$*\tilde{\mathcal{H}}(q, \tilde{p}, t)$	$\partial\tilde{W}/\partial t$	q.m. energy
$\hat{H}(q, \hat{p}, t)$	as in (11)	q.m. energy operator, obtained by substituting $q, \hat{p}$ for x, p in $H(x, p, t)$
$L(x, v, t)$	$pv - H$	classical Lagrangian
$*\tilde{L}(q, \tilde{v}, t)$	$\tilde{p}\tilde{v} - \tilde{H}$	q.m. Lagrangian
m		mass
p	$-\partial W/\partial x$	classical momentum
$*\tilde{p}$	$-\partial\tilde{W}/\partial q$	q.m. momentum
$\hat{p}$	$-i\hbar\partial/\partial q$	q.m. momentum operator
v	$\partial H/\partial p$	classical velocity
$*\tilde{v}$	$\partial\tilde{H}/\partial\tilde{p}$	q.m. velocity
$V(x, t)$		classical potential energy
$*\tilde{V}(q, t)$		obtained by substituting q for x in $V(x, t)$
W	as in (1)	classical action integral with reversed sign
$*\tilde{W}$	as in (6)	q.m. action integral
ρ		density of physical particles on the real axis
ψ	$\exp(\tilde{W}/i\hbar)$	wave function satisfying Schrödinger's equation
$\underset{v \cdot p}{\mathrm{stat}}$		stationary value obtained by varying the functions v, p
$*\underset{\tilde{v}, \tilde{p}}{\mathrm{stat}}$		stationary value obtained by varying the functions $\tilde{v}, \tilde{p}$
$\underset{v, p}{\delta}$		variation of the succeeding expression with respect to v, p
$*\underset{\tilde{v}, \tilde{p}}{\delta}$		variation of the succeeding expression with respect to $\tilde{v}, \tilde{p}$.

REFERENCES

[1] C. Lanczos, *The Variational Principles of Mechanics*: Toronto University Press, 1964.

[2] H. H. Rosenbrock, "A stochastic variational principle for quantum mechanics," *Phys. Lett.*, vol. 110A, pp. 343–346, 1986.

[3] ——, "A variational treatment of quantum mechanics," *Proc. R. Soc. A*, vol. 450, pp. 417–437, 1995.

[4] ——, "A correction to a stochastic variational treatment of quantum mechanics," *Proc. R. Soc. A*, vol. 453, pp. 983–986, 1997.

[5] ——, "The definition of state in the stochastic variational treatment of quantum mechanics," *Phys. Lett. A*, vol. 254, pp. 307–313, 1999.

[6] R. E. Bellman and S. E. Dreyfus, *Applied Dynamic Programming*: Princeton University Press, 1964.

[7] L. Arnold, *Stochastic Differential Equations*. New York, NY: Wiley, 1974.

[8] M. Hesse, *Revolutions and Reconstructions in the Philosophy of Science*: Harvester, 1980.

[9] B. Barnes, *Scientific Knowledge and Sociological Theory*: Routledge and Kegan Paul, 1974.

[10] S. Drake, *Discoveries and Opinions of Galileo*. New York, NY: Doubleday, 1957, p. 164.

[11] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*. New York, NY: McGraw-Hill, 1965.

[12] L. S. Schulman, *Techniques and Applications of Path Integration*. New York, NY: Wiley-Interscience, 1981.

[13] B. D. Bramson, "Stochastic optimal control and command," DERA, Malvern, U.K., Rep. DERA/LSB/2CR980 635/1.0, 1999.

[14] J. Monod, *Chance and Necessity*: Collins, 1971.

[15] W. Yourgrau and S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory*: Pitman, 1968, pp. 173–174.

[16] A. Einstein, *Out of My Later Years*. New York, NY: Thames and Hudson, 1950, p. 114.

[17] R. Dawkins, *The Blind Watchmaker*. White Plains, NY: Longman, 1986.

[18] H. H. Rosenbrock, *Machines with a Purpose*. New York, NY: Oxford University Press, 1990.

Robust H_∞ Control of Uncertain Markovian Jump Systems with Time-Delay

Yong-Yan Cao and James Lam

***Abstract*—This correspondence is concerned with the robust stochastic stabilizability and robust H_∞ disturbance attenuation for a class of uncertain linear systems with time delay and randomly jumping parameters. The transition of the jumping parameters is governed by a finite-state Markov process. Sufficient conditions on the existence of a robust stochastic stabilizing and γ-suboptimal H_∞ state-feedback controller are presented using the Lyapunov functional approach. It is shown that a robust stochastically stabilizing H_∞ state-feedback controller can be constructed through the numerical solution of a set of coupled linear matrix inequalities.**

***Index Terms*—Jumping parameters, linear matrix inequality (LMI), linear uncertain systems, robust control, time-delay systems.**

I. INTRODUCTION

A great deal of attention has recently been devoted to the Markovian jump linear systems. This family of systems is modeled by a set of linear systems with the transitions between the models determined by a Markov chain taking values in a finite set. It was introduced by Krasovskii and Lidskii in 1961 [13] and may represent a large variety of processes, including those in production systems and economic problems. Developments in control engineering regarding applications, stability conditions, and optimal control problems for jump linear systems are reported in [1], [3], [8]–[10], [12], and [17]. On the other hand, time-delay systems have been studied extensively on the subject of stability and control over the years; see [4], [5], and [16] for instance. The problem of robust H_∞ control of linear uncertain systems with time delay has gathered much attention, and some sufficient conditions have been presented [6], [7], [11], [15].

In this correspondence, we study the robust stochastic stabilizability and robust H_∞ disturbance attenuation for a class of uncertain linear

Manuscript received October 5, 1998; revised March 15, 1999. Recommended by Associate Editor, J. C. Spall. This work was supported in part by the University of Hong Kong, under a CRCG Grant, and the National Natural Science Foundation of China under Grant 69604007.

Y.-Y. Cao is with the National Lab of Industrial Control Technology, Institute of Industrial Process Control, Zhejiang University, Hangzhou, 310027 China (e-mail: yycao@iipc.zju.edu.cn).

J. Lam is with the Department of Mechanical Engineering, University of Hong Kong, Hong Kong (e-mail: jlam@hku.hk).

Publisher Item Identifier S 0018-9286(00)00807-2.

time-delay systems with Markovian jumping parameters using the stochastic Lyapunov functional approach first developed by Kushner [14]. Mariton [17] used the stochastic Lyapunov function approach to obtain a sufficient condition for the mean square stability and the almost sure stability of the linear jump systems without delay. In [10] and [12], the stochastic stability, stabilizability, and controllability are studied based on the same methodology, and in [2], the authors discussed stochastic stability for a class of uncertain jump linear systems with time delay. In general, these results are in the form of a set of coupled algebraic Riccati equations that is difficult to solve. To facilitate the solution process, the linear matrix inequality (LMI) approach will be employed in the present development.

II. Problem Statement

In the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions. The notation $M > (\geq, <, \leq) 0$ is used to denote a symmetric positive-definite (positive-semidefinite, negative, negative-semidefinite) matrix. $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalue of the corresponding matrix, respectively. $\|\cdot\|$ denotes the Euclidean norm for vectors or the spectral norm of matrices. $\boldsymbol{E}[\cdot]$ stands for the mathematical expectation.

Consider the class of stochastic, uncertain linear state-delay systems with Markovian jumping parameters

$$\begin{aligned}\dot{x}(t) = {} & \tilde{A}_1(t, r(t))x(t) + \tilde{A}_2(t, r(t))x(t-\tau) \\ & + \tilde{B}_1(t, r(t))u(t) + B_2(r(t))w(t) \qquad (1)\end{aligned}$$

$$\begin{aligned}z(t) = {} & \tilde{C}_1(t, r(t))x(t) + \tilde{C}_2(t, r(t))x(t-\tau) \\ & + \tilde{D}_1(t, r(t))u(t) + D_2(r(t))w(t) \qquad (2)\end{aligned}$$

$$x(t) = \psi(t), \qquad t \in [-\tau, 0], \; r(0) = r_0 \qquad (3)$$

where

$x(t) \in \mathbb{R}^n$	system state;
$w(t) \in \mathbb{R}^q$	exogenous disturbance input;
$u(t) \in \mathbb{R}^m$	control input;
$z(t) \in \mathbb{R}^p$	output to be controlled.

$\tilde{A}_1(t, r(t))$, $\tilde{A}_2(t, r(t))$, $\tilde{B}_1(t, r(t))$, $B_2(r(t))$, $\tilde{C}_1(t, r(t))$, $\tilde{C}_2(t, r(t))$, $\tilde{D}_1(t, r(t))$, and $D_2(r(t))$ are matrix functions of the random jumping process $\{r(t)\}$. $r(t)$ is a finite-state Markov jump process representing the system mode; that is, $r(t)$ takes discrete values in a given finite set $\mathcal{S} = \{1, 2, \cdots, s\}$. Let $\Pi = [\pi_{ij}]$, where $i, j = 1, 2, \cdots, s$, denote the transition probability matrix with

$$\Pr\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta), & i \neq j \\ 1 + \pi_{ii}\Delta + o(\Delta), & i = j \end{cases} \qquad (4)$$

where $\Delta > 0$, $\pi_{ij} \geq 0$ for $i \neq j$, $\pi_i \triangleq \pi_{ii} \leq 0$ with $\sum_{j=1, j\neq i}^{s} \pi_{ij} = -\pi_{ii}$ for each mode i, $i = 1, 2, \cdots, s$, and $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$. τ is the constant delay time of the state in the system. $\psi(t)$ is a vector-valued initial continuous function defined on the interval $[-\tau, 0]$, and $r_0 \in \mathcal{S}$ are the initial conditions of the continuous state and the mode, respectively. To simplify the notation, $M(t, r)$ will be denoted by $M_r(t)$. For instance, $\tilde{A}_1(t, r)$ is denoted by $\tilde{A}_{1r}$, and so on.

Time-varying uncertainties may appear in these matrices; that is

$$\tilde{A}_{1r}(t) = A_{1r} + \Delta A_{1r}(t), \; \tilde{A}_{2r}(t) = A_{2r} + \Delta A_{2r}(t) \qquad (5)$$

$$\tilde{B}_{1r}(t) = B_{1r} + \Delta B_{1r}(t), \; \tilde{C}_{1r}(t) = C_{1r} + \Delta C_{1r}(t) \qquad (6)$$

$$\tilde{C}_{2r}(t) = C_{2r} + \Delta C_{2r}(t), \; \tilde{D}_{1r}(t) = D_{1r} + \Delta D_{1r}(t) \qquad (7)$$

where A_{1r}, A_{2r}, B_{1r}, C_{1r}, C_{2r}, and D_{1r} are governed only by the Markovian jump process and $\Delta A_{1r}(t)$, $\Delta A_{2r}(t)$, $\Delta B_{1r}(t)$, $\Delta C_{1r}(t)$, $\Delta C_{2r}(t)$, and $\Delta D_{1r}(t)$ are real-valued functions representing time-varying parameter uncertainties. We assume that the uncertainties are norm-bounded and can be described as

$$\begin{aligned}&\begin{bmatrix} \Delta A_{1i}(t) & \Delta A_{2i}(t) & \Delta B_{1i}(t) \\ \Delta C_{1i}(t) & \Delta C_{2i}(t) & \Delta D_{1i}(t) \end{bmatrix} \\ &\quad = \begin{bmatrix} E_{1i} \\ E_{2i} \end{bmatrix} F_i(t) [\, H_{1i} \quad H_{2i} \quad H_{3i} \,], \qquad \text{when } r(t) = i \qquad (8)\end{aligned}$$

where $E_{1i} \in \mathbb{R}^{n \times n_f}$, $E_{2i} \in \mathbb{R}^{p \times n_f}$, and H_{1i}, $H_{2i} \in \mathbb{R}^{n_f \times n}$, $H_{3i} \in \mathbb{R}^{n_f \times m}$ are known constant matrices for each $i \in \mathcal{S}$ and $F_i(t) \in \mathbb{R}^{n_f \times n_f}$ are unknown matrix functions satisfying

$$F_i^T(t) F_i(t) \leq I, \qquad \forall\, i \in \mathcal{S}.$$

It is assumed the elements of $F_i(t)$ are Lebesgue measurable. When $F_i(t) \equiv 0$, then system (1)–(4) is referred to as a *nominal jump linear system*. It is said to be a *free* system if $u(t) \equiv 0$.

In this correspondence, we will assume that for all $\delta \in [-\tau, 0]$, a scalar $\varepsilon > 0$ exists such that

$$\|x(t+\delta)\| \leq \varepsilon \|x(t)\|. \qquad (9)$$

As was indicated by [16], this assumption does not represent a restriction because ε can be chosen arbitrarily.

This correspondence is concerned with the design of a robust state-feedback controller

$$u(t) = -K_i x(t), \qquad \text{when } r(t) = i \qquad (10)$$

where K_i is constant for each value $i \in \mathcal{S}$ such that the closed-loop system

$$\begin{aligned}\dot{x}(t) = {} & (\hat{A}_{1i} + E_{1i}F_i\hat{H}_{1i})x(t) + (A_{2i} + E_{1i}F_iH_{2i})x(t-\tau) \\ & + B_{2i}w(t) \qquad (11)\end{aligned}$$

$$\begin{aligned}z(t) = {} & (\hat{C}_{1i} + E_{2i}F_i\hat{H}_{1i})x(t) + (C_{2i} + E_{2i}F_iH_{2i})x(t-\tau) \\ & + D_{2i}w(t) \qquad (12)\end{aligned}$$

where $\hat{A}_{1i} \triangleq A_{1i} - B_{1i}K_i$, $\hat{H}_{1i} \triangleq H_{1i} - H_{3i}K_i$, and $\hat{C}_{1i} \triangleq C_{1i} - D_{1i}K_i$ is stochastically stable with γ-disturbance attenuation. It is assumed that the controller has complete access to the state variables $\{x(t)\}$ and the jumping process $\{r(t)\}$. In the following, we will denote by $x(t, \psi, r_0, u)$ the solution of system (1)–(4) at time t under the initial conditions $\psi(t)$ and r_0, and the control input $u(t)$, and x_0 represents $x(t, \psi, r_0, u)$ at $t = 0$.

Definition 1: The free nominal jump linear system is said to be *stochastically stabilizable* if, when $w(t) \equiv 0$, for all finite $\psi(t) \in \mathbb{R}^n$ defined on $[-\tau, 0]$ and initial mode $r_0 \in \mathcal{S}$, a linear feedback control law (10) exists satisfying

$$\lim_{T \to \infty} \boldsymbol{E}\left\{ \int_0^T x^T(t, \psi, r_0, u) x(t, \psi, r_0, u)\, dt \,\middle|\, \psi, r_0 \right\} \leq x_0^T \tilde{M} x_0$$

for some $\tilde{M} > 0$. The uncertain system (1)–(4) is said to be *robust stochastically stabilizable* if it is stochastically stabilizable for all possible uncertainty $\|F_i(t)\| \leq 1$.

This definition is similar to that of stochastic stabilizability of jump linear systems without time delay [10], [12]. Under the above definition, stochastic stabilizability of a system means that a linear state-feedback control law exists that drives the x state from any given initial condition (ψ, t_0) asymptotically to the origin, in the mean square sense, which implies the asymptotic stability of the closed-loop system.

Definition 2: For a given control law (10) and a real number $\gamma > 0$, the nominal jump linear system is said to be *stochastically stabilizable with γ-disturbance attenuation* if for every $T > 0$ and for every piecewise continuous function $w: [0, \infty) \to \mathbb{R}^q$, the closed-loop system (1)–(4), (11) is asymptotically stable and the response $z: [0, \infty) \to \mathbb{R}^p$ satisfies

$$\int_0^T z^T(t)z(t)\,dt < \gamma^2 \int_0^T w^T(t)w(t)\,dt. \tag{13}$$

The uncertain system (1)–(4) is said to be *robust stochastically stabilizable with γ-disturbance attenuation* if it is stochastically stabilizable with γ-disturbance attenuation for all possible uncertainty $\|F_i(t)\| \le 1$.

Let

$$\|w\|_2 = \boldsymbol{E}\left\{\int_0^T w^T(t)w(t)\,dt\right\}^{1/2}$$

and

$$\|z\|_2 = \boldsymbol{E}\left\{\int_0^T z^T(t)z(t)\,dt\right\}^{1/2}$$

and let T_{zw} denote the system from the exogenous input $w(t)$ to the controlled output $z(t)$; then, the H_∞-norm of T_{zw} is

$$\|T_{zw}\|_\infty = \sup_{w(t)\in\mathcal{L}_2(0,\,\infty)} \frac{\|z\|_2}{\|w\|_2}.$$

Hence, (13) implies $\|T_{zw}\|_\infty < \gamma$. In other words, γ-disturbance attenuation implies γ-suboptimal H_∞ control. The following matrix inequalities are essential for the proofs in the next sections [20].

Lemma 1: For any vectors $x, y \in \mathbb{R}^n$, matrices $A, P \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times n_f}$, $E \in \mathbb{R}^{n_f\times n}$, and $F \in \mathbb{R}^{n_f\times n_f}$ with $P > 0$, $\|F\| \le 1$, and scalar $\varepsilon > 0$, we have:

i) $2x^T y \le x^T P^{-1} x + y^T P y$;

ii) $DFE + E^T F^T D^T \le \varepsilon^{-1} DD^T + \varepsilon E^T E$;

iii) if $\varepsilon I - EPE^T > 0$,

$$\begin{aligned}&(A+DFE)P(A+DFE)^T \\ &\quad\le APA^T + APE^T\left(\varepsilon I - EPE^T\right)^{-1} EPA^T + \varepsilon DD^T;\end{aligned}$$

iv) if $P - \varepsilon DD^T > 0$,

$$\begin{aligned}&(A+DFE)^T P^{-1}(A+DFE) \\ &\quad\le A^T\left(P - \varepsilon DD^T\right)^{-1} A + \varepsilon^{-1} E^T E.\end{aligned}$$

III. Robust Stochastic Stabilizability

In this section, we present a sufficient condition for the robust stochastic stabilizability of the jump linear system (1)–(4) with $w(t) \equiv 0$.

Theorem 1: The nominal jump linear system is stochastically stabilizable if $Q > 0$, $P_i > 0$, and K_i, $i = 1, \cdots, s$ exist, satisfying the coupled matrix inequalities

$$M_i \triangleq \begin{bmatrix} \hat{A}_{1i}^T P_i + P_i \hat{A}_{1i} + \sum_{j=1}^{s} \pi_{ij} P_j + Q & P_i A_{2i} \\ A_{2i}^T P_i & -Q \end{bmatrix} < 0, \quad i = 1, \cdots, s. \tag{14}$$

Proof: Let the mode at time t be i; that is, $r(t) = i \in \mathcal{S}$. Consider feedback control $u(t) = -K_i x(t)$, $t \ge 0$; then, (1) becomes

$$\dot{x}(t) \equiv \hat{A}_{1i} x(t) + A_{2i} x(t-\tau) \tag{15}$$

with $x(t) = \psi(t)$, $t \in [-\tau, 0]$, $r(0) = r_0$. In the following, we will simply use $x(t)$ to denote the solution $x(t, \psi, r_0, u)$ under the initial condition $\psi(t)$ and r_0 with control $u(t)$.

Take the stochastic Lyapunov functional $V(\cdot): \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ to be

$$\begin{aligned}&V(x(t), r(t) = i) \\ &\quad\equiv V(x, i) \\ &\quad\triangleq x^T(t) P_i x(t) + \int_{-\tau}^{0} x^T(t+\theta) Q x(t+\theta)\,d\theta\end{aligned} \tag{16}$$

where

Q constant

P_i constant for each i.

The weak infinitesimal operator $\mathcal{A}$ [12], [14] of the stochastic process $\{r(t), x(t)\}$, $t \ge 0$, is given by

$$\begin{aligned}&\mathcal{A}V(x(t), r(t)) \\ &\quad= \lim_{\Delta\to 0} \frac{1}{\Delta}[\boldsymbol{E}\{V(x(t+\Delta), r(t+\Delta)|x(t), r(t) = i\} \\ &\qquad\quad - V(x(t), r(t) = i)] \\ &\quad= x^T(t)\left[\hat{A}_{1i}^T P_i + P_i \hat{A}_{1i} + \sum_{j=1}^{s} \pi_{ij} P_j\right] x(t) \\ &\qquad + x^T(t-\tau) A_{2i}^T P_i x(t) + x^T(t) P_i A_{2i} x(t-\tau) \\ &\qquad + x^T(t) Q x(t) - x^T(t-\tau) Q x(t-\tau) \\ &\quad= x_e^T(t) M_i x_e(t)\end{aligned}$$

where $x_e(t) \triangleq [x^T(t) \;\; x^T(t-\tau)]^T$. With the assumption in (9), we have $\|x(t)\| \le \|x_e(t)\|$. Note that $M_i < 0$ and $P_i > 0$; hence, we have for $x \ne 0$

$$\begin{aligned}\frac{\mathcal{A}V(x, i)}{V(x, i)} &\le \frac{x_e^T(t) M_i x_e(t)}{x^T(t) P_i x(t) + \int_{-\tau}^{0} x^T(t+\Theta) Q_x(t+\Theta)\,d\Theta} \\ &= -\frac{x_e^T(t)(-M_i) x_e(t)}{x^T(t) P_i x(t) + \int_{-\tau}^{0} x^T(t+\Theta) Q_x(t+\Theta)\,d\Theta} \\ &\le -\min_{i\in\mathcal{S}}\left\{\frac{\lambda_{\min}(-M_i)}{\lambda_{\max}(P_i) + \tau\varepsilon^2 \lambda_{\max}(Q)}\right\}.\end{aligned}$$

Define

$$\alpha \triangleq \min_{i\in\mathcal{S}}\left\{\frac{\lambda_{\min}(-M_i)}{\lambda_{\max}(P_i) + \tau\varepsilon^2\lambda_{\max}(Q)}\right\}.$$

We have $\alpha > 0$ and $\mathcal{A}V(x, i) \le -\alpha V(x, i)$. Then, by Dynkin's formula (see [14])

$$\begin{aligned}&\boldsymbol{E}\{V(x(t), r(t))\} - V(x_0, r_0) \\ &\quad= \boldsymbol{E}\left\{\int_0^t \mathcal{A}V(x(s), r(s))\,ds\right\} \\ &\quad\le -\alpha \int_0^t \boldsymbol{E}\{V(x(s), r(s))\}\,ds.\end{aligned}$$

The Gronwell–Bellman lemma gives $\boldsymbol{E}\{V(x(t), r(t))\} \le \exp(-\alpha t) V(x_0, r_0)$. Because $Q > 0$, then

$$\boldsymbol{E}\left\{\int_{-\tau}^{0} x^T(t+s) Q x(t+s)\,ds\right\} > 0.$$

Thus, for all $r_0 \in \mathcal{S}$

$$\begin{aligned}&\boldsymbol{E}\left\{x^T(t) P_i x(t)|\psi, r_0\right\} \\ &\quad= \boldsymbol{E}\{V(x, i)|\psi, r_0\}\end{aligned}$$

$$
\begin{aligned}
&- \boldsymbol{E}\left\{ \int_{-\tau}^{0} x^T(t+s) Q_i x(t+s)\, ds \,\Big|\, \psi,\, r_0 \right\} \\
&\le \exp(-\alpha t) V(x_0,\, i).
\end{aligned}
$$

Then

$$
\begin{aligned}
\boldsymbol{E}&\left\{ \int_{0}^{T} x^T(t) P_i x(t)\, dt \,\Big|\, \psi,\, r_0 \right\} \\
&< \int_0^T \exp(-\alpha t)\, dt V(x_0,\, i) \\
&= -\frac{1}{\alpha}\, [\exp(-\alpha T) - 1] V(x_0,\, i).
\end{aligned}
$$

Taking the limit as $T \to \infty$, we have

$$
\begin{aligned}
&\lim_{T\to\infty} \boldsymbol{E}\left\{ \int_{0}^{T} x^T(t) P_i x(t)\, dt \,\Big|\, \psi,\, r_0 \right\} \\
&\quad < \lim_{T\to\infty} \left\{ -\frac{1}{\alpha}\, [\exp(-\alpha T) - 1] V(x_0,\, i) \right\} \\
&\quad \le \frac{1}{\alpha}\, x_0^T (\lambda_{\max}(P_i) + \varepsilon^2 \tau \lambda_{\max}(Q)) I) x_0.
\end{aligned}
$$

Because $P_i > 0$ for each $i \in \mathcal{S}$, this means

$$
\lim_{T\to\infty} \boldsymbol{E}\left\{ \int_{0}^{T} x^T(t) x(t)\, dt \,\Big|\, \psi,\, r_0 \right\} \le x_0^T \tilde{P} x_0
$$

where

$$
\tilde{P} = \max_{i\in\mathcal{S}} \left\{ \frac{\lambda_{\max}(P_i) + \varepsilon^2 \tau \lambda_{\max}(Q)}{\alpha \lambda_{\min}(P_i)} \right\} I
$$

which implies that the closed-loop system under control law (10) is stochastically stable. ∎

From Schur complement, $M_i < 0$ if and only if

$$
\begin{aligned}
&(A_{1i} - B_{1i}K_i)^T P_i + P_i(A_{1i} - B_{1i}K_i) \\
&\quad + \sum_{j=1}^{s} \pi_{ij} P_j + Q + P_i A_{2i} Q^{-1} A_{2i}^T P_i < 0.
\end{aligned} \tag{17}
$$

Define $X_i \triangleq P_i^{-1}, Y_i \triangleq K_i X_i$. Pre- and postmultiplying the last matrix inequality by X_i, it is easy to find that $M_i < 0$, $i = 1, \cdots, s$ if and only if for $i = 1, \cdots, s$

$$
\begin{aligned}
&X_i A_{1i}^T + A_{1i} X_i - Y_i^T B_{1i}^T - B_{1i} Y_i \\
&\quad + \sum_{j=1}^{s} \pi_{ij} X_i X_j^{-1} X_i + X_i Q X_i + A_{2i} Q^{-1} A_{2i}^T < 0
\end{aligned} \tag{18}
$$

which are equivalent to the following LMI's

$$
\begin{bmatrix} \Delta_i + A_{2i} R A_{2i}^T & \Xi_i \\ \Xi_i^T & -\Upsilon_i \end{bmatrix} < 0, \qquad i = 1, \cdots, s \tag{19}
$$

where

$$
\Delta_i \triangleq X_i A_{1i}^T + A_{1i} X_i - Y_i^T B_{1i}^T - B_{1i} Y_i + \pi_i X_i \tag{20}
$$

$$
\begin{aligned}
\Xi_i \triangleq [\, &\sqrt{\pi_{i1}} X_i \quad \cdots \quad \sqrt{\pi_{i,\,i-1}} X_i \\
&\sqrt{\pi_{i,\,i+1}} X_i \quad \cdots \quad \sqrt{\pi_{is}} X_i \quad X_i \,]
\end{aligned} \tag{21}
$$

$$
\Upsilon_i \triangleq \operatorname{diag}[\, X_1 \quad \cdots \quad X_{i-1} \quad X_{i+1} \quad \cdots \quad X_s \quad R \,] \tag{22}
$$

$$
R \triangleq Q^{-1}. \tag{23}
$$

The foregoing analysis leads to a stochastic stabilizability result in terms of LMI's.

Theorem 2: The nominal jump linear system is stochastically stabilizable if $R > 0$, $X_i > 0$, and Y_i, $i = 1, \cdots, s$ exist, satisfying the coupled LMI's (19). A stabilizing controller is constructed as $K_i = Y_i X_i^{-1}$.

Remark 1: The proof of Theorem 1 is similar to the corresponding result in [12, Theorem 1]]. If the nominal jump linear system has zero time delay, that is, $A_{2i} = 0$ for all $i \in \mathcal{S}$, condition (14) on stochastic stabilizability becomes

$$
\begin{aligned}
&(A_{1i} - B_{1i}K_i)^T P_i + P_i(A_{1i} - B_{1i}K_i) \\
&\quad + \sum_{j=1}^{s} \pi_{ij} P_j < 0, \qquad i = 1, \cdots, s
\end{aligned} \tag{24}
$$

which are equivalent to the following LMI's

$$
\begin{bmatrix} \Delta_i & \Xi_i \\ \Xi_i^T & -\Upsilon_i \end{bmatrix} < 0, \qquad i = 1, \cdots, s \tag{25}
$$

given in [19]. We can show that it is a necessary and sufficient condition for the stochastic stabilizability of jump linear systems without time delay [12, Theorem 1]. Obviously, if $P_i > 0$ and K_i exist, satisfying the coupled matrix inequality (24), then some $N_i > 0$ exists such that the following Lyapunov equations are solvable:

$$
\begin{aligned}
&(A_{1i} - B_{1i}K_i)^T P_i + P_i(A_{1i} - B_{1i}K_i) \\
&\quad + \sum_{j=1}^{s} \pi_{ij} P_j = -N_i, \qquad i = 1, \cdots, s
\end{aligned} \tag{26}
$$

which is the coupled equations for the stochastically stabilizability in [12]. It is very difficult, however, to solve the above-coupled matrix equations [1], [18]. Fortunately, the LMI's (19) and (25) can be numerically solved efficiently.

Remark 2: If we consider free jump linear systems, then the feasibility of LMI's (19) with $B_{1i} = 0$ is sufficient for stochastic stability of this class of time-delay systems. It can also be extended to test the stochastic stability of free jump linear systems without time delay [10].

Theorem 3: The uncertain jump linear system (1)–(4) is robust stochastically stabilizable if scalars $\alpha_i > 0$, $\beta_i > 0$ and matrices $R > 0$, $X_i > 0$, and Y_i, $i = 1, \cdots, s$ exist, satisfying the coupled matrix inequalities

$$
\begin{bmatrix}
\hat{\Delta}_i & A_{2i} R H_{2i}^T & H_{1i}^T & \Xi_i \\
H_{2i} R A_{2i}^T & -\beta_i I + H_{2i} R H_{2i}^T & 0 & 0 \\
H_{1i} & 0 & -\alpha_i I & 0 \\
\Xi_i^T & 0 & 0 & -\Upsilon_i
\end{bmatrix} < 0, \quad i = 1, \cdots, s \tag{27}
$$

where $\hat{\Delta}_i \triangleq \Delta_i + A_{2i} R A_{2i}^T + \alpha_i E_{1i} E_{1i}^T + \beta_i E_{1i} E_{1i}^T$.

Proof: From Theorem 1, the uncertain closed-loop system is stochastically stable, if for each mode $i \in \mathcal{S}$ and $\|F_i\| \le 1$, the following coupled matrix inequalities hold for $i = 1, \cdots, s$:

$$
\begin{aligned}
&X_i \left(\hat{A}_{1i} + E_{1i} F_i \hat{H}_{1i} \right)^T + \left(\hat{A}_{1i} + E_{1i} F_i \hat{H}_{1i} \right) X_i \\
&\quad + \sum_{j=1}^{s} \pi_{ij} X_i X_j^{-1} X_i + X_i Q X_i + \tilde{A}_{2i} R \tilde{A}_{2i}^T < 0.
\end{aligned} \tag{28}
$$

On the other hand, from Lemma 1, LMI's (28) hold if scalars $\alpha_i > 0$, $\beta_i > 0$ exist such that

$$
\begin{aligned}
&\overline{\Delta}_i + \alpha_i E_{1i} E_{1i}^T + \alpha_i^{-1} H_{1i}^T H_{1i} + A_{2i} R H_{2i}^T \\
&\quad \cdot \left(\beta_i I - H_{2i} R H_{2i}^T \right)^{-1} H_{2i} R A_{2i}^T + \beta_i E_{1i} E_{1i}^T < 0 \\
&\beta_i I - H_{2i} R H_{2i}^T > 0
\end{aligned}
$$

where $\overline{\Delta}_i \triangleq \Delta_i + \sum_{j=1,\, j\neq i}^{s} \pi_{ij} X_i X_j^{-1} X_i + X_i Q X_i + A_{2i} R A_{2i}^T$, which are equivalent to (27) from the Schur complement. ∎

IV. Robust H_∞ Disturbance Attenuation

In this section, we consider robust H_∞ disturbance attenuation for the uncertain jump linear state-delay systems.

Theorem 4: For the nominal jump linear system, a state feedback control (10) exists such that the closed-loop system possesses the γ-disturbance attenuation property; that is, $\|z\|_2 < \gamma\|w\|_2$ for all $w \in \mathcal{L}_2[0, \infty]$, $w \neq 0$, if $Q > 0$ and $P_i > 0$, $i = 1, \cdots, s$ exist, satisfying the following coupled matrix inequalities:

$$\Theta_i \triangleq \begin{bmatrix} \hat{A}_i & P_iA_{2i} + \hat{C}_{1i}^T + C_{2i} & P_iB_{2i} + \hat{C}_{1i}^T + D_{2i} \\ A_{2i}^TP_i + C_{2i}^T + \hat{C}_{1i} & -Q + C_{2i}^T + C_{2i} & C_{2i}^TD_{2i} \\ B_{2i}^TP_i + D_{2i}^T + \hat{C}_{1i} & D_{2i}^TC_{2i} & -(\gamma^2I - D_{2i}^TD_{2i}) \end{bmatrix} < 0 \tag{29}$$

as shown in (29) for $i = 1, \cdots, s$, where $\hat{A}_i \triangleq \hat{A}_{1i}^TP_i + P_i\hat{A}_{1i} + \sum_{j=1}^s \pi_{ij}P_j + Q + \hat{C}_{1i}^T\hat{C}_{1i}$.

Proof: Let the mode at time t be i; that is, $r(t) = i \in \mathcal{S}$. Consider feedback control $u(t) = -K_ix(t)$ for $t \geq 0$. Then, the nominal closed-loop system becomes

$$\dot{x}(t) = \hat{A}_{1i}x(t) + A_{2i}x(t-\tau) + B_{2i}w(t) \tag{30}$$

$$z(t) = \hat{C}_{1i}x(t) + C_{2i}x(t-\tau) + D_{2i}w(t). \tag{31}$$

Choose the stochastic Lyapunov functional $V(\cdot)\colon \mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ as in (16). Then

$$\mathcal{A}V(x(t), r(t)) = x_e^T(t)M_ix_e(t) + w^T(t)B_{2i}^TP_ix(t) + x^T(t)P_iB_{2i}w(t).$$

Notice that, when the disturbance input is zero, that is, $w(t) = 0$, based on Theorem 1 and (29), we have $\mathcal{A}V(x(t), r(t)) < 0$, which ensures the asymptotic stability of the closed-loop system.

In the following, we assume zero initial condition, that is, $x(t) = 0$ for $t \in [-\tau, 0]$, and define

$$J_T \triangleq \boldsymbol{E}\left\{\int_0^T \left[z^T(t)z(t) - \gamma^2w^T(t)w(t)\right]dt\right\}.$$

From Dynkin's formula [14] and the fact that $x_0 = x(0, 0, r_0, u) = 0$, we have

$$\boldsymbol{E}\{V(x(T), r(T)\} = \boldsymbol{E}\left\{\int_0^T \mathcal{A}V(x(s), r(s))\,ds\right\}$$

since, $V(x_0, r_0) = 0$. Then, for any nonzero $w(t) \in \mathcal{L}_2[0, \infty]$

$$J_T = \boldsymbol{E}\left\{\int_0^T \left[z^T(t)z(t) - \gamma^2w^T(t)w(t) + \mathcal{A}V(x(t), r(t)\right]dt\right\} - \boldsymbol{E}\{V(x(T), r(T)\}.$$

So

$$J_T \leq \boldsymbol{E}\left\{\int_0^T \left[z^T(t)z(t) - \gamma^2w^T(t)w(t) + \mathcal{A}V(x(t), r(t))\right]dt\right\} = \boldsymbol{E}\left\{\int_0^T \left[\sigma^T(t)\Theta_i\sigma(t)\right]dt\right\} < 0$$

where $\sigma(t) \triangleq [x^T(t) \;\; x^T(t-\tau) \;\; w^T(t)]^T$. Therefore, the dissipativity inequality (13) holds for all $T > 0$. In other words, we have that $z \in \mathcal{L}_2[0, \infty)$, for any nonzero $w \in \mathcal{L}_2[0, \infty)$, and $\|z\|_2 < \gamma\|w\|_2$. ■

Theorem 4 can be rewritten in the form of LMI's. From the Schur complement, it is easy to find that the coupled matrix inequalities in (29) are equivalent to the following matrix inequalities:

$$\Lambda_i \triangleq \begin{bmatrix} \hat{A}_{1i}^TP_i + P_i\hat{A}_{1i} + \sum_{j=1}^s \pi_{ij}P_j + Q & P_iA_{2i} & P_iB_{2i} & \hat{C}_{1i}^T \\ A_{2i}^TP_i & -Q & 0 & C_{2i}^T \\ B_{2i}^TP_i & 0 & -\gamma^2I & D_{2i}^T \\ \hat{C}_{1i} & C_{2i} & D_{2i} & -I \end{bmatrix} < 0 \tag{32}$$

for $i = 1, \cdots, s$. Obviously, the matrix M_i is a principal submatrix of Λ_i. So, if $\Lambda_i < 0$, then $M_i < 0$, which proves that Theorem 3 ensures Theorem 1 again. Let $X_i \triangleq P_i^{-1}$ and $Y_i \triangleq K_iX_i$, and define $T_i = \mathrm{diag}(X_i, I, I, I)$. Pre- and postmultiplying (32) by T_i, we find

$$\begin{bmatrix} \Omega_i + A_{2i}Q^{-1}A_{2i}^T & (C_{1i}X_i - D_{1i}Y_i)^T + A_{2i}Q^{-1}C_{2i}^T & B_{2i} \\ C_{1i}X_i - D_{1i}Y_i + C_{2i}Q^{-1}A_{2i}^T & -I + C_{2i}Q^{-1}C_{2i}^T & D_{2i} \\ B_{2i}^T & D_{2i}^T & -\gamma^2I \end{bmatrix} < 0 \tag{33}$$

$$\begin{bmatrix} \Delta_i + A_{2i}RA_{2i}^T & (C_{1i}X_i - D_{1i}Y_i)^T + A_{2i}RC_{2i}^T & B_{2i} & \Xi_i \\ C_{1i}X_i - D_{1i}Y_i + C_{2i}RA_{2i}^T & -I + C_{2i}RC_{2i}^T & D_{2i} & 0 \\ B_{2i}^T & D_{2i}^T & -\gamma^2I & 0 \\ \Xi_i^T & 0 & 0 & -\Upsilon_i \end{bmatrix} < 0 \tag{34}$$

$$\begin{bmatrix} \Delta_i + \alpha_iE_{1i}E_{1i}^T + A_{2i}RA_{2i}^T & * & * & * & * & * \\ C_{1i}X_i - D_{1i}Y_i + C_{2i}RA_{2i}^T & Z_{1i} & * & * & * & 0 \\ H_{1i}X_i - H_{3i}Y_i + H_{2i}RA_{2i}^T & H_{2i}RC_{2i}^T & Z_{2i} & * & 0 & 0 \\ H_{1i}X_i - H_{3i}Y_i + H_{2i}RA_{2i}^T & H_{2i}RC_{2i}^T & H_{2i}RH_{2i}^T & Z_{3i} & 0 & 0 \\ B_{2i}^T & D_{2i}^T & 0 & 0 & -\gamma^2I & 0 \\ \Xi_i^T & 0 & 0 & 0 & 0 & -\Upsilon_i \end{bmatrix} < 0. \tag{35}$$

that the coupled matrix inequalities (32) are equivalent to the following matrix inequalities:

$$\begin{bmatrix} \Omega_i & A_{2i} & B_{2i} & (C_{1i}X_i - D_{1i}Y_i)^T \\ A_{2i}^T & -Q & 0 & C_{2i}^T \\ B_{2i}^T & 0 & -\gamma^2 I & D_{2i}^T \\ C_{1i}X_i - D_{1i}Y_i & C_{2i} & D_{2i} & -I \end{bmatrix} < 0, \qquad i = 1, \cdots, s$$

where

$$\Omega_i \triangleq X_i A_{1i}^T + A_{1i}X_i - Y_i^T B_{1i}^T - B_{1i}Y_i + \sum_{j=1}^{s} \pi_{ij} X_i X_j^{-1} X_i + X_i Q X_i.$$

From the Schur complement, we find the above matrix inequalities are equivalent to (33), given at the bottom of the previous page, for $i = 1, \cdots, s$, which are in turn equivalent to the following LMI's as shown in (34), given at the bottom of the previous page, for $i = 1, \cdots, s$, where Δ_i, Ξ_i, Υ_i, and R are defined in (20)–(23), respectively.

From the above derivations, the γ-disturbance attenuation result is summarized in the following theorem involving LMI's.

Theorem 5: For the nominal jump linear system, a state feedback control law (10) exists such that the closed-loop system is stochastically stable with γ-disturbance attenuation; that is, $\|z\|_2 < \gamma\|w\|_2$ for all $w \in \mathcal{L}_2[0, \infty]$, $w \neq 0$, if $R > 0$, $X_i > 0$, and Y_i, $i = 1, \cdots, s$ exist, satisfying the coupled LMI's shown in (34). A stabilizing controller to provide γ-disturbance attenuation can be constructed as $K_i = Y_i X_i^{-1}$.

Theorem 6: For the uncertain jump linear system, a state feedback control (10) exists such that the closed-loop system is robust stochastically stable with γ-disturbance attenuation; that is, $\|z\|_2 < \gamma\|w\|_2$ for all $w \in \mathcal{L}_2[0, \infty]$, $w \neq 0$, if scalars $\alpha_i > 0$, $\delta_i > 0$ and matrices $X_i > 0$, $R > 0$, and Y_i, $s = 1, \cdots, s$ exist, satisfying the coupled LMI's as shown in (35), given at the bottom of the previous page, for $i = 1, \cdots, s$, where $*$ are entries readily inferred by symmetry and

$$\begin{aligned} Z_{1i} &\triangleq -I + C_{2i} R C_{2i}^T + \delta_i E_{2i} E_{2i}^T \\ Z_{2i} &\triangleq -\alpha_i I + H_{2i} R H_{2i}^T \\ Z_{3i} &\triangleq -\delta_i I + H_{2i} R H_{2i}^T. \end{aligned}$$

A stabilizing controller to provide γ-disturbance attenuation can be constructed as $K_i \triangleq Y_i X_i^{-1}$.

Proof: Let the mode at time t be i; that is, $r(t) = i \in \mathcal{S}$. Consider the closed-loop system (11) and (12). From Theorem 5, it is stochastically stable with γ-disturbance attenuation if for all possible uncertainties $\|F_i(t)\| \leq 1$, $Q > 0$ and $X_i > 0$, $i = 1, \cdots, s$ exist, satisfying the following coupled matrix inequalities:

$$\begin{bmatrix} \hat{\Omega}_i & A_{2i} + E_{1i}F_iH_{2i} & B_{2i} \\ (A_{2i} + E_{1i}F_iH_{2i})^T & -Q & 0 \\ B_{2i}^T & 0 & -\gamma^2 I \end{bmatrix} + \overline{C}_i^T \overline{C}_i < 0 \tag{36}$$

for $i = 1, \cdots, s$, where

$$\begin{aligned} \hat{\Omega}_i \triangleq{}& X_i \left(\hat{A}_{1i} + E_{1i}F_i\hat{H}_{1i}\right)^T + \left(\hat{A}_{1i} + E_{1i}F_i\hat{H}_{1i}\right) X_i \\ &+ \sum_{j=1}^{s} \pi_{ij} X_i X_j^{-1} X_i + X_i Q X_i \\ \overline{C}_i \triangleq{}& [\, \hat{C}_{1i}X_i + E_{2i}F_i\hat{H}_{1i}X_i \quad C_{2i} + E_{2i}F_iH_{2i} \quad D_{2i} \,] \\ ={}& G_{2i} + E_{2i}F_iL_{1i} \end{aligned}$$

which can be rewritten as

$$G_0 + G_{1i}F_iL_{1i} + (G_{1i}F_iL_{1i})^T + (G_{2i} + E_{2i}F_iL_{1i})^T(G_{2i} + E_{2i}F_iL_{1i}) < 0 \tag{37}$$

for $i = 1, \cdots, s$, where

$$G_0 = \begin{bmatrix} X_i\hat{A}_{1i}^T + \hat{A}_{1i}X_i + \sum_{j=1}^{s}\pi_{ij}X_iX_j^{-1}X_i + X_iQX_i & A_{2i} & B_{2i} \\ A_{2i}^T & -Q & 0 \\ B_{2i}^T & 0 & -\gamma^2 I \end{bmatrix}$$

$$\begin{aligned} G_{1i} &= [\, E_{1i}^T \quad 0 \quad 0 \,]^T, \\ G_{2i} &= [\, \hat{C}_{1i}X_i \quad C_{2i} \quad D_{2i} \,] \\ L_{1i} &= [\, \hat{H}_{1i}X_i \quad H_{2i} \quad 0 \,]. \end{aligned}$$

From Lemma 1, the matrix inequalities in (37) hold if scalars $\alpha_i > 0$ and $\delta_i > 0$ exist such that, for $i = 1, \cdots, s$

$$\begin{aligned} &I - \delta_i E_{2i} E_{2i}^T > 0 \\ &G_0 + \alpha_i G_{1i} G_{1i}^T + \alpha_i^{-1} L_{1i}^T L_{1i} + G_{2i}^T \\ &\quad \cdot \left(I - \delta_i E_{2i} E_{2i}^T\right)^{-1} G_{2i} + \delta_i^{-1} L_{1i}^T L_{1i} < 0 \end{aligned}$$

which are equivalent to the LMI's in (35). ■

Remark 3: Theorems 5 and 6 can be easily adapted to γ-suboptimal H_∞ control of jump linear systems without delay. In [9], the γ-suboptimal H_∞ control was addressed based on a set of coupled algebraic Riccati equations for a special class of jump linear systems without delay. No solution method for these coupled equations is presented, however.

V. Conclusion

In this correspondence, we have studied the robust stochastic stabilizability and H_∞ disturbance attenuation for a class of uncertain jump linear systems with time delay. Sufficient conditions on robust stochastic stabilizability and robust γ-disturbance attenuation are presented based on coupled LMI's. All of these results established are independent of size of the delay time and applicable to situations in which *a priori* knowledge of delay time is available. The results can also be extended to the jump linear systems with multiple time delays using the method of [5]. A possible direction for future work is to obtain delay-dependent conditions that are expected to be less conservative.

References

[1] H. Abou-Kandil, G. Freiling, and G. Jank, "Solution and asymptotic behavior of coupled Riccati equations in jump linear systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1631–1635, Aug. 1994.

[2] K. Benjelloun and E. K. Boukas, "Independent delay sufficient conditions for robust stability of uncertain linear time-delay systems with jumps," in *Proc. Amer. Contr. Conf.*, Albuquerque, NM, June 1997, pp. 3814–3815.

[3] E. K. Boukas, "Control of systems with controlled jump Markov disturbances," *Contr. Theory Adv. Tech.*, vol. 9, pp. 577–595, 1993.

[4] S. D. Brierley, J. N. Chiasson, E. B. Lee, and S. H. Zak, "On stability independent of delay for linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 252–254, Jan. 1982.

[5] Y.-Y. Cao and Y.-X. Sun, "Robust stabilization of uncertain systems with time-varying multi-state-delay," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 1484–1488, Oct. 1998.

[6] Y.-Y. Cao, Y.-X. Sun, and J. Lam, "Delay-dependent robust H_∞ control for uncertain systems with time-varying delays," *Proc. Inst. Elect. Eng.*, vol. 145, pt. D, pp. 338–344, 1998.

[7] H. H. Choi and M. J. Chung, "Memoryless H_∞ controller design for linear systems with delayed state and control," *Automatica*, vol. 31, pp. 917–919, 1995.

[8] O. L. V. Costa and R. P. Marques, "Mixed H_2/H_∞ control of discrete-time Markovian jump linear systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 95–100, Jan. 1998.
[9] C. E. De Souza and M. D. Fragoso, "H_∞ control for linear systems with Markovian jumping parameters," *Contr. Theory Adv. Tech.*, vol. 9, pp. 457–466, 1993.
[10] X. Feng, K. A. Loparo, Y. Ji, and H. J. Chizeck, "Stochastic stability properties of jump linear system," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 38–53, 1992.
[11] J.-B. He, Q.-G. Wang, and T.-H. Lee, "H_∞ disturbance attenuation for state delayed systems," *Syst. Contr. Lett.*, vol. 33, pp. 105–114, 1998.
[12] Y. Ji and H. J. Chizeck, "Controllability, stabilizability, and continuous-time Markovian jump linear quadratic control," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 777–788, July 1990.
[13] N. N. Krasovskii and E. A. Lidskii, "Analytical design of controllers in systems with random attributes," *Automat. Remote Contr.*, vol. 22, pp. 1021–1025, 1961.
[14] *Stochastic Stability and Control*, New York: Academic, 1967.
[15] J. H. Lee, S. W. Kim, and W. H. Kwon, "Memoryless H_∞ controllers for state delayed systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 159–161, Jan. 1994.
[16] M. S. Mahmoud and N. F. Al-Muthairi, "Design of robust controllers for time-delay systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 995–999, May 1994.
[17] M. Mariton, *Jump Linear Systems in Automatic Control*, New York: Marcel Dekker, 1990.
[18] M. Mariton and P. Bertrand, "A homotopy algorithm for solving coupled Riccati equations," *Opt. Contr. Appl. Meth.*, vol. 6, pp. 351–357, 1984.
[19] M. A. Rami and L. E. Ghaoui, "LMI optimization for nonstandard Riccati equations arising in stochastic control," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1666–1671, Nov. 1996.
[20] Y. Wang, L. Xie, and C. E. De Souza, "Robust control of a class of uncertain nonlinear systems," *Syst. Contr. Lett.*, vol. 19, pp. 139–149, 1992.

Exponential Stability of Constrained Receding Horizon Control with Terminal Ellipsoid Constraints

Jae-Won Lee

***Abstract*—In this correspondence, state- and output-feedback receding horizon controllers are proposed for linear discrete time systems with input and state constraints. The proposed receding horizon controllers are obtained from the finite horizon optimization problem with the finite terminal weighting matrix and the artificial invariant ellipsoid constraint, which is less restrictive than the conventional terminal equality constraint. Both hard constraints and mixed constraints are considered in the state-feedback case, and mixed constraints are considered in the output-feedback case. It is shown that all proposed state- and output-feedback receding horizon controllers guarantee the exponential stability of closed-loop systems for all feasible initial sets using the Lyapunov approach.**

***Index Terms*—Discrete linear system with input and state constraints, exponential stability, output-feedback control, receding horizon control.**

I. Introduction

The receding horizon control has emerged as a powerful strategy for constrained systems with limitations on inputs, states, and outputs [4]–[6], [8]–[11]. Especially, the stability issue of the receding horizon control for constrained systems has been focused on in recent literatures [4], [6], [8], [11].

For the state-feedback case, the terminal equality constraint has been utilized to guarantee the closed-loop stability of the receding horizon controller for unconstrained systems [2], [3] and for constrained systems [8], [11]. This artificial constraint is satisfied by driving the state (or unstable mode) to the origin at the finite terminal time. This terminal equality constraint, however, is rather restrictive, because it is generally more difficult to drive a state to a specified point than into a specified set such as an ellipsoid or a ball. Moreover, this approach may make the optimization problem infeasible under the hard state constraint. Hence, the horizon size may have to be made longer so as to make the problem feasible. Even though the mixed constraint has been introduced to relax the hard state constraint [11], issues still need to be covered regarding feasibility and stability, because a somewhat restrictive terminal equality constraint should still be satisfied under the input constraint, and only "attractivity" rather than "asymptotic" or "exponential" stability has been shown in existing results [8], [11]. Recently, efforts have been made to overcome restriction of the terminal equality constraint [4], [7]. In [4], the invariant ellipsoid constraint has been introduced. This artificial constraint is satisfied by putting the state (or unstable mode) into an invariant ellipsoid. In this result, however, exponential stability is shown only for initial states inside the invariant ellipsoid defined by the terminal weighting matrix. In [7], it is shown that the receding horizon control with a sufficiently long horizon size can guarantee attractivity without any artificial constraint. The long horizon size is, however, needed to make the state sufficiently small at the final time, which may increase computational burden and may be regarded as a kind of artificial constraint. Moreover, in this result, only attractivity is shown.

Manuscript received January 6, 1998; revised July 20, 1998, and January 15, 1999. Recommended by Associate Editor, A. Rantzer.

The author is with the Systems and Control Sector, Samsung Advanced Institute of Technology, San 14-1, Nongseo-Ri, Kiheung-Eup, Yongin-City, Kyungki-Do 449-712, Korea (e-mail: ljw@sait.samsung.co.kr).

Publisher Item Identifier S 0018-9286(00)00809-6.

0018–9286/00$10.00 © 2000 IEEE