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## NOTE

# FURTHER INSPECTION OF THE STOCHASTIC GROWTH MODEL BY AN ANALYTICAL APPROACH

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It has been argued that a clear understanding of the stochastic growth model can best be achieved by working out an approximate analytical solution. This paper follows that idea but streamlines the presentation of the loglinear approximate solution for the neoclassical model of capital accumulation. By focusing on the partial elasticity of capital stock with respect to its lag term, this paper is able to confirm analytically some conclusions based on numerical calculations in previous papers, and to clarify why a simpler solution arises in several special cases.

**Keywords:** Stochastic Growth, Approximate Analytical Solution

## 1. INTRODUCTION

The neoclassical model of capital accumulation has been the main framework for examining long-run growth issues for several decades. Both the version with constant saving rates [Solow (1956), Swan (1956)] and the version with a representative agent maximizing an intertemporal utility function [Cass (1965), Koopmans (1965)] have been extensively examined and applied. The model has gained further popularity recently as the advocates of the real-business-cycle approach [e.g., Prescott (1986), King et al. (1988)] suggest applying the dynamic stochastic perfectly competitive general equilibrium framework (usually extended to include labor–leisure choice and other relevant variables such as productivity shock and government expenditure) to examine short-run fluctuations together with long-run growth.

Although stochastic growth models have become more and more popular in macroeconomics, non-economists and perhaps even economists in other specialities do not seem to become more familiar with the details of the real-business-cycle approach. Part of the reason is that most papers adopting this approach rely on computational methods to obtain the equilibrium solution. As mentioned by Campbell

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(1994) and Romer (1996), the mixture of linear and loglinear elements in the model makes it impossible to obtain an analytical solution in general, and thus necessary to resort to the use of computational methods for quantitative analysis. The only known case with an exact analytical solution occurs when intertemporal elasticity of substitution is one and capital depreciates completely in one period [Long and Plosser (1983)]. Unfortunately, the usefulness of this example is limited because the assumption of complete depreciation of capital in one period is extremely unrealistic.

Campbell (1994) argues that an analytical approach would convey better than a purely computational approach the usefulness of the stochastic growth model to the reader. To overcome the impossibility of obtaining an exact analytical solution for more general cases, he uses the loglinear approximation, which gives the correct solution in the special case that can be solved exactly. Specifically, he derives an approximate analytical solution by loglinearizing the Euler equation and the intertemporal resource constraint.

This paper follows the preceding idea but streamlines the derivation of the loglinear approximate solution for the neoclassical model of capital accumulation. Besides having a shorter presentation (which is, arguably, also easier to follow), the slightly modified method used in this paper enables (i) the derivation of further analytical results to confirm two conclusions based on computational methods in previous papers and (ii) the clarification of why simpler analytical solution of the stochastic growth model arises in several special cases.

The remaining parts of this note are organized as follows: Section 2 investigates the neoclassical model of capital accumulation (with fixed labor supply), Section 3 focuses on the analytical results, and Section 4 concludes.

## 2. LOGLINEAR APPROXIMATE SOLUTION TO NEOCLASSICAL MODEL OF CAPITAL ACCUMULATION

This section considers the loglinear approximation [as in King et al. (1988), Campbell (1994)] to the neoclassical growth model with optimizing agents [Cass (1965), Koopmans (1965)]. In particular, with some standard assumptions on the functional forms about technology and preference, it derives an approximate analytical solution by loglinearizing the Euler equation and the intertemporal resource constraint around the steady-state growth path, so that quantitative issues can be examined.

The closed economy is populated by a constant and large number of identical agents (consumer-producers). The production technology is represented by a standard Cobb–Douglas production function,

$$Y_t = A_t F\left(K_t, (1+\gamma_x)^t N_t\right) = A_t\left[(1+\gamma_x)^t N_t\right]^{\alpha} K_t^{1-\alpha} = (1+\gamma_x)^{\alpha t} A_t K_t^{1-\alpha}, \tag{1}$$

where $\gamma_x \geq 0$, $\alpha (0 < \alpha < 1)$ is the exponent on labor in the production function, the (inelastically supplied) labor input at each period $N_t$ has been normalized to be 1, and $Y_t$ and $K_t$ are, respectively, the output and the associated capital input at period $t$.[1] (Because labor input has been normalized to 1, $Y_t$ and $K_t$ can also be interpreted, respectively, as output per worker and capital per worker.) Technological changes have been decomposed into two components: deterministic and stochastic. The deterministic component is labor augmenting, with the rate of exogenous technological progress given by $\gamma_x$. On the other hand, $A_t$ is a random shock to technology, which is assumed to be a first-order autoregressive process,

$$(\ln A_t - \ln A) = \phi(\ln A_{t-1} - \ln A) + \varepsilon_t, \tag{2}$$

where $-1 \le \phi \le 1$. The coefficient $\phi$ measures the persistence of technology shocks, and $\ln A$ is the unconditional mean of $\ln A_t$.

On the preference side, it is assumed that the representative agent chooses a consumption path to maximize expected lifetime utility,

$$E_t \sum_{j=0}^{\infty} \beta^j U(C_{t+j}) = E_t \sum_{j=0}^{\infty} \beta^j \frac{C_{t+j}^{1-\frac{1}{\sigma}} - 1}{1 - \dfrac{1}{\sigma}}, \tag{3}$$

where $E_t$ is the expectation operator conditional on the information set at period $t$, $\beta$ is the subjective time discount factor, $\sigma\,(\sigma > 0)$ is the intertemporal elasticity of substitution, and $C_t$ is consumption at period $t$. The intertemporal resource constraint is described by

$$K_{t+1} = (1 - \delta)K_t + Y_t - C_t, \tag{4}$$

where $\delta$ $(0 \le \delta \le 1)$ is the depreciation rate per period, and the initial level of capital is given.

It is well known that along the (stochastic) steady-state growth path, output, consumption, and capital all grow at a average rate of $\gamma_x$. To transform a growing economy (when $\gamma_x > 0$) to a "no-growth" economy, define the variable per unit of effective labor as the variable per worker divided by $(1 + \gamma_x)^t$, and denote it by lowercase letter [such as $c_t = C_t/(1 + \gamma_x)^t$]. With this transformation, (1) and (4) lead to

$$(1 + \gamma_x)k_{t+1} = (1 - \delta)k_t + A_t k_t^{1-\alpha} - c_t. \tag{5}$$

Moreover, it is easy to show that the representative agent is maximizing an equivalent intertemporal utility function defined in terms of consumption per unit of effective labor,

$$E_t \sum_{j=0}^{\infty} (\beta_x)^j \frac{c_{t+j}^{1-\frac{1}{\sigma}} - 1}{1 - \dfrac{1}{\sigma}}, \tag{6}$$

where the effective discount factor $\beta_x$ differs from $\beta$ due to the transformation of the preference specification and is given by

$$\beta_x = \beta(1 + \gamma_x)^{1-\frac{1}{\sigma}}. \tag{7}$$

[See, e.g., King et al. (1988, Sec. 2.4) regarding the transformation of the preference specification.] To guarantee finiteness of lifetime utility, the restriction $\beta_x < 1$ is required. Note that, when $\sigma = 1$, the constant intertemporal elasticity-of-substitution utility function in (3) becomes $\ln C_{t+j}$, and the corresponding utility function in (6) becomes $\ln c_{t+j}$.

The first-order condition for optimal choice of this model, given the objective function (6), and the constraints (2) and (5), is given by

$$\beta_x E_t \left\{ c_{t+1}^{-\frac{1}{\sigma}} \left[ 1 - \delta + (1-\alpha) A_{t+1} k_{t+1}^{-\alpha} \right] \right\} = (1+\gamma_x) c_t^{-\frac{1}{\sigma}}. \tag{8}$$

The optimal solution of the economy, in terms of consumption, capital, and technology shock, is described by the system of three nonlinear expectational difference equations: (2), (5), and (8).[2] The optimal choice also satisfies a transversality condition.

For the nonlinear system of optimal choice in this economy, an exact analytical solution is not possible except for the special case that $\delta = 1$ and $\sigma = 1$ [Long and Plosser (1983), McCallum (1989)]. Campbell (1994) argues that an analytical approach to the stochastic growth model generates important insights about the dynamic effects of different shocks to the economy, and suggests loglinear approximations of the Euler equation and the intertemporal resource constraint. The following analysis also adopts this approach, but modifies the method slightly and presents further results based on an analytical approach. In particular, the following analysis expresses the approximate solution in terms of fundamental preference and technology parameters, instead of the real interest rate used by Campbell (1994). Although expressing the approximate solution in terms of the real interest rate is useful for numerical calculations [as done by Campbell (1994, Sec. 2.6) and, indirectly, in Table 1 of this paper], expressing the solution in terms of fundamental parameters allows the derivation of (23), below. That equation makes it possible to obtain analytical results regarding the effects of changes in fundamental parameters, and to show that simpler solutions of the stochastic growth model arise in some special cases.

To obtain the loglinear approximation, first use the Euler equation and the intertemporal resource constraint to get the (nonstochastic) steady-state values of $k$ and $c$, when $A_t = A$. Equations (5) and (8) imply that the steady-state values are related by

$$1 - \delta + (1-\alpha) A k^{-\alpha} = \frac{1+\gamma_x}{\beta_x}, \tag{9}$$

and

$$c = \left[ \frac{1 + \gamma_x - \beta_x(1-\delta)}{\beta_x(1-\alpha)} - (\delta + \gamma_x) \right] k. \tag{10}$$

Next, linearize (5) and (8) around $\ln k$, $\ln c$, and $\ln A$. Linearizing (8) leads to

$$E_t[(\ln c_{t+1} - \ln c) + \theta_{ck}(\ln k_{t+1} - \ln k) - \theta_{cA}(\ln A_{t+1} - \ln A)] = (\ln c_t - \ln c), \tag{11}$$

where

$$\theta_{ck} = \sigma\alpha[1 - \beta_x(1 - \delta_x)], \tag{12}$$

$$\theta_{cA} = \sigma[1 - \beta_x(1 - \delta_x)], \tag{13}$$

and $\delta_x$, which can be interpreted as the effective depreciation rate for the transformed problem, is defined as

$$\delta_x = \frac{\delta + \gamma_x}{1 + \gamma_x}. \tag{14}$$

It is easy to see that $0 \le \delta_x \le 1$. Similarly, linearizing (5) leads to

$$\ln k_{t+1} - \ln k = \theta_{kk}(\ln k_t - \ln k) - \theta_{kc}(\ln c_t - \ln c) + \theta_{kA}(\ln A_t - \ln A), \tag{15}$$

where

$$\theta_{kk} = \frac{1}{\beta_x}, \tag{16}$$

$$\theta_{kc} = \left[\frac{1 - \beta_x(1 - \delta_x\alpha)}{\beta_x(1 - \alpha)}\right], \tag{17}$$

$$\theta_{kA} = \left[\frac{1 - \beta_x(1 - \delta_x)}{\beta_x(1 - \alpha)}\right]. \tag{18}$$

Equations (2), (11), and (15) form a system of *loglinear* expectational difference equations in consumption, capital stock, and the technology shock. This system represents an *approximation* to the system of *nonlinear* expectational difference equations of the same variables: (2), (8), and (5).

There are many ways to solve the above loglinear dynamic system. To compare with the results of Campbell (1994), this section uses the method of undetermined coefficients. However, the emphasis is on the partial elasticity of capital stock with respect to its lag term, $\eta_{kk}$, defined as (20) below, or as Equation (20) in Campbell (1994). This coefficient is focused because the results of Lau (1999) regarding the presence of unit roots in stochastic endogenous growth models suggest that $\eta_{kk} = 1$ as $\alpha$ tends to 0. (The limiting case of the neoclassical model of capital accumulation with no exogenous technological progress as $\alpha$ tends to zero is an endogenous growth model; see note 1 also.) Moreover, it is observed from Table 1 of Campbell (1994) that $\eta_{kk}$ tends to 1 as $\sigma$ tends to 0, but it is not clear whether there are similar special results for other coefficients such as the partial elasticity of consumption with respect to current capital, as given in Equation (19) of Campbell (1994). Focusing on the equation of motion of the capital stock is likely to yield more interesting results.

To this purpose, the following analysis first transforms the system of first-order difference equations in capital and consumption, (11) and (15), into a second-order difference equation in capital:

$$E_t\left\{\frac{1}{\theta_{kc}}[(\ln k_{t+2}-\ln k)-\theta_{kk}(\ln k_{t+1}-\ln k)-\theta_{kA}(\ln A_{t+1}-\ln A)]\right.$$
$$\left.-\theta_{ck}(\ln k_{t+1}-\ln k)+\theta_{cA}(\ln A_{t+1}-\ln A)\right\}$$
$$=\frac{1}{\theta_{kc}}[(\ln k_{t+1}-\ln k)-\theta_{kk}(\ln k_t-\ln k)-\theta_{kA}(\ln A_t-\ln A)]. \quad \textbf{(19)}$$

Second, guess that the solution of capital in terms of its lag and the technology shock as

$$(\ln k_{t+1}-\ln k)=\eta_{kk}(\ln k_t-\ln k)+\eta_{kA}(\ln A_t-\ln A), \quad \textbf{(20)}$$

where $\eta_{kk}$ and $\eta_{kA}$ are unknown coefficients to be determined.

Substituting (20) into (19) and using the result $E_t(\ln A_{t+1}-\ln A)=\phi(\ln A_t-\ln A)$, which can be derived from (2), lead to the following equation:

$$\eta_{kk}^2(\ln k_t-\ln k)+\eta_{kk}\eta_{kA}(\ln A_t-\ln A)+\eta_{kA}\phi(\ln A_t-\ln A)$$
$$-\theta_{kk}[\eta_{kk}(\ln k_t-\ln k)+\eta_{kA}(\ln A_t-\ln A)]-\theta_{kA}\phi(\ln A_t-\ln A)$$
$$-\theta_{kc}\theta_{ck}[\eta_{kk}(\ln k_t-\ln k)+\eta_{kA}(\ln A_t-\ln A)]+\theta_{kc}\theta_{cA}\phi(\ln A_t-\ln A)$$
$$=\eta_{kk}(\ln k_t-\ln k)+\eta_{kA}(\ln A_t-\ln A)-\theta_{kk}(\ln k_t-\ln k)$$
$$-\theta_{kA}(\ln A_t-\ln A). \quad \textbf{(21)}$$

The two unknown coefficients are obtained from (21) as follows: First equate coefficients on $(\ln k_t-\ln k)$ to find $\eta_{kk}$, and then equate coefficients on $(\ln A_t-\ln A)$ to find $\eta_{kA}$, given $\eta_{kk}$.

Equating the coefficients on $(\ln k_t-\ln k)$ gives the following quadratic equation:

$$\eta_{kk}^2-(1+\theta_{kk}+\theta_{kc}\theta_{ck})\eta_{kk}+\theta_{kk}=0. \quad \textbf{(22)}$$

It can be shown that there are two unequal positive real roots to this quadratic equation. Moreover, the larger root is excluded since the transversality condition would otherwise be violated.[3] Therefore, $\eta_{kk}$ is given by

$$\eta_{kk}=\frac{(1+\theta_{kk}+\theta_{kc}\theta_{ck})-\sqrt{(1+\theta_{kk}+\theta_{kc}\theta_{ck})^2-4\theta_{kk}}}{2}. \quad \textbf{(23)}$$

Since $\sigma>0$, $0<\beta_x<1$, $0<\alpha<1$, and $0\leq\delta_x\leq 1$, it is easy to show from (12), (16), and (17) that $\theta_{ck}>0$, $\theta_{kk}>0$, and $\theta_{kc}>0$. As a result, $\eta_{kk}$ in (23) is between 0 and 1. Once $\eta_{kk}$ is found, $\eta_{kA}$ can be obtained by

**TABLE 1.** Numerical values of $\eta_{kk}$ for a quarterly model[a]

| | $\sigma$ | | | | | |
|---|---|---|---|---|---|---|
| $\alpha$ | 0.2 | 0.5 | 1 | 1.5 | 2 | 5 |
| 0.2 | 0.997 | 0.995 | 0.992 | 0.989 | 0.987 | 0.977 |
| 0.33 | 0.995 | 0.990 | 0.985 | 0.981 | 0.977 | 0.962 |
| 0.58 | 0.987 | 0.978 | 0.967 | 0.959 | 0.952 | 0.922 |
| 0.67 | 0.983 | 0.971 | 0.957 | 0.947 | 0.938 | 0.902 |

[a]The number in each cell is $\eta_{kk}$. Parameter $\sigma$ is the intertemporal elasticity of substitution and $\alpha$ is the exponent on labor in the production function. The assumed values of other parameters are $\delta = 0.025$ and $\gamma_x = 0.005$. Moreover, to make the steady-state real interest rate equal to 0.015, the implied value of $\beta_x$ is 0.990.

$$\eta_{kA} = \frac{\theta_{kc}\theta_{cA}\phi + \theta_{kA}(1-\phi)}{1-\phi+\theta_{kk}-\eta_{kk}+\theta_{kc}\theta_{ck}}. \tag{24}$$

Equation (23) shows that $\eta_{kk}$ depends on parameters $\beta_x$, $\sigma$, $\alpha$, and $\delta_x$ (with $\delta_x$ further depending on $\delta$ and $\gamma_x$).[4] Table 1 reports some numerical values of $\eta_{kk}$ for a quarterly model. The focus is on the variation of $\eta_{kk}$ with respect to $\sigma$ and $\alpha$. The value of the intertemporal elasticity of substitution ($\sigma$) is set at different values to cover a wide range of possibilities. Regarding the exponent on labor in the production function ($\alpha$), the baseline value is 0.67. Other values examined in Table 1 include 0.58 [as used by King et al. (1988)], 0.33 [as suggested by Mankiw et al. (1992) for a neoclassical growth model augmented with human capital], and 0.2 [for a model with a broad concept of capital, so as to be consistent with the convergence speed of 0.02 at an annual rate reported by Barro and Sala-i-Martin (1992)]. The other parameters are chosen to resemble the long-run behavior of the U.S. economy; see Campbell (1994) also. Specifically, it is assumed that $\delta = 0.025$ (10% at an annual rate) and $\gamma_x = 0.005$ (2% at an annual rate). Moreover, $\beta_x$ is chosen so as to make the steady-state real interest rate equal to 0.015 (6% at an annual rate).

First, it is observed from Table 1 that $\eta_{kk}$ decreases with respect to $\sigma$. Campbell (1994, p. 474) also reports this pattern based on numerical calculations (which are the same as those in row 4 of Table 1, with $\alpha = 0.67$). He discusses the responses of various coefficients (including the partial elasticity of capital stock with respect to lag capital) in terms of income and substitution effects as the intertemporal elasticity of substitution changes. Second, it is observed from Table 1 that $\eta_{kk}$ decreases with respect to $\alpha$. For the neoclassical growth model with nonstochastic technology, Barro and Sala-i-Martin (1992, p. 226) present some numerical calculations to show that the rate of convergence to the steady-state growth path is increasing in the exponent of labor in the production function. Because $\eta_{kk}$ is related negatively to the rate of convergence when technology is nonstochastic, the numerical calculations presented in Table 1 are consistent with their results.

## 3. FURTHER ANALYTICAL RESULTS

One advantage of the analytical approach used in this paper is its potential ability to confirm conjectures suggested by computational results. Specifically, since $\eta_{kk}$ in (23) is expressed in terms of the underlying parameters, it can be shown that

$$\frac{\partial \eta_{kk}}{\partial \sigma} = \frac{1}{2}\left[1 - \frac{1+\theta_{kk}+\theta_{kc}\theta_{ck}}{\sqrt{(1+\theta_{kk}+\theta_{kc}\theta_{ck})^2 - 4\theta_{kk}}}\right]\theta_{kc}\frac{\partial \theta_{ck}}{\partial \sigma}. \quad \textbf{(25)}$$

Since $\theta_{kk} > 0, \theta_{kc} > 0, \theta_{ck} > 0$, and $\partial\theta_{ck}/\partial\sigma = \alpha[1-\beta_x(1-\delta_x)] > 0$, it is easy to see that $\partial\eta_{kk}/\partial\sigma < 0$. The partial elasticity of capital with respect to its lag term decreases monotonically in the intertemporal elasticity of substitution.

Similarly,

$$\frac{\partial \eta_{kk}}{\partial \alpha} = \frac{1}{2}\left[1 - \frac{1+\theta_{kk}+\theta_{kc}\theta_{ck}}{\sqrt{(1+\theta_{kk}+\theta_{kc}\theta_{ck})^2 - 4\theta_{kk}}}\right]\left(\theta_{kc}\frac{\partial \theta_{ck}}{\partial \alpha} + \theta_{ck}\frac{\partial \theta_{kc}}{\partial \alpha}\right), \quad \textbf{(26)}$$

which is negative, since $\theta_{kk} > 0, \theta_{kc} > 0, \theta_{ck} > 0, \partial\theta_{ck}/\partial\alpha = \sigma[1-\beta_x(1-\delta_x)] > 0$, and $\partial\theta_{kc}/\partial\alpha = [1-\beta_x(1-\delta_x)]/[\beta_x(1-\alpha)^2] > 0$. The partial elasticity of capital with respect to its lag term decreases monotonically in the exponent of labor in the production function. The pattern of the variation of $\eta_{kk}$ with respect to $\sigma$ and $\alpha$ based on numerical calculations in Table 1 is confirmed analytically.

The analysis of the stochastic growth model also leads to simple solutions in some special cases. The first case is that as $\sigma$ tends to 0, $\eta_{kk}$ in (23) tends to 1. This (zero intertemporal elasticity of substitution) case has been shown numerically in Table 1 of Campbell (1994), and its relation to the permanent-income hypothesis has been discussed. A similar second case is when $\alpha$ tends to zero (and $\gamma_x = 0$). In this (endogenous growth) case, $\eta_{kk}$ also tends to 1. When either $\sigma$ or $\alpha$ tends to 0, it can easily be observed from (12) that $\theta_{ck}$ tends to 0. As a result, $1+\theta_{kk}+\theta_{kc}\theta_{ck}$ tends to $\theta_{kk}+1$, $\sqrt{(1+\theta_{kk}+\theta_{kc}\theta_{ck})^2-4\theta_{kk}}$ tends to $\theta_{kk}-1$, and $\eta_{kk}$ in (23) tends to 1.

The third case is when $\delta = 1$ and $\sigma = 1$ [as considered by Long and Plosser (1983)]. In this case, an exact analytical solution is possible because the equilibrium consumption/output ratio turns out to be constant. Using various formulas in Section 2, it can be shown that $\theta_{ck} = \alpha$ and $\theta_{kc} = [1-\beta_x(1-\alpha)]/[\beta_x(1-\alpha)]$. Therefore,

$$1+\theta_{kk}+\theta_{kc}\theta_{ck} = \frac{1}{\beta_x(1-\alpha)} + (1-\alpha),$$

and

$$\sqrt{(1+\theta_{kk}+\theta_{kc}\theta_{ck})^2 - 4\theta_{kk}} = \frac{1}{\beta_x(1-\alpha)} - (1-\alpha).$$

As a result, the complicated formula of (23) is reduced to a simple result that $\eta_{kk} = 1-\alpha$. Moreover, formula (24) gives $\eta_{kA} = 1$.[5]

## 4. CONCLUSION

Campbell (1994) suggests that an analytical approach would convey better the usefulness of stochastic growth models. An approximate analytical solution is able to provide more insights about the dynamic effects of the underlying shocks. It is also useful to provide analytical results regarding some economic questions, such as how the convergence rate of a nonstochastic growth model is related to the fundamental parameters.

This paper continues on working from an analytical approach, and streamlines the presentation of the loglinear approximate solution for the basic neoclassical model of capital accumulation by modifying the analysis of King et al. (1988) and Campbell (1994). It first derives the nonlinear system of optimal choice, and then loglinearizes the Euler equation and the intertemporal resource constraint to obtain the approximate system. In this respect, the approach follows that of Campbell (1994). However, it differs from Campbell's paper in (i) focusing on the partial elasticity of capital stock with respect to its lag term instead of the partial elasticity of consumption with respect to current capital, and (ii) expressing the various elasticities in terms of the underlying preference and technology parameters instead of the real interest rate. As a result, this paper is able to show that simpler analytical solution arises in several special cases (because the term inside the square root in the right-hand side of (23) can be simplified). Furthermore, it shows analytically that the partial elasticity of capital with respect to its lag term (which is negatively related to the convergence rate of the nonstochastic neoclassical growth model) is decreasing in the exponent of labor in the production function ($\alpha$) and in the intertemporal elasticity of substitution ($\sigma$). These analytical results provide confirmation to some conclusions based on numerical calculations in, among others, Barro and Sala-i-Martin (1992) and Campbell (1994).

### *NOTES*

1. The limiting case of this model with no exogenous technological progress ($\gamma_x = 0$) as $\alpha$ tends to zero is a stochastic version of the AK endogenous growth model [Rebelo (1991)]. Note that the specification in (1) is similar, but not identical, to Equation (1) of Campbell (1994). If the latter specification is used instead, then the limiting case as $\alpha$ tends to zero and $\gamma_x = 0$ is the nonstochastic AK model.

2. To economize on the use of words, consumption per worker, consumption per effective unit of labor, deviation of consumption per effective unit of labor from the steady-state value, etc., are simply represented by the term "consumption" in this paper, whenever the precise meaning can easily be understood in the context.

3. It can be shown that the larger root of (22) is larger than $\theta_{kk} = (\beta_x)^{-1}$; see also King et al. (1988, p. 206). On the other hand, Campbell (1994, p. 471) mentions that his Equation (26) is chosen because the resulting steady state is locally stable. Equivalently, the other (unstable) root is ruled out so that the transversality condition would not be violated.

4. On the other hand, $\eta_{kA}$ in (24) depends on these parameters as well as $\phi$. In unreported numerical calculations as well as in Campbell (1994, Table 1), it can be observed that $\eta_{kA}$ is not monotonic with respect to either $\sigma$ or $\phi$. As there is no clear pattern in the variation of $\eta_{kA}$ with respect to the parameters, the remaining analysis only focuses on $\eta_{kk}$.

5. Combining (15) and (20) gives $(\ln c_t - \ln c) = [(\theta_{kk} - \eta_{kk})/\theta_{kc}](\ln k_t - \ln k) + [(\theta_{kA} - \eta_{kA})/\theta_{kc}](\ln A_t - \ln A)$. When $\delta = 1$ and $\sigma = 1$, this equation is simplified to $(\ln c_t - \ln c) = (1-\alpha)(\ln k_t - \ln k) + (\ln A_t - \ln A)$. Since the production function (1) implies $(\ln y_t - \ln y) = (\ln A_t - \ln A) + (1-\alpha)(\ln k_t - \ln k)$, it is easy to see that consumption/output ratio is constant.

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