Improvement of Fourier Series Analysis Technique by Time-Domain Window Function

P. Shum and S. F. Yu

Abstract— **We demonstrate that by adding a time-domain window function to the recently developed Fourier series analysis technique can reduce the propagation error in solving the nonlinear soliton propagation equation. With suitable modification of window function parameters, the number of sampling points as well as computational time required for the calculation can be minimized even with higher order dispersion terms taken into consideration.**

INTENSIVE RESEARCH work has been carried out to study the enormous potential of using soliton in long-haul communication since the first successful demonstration of using optical soliton in dielectric fiber [1]. Split-step Fourier method (SSFM) [2] is the commonly used numerical technique to study the propagation of solitons in an optical fiber. Apart from this method, a general Fourier series analysis technique (FSAT) is also developed [3] to analyze the soliton propagation behavior. The main advantage of FSAT over SSFM is that much fewer sampling points are required to calculate the same problem. However, the manipulation efficiency of FSAT is limited if the higher order dispersion is taken into account. In this letter, we propose a time-domain window function (TDWF) to enhance the performance of FSAT. The effects of higher order dispersion are considered and the results are compared with the original FSAT (without TDWF).

The general soliton equation, which includes fiber loss, higher order dispersive, and nonlinear effects, can be expressed as follows [4]:

$$\frac{\partial u}{\partial x} = j\frac{1}{2}\frac{\partial^2 u}{\partial T^2} + B\frac{\partial^3 u}{\partial T^3} - C\frac{\partial |u|^2 u}{\partial T} - jT_R u\frac{\partial |u|^2}{\partial T} + j|u|^2 u - \Gamma u \tag{1}$$

where $u(x, T)$ is the normalized complex amplitude of the soliton pulse, x represents the normalized distance along the direction of propagation, T represents the normalized time, B is the third-order dispersion coefficient, Γ is the normalized loss factor, and C and T_R are higher order nonlinear coefficients. The first and second terms on the right-hand side of (1) are, respectively, the second- and third-order group velocity dispersions. The third to fifth terms are nonlinear terms governing the effects of self-steeping, retarded nonlinear response and self-phase modulation. The last term is the attenuation term corresponding to the fiber loss. By transforming (1) into another form of expression through the FSAT, we obtained a set of $2N+1$ first-order partial differential equations

$$\begin{aligned}\frac{\partial \hat{u}_n(x)}{\partial x} = &\underbrace{[-j\sigma(n) - \Gamma]\hat{u}_n(x)}_{\text{linear term}} \\ &\underbrace{+ j\sum_{\forall \mu-\nu+\lambda=n} \hat{u}_\mu(x)\hat{u}_\nu^*(x)\hat{u}_\lambda(x)}_{\text{nonlinear term}} \\ &\underbrace{- jC\varepsilon n \sum_{\forall k-l+m=n} \hat{u}_k(x)\hat{u}_l^*(x)\hat{u}_m(x)}_{\text{nonlinear term}} \\ &\underbrace{+ T_R\varepsilon \sum_{\forall p+q=n} q\hat{u}_p(x)\Theta_q(x)}_{\text{nonlinear term}}\end{aligned} \tag{2}$$

where

$$\Theta_q(x) = \sum_{\forall a-b=q} \hat{u}_a(x)\hat{u}_b^*(x)$$

$n, \mu, \nu, \lambda, k, l, m, p, q, a$, and b are integers between $-N$ and N, and is the fundamental frequency of Fourier series. (2) can be solved by the Merson form of Runge–Kutta Method. The effects of higher order nonlinear terms are to shift the soliton pulses away from their center position and to cause the decay of higher order solitons. However, in solving of (2) with higher order dispersion takes into account, dispersive tail is formed at the trailing edge of the soliton pulse. This dispersive tail gets longer as soliton pulse propagates further away along the optical fiber. As a result, separation with adjacent pulses at another bit period should increase in order to prevent any interaction happens. Therefore, a large number of sampling points is required to avoid the influence of the dispersive tail.

A TDWF can be utilized to attenuate the dispersive tail of the soliton pulse in order to improve the accuracy and efficiency of the FSAT. The TDWF of the soliton envelope is given by the following expression:

$$W(x, T) = G \cdot H(T - T_0) \cdot (T - T_0) \cdot u(x, T) \tag{3}$$

where T_0 is a cut-off point of the window function, G is the slope of the TDWF, $H(T - T_0)$ is a unit step function and $u(x, T)$ is the soliton envelope. We can express each term on the right-hand side of (3) as a Fourier series, which can then be multiplied together to give the Fourier series of the whole

Manuscript received April 22, 1996; revised June 25, 1996.
The authors are with the Department of Electrical and Electronic Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong.
Publisher Item Identifier S 1041-1135(96)07413-7.

function. The Fourier series of the term $(T-T_0)$ is given by

$$\begin{aligned}\frac{\varepsilon}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}&(T-T_0)j\exp(-jL\varepsilon T)\cdot dT\\&=\frac{\varepsilon}{2\pi}\left[\frac{T\exp(-jL\varepsilon T)}{-jL\varepsilon}+\frac{\exp(-jL\varepsilon T)}{(L\varepsilon)^2}\right.\\&\quad\left.+\frac{T_0\exp(-jL\varepsilon T)}{jL\varepsilon}\right]_{T=-\pi/\varepsilon}^{\pi/\varepsilon}.\end{aligned}\tag{4}$$

Because L is an integer, using $\sin(L\pi)=0$ and $\cos(L\pi)=(-1)^L$, we can simplify the solution to:

$$\frac{\varepsilon}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}(T-T_0)\exp(-jL\varepsilon T)\cdot dT=\begin{cases}j\dfrac{(-1)^L}{L\varepsilon} & \text{for } L\neq 0\\ -T_0 & \text{for } L=0\end{cases}.\tag{5}$$

Finally, the Fourier series of the time-delayed unit step function $H(T-T_0)$ is given by

$$\frac{\varepsilon}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}H(T-T_0)\exp(-jm\varepsilon T)\cdot dT=\begin{cases}j\dfrac{(-1)^m-\cos(m\varepsilon T_0)+j\sin(m\varepsilon T_0)}{2\pi m} & \text{for } m\neq 0\\ \dfrac{\varepsilon}{2\pi}\left(\dfrac{\pi}{\varepsilon}-T_0\right) & \text{for } m=0\end{cases}.\tag{6}$$

From (3), (5), and (6), the Fourier series of the TDWF can be obtained as

$$\begin{aligned}W(x,T)&=\sum_{n=-N}^{N}\hat{W}_n(x)\exp(jn\varepsilon T)\\&=\left(G\sum_{k=-N}^{N}\hat{u}_k(x)\exp(jk\varepsilon T)\right)\\&\quad\cdot\left(\sum_{L=-N}^{N}j\frac{(-1)^L}{L\varepsilon}\exp(jL\varepsilon T)\right)\left(\sum_{m=-N}^{N}\right.\\&\quad\left.\cdot\, j\frac{(-1)^m-\cos(m\varepsilon T_0)+j\sin(m\varepsilon T_0)}{2\pi m}\exp(jm\varepsilon T)\right)\end{aligned}\tag{7}$$

where $k+L+m=n$. Making use of the orthogonal properties of the complex exponential terms, (7) can be simplified as

$$\begin{aligned}\hat{W}_n(x)=\sum_{\forall k+L+m=n}&\hat{u}_k(x)\cdot\frac{(-1)^L}{L\varepsilon}\\&\cdot\frac{-(-1)^m+\cos(m\varepsilon T_0)-j\sin(m\varepsilon T_0)}{2\pi m}.\end{aligned}\tag{8}$$

As a result, by applying the TDWF to the soliton propagation equation a system of first-order partial differential equations is obtained as (9), found at the bottom of the page.

The computation of the FSAT with TDWF together with the nonlinear terms can be performed by solving (9). Comparing with the original FSAT given in (2), there is no increase in the number of first-order partial differential equations. The major effects of the higher order nonlinear terms are to shift the soliton position and to cause the decay of higher order solitons [6]. In order to focus on the elimination of third-order dispersion tails by the use of TDWF, the coefficients of higher order nonlinear terms C and T_R are set to zero for simplicity. Fig. 1 shows the propagation profile of a fundamental soliton pulse for a distance of $x=3$ (i.e., 150 m) in an optical fiber. The step size of propagation is automatically adjusted in order to maintain the calculation accuracy of 10^{-6}. The parameters used in the calculation are $\alpha=0.2$ dB/km, $\beta_2=-2$ ps^2/ km, $\beta_3=0.1$ ps^3/km, $t_0=0.01$ ps, $\varepsilon=0.16$, $2N+1=101, G=1, T_0=8$. The normalized peak power is defined as $|u(x,T)|^2$. It is observed from Fig. 1 that there is an exponential decrease in the peak power of the soliton pulse due to the fiber absorption loss and third-order dispersion. Dispersive tail is obtained at the trailing edge of the soliton pulse and gets longer as the soliton pulse propagates.

In order to compare with the case without using TDWF in the calculation, the results of solving (1) with the original FSAT is shown in Fig. 2. The parameters used in the calculation are identical to that given in Fig 1. It is observed that at the beginning of the propagation, the soliton profiles are the same for both cases. As the soliton pulse further propagates along the fiber ($x>1.5$), the results obtained in Fig. 2 become inaccurate. Instead of a steady change in the soliton peak power, the peak power is observed to be moving up and down and shifting left and right. The sampling density in both cases is $\rho=(2N+1)/(2\pi/\varepsilon)=2.57$ points per unit time. In the Fourier series analysis, there exists a bit period, $T_p=2\pi/\varepsilon$, such that the soliton pulse repeat itself after T_p. The error obtained in Fig. 2 is due to the dispersive tail propagates to the adjacent pulse (i.e., the dispersive tail passed the point at $T=\pi/\varepsilon$). Using the same number of sampling points,

$$\begin{aligned}\frac{\partial\hat{u}_n(x)}{\partial x}=&\underbrace{\hat{u}_n(x)\left[-j\left(\frac{n^2\varepsilon^2}{2}+Bn^3\varepsilon^3\right)-\Gamma\right]}_{\text{linear term}}+\underbrace{j\sum_{\forall\mu-\nu+\lambda=n}\hat{u}_\mu(x)\hat{u}_\nu^*(x)\hat{u}_\lambda(x)}_{\text{nonlinear term}}\underbrace{-jC\varepsilon n\sum_{\forall k-l+m=n}\hat{u}_k(x)\hat{u}_l^*(x)\hat{u}_m(x)}_{\text{nonlinear term}}\\&\underbrace{+T_R\varepsilon\sum_{\forall p+q=n}q\hat{u}_p(x)\Theta_q(x)}_{\text{nonlinear term}}\underbrace{-\sum_{\forall k+L+m=n}G\hat{u}_k(x)\cdot\frac{(-1)^L}{L\varepsilon}\cdot\frac{-(-1)^m+\cos(m\varepsilon T_0)-j\sin(m\varepsilon T_0)}{2\pi m}}_{\text{time window function term}}\end{aligned}\tag{9}$$

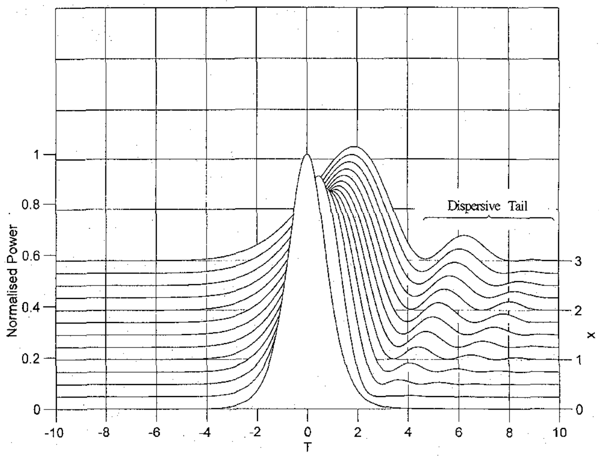

Fig. 1. The 2-D plot of the fundamental soliton propagation in a dispersion-shifted optical fiber with the TDWF employed.

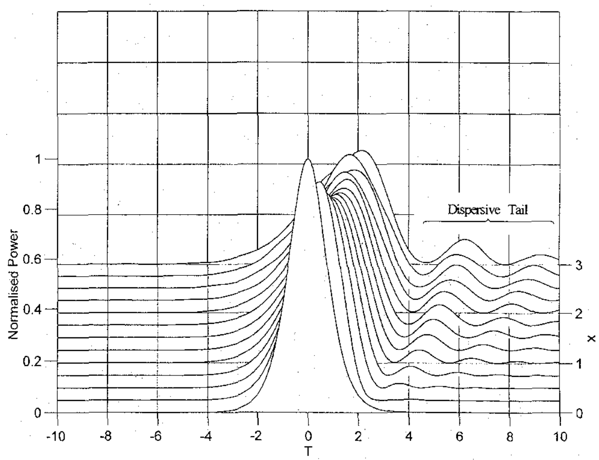

Fig. 2. The 2-D plot of the fundamental soliton propagation in a dispersion-shifted optical fiber without using the TDWF.

FSAT with and without using the TDWF gives roughly the same computational time. However, the FSAT gives a wrong result without using the TDWF unless changing the value of ε to 0.1 and $2N+1$ to 161 (i.e., increase the bit period and maintain the sampling density) correct result can be obtained. The computational time of a 133 MHz Pentium PC requires for this calculation is 296 min whereas only 72 min is required for the same accuracy with the TDWF employed. As a result, the computational time can be reduced by 75% as TDWF is used.

The parameter G given in (3) is defined as the slope of the TDWF, the larger the value of G, the larger the attenuation on the dispersive tail can be obtained. Fig. 3 shows the soliton profile obtained at $x = 3$ (after 150 m of propagation) for various magnitudes of G. For the case without the TDWF (i.e., $G = 0$), the calculated soliton profile is distorted. By increasing the value of G to 0.1, the distortion level is found

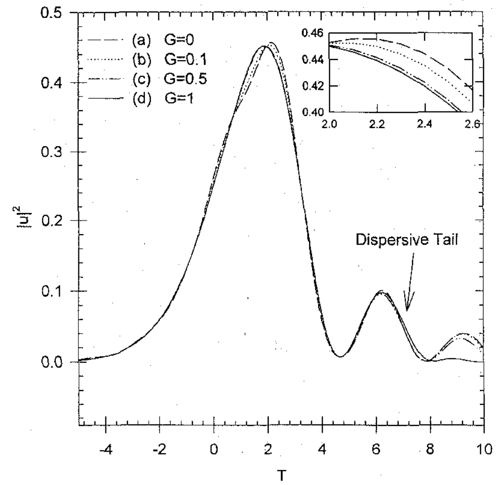

Fig. 3. The soliton waveforms obtained after a propagation distance of $x = 3$ in a dispersion-shifted fiber of (a) $G = 0$, (b) $G = 0.1$, (c) $G = 0.5$ and (d) $G = 1$ with $\varepsilon = 0.16$ for all cases.

to be reduced. At $G = 1$, correct result can be obtained and the dispersive tail is almost vanished at $T = 10$ and the soliton profile remains unchanged for G greater than 1. As a result, $G = 1$ is the optimum value that we should choose for the calculation.

In conclusion, the efficiency of FSAT in solving the soliton propagation along an optical fiber with absorption loss and higher order dispersion is improved significantly by the TDWF. The advantage of using TDWF is that the computational time can be reduced by 75% or more with the increase of propagation distance. This is because original FSAT requires more sampling points as well as computational time to achieve the same accuracy. Furthermore, any dispersive component due to the fiber absorption loss and third-order dispersion can also be eliminated by the TDWF. The slope G of the TDWF determines the attenuation strength on the dispersive tail of the soliton pulse and the optimum value of G is found to be 1. This is because for G less than 1 gives inaccurate result, however, for G greater than 1 the calculation efficiency does not improve significantly. Therefore the efficiency and accuracy of FSAT can both be improved by using the TDWF.

REFERENCES

[1] L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, "Experimental observation of picosecond pulse narrowing and solitons in optical fibers," *Phys. Rev. Lett.*, vol. 45, no. 13, pp. 1095–1098, 1980.

[2] R. H. Hardin and F. D. Tappert, "Applications of the split-step fourier method to the numerical solution of nonlinear and variable coefficient wave equations," *SIAM Rev.*, vol. 15, pp. 423–423, 1973.

[3] H. Ghafouri-Shiraz and P. Shum, "A novel method for analysis of soliton propagation in optical fibers," *IEEE J. Quantum Electron.*, vol. 31, no. 1, pp. 190–200, 1995.

[4] G. P. Agrawal, "Effect of intrapulse stimulated raman scattering on soliton-effect pulse compression in optical fibers," *Opt. Lett.*, vol. 15, no. 4, pp. 224–226, 1990.