Maximum mass-radius ratios for charged compact general relativistic objects

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Abstract. – Upper limits for the mass-radius ratio and total charge are derived for stable charged general relativistic matter distributions. For charged compact objects the mass-radius ratio exceeds the value 4/9 corresponding to neutral stars. General restrictions for the red shift and total energy (including the gravitational contribution) are also obtained.

It is generally accepted today that black holes are uniquely characterized by their total mass energy E, charge Q and angular momentum J [1]. Most of the investigations of the astrophysical objects have been done under the assumption of the electric charge neutrality of the stellar matter. However, as a result of accretion neutron stars can acquire a net charge, if accretion produces luminosity close to the Eddington limit L_E [2]. Let us consider a star of mass M that undergoes spherical accretion and assume, for simplicity, that the accreting material is ionized hydrogen. If the accreting luminosity is L, the infalling electrons experience a radiative force $F_R = \frac{\sigma_T L}{4\pi c r^2}$, where σ_T is the Thomson cross-section. Since the radiation drag acting on the protons is a factor $(m_e/m_p)^2$ smaller, electrons and protons are subject to different accelerations, and the star acquires a net positive charge $Q = \frac{GMm_p}{e}\frac{L}{L_E}$ [3], where $L_E = 4\pi GMm_p c/\sigma_T$ is the Eddington luminosity. The astrophysical conditions under which this phenomenon can take place are rather extreme but in principle they could lead to a charged astrophysical configuration. This mechanism has been recently proposed, via vacuum breakdown near a charged black hole, as a source of γ-ray bursts [4]. A phase transition of neutron matter to quark matter at zero temperature or temperatures small compared to degeneracy temperature allows the existence of hybrid stars, *i.e.* stars having a quark core and a crust of neutron matter [5]. In fact, quark matter with electrically charged constituents rather than neutron matter could hold the large magnetic field of the pulsars [6] and hence it

is possible that for strange-matter–made stars the effects of the non-zero electrical charge be important.

By using the static spherically symmetric gravitational field equations, Buchdahl [7] has obtained an absolute constraint of the maximally allowable mass M-radius R for isotropic fluid spheres of the form $\frac{2M}{R} < \frac{8}{9}$ (where natural units $c = G = 1$ have been used).

It is the purpose of the present letter to obtain the maximum allowable mass-radius ratio in the case of stable charged compact general relativistic objects. This is achieved by generalizing to the charged case the method described for neutral stars in Buchdahl [7] and Straumann [8].

For a static general relativistic spherically symmetric configuration the interior line element is given by $\mathrm{d}s^2 = e^\nu \, \mathrm{d}t^2 - e^\lambda \, \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2)$.

The properties of the charged compact object can be completely described by the structure equations, which are given by [9]:

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\rho r^2 + \frac{Q}{r}\frac{\mathrm{d}Q}{\mathrm{d}r}, \tag{1}$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(\rho + p)\left(m + 4\pi r^3 p - \frac{Q^2}{r}\right)}{r^2\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)} + \frac{Q}{4\pi r^4}\frac{\mathrm{d}Q}{\mathrm{d}r}, \tag{2}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = \frac{2\left(m + 4\pi r^3 p - \frac{Q^2}{r}\right)}{r^2\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)}, \tag{3}$$

where ρ is the energy density of the matter, p is the thermodynamic pressure, $m(r)$ is the mass and $Q(r) = 4\pi \int_0^r e^{\frac{\nu+\lambda}{2}} r^2 j^0 \, \mathrm{d}r$ is the charge inside radius r, respectively. The electric current inside the star is given by $j^i = (j^0, 0, 0, 0)$.

The structure equations (1)–(3) must be considered together with the boundary conditions $p(R) = 0$, $p(0) = p_\mathrm{c}$ and $\rho(0) = \rho_\mathrm{c}$, where ρ_c, p_c are the central density and pressure, respectively.

With the use of eqs. (1)–(3), it is easy to show that the function $\zeta = e^{\frac{\nu}{2}} > 0$, $\forall r \in [0, R]$ obeys the equation

$$\sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}\frac{1}{r}\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right] = \frac{\zeta}{r}\left[\frac{\mathrm{d}}{\mathrm{d}r}\frac{m}{r^3} + \frac{Q^2}{r^5}\right]. \tag{4}$$

For $Q = 0$ we obtain the equation considered in [8]. Since the density ρ does not increase with increasing r, the mean density of the matter $\langle\rho\rangle = \frac{3m}{4\pi r^3}$ inside radius r does not increase either. Therefore we assume that inside a compact general relativistic object the condition $\frac{\mathrm{d}}{\mathrm{d}r}\frac{m}{r^3} < 0$ holds independently of the equation of state of dense matter. By defining a new function

$$\eta(r) = \int_0^r \frac{r'}{\sqrt{1 - \frac{2m(r')}{r'} + \frac{Q^2(r')}{r'^2}}}\left[\int_0^{r'} \frac{Q^2(r'')\zeta(r'')}{r''^5\sqrt{1 - \frac{2m(r'')}{r''} + \frac{Q^2(r'')}{r''^2}}}\, \mathrm{d}r''\right]\mathrm{d}r', \tag{5}$$

denoting $\Psi = \zeta - \eta$, and introducing a new independent variable $\xi = \int_0^r r'\left(1 - \frac{2m(r')}{r'} + \frac{Q^2(r')}{r'^2}\right)^{-\frac{1}{2}}\mathrm{d}r'$ [8], from eq. (5) we obtain the basic result that inside all stable stellar-type charged general relativistic matter distributions the condition $\frac{\mathrm{d}^2\Psi}{\mathrm{d}\xi^2} < 0$ must hold for all $r \in [0, R]$. Using the mean-value theorem, we conclude $\frac{\mathrm{d}\Psi}{\mathrm{d}\xi} \leq \frac{\Psi(\xi) - \Psi(0)}{\xi}$, or, taking into

account that $\Psi(0) > 0$ it follows that:

$$\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}\xi} \leq \frac{1}{\xi}.$$ (6)

In the initial variables inequality (6) takes the form

$$\frac{1}{r}\left(1 - \frac{2m(r)}{r} + \frac{Q^2(r)}{r^2}\right)^{\frac{1}{2}}\left[\frac{1}{2}\frac{\mathrm{d}\nu}{\mathrm{d}r}e^{\frac{\nu(r)}{2}} - \frac{r}{\sqrt{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}}\int_0^r \frac{Q^2(r')e^{\frac{\nu(r')}{2}}}{r'^5\sqrt{1 - \frac{2m(r')}{r'} + \frac{Q^2(r')}{r'^2}}}\mathrm{d}r'\right] \leq$$

$$\leq \frac{e^{\frac{\nu(r)}{2}} - \int_0^r r'\left(1 - \frac{2m(r')}{r'} + \frac{Q^2(r')}{r'^2}\right)^{-\frac{1}{2}}\left[\int_0^{r'}\left(1 - \frac{2m(r'')}{r''} + \frac{Q^2(r'')}{r''^2}\right)^{-\frac{1}{2}}\frac{Q^2(r'')e^{\frac{\nu(r'')}{2}}}{r''^5}\mathrm{d}r''\right]\mathrm{d}r'}{\int_0^r r'\left(1 - \frac{2m(r')}{r'} + \frac{Q^2(r')}{r'^2}\right)^{-\frac{1}{2}}\mathrm{d}r'}.$$ (7)

In the following we denote $\alpha(r) = 1 - \frac{Q^2(r)}{2m(r)r}$. For stable stellar-type compact objects, $\frac{m}{r^3}$ does not increase outwards. We suppose that for all $r' \leq r$ we have $\frac{\alpha(r')m(r')}{r'} \geq \frac{\alpha(r)m(r)}{r}(\frac{r'}{r})^2$ or, equivalently, $\frac{2m(r')}{r'} - \frac{2m(r)}{r}(\frac{r'}{r})^2 \geq \frac{Q^2(r')}{r'^2} - \frac{Q^2(r)}{r^2}(\frac{r'}{r})^2$.

We assume that inside the compact stellar object the charge $Q(r)$ satisfies the general condition

$$\frac{Q(r'')e^{\frac{\nu(r'')}{2}}}{r''^5} \geq \frac{Q(r')e^{\frac{\nu(r')}{2}}}{r'^5} \geq \frac{Q(r)e^{\frac{\nu(r)}{2}}}{r^5}, \quad r'' \leq r' \leq r.$$ (8)

Therefore we can evaluate the terms in eq. (7) as follows. For the term in the denominator of the r.h.s. of eq. (7) we obtain

$$\left[\int_0^r r'\left(1 - \frac{2m(r')}{r'} + \frac{Q^2(r')}{r'^2}\right)^{-\frac{1}{2}}\mathrm{d}r'\right]^{-1} \leq \frac{2\alpha(r)m(r)}{r^3}\left[1 - \left(1 - \frac{2\alpha(r)m(r)}{r}\right)^{\frac{1}{2}}\right]^{-1}.$$ (9)

For the second term in the bracket of the l.h.s. of eq. (7) we have

$$\int_0^r \left(1 - \frac{2m(r')}{r'} + \frac{Q^2(r')}{r'^2}\right)^{-\frac{1}{2}}\frac{Q^2(r')e^{\frac{\nu(r')}{2}}}{r'^5}\mathrm{d}r' \geq$$

$$\geq \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}\int_0^r \left(1 - \frac{2m(r)}{r^3}r'^2 + \frac{Q^2(r)}{r^6}r'^2\right)^{-\frac{1}{2}}\mathrm{d}r' =$$

$$= \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}\left(\frac{2\alpha(r)m(r)}{r^3}\right)^{-\frac{1}{2}}\arcsin\left(\sqrt{\frac{2\alpha(r)m(r)}{r}}\right).$$ (10)

The second term in the bracket of the r.h.s. of eq. (7) can be evaluated as

$$\int_0^r r' \left(1 - \frac{2m(r')}{r'} + \frac{Q^2(r')}{r'^2}\right)^{-\frac{1}{2}} \left[\int_0^{r'} \left(1 - \frac{2m(r'')}{r''} + \frac{Q^2(r'')}{r''^2}\right)^{-\frac{1}{2}} \frac{Q^2(r'')e^{\frac{\nu(r'')}{2}}}{r''^5} \, \mathrm{d}r''\right] \mathrm{d}r' \ge$$

$$\ge \int_0^r r'\left(1 - \frac{2m(r')}{r'} + \frac{Q^2(r')}{r'^2}\right)^{-\frac{1}{2}} \frac{Q^2(r')e^{\frac{\nu(r')}{2}}}{r'^4}\left(\frac{2\alpha(r')m(r')}{r'}\right)^{-\frac{1}{2}} \arcsin\left(\sqrt{\frac{2\alpha(r')m(r')}{r'}}\right) \mathrm{d}r' \ge$$

$$\ge \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5} \int_0^r r'\left(1 - \frac{2\alpha(r)m(r)}{r}r'^2\right)^{-\frac{1}{2}}\left(\frac{2\alpha(r)m(r)}{r^3}r'^2\right)^{-\frac{1}{2}} \arcsin\left(\sqrt{\frac{2\alpha(r)m(r)}{r^3}}r'\right) \mathrm{d}r' =$$

$$= \frac{Q^2(r)e^{\frac{\nu(r)}{2}}}{r^5}r^2\left(\frac{2\alpha(r)m(r)}{r}\right)^{-\frac{3}{2}}\left[\sqrt{\frac{2(r)m(r)}{r}} - \sqrt{1 - \frac{2\alpha(r)m(r)}{r}} \arcsin\left(\sqrt{\frac{2\alpha(r)m(r)}{r}}\right)\right]. \tag{11}$$

In order to obtain the inequality (11), we have also used the property of monotonic increase in the interval $x \in [0, 1]$ of the function $\frac{\arcsin x}{x}$.

Using eqs. (9)–(11), eq. (7) becomes

$$\left[1 - \left(1 - \frac{2\alpha(r)m(r)}{r}\right)^{\frac{1}{2}}\right] \frac{m(r) + 4\pi r^3 p - \frac{Q^2}{r}}{r^3\sqrt{1 - \frac{2\alpha(r)m(r)}{r}}} \le \frac{2m(r)}{r^3} + \frac{Q^2}{r^4}\left[\frac{\arcsin\left(\sqrt{\frac{2\alpha(r)m(r)}{r}}\right)}{\sqrt{\frac{2\alpha(r)m(r)}{r}}} - 1\right]. \tag{12}$$

Equation (12) is valid for all r inside the star. Consider first the neutral case $Q = 0$. By evaluating (12) for $r = R$, we obtain $\frac{1}{\sqrt{1 - \frac{2M}{R}}} \le 2\left[1 - \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}}\right]^{-1}$, leading to the well-known result $\frac{2M}{R} \le \frac{8}{9}$ [8].

Next consider the case $Q \ne 0$. We denote

$$f(M, R, Q) = \frac{Q^2(R)}{R^4}\left(\frac{\alpha(R)M}{R^3}\right)^{-1}\sqrt{1 - \frac{2\alpha(R)M}{R}}\left[\frac{\arcsin\left(\sqrt{\frac{2\alpha(R)M}{R}}\right)}{\sqrt{\frac{2\alpha(R)M}{R}}} - 1\right]. \tag{13}$$

The function $f(M, R, Q) \ge 0$, $\forall M, R, Q$. Then (12) leads to the following restriction on the mass-radius ratio for compact charged general relativistic objects:

$$\frac{2M}{R} \le \frac{8}{9} + \frac{2f(M, R)}{9} - \frac{f^2(M, R)}{9}. \tag{14}$$

The variation of the maximum mass-radius ratio $u = \frac{M}{R}$ for a charged compact object as a function of the charge-mass ratio $q = \frac{Q}{M}$ is represented in fig. 1.

Due to the presence of the charge, the maximum mass-radius ratio is only slightly modified as compared to the non-charged case. In the uncharged case the bound $2\frac{M}{R} < \frac{8}{9}$ is very close to the limit $2\frac{M}{R} < 1$ arising from black-hole considerations. But for a charged compact general relativistic object the bound $\frac{M}{R} < 1$ obtained from horizon considerations is much larger than the limit following from eq. (14).

In order to find a general restriction for the total charge Q a compact stable object can acquire, we shall consider the behavior of the Ricci invariants $r_0 = R_i^i = R$, $r_1 = R_{ij}R^{ij}$
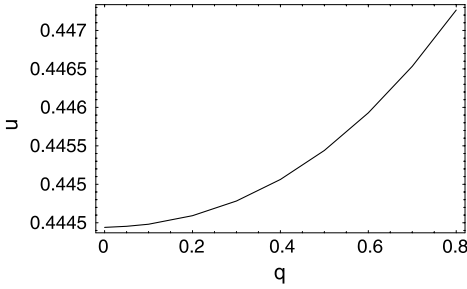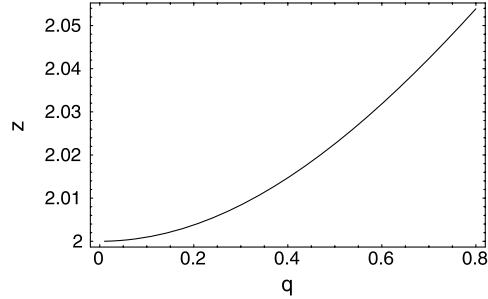
Fig. 1

Fig. 2

Fig. 1 – Variation of the maximum value u of the mass-radius ratio M/R for charged compact general-relativistic objects as a function of the total charge-mass ratio $q = Q/M$.

Fig. 2 – Variation of the maximum red shift z of a compact general relativistic object as a function of the total charge-mass ratio $q = Q/M$ for a value of the mass-radius ratio of $u = 0.4$.

and $r_2 = R_{ijkl}R^{ijkl}$. If the general static line element is regular, satisfying the conditions $e^{\nu(0)} = \text{const} \neq 0$ and $e^{\lambda(0)} = 1$, then the Ricci invariants are also non-singular functions throughout the star. In particular for a regular space-time the invariants are non-vanishing at the origin $r = 0$. For the invariant r_2 we find

$$r_2 = \left[8\pi(\rho + p) - \frac{4m}{r^3} + \frac{6Q^2}{r^4}\right]^2 + 2\left(8\pi p + \frac{2m}{r^3} - \frac{2Q^2}{r^4}\right)^2 +$$

$$+ 2\left(8\pi\rho - \frac{2m}{r^3} + \frac{2Q^2}{r^4}\right)^2 + 4\left(\frac{2m}{r^3} - \frac{Q^2}{r^4}\right)^2. \tag{15}$$

For a monotonically decreasing interior electric field $\frac{Q^2}{8\pi r^4}$, the function r_2 is regular and motonically decreasing throughout the star. Therefore it satisfies the condition $r_2(R) < r_2(0)$, leading to the following general constraint on the value of the electric field at the surface of the compact object:

$$6\frac{M^2}{R^6} < \frac{12M}{R^3}\frac{Q^2}{R^4} - 7\left(\frac{Q^2}{R^4}\right)^2 + 4\pi^2\left(6\rho_c^2 + 4\rho_c p_c + 6p_c^2\right), \tag{16}$$

where we assumed that at the surface of the star the matter density vanishes, $\rho(R) = 0$.

Another condition on $Q(R)$ can be obtained from the study of the scalar

$$r_1 = \left(8\pi\rho + \frac{Q^2}{r^4}\right)^2 + 3\left(8\pi p - \frac{Q^2}{r^4}\right)^2 + \frac{64\pi p Q^2}{r^4}. \tag{17}$$

Under the same assumptions of regularity and monotonicity for the function r_1 and considering that the surface density is vanishing, we obtain for the surface value of the monotonically decreasing electric field the upper bound

$$\frac{Q^2}{R^4} < 4\pi\rho_c\sqrt{1 + 3\left(\frac{p_c}{\rho_c}\right)^2}. \tag{18}$$

The invariant r_0 leads to the trace condition $\rho_c > 3p_c$ of the energy-momentum tensor that holds at the center of the fluid spheres.

The existence of a limiting value of the mass-radius ratio leads to limiting values for other physical quantities of observational interest. One of these quantities is the surface red shift z of the compact object, defined according to $z = (1 - \frac{2M}{R})^{-\frac{1}{2}} - 1$. For an electrically neutral star eq. (7) leads to the well-known constraint $z \leq 2$. For the charged star the surface red shift must obey the more general restriction

$$z \leq 2 + \frac{Q^2}{R^4}\left(\frac{\alpha(R)M}{R^3}\right)^{-1}\left[\frac{\arcsin\left(\sqrt{\frac{2\alpha(R)M}{R}}\right)}{\sqrt{\frac{2\alpha(R)M}{R}}} - 1\right]. \tag{19}$$

The variation as a function of the charge-mass ratio of the maximum red shift for charged compact objects for a given mass-radius ratio is presented in fig. 2.

Therefore surface red shifts higher than 2 could be observational criteria indicating the presence of electrically charged ultra compact matter distributions.

As another application of obtained limiting mass-radius ratios for charged stars, we shall derive an explicit limit for the total energy of compact general relativistic objects. The total energy (including the gravitational field contribution) inside an equipotential surface S can be defined to be [10]

$$E = E_M + E_F = \frac{1}{8\pi}\xi_s\int_S [K]\,\mathrm{d}S, \tag{20}$$

where ξ^i is a Killing field of time translation, ξ_s its value at S and $[K]$ is the jump across the shell of the trace of the extrinsic curvature of S, considered as embedded in the 2-space $t = \text{const.}$ $E_M = \int_S T_i^k \xi^i \sqrt{-g}\,\mathrm{d}S_k$ and E_F are the energy of the matter and of the gravitational field, respectively. This definition is manifestly coordinate invariant. In the case of static spherically symmetric matter distribution we obtain for the total energy (also including the gravitational contribution) the exact expression $E = -re^{\frac{\nu-\lambda}{2}}$ [10]. Hence the total energy (including the gravitational contribution) of a charged compact general relativistic object is $E = -R\left[1 - \frac{2M}{R} + \frac{Q^2(R)}{R^2}\right]$.

For a neutral matter distribution $Q = 0$ and for the total energy of the star, we find the upper limit $E \leq -\frac{R}{9}$. In the charged case we obtain

$$E \leq -\frac{R}{9} + \frac{2f}{9}R - \frac{f^2}{9}R, \tag{21}$$

with the function f defined in eq. (13).

In the present letter we have considered the mass-radius ratio limit for charged stable compact general relativistic objects. Also in this case it is possible to obtain explicit inequalities involving $\frac{2M}{R}$ as an explicit function of the charge Q. The surface red shift and the total energy (including the gravitational one) are modified due to the presence of a strong electric field inside the compact object. The mass-radius ratio depends on the value of the total charge of the star, with the increases in mass, red shift or total energy proportional to the charge parameter.

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