad + 2(x(t) - A_d x(t-d))^T (Q + S + PA) A_d x(t-d) \\ &\quad + 2(x(t) - A_d x(t-d))^T P A_h x(t-h) \\ &\quad - x(t-h)^T S x(t-h). \end{aligned}$$

Noting the definition of the operator $\mathbb{D}$, this equality can be rewritten as

$$\begin{aligned} \dot{V}(x_t) &= \mathbb{D}(x_t)^T (PA + A^T P + Q + S)\mathbb{D}(x_t) - x(t-d)^T \\ &\quad \cdot W x(t-d) + 2\mathbb{D}(x_t)^T (Q + S + PA) A_d x(t-d) \\ &\quad + 2\mathbb{D}(x_t)^T P A_h x(t-h) - x(t-h)^T S x(t-h). \end{aligned}$$

By considering (12), it follows that:

$$\begin{aligned} \dot{V}(x_t) &= \mathbb{D}(x_t)^T (PA + A^T P + Q + S + (Q + S + PA) \\ &\qquad \cdot A_d W^{-1} A_d^T (Q + S + PA)^T \\ &\qquad + PA_h S^{-1} A_h^T P)\mathbb{D}(x_t) \\ &\quad - [x(t-d)^T - \mathbb{D}(x_t)^T (Q + S + PA) A_d W^{-1}] W \\ &\quad \cdot [x(t-d) - W^{-1} A_d^T (Q + S + PA)^T \mathbb{D}(x_t)] \\ &\quad - [x(t-h)^T - \mathbb{D}(x_t)^T P A_h S^{-1}] S \\ &\quad \cdot [x(t-h) - S^{-1} A_h^T P \mathbb{D}(x_t)]. \end{aligned}$$

This equality, together with (11), implies that there exists a scalar $c > 0$ such that

$$\dot{V}(x_t) \le -c\|\mathbb{D}(x_t)\|^2.$$

Finally, noting the stability of the operator $\mathbb{D}$ and the above inequality and (15), the desired result follows immediately from [6, Th. 7.1]. □

Remark 1: Lemma 1 provides a delay-independent stability condition for the neutral delay system (9) and (10), it is worth noting that, for any $d > 0$ and $h > 0$, Lemma 1 is always applicable. However, [5, Th. 1, p. 93] is only applicable to the case when $d = h$. In this sense, our stability result extends that of [5] and is more general.

The following result will play an important role in solving the H_∞ control problem in this section.

Theorem 1: Consider the neutral delay system (1)–(3) with $u(t) \equiv 0$, that is

$$\dot{x}(t) = Ax(t) + A_h x(t-h) + A_d \dot{x}(t-d) + E\omega(t) \tag{16}$$

$$z(t) = Cx(t) + D\omega(t) \tag{17}$$

$$x(t_0 + \theta) = \phi(\theta) \qquad \forall\,\theta \in [-l, 0]. \tag{18}$$

If there exist matrices $P > 0$, $Q > 0$, and $S > 0$ such that

$$\begin{aligned} &PA + A^T P + C^T C + Q + S + (PE + C^T D) \\ &\quad \cdot V^{-1} (PE + C^T D)^T + PA_h S^{-1} A_h^T P \\ &\quad + M A_d W^{-1} A_d^T M^T < 0 \end{aligned} \tag{19}$$

with

$$V = \gamma^2 I - D^T D > 0 \tag{20}$$

$$W = Q - A_d^T [C^T (I + DV^{-1} D^T) C + Q + S] A_d > 0 \tag{21}$$

$$M = C^T (I + DV^{-1} D^T) C + Q + S + P(A + EV^{-1} D^T C) \tag{22}$$

then the system (16)–(18) is asymptotically stable, and

$$\left\| C[s(I - A_d e^{-ds}) - (A + A_h e^{-hs})]^{-1} E + D \right\|_\infty < \gamma.$$

For the proof of Theorem 1, the following two lemmas will be used.

Lemma 2: If there exist matrices $P > 0$, $Q > 0$, and $S > 0$ such that

$$\begin{aligned} &PA + A^T P + C^T C + Q + S + \gamma^{-2} PEE^T P + PA_h S^{-1} A_h^T P \\ &\quad + (C^T C + Q + S + PA) A_d W^{-1} \\ &\quad \cdot A_d^T (C^T C + Q + S + PA)^T < 0 \end{aligned} \tag{23}$$

$$W = Q - A_d^T (C^T C + Q + S) A_d > 0 \tag{24}$$

then system (9) is asymptotically stable, and

$$\left\| C[s(I - A_d e^{-ds}) - (A + A_h e^{-hs})]^{-1} E \right\|_\infty < \gamma.$$

Proof: Let

$$S_1 = S + C^T C.$$

From (23), it can be deduced that

$$\begin{aligned} &PA + A^T P + Q + S_1 + (Q + S_1 + PA) \\ &\quad \cdot A_d W^{-1} A_d^T (Q + S_1 + PA)^T + PA_h S_1^{-1} A_h^T P < 0. \end{aligned}$$

By Lemma 1, this inequality, together with (24), implies the asymptotic stability of the neutral delay system (9). Next, we will show that the H_∞-norm bound constraint is satisfied. To this end, we set

$$\Psi(j\omega) = I - A_d e^{-j\omega d}.$$

From (24), we have

$$A_d^T Q A_d - Q < 0.$$

Therefore, $\rho(A_d) < 1$. This implies that for all $\omega \in \mathbb{R}$, $\Psi(j\omega)$ is invertible.

Now, through some routine algebraic manipulations, we obtain

$$\begin{aligned} &C^T C + Q + S \\ &\quad = \Psi(j\omega)^{-*}[\Psi(j\omega)^*(C^T C + Q + S)\Psi(j\omega)]\Psi(j\omega)^{-1} \\ &\quad = \Psi(j\omega)^{-*}(S + C^T C + W)\Psi(j\omega)^{-1} + X(j\omega) + X(j\omega)^* \end{aligned} \tag{25}$$

for all $\omega \in \mathbb{R}$, where

$$X(j\omega) = -e^{-j\omega d}(C^T C + Q + S) A_d \Psi(j\omega)^{-1}.$$

Then, (23) can be rewritten as

$$\begin{aligned} &PA + A^T P + \gamma^{-2} PEE^T P + PA_h S^{-1} A_h^T P + X(j\omega) \\ &\quad + X(j\omega)^* + \Psi(j\omega)^{-*}(S + C^T C + W)\Psi(j\omega)^{-1} \\ &\quad + (C^T C + Q + S + PA) A_d W^{-1} \\ &\quad \cdot A_d^T (C^T C + Q + S + PA)^T < 0. \end{aligned} \tag{26}$$

Define

$$\begin{aligned} Y(j\omega) &= e^{-j\omega d}(C^T C + Q + S + PA) A_d \Psi(j\omega)^{-1}, \\ Z(j\omega) &= e^{-j\omega h} PA_h \Psi(j\omega)^{-1}. \end{aligned}$$

Recalling that for any matrices K_1, K_2 and K_3 of appropriate dimensions with $K_2 > 0$

$$K_1^* K_3 + K_3^* K_1 \le K_1^* K_2 K_1 + K_3^* K_2^{-1} K_3.$$

Therefore

$$\begin{aligned} &Y(j\omega) + Y(j\omega)^* \\ &\quad \le (C^T C + Q + S + PA) A_d W^{-1} A_d^T \\ &\quad\quad \cdot (C^T C + Q + S + PA)^T + \Psi(j\omega)^{-*} W \Psi(j\omega)^{-1} \end{aligned} \tag{27}$$

and

$$Z(j\omega) + Z(j\omega)^* \le PA_h S^{-1} A_h^T P + \Psi(j\omega)^{-*} S \Psi(j\omega)^{-1}. \tag{28}$$

From (26)–(28), it follows that for all $\omega \in \mathbb{R}$:

$$\begin{aligned} &PA + A^T P + \gamma^{-2} PEE^T P + X(j\omega) + X(j\omega)^* + Y(j\omega) \\ &\quad + Y(j\omega)^* + Z(j\omega) + Z(j\omega)^* + \Psi(j\omega)^{-*} C^T C \Psi(j\omega)^{-1} < 0. \end{aligned} \tag{29}$$

Observing that

$$\begin{aligned} &P(A + A_h e^{-j\omega h})\Psi(j\omega)^{-1} \\ &\quad = PA + e^{-j\omega d} P \left(AA_d + e^{-j\omega(h-d)} A_h\right) \Psi(j\omega)^{-1} \\ &\quad = PA + X(j\omega) + Y(j\omega) + Z(j\omega). \end{aligned}$$

Substituting this equality into (29) yields

$$\begin{aligned} &P(A + A_h e^{-j\omega h})\Psi(j\omega)^{-1} + \Psi(j\omega)^{-*}(A^T + A_h^T e^{j\omega h})P \\ &\quad + \gamma^{-2} PEE^T P + \Psi(j\omega)^{-*} C^T C \Psi(j\omega)^{-1} < 0. \end{aligned}$$

That is

$$\begin{aligned} &P[j\omega I - (A + A_h e^{-j\omega h})\Psi(j\omega)^{-1}] \\ &\quad + [j\omega I - (A + A_h e^{-j\omega h})\Psi(j\omega)^{-1}]^* P \\ &\quad - \gamma^{-2} PEE^T P - \Psi(j\omega)^{-*} C^T C \Psi(j\omega)^{-1} > 0. \end{aligned} \tag{30}$$

Let $\Gamma(j\omega) = j\omega I - (A + A_h e^{-j\omega h})\Psi(j\omega)^{-1}$, then for all $\omega \in \mathbb{R}$, $\Gamma(j\omega)$ is invertible since the neutral delay system (9) is stable.

Premultiplying (30) by $E^T \Gamma(j\omega)^{-*}$ and postmultiplying (30) by $\Gamma(j\omega)^{-1} E$ give

$$\begin{aligned} &E^T \Gamma(j\omega)^{-*} PE + E^T P \Gamma(j\omega)^{-1} E \\ &\quad - \gamma^{-2} E^T \Gamma(j\omega)^{-*} PEE^T P \Gamma(j\omega)^{-1} E \\ &\quad - E^T \Gamma(j\omega)^{-*} \Psi(j\omega)^{-*} C^T C \Psi(j\omega)^{-1} \Gamma(j\omega)^{-1} E > 0. \end{aligned}$$

In completing the squares in this inequality, it follows that, for all $\omega \in \mathbb{R}$,

$$\begin{aligned} &\gamma^2 I - \gamma^{-2}[E^T \Gamma(j\omega)^{-*} PE - \gamma^2 I][E^T P \Gamma(j\omega)^{-1} E - \gamma^2 I] \\ &\quad - E^T \Gamma(j\omega)^{-*} \Psi(j\omega)^{-*} C^T C \Psi(j\omega)^{-1} \Gamma(j\omega)^{-1} E > 0. \end{aligned}$$

Thus, for all $\omega \in \mathbb{R}$

$$E^T \Gamma(j\omega)^{-*} \Psi(j\omega)^{-*} C^T C \Psi(j\omega)^{-1} \Gamma(j\omega)^{-1} E < \gamma^2 I. \tag{31}$$

Finally, by noting that

$$\Psi(j\omega)^{-1} \Gamma(j\omega)^{-1} = [j\omega(I - A_d e^{-j\omega d}) - (A + A_h e^{-j\omega h})]^{-1}$$

and using (31), we conclude that $\|C[s(I - A_d e^{-ds}) - (A + A_h e^{-hs})]^{-1} E\|_\infty < \gamma$ holds. □

Lemma 3: Let $H > 0$, $\Psi(s) = I - A_d e^{-ds}$

$$\begin{aligned} \Phi(s) = &\left[\, E^T(-s\Psi(-s)^T - (A + A_h e^{sh})^T)^{-1} \quad I \,\right] \\ &\cdot \begin{bmatrix} P & S \\ S^T & H \end{bmatrix} \begin{bmatrix} (s\Psi(s) - (A + A_h e^{-sh}))^{-1} E \\ I \end{bmatrix} \end{aligned}$$

and

$$\hat{\Phi}(s) = \left[\hat{E}^T \left(-s\Psi(-s)^T - (\hat{A} + A_h e^{sh})^T\right)^{-1} \quad I\right] \cdot \begin{bmatrix} \hat{P} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \left(s\Psi(s) - (\hat{A} + A_h e^{-sh})\right)^{-1} \hat{E} \\ I \end{bmatrix}$$

where

$$\hat{A} = A - EH^{-1}S^T, \quad \hat{E} = EH^{-(1/2)}, \quad \hat{P} = P - SH^{-1}S^T.$$

Then, $\Phi(j\omega) > 0$ if and only if $\hat{\Phi}(j\omega) > 0$.

Proof: The proof follows the same idea as in [17, Lemma 13.18] and is thus omitted. □

Proof of Theorem 1: The asymptotic stability of the neutral delay system (9) can be inferred from (19) and Lemma 1. To show the H_∞ norm bound constraint is satisfied, we rewrite (19) as

$$\begin{aligned} &P(A + EV^{-1}D^TC) + (A + EV^{-1}D^TC)^TP \\ &\quad + C^T(I + DV^{-1}D^T)C + Q + S + PEV^{-1}E^TP \\ &\quad + PA_hS^{-1}A_h^TP + MA_dW^{-1}A_d^TM^T < 0. \end{aligned}$$

Applying Lemma 2 to this inequality yields

$$\begin{aligned} &\left(\hat{C}[j\omega(I - A_de^{-j\omega d}) - (\hat{A} + A_he^{-j\omega h})]^{-1}\hat{E}\right)^T \\ &\quad \cdot (\hat{C}[j\omega(I - A_de^{-j\omega d}) - (\hat{A} + A_he^{-j\omega h})]^{-1}\hat{E}) < 1 \end{aligned} \tag{32}$$

where

$$\begin{aligned} \hat{A} &= A + EV^{-1}D^TC, \\ \hat{C} &= (I + DV^{-1}D^T)^{1/2}C, \\ \hat{E} &= EV^{-(1/2)}. \end{aligned}$$

By Lemma 3, it is easy to show that (32) is equivalent to

$$\begin{aligned} &\left(C[j\omega(I - A_de^{-j\omega d}) - (A + A_he^{-j\omega h})]^{-1}E + D\right)^T \\ &\quad \cdot \left(C[j\omega(I - A_de^{-j\omega d}) - (A + A_he^{-j\omega h})]^{-1}E + D\right) < \gamma^2 I. \end{aligned}$$

This completes the proof. □

In the case when $A_d = 0$, from Theorem 1, we have the following result.

Corollary 1: Consider the following delay system:

$$\dot{x}(t) = Ax(t) + A_hx(t - h) + E\omega(t) \tag{33}$$

$$z(t) = Cx(t) + D\omega(t) \tag{34}$$

$$x(t_0 + \theta) = \phi(\theta) \qquad \forall\, \theta \in [-h, 0]. \tag{35}$$

If there exist matrices $P > 0$ and $S > 0$ such that

$$\begin{aligned} &PA + A^TP + C^TC + (PE + C^TD)V^{-1}(PE + C^TD)^T \\ &\quad + PA_hS^{-1}A_h^TP + S < 0 \end{aligned} \tag{36}$$

with

$$V = \gamma^2 I - D^TD > 0$$

then the system (33)–(35) is asymptotically stable when $\omega(t) \equiv 0$, and the H_∞ norm of the transfer function satisfies

$$\left\| C[sI - A - A_he^{-hs}]^{-1}E + D \right\|_\infty < \gamma.$$

Remark 2: It is easy to see that Corollary 1 is the same as [7, Th. 1] when $D = 0$, thus Theorem 1 here can be viewed as an extension of the existing results on H_∞ disturbance attenuation for delay systems to neutral delay systems.

Now we are in a position to give a solution to the H_∞ control problem specified above.

Theorem 2: Suppose that there exist matrices $X > 0$, Y, $Q > 0$, $S > 0$ satisfying the LMI, as shown in (37) at the bottom of the page, where

$$\begin{aligned} L &= [E(I + D^TV^{-1}D)E^T + Q + S + XC^TV^{-1}DE^T \\ &\qquad + XA^T + Y^TB^T]A_d^T \\ H &= [\, XA_h^T \quad XC^T + ED^T \,] \\ J &= \mathrm{diag}(S, V) \\ V &= \gamma^2 I - DD^T > 0 \end{aligned}$$

then the memoryless state feedback controller

$$u(t) = YX^{-1}x(t), \tag{38}$$

stabilizes system (1)–(3) and guarantees that the H_∞ norm bound of the closed-loop transfer function constraint has a prescribed level $\gamma > 0$.

Proof: Applying the controller (38) to the neutral delay system (1)–(3), we obtain the resulting closed-loop system in the form of (5) and (6) with

$$A_c = A + BYX^{-1}.$$

By Schur complement, (37) implies

$$\begin{aligned} &XA_c^T + A_cX + EE^T + Q + S + (XC^T + ED^T) \\ &\quad \cdot V^{-1}(XC^T + ED^T)^T + XA_h^TS^{-1}A_hX \\ &\quad + M_1A_d^TW_1^{-1}A_dM_1^T < 0 \end{aligned} \tag{39}$$

where

$$\begin{aligned} M_1 &= E(I + D^TV^{-1}D)E^T + Q + S + X(A_c^T + C^TV^{-1}DE^T) \\ W_1 &= Q - A_d[E(I + D^TV^{-1}D)E^T + Q + S]A_d^T > 0. \end{aligned}$$

Using Theorem 1, we have that the following neutral delay system:

$$\dot{x}_1(t) = A_c^Tx_1(t) + A_h^Tx(t - h) + A_d^T\dot{x}(t - d) + C^T\omega_1(t) \tag{40}$$

$$z_1(t) = E^Tx_1(t) + D^T\omega_1(t) \tag{41}$$

is asymptotically stable when $\omega_1(t) \equiv 0$, and

$$\left\| E^T[s(I - A_d^Te^{-sd}) - (A_c^T + A_h^Te^{-sh})]^{-1}C^T + D^T \right\|_\infty < \gamma \tag{42}$$

where the state $x_1(t) \in \mathbb{R}^n$, the disturbance input $\omega_1(t) \in \mathbb{R}^s$, and the controlled output $z(t) \in \mathbb{R}^q$. It is easy to see that the system

$$\begin{bmatrix} XA^T + AX + Y^TB^T + BY + EE^T + Q + S & L & H \\ L^T & A_d\left[E\left(I + D^TV^{-1}D\right)E^T + Q + S\right]A_d^T - Q & 0 \\ H^T & 0 & -J \end{bmatrix} < 0 \tag{37}$$

$$\begin{bmatrix} XA^T + AX + Y^TB^T + BY + Q + S & L & H \\ L^T & A_d\left[Q + S + EU^{-1}E^T\right]A_d^T - Q & 0 \\ H^T & 0 & -J \end{bmatrix} < 0 \tag{46}$$

(40) and (41) is asymptotically stable and (42) holds, if and only if the closed-loop system (5) and (6) is asymptotically stable and

$$\left\|C[s(I - A_d e^{-sd}) - (A_c + A_h e^{-sh})]^{-1}E + D\right\|_\infty < \gamma$$

holds. This completes the proof. □

By considering the connections between bounded realness and positive realness, next we shall consider the positive-real control problem for the linear neutral delay system (1)–(3). we first introduce the following concepts of bounded realness and positive realness.

Definition 1 [1]: A transfer function $G(s)$ is bounded real if all elements of $G(s)$ are analytic for $\mathrm{Re}(s) \geq 0$ and $\|G(s)\|_\infty < 1$.

Definition 2 [13]: A system [or its transfer function $G(s)$] is said to be extended strictly positive real (ESPR) if $G(s)$ is analytic in $\mathrm{Re}(s) \geq 0$ and satisfies $G(j\omega) + G(j\omega)^* > 0$ for $\omega \in [0, \infty]$.

The problem to be addressed is to determine the state feedback controller (4) such that the resulting closed-loop system (5) and (6) is stable and the transfer function $G_{z\omega}(s)$ is ESPR. To solve this problem, we first give the relationship between bounded realness and positive realness stated in the following lemma.

Lemma 4: Let $G(s)$ be a square transfer function with $\det(G(s) + I) \neq 0$ for $\mathrm{Re}(s) \geq 0$, and $G(j\infty) + G(-j\infty)^T > 0$. Then the bounded realness of $H(s) = (G(s) - I)(G(s) + I)^{-1}$ implies that $G(s)$ is ESPR.

Proof: By the definitions of bounded realness and ESPR, the desired result follows immediately. □

The following result will play an important role in solving the positive-real control problem.

Theorem 3: Consider the neutral delay system (16)–(18). If there exist matrices $P > 0$, $Q > 0$ and $S > 0$ such that the following matrix inequalities hold:

$$\begin{aligned} &PA + A^TP + Q + S + (PE - C^T)U^{-1}(PE - C^T)^T \\ &\quad + PA_hS^{-1}A_h^TP + [PA + Q + S - (PE - C^T)U^{-1}C] \\ &\quad \cdot A_dW^{-1}A_d^T[PA + Q + S - (PE - C^T)U^{-1}C]^T \\ &\quad < 0 \end{aligned} \tag{43}$$

$$U = D + D^T > 0 \tag{44}$$

$$W = Q - A_d^T(Q + S + C^TU^{-1}C)A_d > 0 \tag{45}$$

then, system (16)–(18) is asymptotically stable and ESPR.

Proof: The proof can be carried out by using Theorem 1 and Lemma 4. □

Remark 3: In the case when $A_d = 0$, that is, the neutral delay system (16)–(18) reduces to a usual delay system. It is easy to see that Theorem 3 coincides with [8, Th. 1]. Moreover, if both $A_d = 0$ and $A_h = 0$, the neutral delay system (16)–(18) becomes a system without any delays, then we can see that Theorem 3 corresponds to the result of positive realness for usual state-space systems with delay-free (see, e.g., [13]). In view of this, Theorem 3 can be regarded as an extension of the existing results on positive realness for systems with or without delays.

Now we are in a position to present our result on positive-real control problem for neutral delay systems.

Theorem 4: Consider the linear neutral delay system (1)–(3). If there exist matrices $X > 0$, $Q > 0$, $S > 0$ and a matrix Y, satisfying the LMI shown in (46) at the top of the page, where

$$\begin{aligned} U &= D + D^T > 0 \\ L &= [XA^T + Y^TB^T + Q + S - (XC^T - E)U^{-1}E^T]A_d^T \\ H &= [\,XA_h^T \quad XC^T - E\,] \\ J &= \mathrm{diag}(S, U) \end{aligned}$$

then the memoryless state feedback controller

$$u(t) = YX^{-1}x(t) \tag{47}$$

will be such that the resulting closed-loop system is asymptotically stable and ESPR.

Proof: Following a similar line as in the proof of Theorem 2 and using Theorem 4, the desired result follows immediately. □

Remark 4: Theorems 2 and 4 provide sufficient conditions for solvability of the problems of H_∞ and positive-real control for neutral delay systems, respectively. It is worth pointing out that the LMI (37) in Theorem 2 and the LMI (46) in Theorem 4 can be solved efficiently, and no tuning of parameters is required [2].

III. Conclusion

In this note, we have studied the H_∞ and positive-real control problem for linear-neutral delay systems. Based on the LMI approach, sufficient conditions for the solvability of these two problems have been presented. Our results on H_∞ control and positive-real control for neutral delay systems encompass earlier ones for delay systems.

References

[1] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*. Upper Saddle River, NJ: Prentice-Hall, 1973.

[2] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*. Philadelphia, PA: SIAM, 1994, SIAM Studies in Applied Mathematics.

[3] H. H. Choi and M. J. Chung, "Memoryless H_∞ controller design for linear systems with delayed state and control," *Automatica*, vol. 31, pp. 917–919, 1995.

[4] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–847, Aug. 1989.

[5] L. Dugard and E. I. Verriest, *Stability and Control of Time-Delay Systems*. London, U.K.: Springer-Verlag, 1998.

[6] J. K. Hale, *Theory of Functional Differential Equations*. New York: Springer-Verlag, 1977.

[7] J. H. Lee, S. W. Kim, and W. H. Kwon, "Memoryless H_∞ controllers for state delayed systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 159–162, Jan. 1994.

[8] M. S. Mahmoud and L. Xie, "Stability and positive realness of time-delay systems," *J. Math. Anal. Appl.*, vol. 239, pp. 7–19, 1999.

[9] M. S. Mahmoud and M. Zribi, "H_∞ controllers for time-delay systems using linear matrix inequalities," *J. Optim. Theory Appl.*, vol. 100, pp. 89–121, 1999.

[10] J.-H. Park and S. Won, "Asymptotic stability of neutral systems with multiple delays," *J. Optim. Theory Appl.*, vol. 103, pp. 183–200, 1999.

[11] Y. C. Soh and L. Xie, "Positive real control of linear uncertain systems," in *Proc. 3rd Euro. Control Conf.*, Rome, Italy, 1995, pp. 1032–1036.

[12] S.-H. Song and J.-K. Kim, "H_∞ control of discrete-time linear systems with norm-bounded uncertainties and time delay in state," *Automatica*, vol. 34, pp. 137–139, 1998.

[13] W. Sun, P. P. Khargonekar, and D. Shim, "Solution to the positive real control problem for linear time-invariant systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2034–2046, Oct. 1994.
[14] S. Tarbouriech and G. Garcia, "Stabilization of neutral linear time-delay systems with saturating actuators," in *Proc. 38th IEEE Conf. Decision Control*, Phoenix, AZ, USA, Dec. 1999, pp. 2011–2016.
[15] M. Vidyasagar, *Nonlinear Systems Analysis*. Upper Saddle River, NJ: Prentice-Hall, 1993.
[16] L. Xie and Y. C. Soh, "Positive real control for uncertain linear time-invariant systems," *Syst. Control Lett.*, vol. 24, pp. 265–271, 1992.
[17] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.
[18] K. Zhou and P. P. Khargonekar, "An algebraic Riccati equation approach to H_∞ optimization," *Syst. Control Lett.*, vol. 11, pp. 85–91, 1988.

A Note on Uniform Observability

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***Abstract*—We prove in this note that the classical inequality $P_t \leq \mathcal{O}_t^{-1} + \mathcal{C}_t$ relating the variance of the Kalman filter estimate, the observability matrix, and the controllability matrix is not true. This inequality is the cornerstone of the asymptotic stability theory of the Kalman filter for time-varying systems. We provide another inequality of the same type.**

***Index Terms*—Kalman filter, time-varying, uniform observability.**

I. Introduction

We consider the following system:

$$\dot{x}_t = A_t x_t + v_t$$

$$\dot{y}_t = C_t x_t + w_t$$

$$E\begin{pmatrix} v_s v_t^T & v_s w_t^T \\ w_s v_t^T & w_s w_t^T \end{pmatrix} = \begin{pmatrix} Q_t & R_t \\ R_t^T & S_t \end{pmatrix}\delta(t-s).$$

with an initial value with Gaussian distribution $x_0 \sim \mathcal{N}(\hat{x}_0, P_0)$. The corresponding Kalman filter is

$$\dot{\hat{x}}_t = A_t\hat{x}_t + \left(P_t C_t^T + R_t\right) S_t^{-1}\left(\dot{y}_t - C_t\hat{x}_t\right)$$

$$\dot{P}_t = A_t P_t + P_t A_t^T + Q_t - \left(P_t C_t^T + R_t\right) S_t^{-1}\left(C_t P_t + R_t^T\right).$$

The matrix P_t is the variance of the estimation error $\hat{x}_t - x_t$. Bounding P_t is, for obvious reasons, an important issue. In [2, p. 359], R. E. Kalman considers the case where $R_t = 0$ and states the following lemma (we set $W_t = C_t^T S_t^{-1} C_t$).

Lemma 1 (Case $R_t = 0$): Let $P_t, \mathcal{O}_t, \mathcal{C}_t$ be the solutions to

$$\dot{P}_t = A_t P_t + P_t A_t^T + Q_t - P_t W_t P_t \quad P_0 = P_0^T \geq 0$$

$$\dot{\mathcal{O}}_t = -\mathcal{O}_t A_t - A_t^T\mathcal{O}_t + W_t \quad \mathcal{O}_0 = 0$$

$$\dot{\mathcal{C}}_t = A_t\mathcal{C}_t + \mathcal{C}_t A_t^T + Q_t \quad \mathcal{C}_0 = 0$$

then

$$P_t \leq \mathcal{O}_t^{-1} + \mathcal{C}_t$$

as soon as $\mathcal{O}_t^{-1}$ exists.

This lemma is more explicitly stated and "proved" in [1, p. 234, 243]. The flaw in the proof is apparent in the last correlation inequality at the end of [1, p. 234].

We show here in Section II that this lemma is untrue and prove in Section III the following modified version of it.

Lemma 2 (Case $R_t = 0$): Let A_t, Q_t, W_t be arbitrary square matrices with same dimensions, piecewise continuous w.r.t. t, such that Q_t and W_t are symmetric and W_t is nonnegative for all t. Let $P_t, \mathcal{O}_t, \mathcal{D}_t$ be the solutions to

$$\dot{P}_t = A_t P_t + P_t A_t^T + Q_t - P_t W_t P_t \quad P_0 = P_0^T \geq 0$$

$$\dot{\mathcal{O}}_t = -\mathcal{O}_t A_t - A_t^T\mathcal{O}_t + W_t \quad \mathcal{O}_0 = 0 \tag{1}$$

$$\dot{\mathcal{D}}_t = -\mathcal{D}_t A_t - A_t^T\mathcal{D}_t + \mathcal{O}_t Q_t \mathcal{O}_t \quad \mathcal{D}_0 = 0 \tag{2}$$

then

$$P_t \leq \mathcal{O}_t^{-1} + \mathcal{O}_t^{-1}\mathcal{D}_t\mathcal{O}_t^{-1} \tag{3}$$

as soon as $\mathcal{O}_t^{-1}$ exists. Furthermore, one has

$$\mathcal{O}_t \leq \int_0^t e^{2\alpha(t-s)}\|W_s\|\,ds \quad \alpha = \sup_{0\leq s\leq t}\|A_s\|$$

$$\mathcal{D}_t \leq \int_0^t e^{2\alpha(t-s)}\|\mathcal{O}_s\|^2\|Q_s\|\,ds.$$

Some classical comments, which are shared by both lemmas, are in order.

2) The important point here is that the bound is independent of P_0. This allows indeed to get bounds for P_t, $t \geq 0$, by considering the system on a finite-time interval $(t-\sigma, t)$:
   P_t remains bounded if for all t the solutions to (1) and (2) over $(t-\sigma, t)$ with initial condition $\mathcal{O}_{t-\sigma} = \mathcal{D}_{t-\sigma} = 0$ satisfy $\|\mathcal{O}_t^{-1} + \mathcal{O}_t^{-1}\mathcal{D}_t\mathcal{O}_t^{-1}\| < C$.
3) A bound on $\mathcal{D}_t$ is easily obtained assuming boundedness of A_t and integrability of $\|W_t\| + \|Q_t\|$ over finite intervals. The main condition is the invertibility of $\mathcal{O}_t$.
4) The case $R_t \neq 0$ is actually covered via simple changes in A_t and Q_t.
5) Since P_t^{-1} satisfies
   $$\dot{P}_t^{-1} = -P_t^{-1}A_t - A_t^T P_t^{-1} + W_t - P_t^{-1}Q_t P_t^{-1}$$
   another application of this theorem leads to a lower bound on P_t based on the invertibility of the controllability matrix $\mathcal{C}_t$.
6) The solution to the equation for P_t^{-1} with initial value $P_0^{-1} = 0$ is smaller than $\mathcal{O}_t$; this implies that in the limit $P_0 \to \infty$, one has $P_t \geq \mathcal{O}_t^{-1}$. This is why the term O_t^{-1} cannot be avoided.

II. Counterexample

This example is made with $A_t = 0$. Consider

$$P_t = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{t+1} \end{pmatrix} \quad Q_t = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$W_t = \begin{pmatrix} 1 & t+1 \\ t+1 & (t+1)^2 + 1 \end{pmatrix}$$

then we have

$$P_t W_t P_t = \begin{pmatrix} 1 & 1 \\ 1 & 1 + (t+1)^{-2} \end{pmatrix} = Q_t - \dot{P}_t.$$

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