$[-4.0, 4.0] \times [-6.0, -0.1] \times [0.1, 6.0]$. Now, choosing $\gamma = 1$, $\delta_0 = 0.1$, $\theta(0) = [-2, 2, -2, 2]^T$, and all other initial conditions equal to zero, we implemented the adaptive law (3.4)–(3.8).

Let $\theta(t) = [\theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t)]^T$ denote the estimate of θ^* obtained from the robust adaptive law (3.4)–(3.8). Then the frozen-time estimated modeled part of the plant is given by

$$\hat{P}_0(s, t) = \frac{\theta_3(t)s + \theta_4(t)}{s^2 + (2 - \theta_1(t))s + (2 - \theta_2(t))}$$

and the right half-plane zero is given by $\hat{b}_1 = -(\theta_4/\theta_3)$.

Choosing $W(s) = 0.01/(s + 0.01)$, from (3.10), we have

$$\begin{aligned}\hat{Q}(s,t) &= \left[1 - \frac{s + 0.01}{\hat{b}_1 + 0.01}\right] \frac{s^2 + (2 - \theta_1)s + (2 - \theta_2)}{\theta_3 s + \theta_4} F(s) \\ &= \frac{\hat{b}_1 - s}{\theta_3 s + \theta_4} \frac{(s^2 + (2 - \theta_1)s + (2 - \theta_2))}{\hat{b}_1 + 0.01} F(s) \\ &= -\frac{1}{\theta_3} \frac{(s^2 + (2 - \theta_1)s + (2 - \theta_2))}{\hat{b}_1 + 0.01} F(s) \\ &\qquad \left(\text{using } \hat{b}_1 = -\frac{\theta_4}{\theta_3}\right).\end{aligned}$$

It is now clear that, to make $\hat{Q}(s,t)$ proper, $F(s)$ must be of relative degree 2. So, let us choose $F(s) = (1/(0.15s + 1)^2)$, which results in $n_d = 2$. We now choose $\Lambda_1(s) = s^2 + 2s + 2$, and implement the control law (3.9). Choosing $r(t) = 1.0$ and $r(t) = 0.8 \sin(0.2t)$, we obtained the plots shown in Fig. 3. From these plots, we see that the robust adaptive H_∞ optimal controller does produce reasonably good tracking.

V. Concluding Remarks

In this paper, we have presented the design of an adaptive H_∞ optimal controller based on the IMC structure. The certainty equivalence approach of adaptive control was used to combine a robust adaptive law with a robust H_∞ internal model controller structure to obtain a robust adaptive H_∞ internal model control scheme with provable guarantees of robustness. We do believe that the results of this paper complete our earlier work [4], [5] on adaptive internal model control of single-input single-output stable systems. The extension of these results to multiinput multioutput plants, as well as plants with significant nonlinearities and time delays, is not clear at this stage, and is a topic for further investigation.

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Lyapunov and Riccati Equations of Discrete-Time Descriptor Systems

Liqian Zhang, James Lam, and Qingling Zhang

***Abstract*—In this paper, we further develop the generalized Lyapunov equations for discrete-time descriptor systems given by Bender. We associate a stable discrete-time descriptor system with a Lyapunov equation which has a unique solution. Furthermore, under the assumptions of reachability and observability, the solutions are guaranteed to be positive definite. All results are valid for causal and noncausal descriptor systems. This provides a unification of Lyapunov equations and theories established for both normal and descriptor systems. Based on the developed Lyapunov equation, a Riccati equation is also obtained for solving the state-feedback stabilization problem.**

I. Introduction

In recent years, there has been much research work aimed at generalizing existing theories, especially in the time domain, from normal systems to descriptor systems via different approaches. An approach often used is to transform the system into the Weierstrass canonical form, which provides deep insight into the underlying structure of singular systems. Controllability, observability, stability, and pole placement by state feedback have been considered in [2], [3], and [17]. However, this approach does not always provide a convenient framework for actual computation, not to mention the possible numerical difficulties. The canonical form also entails a change of the internal variables. In many practical situations, designers are often reluctant to execute a variable change due to the contextual and structural significance of the original variables [11], [21]. Hence, many attempts have been made to analyze the systems without a change of variables [11], [15], [16], [21]. The Laurent parameters (or fundamental matrix) play an important role in analyzing descriptor systems [9], [12]. Using this approach, many notions and theories

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The authors are with the Department of Mechanical Engineering, University of Hong Kong, Hong Kong (e-mail: jlam@hku.hk).

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for normal systems are easily extended to descriptor systems, such as the expression of the solution [9], [12], the Cayley–Hamilton theorem [7], [8], reachability and observability [7], [9], the semistate-transition matrix, and the Tschirnhausen polynomials [12]. The prerequisite of this approach is that the Laurent parameters have to be computed.

It is well known that Lyapunov equations have been widely applied to normal systems in controller design [5] and system analysis [13], [20]. Lewis [8] applied the Lyapunov theory to solve optimal control problems for descriptor systems. Zhang *et al.* [19] used generalized Lyapunov methods to analyze structural stability, and solved the linear quadratic control problems. The applications of generalized Lyapunov methods to discuss asymptotic stability can be found in [15] and [16]. For these reasons, the significance of developing Lyapunov equations for descriptor systems is evident. On the other hand, discrete-time descriptor systems may possess anticipation or noncausal behavior which, in the continuous-time case, corresponds to the impulsive behavior. These properties distinguish descriptor systems from normal systems. However, the aforementioned results related to generalized Lyapunov theories were developed only for the causal or impulse-free case. For the noncausal or impulsive situation, Bender *et al.* [1] defined reachability and observability Grammians based on the Laurent parameters, and the associated Lyapunov-like equations are analyzed in terms of reachability, observability, and stability. Zhang *et al.* [18] gave generalized Lyapunov and Riccati equations to examine asymptotic stability and stabilizability of descriptor systems without the impulse-free restriction. Unfortunately, from a computational point of view, it is difficult to obtain the solutions of the already established generalized Lyapunov equations due to the nonuniqueness or the associated constraints for their solutions. This presents a major difficulty in applying the solutions of these equations to develop synthesis and analysis techniques similar to the case of normal systems.

The present paper proposes a kind of Lyapunov equations for discrete-time descriptor systems based on those given in [1]. All results to be established are valid for causal and noncausal descriptor systems. The Lyapunov equations are very similar to those of normal systems in either appearance or theories. The positive definiteness of the solutions implies asymptotic stability of the descriptor systems. Moreover, it is numerically easy to compute the solutions. The corresponding Riccati equation is also developed for stabilization problems.

II. Preliminaries

Throughout the paper, if not explicitly stated, all matrices are assumed to have compatible dimensions. We use $M > 0$ (resp. $M \geq 0$) to denote a symmetric positive-definite (resp. semidefinite) matrix M. The ith eigenvalue of M is denoted by $\lambda_i(M)$.

Consider a linear time-invariant discrete-time descriptor system of the form

$$E x_{k+1} = A x_k + B u_k, \qquad y_k = C x_k \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $(zE - A)$ is a regular pencil. The above system is also identified by the realization quadruple (E, A, B, C). The Laurent parameters ϕ_k, $-\mu \leq k < \infty$, specify the unique series expansion of the resolvent matrix about $z = \infty$:

$$(zE - A)^{-1} = z^{-1} \sum_{k=-\mu}^{\infty} \phi_k z^{-k}, \qquad \mu \geq 0 \tag{2}$$

which is valid in the set $0 < |z| \leq \delta$ for some $\delta > 0$. The positive integer μ is the nilpotent index. There exist two square invertible matrices U and V such that (E, A, B, C) is transformed to the Weierstrass canonical form $(\overline{E}, \overline{A}, \overline{B}, \overline{C}) \equiv (U^{-1}EV^{-1}, U^{-1}AV^{-1}, U^{-1}B, CV^{-1})$ with

$$\begin{aligned} z\overline{E} - \overline{A} &= \begin{bmatrix} zI - J & 0 \\ 0 & zN - I \end{bmatrix}, \\ \overline{B} &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ \overline{C} &= [C_1 \quad C_2] \end{aligned} \tag{3}$$

where J and N are in Jordan canonical forms and N is nilpotent. Also,

$$\overline{\phi}_k := V\phi_k U = \begin{cases} \begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix}, & k \geq 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{bmatrix}, & k < 0. \end{cases} \tag{4}$$

The solution of a discrete-time system can be expressed directly in terms of the Laurent parameters [1] as

$$\begin{aligned} x_i = (\phi_0 A)^i x_0 + \sum_{k=0}^{i-1} (\phi_0 A)^{i-k-1} \phi_0 B u_k \\ - \left((-\phi_{-1}E)^m x_{i+m} + \sum_{k=0}^{m-1} (-\phi_{-1}E)^k \phi_{-1} B u_{i+k} \right). \end{aligned} \tag{5}$$

A descriptor system is *asymptotically stable* if and only if its causal subsystem (I, J, B_1, C_1) is *asymptotically stable*. The *reachability* (*observability*) of a descriptor system is equivalent to both its causal subsystem and noncausal subsystem (N, I, B_2, C_2) being *reachable* (*observable*) [4].

Definition 1—Reachability/Observability Grammian [1]: For the discrete-time descriptor system (E, A, B, C), the causal reachability (resp. observability) Grammian is

$$P_c^r = \sum_{k=0}^{\infty} \phi_k B B^T \phi_k^T \left(\text{resp. } P_c^o = \sum_{k=0}^{\infty} \phi_k^T C^T C \phi_k \right),$$

provided that the series converges; the noncausal reachability (resp. observability) Grammian is

$$P_{nc}^r = \sum_{k=-\mu}^{-1} \phi_k B B^T \phi_k^T \left(\text{resp. } P_{nc}^o = \sum_{k=-\mu}^{-1} \phi_k^T C^T C \phi_k \right).$$

The reachability (resp. observability) Grammian is

$$P^r = P_c^r + P_{nc}^r \qquad (\text{resp. } P^o = P_c^o + P_{nc}^o).$$

In Weierstrass canonical form (3), the corresponding Grammians of P_c^r, P_{nc}^r, P_c^o, and P_{nc}^o are denoted by $\overline{P}_c^r$, $\overline{P}_{nc}^r$, $\overline{P}_c^o$, and $\overline{P}_{nc}^o$, respectively. From (3) and (4), it can be easily shown that

$$\begin{aligned} \overline{P}_c^r &= V P_c^r V^T, \qquad \overline{P}_{nc}^r = V P_{nc}^r V^T, \\ \overline{P}_c^o &= U^T P_c^o U, \qquad \overline{P}_{nc}^o = U^T P_{nc}^o U. \end{aligned} \tag{6}$$

Lemma 1 [1], [9]:

$$\phi_0 E \phi_k = \begin{cases} \phi_k, & k \geq 0 \\ 0, & k < 0 \end{cases} \tag{7}$$

$$\phi_{-1} A \phi_k = \begin{cases} 0, & k \geq 0 \\ -\phi_k, & k < 0. \end{cases} \tag{8}$$

Proposition 1:

i)

$$\phi_0 E P_c^r E^T \phi_0^T = P_c^r, \qquad \phi_0^T E^T P_c^o E \phi_0 = P_c^o. \tag{9}$$

ii)

$$\phi_{-1} A P_{nc}^r A^T \phi_{-1}^T = P_{nc}^r, \quad \phi_{-1}^T A^T P_{nc}^o A \phi_{-1} = P_{nc}^o. \tag{10}$$

Proof:

i) From (7), we have

$$\begin{aligned}\phi_0 E P_c^r E^T \phi_0^T &= \sum_{k=0}^{\infty} \phi_0 E \phi_k B B^T \phi_k^T E^T \phi_0^T \\ &= \sum_{k=0}^{\infty} \phi_k B B^T \phi_k^T = P_c^r \\ \phi_0^T E^T P_c^o E \phi_0 &= \sum_{k=0}^{\infty} \phi_0^T E^T \phi_k^T C^T C \phi_k E \phi_0 \\ &= \sum_{k=0}^{\infty} \phi_k^T C^T C \phi_k = P_c^o.\end{aligned}$$

ii) From (8), we have

$$\begin{aligned}\phi_{-1} A P_{nc}^r A^T \phi_{-1}^T &= \sum_{k=-\mu}^{-1} \phi_{-1} A \phi_k B B^T \phi_k^T A^T \phi_{-1}^T \\ &= \sum_{k=-\mu}^{-1} \phi_k B B^T \phi_k^T = P_{nc}^r\end{aligned}$$

and

$$\begin{aligned}\phi_{-1}^T A^T P_{nc}^o A \phi_{-1} &= \sum_{k=-\mu}^{-1} \phi_{-1}^T A^T \phi_k^T C^T C \phi_k A \phi_{-1} \\ &= \sum_{k=-\mu}^{-1} \phi_k^T C^T C \phi_k = P_{nc}^o.\end{aligned}$$

■

III. Lyapunov Equations and Asymptotic Stability

In relation to the Grammians defined in Definition 1 for (1), the corresponding Lyapunov equations will be stated. The following theorem gives properties of the Lyapunov equations in terms of asymptotic stability and reachability.

Theorem 1:

i) P_c^r satisfies

$$P_c^r - \phi_0 A P_c^r A^T \phi_0^T = \phi_0 B B^T \phi_0^T. \tag{11}$$

ii) P_{nc}^r satisfies

$$P_{nc}^r - \phi_{-1} E P_{nc}^r E^T \phi_{-1}^T = \phi_{-1} B B^T \phi_{-1}^T. \tag{12}$$

iii) $P^r = P_c^r + P_{nc}^r$ satisfies

$$\begin{aligned}&P^r - (\phi_0 A - \phi_{-1} E) P^r (\phi_0 A - \phi_{-1} E)^T \\ &\quad = \phi_0 B B^T \phi_0^T + \phi_{-1} B B^T \phi_{-1}^T.\end{aligned} \tag{13}$$

iv) If (1) is asymptotically stable, then $P_c^r \geq 0$, $P_{nc}^r \geq 0$, and $P^r \geq 0$ are the unique solutions of (11)–(13), respectively.

v) If (1) is asymptotically stable, then (1) is reachable if and only if $P^r > 0$ is the unique solution of (13).

Proof: i) and ii) can be easily established from [1] with (9).

iii) Notice that

$$\begin{aligned}\phi_{-1} E P_c^r E^T \phi_{-1}^T &= \sum_{k=0}^{\infty} \phi_{-1} E \phi_k B B^T \phi_k^T E^T \phi_{-1}^T = 0 \\ \phi_0 A P_c^r E^T \phi_{-1}^T &= \sum_{k=0}^{\infty} \phi_0 A \phi_k B B^T \phi_k^T E^T \phi_{-1}^T = 0 \\ \phi_{-1} E P_c^r A^T \phi_0^T &= \sum_{k=0}^{\infty} \phi_{-1} E \phi_k B B^T \phi_k^T A^T \phi_0^T = 0\end{aligned}$$

follow from (8). Similarly, from (7), we have

$$\begin{aligned}\phi_0 A P_{nc}^r A^T \phi_0^T &= \sum_{k=-\mu}^{-1} \phi_0 A \phi_k B B^T \phi_k^T A^T \phi_0^T = 0 \\ \phi_0 A P_{nc}^r E^T \phi_{-1}^T &= \sum_{k=-\mu}^{-1} \phi_0 A \phi_k B B^T \phi_k^T E^T \phi_{-1}^T = 0 \\ \phi_{-1} E P_{nc}^r A^T \phi_0^T &= \sum_{k=-\mu}^{-1} \phi_{-1} E \phi_k B B^T \phi_k^T A^T \phi_0^T = 0.\end{aligned}$$

Hence, we can rewrite (11) and (12) as follows:

$$\begin{aligned}P_c^r - (\phi_0 A - \phi_{-1} E) P_c^r (A^T \phi_0^T - E^T \phi_{-1}^T) &= \phi_0 B B^T \phi_0^T \\ P_{nc}^r - (\phi_0 A - \phi_{-1} E) P_{nc}^r (A^T \phi_0^T - E^T \phi_{-1}^T) &= \phi_{-1} B B^T \phi_{-1}^T.\end{aligned}$$

This proves the validity of (13).

iv) From (3) and (4),

$$\overline{\phi}_0 \overline{A} = V \phi_0 U U^{-1} A V^{-1} = V \phi_0 A V^{-1}.$$

Hence, $\overline{\phi}_0 \overline{A}$ and $\phi_0 A$ have the same eigenvalues. We know that

$$\overline{\phi}_0 \overline{A} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$$

and (1) is asymptotically stable, that is, $|\lambda_i(J)| < 1$ for all i. Consequently, $|\lambda_i(\phi_0 A)| < 1$ for all i, and this guarantees (11) to have a unique solution. The uniqueness of the solution of (12) follows from the fact that $\phi_{-1} E$ is nilpotent.

Notice that, in Weierstrass canonical form, we have

$$\begin{aligned}\overline{\phi}_0 \overline{A} - \overline{\phi}_{-1} \overline{E} &= V \phi_0 A V^{-1} - V \phi_{-1} E V^{-1} \\ &= \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \\ &= \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}.\end{aligned}$$

As $|\lambda_i(J)| < 1$ for all i and N is nilpotent, so $|\lambda_i(\overline{\phi}_0 \overline{A} - \overline{\phi}_{-1} \overline{E})| = |\lambda_i(\overline{\phi}_0 \overline{A} - \overline{\phi}_{-1} \overline{E})| < 1$ for all i. This implies that the solution of (13) is also unique.

v) When (1) is in Weierstrass canonical form (3), (11), and (12) reduce to

$$\begin{bmatrix} P_{11}^r & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} J P_{11}^r J^T & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1 B_1^T & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & P_{22}^r \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & N P_{22}^r N^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_2 B_2^T \end{bmatrix}$$

or, equivalently,

$$P_{11}^r - J P_{11}^r J^T = B_1 B_1^T \tag{14}$$

$$P_{22}^r - N P_{22}^r N^T = B_2 B_2^T. \tag{15}$$

On the other hand,

$$\overline{P}_c^r = \sum_{k=0}^{\infty} \overline{\phi}_k \overline{B} \overline{B}^T \phi_k^T = \begin{bmatrix} P_{11}^r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\overline{P}_{nc}^r = \sum_{k=-\mu}^{-1} \overline{\phi}_k \overline{B} \overline{B}^T \phi_k^T = \begin{bmatrix} 0 & 0 \\ 0 & P_{22}^r \end{bmatrix}$$

then

$$\overline{P}^r = \overline{P}_c^r + \overline{P}_{nc}^r = \begin{bmatrix} P_{11}^r & 0 \\ 0 & P_{22}^r \end{bmatrix}. \tag{16}$$

As noted in [4], (1) is reachable if and only if (J, B_1) and (N, B_2) are reachable. Hence, J is asymptotically stable [as (1) is asymptotically stable] if and only if (14) has a unique solution $P_{11}^r > 0$ (see

[6]). Equation (15) always has the unique solution $P_{22}^r > 0$ since N is nilpotent. Then from (16), $\overline{P}^r > 0$ is unique if and only if (1) is asymptotically stable under the assumption of the reachability of (1). Notice that

$$\overline{P}^r = \overline{P}_c^r + \overline{P}_{nc}^r = V(P_c^r + P_{nc}^r)V^T = VP^rV^T.$$

Then the proof is completed. ∎

The results for the dual case concerning the observability Grammian are summarized in the next theorem with the proof omitted.

Theorem 2:

i) P_c^o satisfies

$$P_c^o - \phi_0^T A^T P_c^o A\phi_0 = \phi_0^T C^T C\phi_0. \tag{17}$$

ii) P_{nc}^o satisfies

$$P_{nc}^o - \phi_{-1}^T E^T P_{nc}^o E\phi_{-1} = \phi_{-1}^T C^T C\phi_{-1}. \tag{18}$$

iii) $P^o = P_c^o + P_{nc}^o$ satisfies

$$\begin{aligned} &P^o - (\phi_0^T A^T - \phi_{-1}^T E^T)P^o(A\phi_0 - E\phi_{-1}) \\ &\quad = \phi_0^T C^T C\phi_0 + \phi_{-1}^T C^T C\phi_{-1}. \end{aligned} \tag{19}$$

iv) If (1) is asymptotically stable, then $P_c^o \geq 0$, $P_{nc}^o \geq 0$, and $P^o \geq 0$ are the unique solutions of (17)–(19), respectively.

v) If (1) is asymptotically stable, then (1) is observable if and only if $P^o > 0$ is the unique solution of (19).

Remark 1: If E is nonsingular, then $\phi_0 = I$ and $\phi_{-1} = 0$ (see [1]). In this case, the reachability and observability Grammians P^r and P^o become

$$P^r = \sum_{k=0}^{\infty} A^k BB^T (A^k)^T, \qquad P^o = \sum_{k=0}^{\infty} (A^k)^T C C^T A^k.$$

It can be seen from (13) and (19) that P^r and P^o satisfy

$$P^r - AP^rA^T = BB^T, \qquad P^o - A^TP^oA = C^TC.$$

Thus, normal systems and descriptor systems have a unified Grammian form and Lyapunov equations.

IV. Riccati Equation and Stabilizability

Consider a generalized state-feedback control

$$u_k = -Kx_k$$

applied to (1) such that the closed-loop system is given by

$$Ex_{k+1} = (A - BK)x_k. \tag{20}$$

If K is such that (20) is asymptotically stable, then (1) is said to be *stabilizable*. Based on Lyapunov equation (11), a corresponding Riccati equation for descriptor system (1) is defined as

$$\begin{aligned} &A^T\phi_0^T P\phi_0 A - P - A^T\phi_0^T P\phi_0 B(R + B^T\phi_0^T P\phi_0 B)^{-1} \\ &\quad \cdot B^T\phi_0^T P\phi_0 A = -W \end{aligned} \tag{21}$$

where $R > 0$ and $W > 0$.

Lemma 2: Equation (1) is stabilizable if and only if normal system $(I, \phi_0 A, \phi_0 B)$ is stabilizable.

Proof: If (1) is stabilizable, then there exists a feedback K_1 such that $J - B_1K_1$ is asymptotically stable [4, Theorem 3-1.2]. This is equivalent to having

$$x_{k+1} = \begin{bmatrix} J - B_1K_1 & 0 \\ 0 & 0 \end{bmatrix} x_k$$

asymptotically stable. Now, we consider the closed-loop system of $(I, \phi_0 A, \phi_0 B)$ with the feedback $\hat{K} = [K_1 \quad 0]V$:

$$x_{k+1} = (\phi_0 A - \phi_0 B\hat{K})x_k. \tag{22}$$

From

$$\begin{aligned} V(\phi_0 A - \phi_0 B\hat{K})V^{-1} &= \overline{\phi}_0\overline{A} - \overline{\phi}_0\overline{B}[K_1 \quad 0] \\ &= \begin{bmatrix} J - B_1K_1 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

it can be seen that (22) is asymptotically stable, and hence $(I, \phi_0 A, \phi_0 B)$ is stabilizable.

On the other hand, if $(I, \phi_0 A, \phi_0 B)$ is stabilizable, then there exists K such that $\phi_0 A - \phi_0 BK$ is stable. If we denote

$$KV^{-1} = [K_1 \quad K_2],$$

then

$$V(\phi_0 A - \phi_0 BK)V^{-1} = \begin{bmatrix} J - B_1K_1 & 0 \\ 0 & 0 \end{bmatrix}$$

is asymptotically stable. That is, (1) is stabilizable. ∎

Lemma 3: Suppose (1) is stabilizable. For any given $W > 0$, let P be the unique solution of (21). If $\overline{P}$ is the unique solution of the Riccati equation of (1) in Weierstrass canonical form (3),

$$\begin{aligned} &\overline{A}^T\overline{\phi}_0^T\overline{P\phi}_0\overline{A} - \overline{P} - \overline{A}^T\overline{\phi}_0^T\overline{P\phi}_0\overline{B}(R + \overline{B}^T\overline{\phi}_0^T\overline{P\phi}_0\overline{B})^{-1} \\ &\quad \cdot \overline{B}^T\overline{\phi}_0^T\overline{P\phi}_0\overline{A} = -(V^T)^{-1}WV^{-1} \end{aligned} \tag{23}$$

then $\overline{P} = (V^T)^{-1}PV^{-1}$.

Proof: By substituting

$$\phi_0 A = V^{-1}\overline{\phi}_0\overline{A}V, \qquad \phi_0 B = V^{-1}\overline{\phi}_0\overline{B}$$

into (21), we have

$$\begin{aligned} &\overline{A}^T\overline{\phi}_0^T(V^T)^{-1}PV^{-1}\overline{\phi}_0\overline{A} - (V^T)^{-1}PV^{-1} - \overline{A}^T\overline{\phi}_0^T(V^T)^{-1} \\ &\quad \cdot PV^{-1}\overline{\phi}_0\overline{B}(R + \overline{B}^T\overline{\phi}_0^T(V^T)^{-1}PV^{-1}\overline{\phi}_0\overline{B})^{-1} \\ &\quad \cdot \overline{B}^T\overline{\phi}_0^T(V^T)^{-1}PV^{-1}\overline{\phi}_0\overline{A} = -(V^T)^{-1}WV^{-1}. \end{aligned} \tag{24}$$

Since $(I, \phi_0 A, \phi_0 B)$ is stabilizable and $W > 0$, (21) and (23) have unique solutions P and $\overline{P}$. From (23) and (24), the result follows. ∎

Theorem 3: For any given $W > 0$, if (1) is stabilizable, then the closed-loop system (20) with K given by

$$K = (R + B^T\phi_0^T P\phi_0 B)^{-1}B^T\phi_0^T P\phi_0 A \tag{25}$$

is asymptotically stable where $P > 0$ is the unique solution of the Riccati equation (21).

Proof: When (1) is in the Weierstrass canonical form (3), K can be represented as

$$\overline{K} = \overline{\phi}_0\overline{B}(R + \overline{B}^T\overline{\phi}_0^T\overline{P\phi}_0\overline{B})^{-1}\overline{B}^T\overline{\phi}_0^T\overline{P\phi}_0\overline{A} = [\overline{K}_1 \quad 0].$$

With

$$\overline{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \qquad (V^T)^{-1}WV^{-1} \equiv \begin{bmatrix} \overline{W}_{11} & \overline{W}_{12} \\ \overline{W}_{12}^T & \overline{W}_{22} \end{bmatrix},$$

the Riccati equation (21) becomes

$$\begin{aligned} &\begin{bmatrix} J^T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \\ &\quad - \begin{bmatrix} J^T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}[\overline{K}_1 \quad 0] \\ &\quad = -\begin{bmatrix} \overline{W}_{11} & \overline{W}_{12} \\ \overline{W}_{12}^T & \overline{W}_{22} \end{bmatrix}. \end{aligned}$$

That is,

$$\begin{bmatrix} J^T P_{11} J & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} - \begin{bmatrix} J^T P_{11} B_1 \overline{K}_1 & 0 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} \overline{W}_{11} & \overline{W}_{12} \\ \overline{W}_{12}^T & \overline{W}_{22} \end{bmatrix}.$$

Obviously, $P_{12} = \overline{W}_{12}$, $P_{22} = \overline{W}_{22} > 0$, and $P_{11} > 0$ is the unique solution of the Riccati equation

$$J^T P_{11} J - P_{11} - J^T P_{11} B_1 \overline{K}_1 = -\overline{W}_{11} < 0$$

and $(J - B_1 \overline{K}_1)$ is asymptotically stable (see [14]), which implies that $(\overline{E}, \overline{A}, \overline{B})$ is stabilizable by the feedback $\overline{K}$. Now, we prove the stability of (20) with K given by (25). Consider an equivalent system of (20):

$$U^{-1} E V^{-1} x_{k+1} = U^{-1}(A - BK)V^{-1} x_k \Rightarrow$$
$$\overline{E} x_{k+1} = (\overline{A} - \overline{B} K V^{-1}) x_k.$$

Since (1) is stabilizable and $W > 0$, then from Lemmas 2 and 3, we have

$$K V^{-1} = (R + B^T \phi_0^T P \phi_0 B)^{-1} B^T \phi_0^T P \phi_0 A V^{-1} = \overline{K}.$$

It follows that (1) is equivalent to

$$\overline{E} x_{k+1} = (\overline{A} - \overline{BK}) x_k$$

which is asymptotically stable. The uniqueness and positive definiteness of P follow from Lemma 3 and $W > 0$. ■

Remark 2: It is observed that K given by (25) is the optimal state feedback matrix for $(I, \phi_0 A, \phi_0 B)$ under the linear-quadratic cost function [6]

$$\sum_{k=0}^{\infty} x_k^T W x_k + u_k^T R u_k.$$

If the system is causal, which means $\phi_{-1} E = 0$, then the solution of (1) is given by [see (5)]

$$x_i = (\phi_0 A)^i x_0 + \sum_{k=0}^{m-1} (\phi_0 A)^{i-k-1} \phi_0 B u_k,$$

which corresponds to the solution for the system $(I, \phi_0 A, \phi_0 B)$. Consequently, in the causal case, K given by (25) is also the optimal state feedback matrix of (1).

V. Numerical Example

From the given Lyapunov and Riccati equations, it is easy to obtain their solutions after computing ϕ_0 and ϕ_{-1}. In [12], numerically reliable and stable recursive algorithms were provided for calculating ϕ_0 and ϕ_{-1}.

Example 1: Consider the dynamic Leontief model which describes the time pattern of production sectors [4], [10] given by

$$x_k = F x_k + G(x_{k+1} - x_k) + d_k. \tag{26}$$

The elements of $x_k \in \mathbb{R}^{n \times 1}$ are the levels of production in the sectors at time k. $F \in \mathbb{R}^{n \times n}$ is the input–output matrix, and $F x_k$ is the amount required as direct input for the current production. $G \in \mathbb{R}^{n \times n}$ is the capital coefficient matrix, and $G(x_{k+1} - x_k)$ is the amount required for capacity expansion to be able to produce x_{k+1} in the next period. d_k is the amount of production going to current demand. It is assumed that the amount of production d_k is, in turn, controlled by u_k such that $d_k = H u_k$ where $u_k \in \mathbb{R}^{p \times 1}$ where $1 \le p < n$. In multisector economic systems, both F and G have nonnegative elements. Typically, the capital coefficient matrix G has nonzero elements in only a few rows, corresponding to the fact that capital is formed from only a few sectors. Thus, (26) is a practical discrete-time descriptor system since G is often singular. Here, we consider a Leontief model described by

$$F = \begin{bmatrix} 1.25 & 0.5 & 1.5 \\ 0.75 & 0.5 & 1.1 \\ 0.25 & 0 & 1.5 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0.5 & 0.75 \\ 0.25 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix},$$
$$H = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then (26) can be rewritten as

$$\begin{bmatrix} 1 & 0.5 & 0.75 \\ 0.25 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0.75 & 0 & -0.75 \\ -0.5 & 0.5 & -0.6 \\ -0.25 & 0 & -0.5 \end{bmatrix} x_k - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u_k \tag{27}$$

which is a reachable, noncausal descriptor system. Its finite pole is located at $z = 1.3636$, which implies that (27) is an unstable system. The Laurent parameters ϕ_0 and ϕ_{-1} are

$$\phi_0 = \begin{bmatrix} 1.2121 & 1.6529 & -8.5349 \\ 0.4848 & 0.6612 & -3.4140 \\ -0.6061 & -0.8264 & 4.2675 \end{bmatrix}$$
$$\phi_{-1} = \begin{bmatrix} 0 & 1.2121 & -4.9256 \\ 0 & -1.5152 & 0.4298 \\ 0 & -0.6061 & 4.4628 \end{bmatrix}.$$

The reachability Grammian P^r is then obtained from (13) as

$$P^r = \begin{bmatrix} -22.1437 & -12.7671 & 3.6448 \\ -12.7671 & -2.5107 & 4.2128 \\ 3.6448 & 4.2128 & 5.8911 \end{bmatrix}$$

which is an indefinite matrix. Since the system is reachable, this result implies the instability of the system.

To consider the stabilization of (27) based on Theorem 3, let $R = 1$ and $W = I$ in (21), then P is obtained as

$$P = \begin{bmatrix} 1.1496 & 0.0558 & 0.1598 \\ 0.0558 & 1.0208 & 0.0596 \\ 0.1598 & 0.0596 & 1.1706 \end{bmatrix}$$

and the feedback matrix K following from (25) is

$$K = [0.3828 \quad 0.1427 \quad 0.4087].$$

The resulting closed-loop system is

$$\begin{bmatrix} 1 & 0.5 & 0.75 \\ 0.25 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} x_{k+1} = \begin{bmatrix} 1.1328 & 0.1427 & -0.3413 \\ -0.1172 & 0.6427 & -0.1913 \\ 0.1328 & 0.1427 & -0.0913 \end{bmatrix} x_k$$

which has one stable finite pole at 0.02832. Thus, the system is stabilized.

VI. Conclusion

In this paper, Lyapunov equations have been obtained for discrete-time descriptor systems. The Lyapunov equations are applicable to causal and noncausal descriptor systems. Since they have the same form as those for the normal systems, and the solutions are unique if the systems are asymptotically stable, it is easy to obtain numerical solutions. These features make the proposed Lyapunov equations suitable for asymptotic stability analysis as well as control synthesis. A Riccati equation is also considered, from which a static state feedback can be obtained to stabilize the systems. Finally, numerical examples are used to illustrate the results established.

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Control of Markovian Jump Discrete-Time Systems with Norm Bounded Uncertainty and Unknown Delay

Peng Shi, El-Kébir Boukas, and Ramesh K. Agarwal

***Abstract*—This paper studies the problem of control for discrete time delay linear systems with Markovian jump parameters. The system under consideration is subjected to both time-varying norm-bounded parameter uncertainty and unknown time delay in the state, and Markovian jump parameters in all system matrices. We address the problem of robust state feedback control in which both robust stochastic stability and a prescribed H_∞ performance are required to be achieved irrespective of the uncertainty and time delay. It is shown that the above problem can be solved if a set of coupled linear matrix inequalities has a solution.**

***Index Terms*—Discrete-time system, Markovian jump parameter, Riccati inequality, time delay, uncertainty.**

I. INTRODUCTION

During the past years, the study of time delay systems has received considerable interest. Time delay is commonly encountered in various engineering systems, and is frequently a source of instability and poor performance [1]. In [2], nonlinear state feedback controllers have been considered, whereas [3] has focused on memoryless linear state feedback. Recently, memoryless stabilization and H_∞ control of uncertain continuous-time delay systems have been extensively investigated. For some representative prior work on this general topic, we refer the reader to [4]–[6] and the references therein. More recently, optimal quadratic guaranteed cost control for a class of uncertain linear time delay systems with norm-bounded uncertainty has been designed in [7]. The issue of delay-dependent robust stability and stabilization of uncertain linear delay systems has been tackled in [6] via a linear matrix inequality approach. On the other hand, stochastic linear uncertain systems also have been extensively studied for the last ten years, in particular, linear systems with Markovian jumping parameters; see, for example, [8]–[13]. The problems of designing state feedback controllers for uncertain Markovian jumping systems to achieve both stochastic stability and a prescribed H_∞ performance, and guaranteed cost control for Markovian jumping systems have been investigated in [12] and [13]. However, to the best of the authors' knowledge, the problems of robust stochastic stability and H_∞ control of uncertain discrete-time delay systems with Markovian jump parameters have not been fully investigated yet.

In this paper, the problems of stochastic stability and control of a class of uncertain systems with unknown time delay in the state variables, and with Markovian jump parameters in all system matrices are studied. We consider uncertain systems with norm-bounded time-varying parameter uncertainty in both the state and control. We deal

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P. Shi is with the Centre for Industrial and Applicable Mathematics, School of Mathematics, University of South Australia, Mawson Lakes, SA 5095, Australia (e-mail: peng.shi@unisa.edu.au).

E.-K. Boukas is with the Département de Génie Mécanique, École Polytechnique de Montréal, Montréal, P.Q. H3C 3A7, Canada.

R. K. Agarwal is with the Department of Aerospace Engineering, National Institute for Aviation Research, Wichita State University, Wichita, KS 67260-0044 USA.

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