Contemporary Mathematics
Volume **222**, 1999

G-structures on Irreducible Hermitian Symmetric Spaces of Rank ≥ 2 and Deformation Rigidity

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In June 1997 the author gave a series of three lectures in the Postech International Conference in Several Complex Variables. The first two lectures were introductory in nature, entitled "Cones of minimal rational curves and complex structure", building up towards the third lecture, on "Deformation rigidity of irreducible Hermitian symmetric spaces of the compact type". One objective of the lectures was to provide the background for joint works of the author with Jun-Muk Hwang, especially our work on deformation rigidity. The present article is an amplified version of these lectures. Consistent with the focus of the lectures and the background of the audience, which is primarily in the area of Several Complex Variables, we emphasize the complex-analytic and differential-geometric aspects, notably analytic continuation, Hartogs extension, holomorphic distributions and the Gauss map. These techniques, taken together with results from the deformation theory of rational curves, will lead to our proof of deformation rigidity at the end of the article.

Here is a more detailed description of the contents. In §1 we present first of all irreducible Hermitian symmetric spaces S as compactifications of Euclidean spaces, introduce Harish-Chandra coordinates, leading to special G-structures, called S-structures, in the case of *rank* ≥ 2. The relevant general theory of Hermitian symmetric spaces will only be mentioned and illustrated by examples. In §2 we study the varieties $\mathcal{W}_x$ of highest weight tangents of the associated S-structures, introduce the method of analytic continuation to give a complex-analytic and geometric proof of Ochiai's theorem characterizing S in the case of *rank* ≥ 2 as the only compact simply-connected manifolds admitting flat S-structures. An important element of the proof is the use of the Gauss map on $\mathcal{W}_x \subset \mathbb{P}T_x(S)$. The projective varieties $\mathcal{W}_x$ agree with the varieties $\mathcal{C}_x$ of minimal rational tangents, which in the case of S consist simply of tangents of degree-1 rational curves. The first two sections can be understood with a minimal background in Algebraic Geometry. The deformation theory of rational curves in the context of deformation rigidity [HM2] will be taken up in §3, where we discuss the notions of minimal rational curves, varieties of minimal rational tangents $\mathcal{C}_x$ and distributions W spanned by $\mathcal{C}_x$ at generic points. In §4, the last section, we present the basic ideas of the proof of the rigidity of S under Kähler deformation, which consists of recovering

MR Classification Numbers: 32F25, 32G08, 32L05

*Partially supported by a grant of the Research Grants Council, Hong Kong.

flat S-structures on the central fiber from limits of S-structures at generic fibers. The crux of the proof is the linear non-degeneracy of the generic variety of minimal rational tangents $\mathcal{C}_x$ on the central fiber, which we obtain by means of results on the integrability of distributions spanned by minimal rational tangents.

Besides giving a proof of deformation rigidity, one primary goal of the article is to illustrate the role played by complex-analytic and differential-geometric methods in the general study of problems regarding complex structures of Fano manifolds. Such techniques enter into play in a variety of problems. In the broader context, these methods should be understood as tools for the study of deformation theory of curves, notably of rational curves, on projective manifolds. The case of irreducible Hermitian symmetric spaces [HM1,2] serve as first examples, to be followed by rational homogeneous spaces of Picard number 1 [HM4,5; Hw]. They deal with deformation rigidity, algebro-geometric characterizations and holomorphic mappings. Such methods, taken together with techniques from projective geometry, have begun to be applied in [HM5] to a class of Fano manifolds that include rational homogeneous spaces and most Fano complete intersections. For a more thorough overview on varieties of minimal rational tangents, especially for an in-depth discussion of algebro-geometric techniques, a substantial part of which is missing from the present article, we refer the reader to [HM5]. For a survey of S-structures and algebro-geometric characterizations of S, we refer the reader to [HM3].

The author would like to thank the organizers of the conference, especially Professor Kang-Tae Kim, for their invitation and hospitality. Needless to say, the contents of the lectures and the article were fruit of collaboration and numerous discussions with Jun-Muk Hwang, to whom the author would like to express his gratitude.

§1 Harish-Chandra coordinates and S-structures

(1.1) The present article, which follows up on lectures given by the author in the Postech International Conference in Several Complex Variables in June 1997, is meant to provide some background and illustration for a series of recent joint works of the author with Jun-Muk Hwang [HM1-5], which surrounds the question of recapturing complex-analytic properties of Fano manifolds from their varieties of minimal rational tangents. To focus attention we concentrate on the first of the series [HM2], on deformation rigidity, with an eye on explaining basic notions in a concrete context. The interested reader can then read the surveys [HM3,5], where problems and techniques are treated in much broader contexts. To start with we state the Main Theorem of [HM2].

Main Theorem. *Let S be an irreducible Hermitian symmetric space of the compact type. Let $\pi : \mathcal{X} \to \Delta$ be a regular family of compact complex manifolds over the unit disk Δ. Suppose $X_t := \pi^{-1}(t)$ is biholomorphic to S for $t \neq 0$ and the central fiber $X = X_0$ is Kähler. Then, X is also biholomorphic to S.*

The proof of the Main Theorem relies on the one hand on the study of rational curves in the central fiber, on the other hand on the study of G-structures arising from irreducible Hermitian symmetric spaces S of the compact type and of *rank* $\geq$

2. We will start with an illustration of the latter in the case of Grassmannian $G(p,q)$; $p,q \geq 2$.

(1.2) Let $p,q \geq 2$ and W be a $(p+q)$-dimensional complex vector space. We denote by $G(p,q)$ the Grassmannian of all p-planes $E \subset W$. Given any $[E] \in G(p,q)$, the annihilator $E^{\perp} \subset W^*$ is a q-plane in the dual space W^*. This gives an isomorphism $G(p,q) \cong G(q,p)$.

As is well-known $G(p,q)$ is covered by a finite number of charts $\varphi : \mathbb{C}^n \to G(p,q)$, as below. Denote by $M(q,p;\mathbb{C})$ the complex vector space of q-by-p matrices, and by I_p the p-by-p identity matrix. Consider the subset $\Omega \subset G(p,q)$ consisting of all p-planes E_Z generated by the column vectors of $\begin{bmatrix} Z \\ I_p \end{bmatrix}$, $Z \in M(q,p;\mathbb{C})$, with respect to a fixed ordered basis $(e_{p+1},\dots,e_{p+q};e_1,\dots,e_p)$. Then the map $\varphi : M(q,p;\mathbb{C}) \cong \Omega \subset G(p,q)$; $M(p,q;\mathbb{C}) \cong \mathbb{C}^{pq}$; is such a chart. $G(p,q)$ is then covered by a finite number of such charts, obtained by permuting the basis vectors in the ordered basis. An open subset $\Omega \subset G(p,q)$ obtained with respect to some choice of ordered basis of W will be called a Euclidean cell.

Any linear automorphism of W induces a biholomorphic automorphism of $G(p,q)$. All biholomorphic automorphisms of $G(p,q)$ in the identity component $Aut_o(G(p,q))$ are obtained this way, and we have $Aut_o(G(p,q)) \cong GL(p+q;\mathbb{C})/\mathbb{C}^*$. For $\Phi \in Aut_o(G(p,q))$, the restriction of Φ to a Euclidean cell $\Omega \cong M(q,p;\mathbb{C})$ can be described as fractional linear transformations, as follows. $\Phi \in Aut_o(G(p,q))$ is defined by a linear transformation $\Phi_o \in GL(W)$, represented by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with respect to the ordered basis $(e_{p+1},\dots,e_n;e_1,\dots,e_p)$, where $A \in GL(q,\mathbb{C})$, etc. Then Φ_o transforms the p-plane E_Z, represented by $\begin{bmatrix} Z \\ I_p \end{bmatrix}$, to the p-plane spanned by the column vectors of $\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} Z \\ I \end{bmatrix} = \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix}$. Provided that $CZ+D$ is invertible, $\Phi_o(E_Z) = E_{\Phi(Z)}$, where $\Phi(Z) = (AZ+B)(CZ+D)^{-1}$.

Recall that $G(p,q)$ is covered by a finite number of charts consisting of Euclidean cells Ω. On the overlapping regions the transition maps are induced by automorphisms of W corresponding to a change of basis, and thus given by fractional linear transformations, as described. The Jacobian matrices of the transition maps are of a particular type, as follows. Let $Z' = \Phi(Z) = (AZ+B)(CZ+D)^{-1}$, $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in GL(p+q,\mathbb{C})$, be a fractional linear transformation. The tangent space at each point of a Euclidean cell can be identified with the vector space $M(q,p;\mathbb{C})$ of q-by-p matrices. Whenever $det(CZ+D) \neq 0$, Φ is holomorphically defined at Z and $d\Phi(Z)$ is invertible. For the differential $d\Phi(Z)$, identified as a Jacobian matrix, we have

$$\begin{aligned} d\Phi(Z)(X) &= AX(CZ+D)^{-1} - (AZ+B)(CZ+D)^{-1}CX(CZ+D)^{-1} \\ &= [A - (AZ+B)(CZ+D)^{-1}C]X(CZ+D)^{-1} . \end{aligned}$$

Hence, $d\Phi(Z)(X) = Q(Z)XP(Z)$, where $Q(Z) \in GL(q,\mathbb{C})$, $P(Z) \in GL(p,\mathbb{C})$. Thus the charts consisting of Euclidean cells endow $G(p,q)$ with a special structure as a complex manifold. It gives in particular a trivialization of the holomorphic tangent bundle over each Euclidean cell, so that the transition functions for the holomorphic tangent bundle takes values in a proper subgroup $G \subsetneq GL(pq, \mathbb{C})$ where G consists of linear transformations γ on $M(p,q;\mathbb{C}) \cong \mathbb{C}^{pq}$ of the form $\gamma(X) = QXP$, $Q \in GL(q,\mathbb{C})$, $P \in GL(p,\mathbb{C})$. This gives an example of a holomorphic G-structure, which we formalize, as follows.

Let n be a positive integer. Fix an n-dimensional complex vector space V and let M be any n-dimensional complex manifold. In what follows all bundles are understood to be holomorphic. The frame bundle $\mathcal{F}(M)$ is a principal $GL(V)$-bundle with the fiber at x defined as $\mathcal{F}(M)_x = Isom(V, T_x(M))$, the set of linear isomorphisms from V to the holomorphic tangent space at x.

Definition 1.2.1 (G-structures). *Let $G \subset GL(V)$ be any complex Lie subgroup. A holomorphic G-structure is a G-principal subbundle $\mathcal{G}(M)$ of $\mathcal{F}(M)$. An element of $\mathcal{G}_x(M)$ will be called a G-frame at x. For $G \neq GL(V)$ we say that $\mathcal{G}(M)$ defines a holomorphic reduction of the tangent bundle to G.*

On an m-dimensional smooth manifold, the choice of a Riemannian metric corresponds to a reduction of the structure group of the tangent bundle from the general linear group $GL(m,\mathbb{R})$ to the orthogonal group $O(m)$. Riemannian geometry may be regarded as the geometry of smooth $O(m)$-structures. A Riemannian manifold is locally isometric to the Euclidean space if and only if there exists a covering by smooth coordinate charts on which orthonormal frames can be chosen to consist of the same basis vectors at each point, when tangent vectors at different points are identified by the standard trivialization on coordinate charts. We may call this a flat smooth $O(m)$-structure. On complex manifolds we have the following analogous notion of flat holomorphic G-structures.

Definition 1.2.2. *Let $\varphi_\alpha : U_\alpha \to V$ be a chart on M. In terms of Euclidean coordinates we identify $\mathcal{F}(U_\alpha)$ with the product $GL(V) \times U_\alpha$. We say that a G-structure $\mathcal{G}(M)$ on M is flat if and only if there exists an atlas of charts $\{\varphi_\alpha : U_\alpha \to V\}$ such that the restriction $\mathcal{G}(U_\alpha)$ of $\mathcal{G}(M)$ to U_α is the product $G \times U_\alpha \subset GL(V) \times U_\alpha$.*

In the example of the Grassmannian we discussed a moment ago, we have in fact a flat holomorphic G-structure defined by the charts consisting of Euclidean cells. Here we take $V = M(q,p;\mathbb{C})$, and $G \subset GL(V,\mathbb{C})$ to be the image under the homomorphism $\Theta : GL(q,\mathbb{C}) \times GL(p,\mathbb{C}) \to GL(M(q,p;\mathbb{C}))$ defined by $\Theta(Q,P)(X) = QXP$ for all X. G is isomorphic to the quotient of $GL(q,\mathbb{C}) \times GL(p,\mathbb{C})$ by a copy of $\mathbb{C}^*$, and is reductive, with the semisimple part isomorphic to the quotient of $SL(q,\mathbb{C}) \times SL(p,\mathbb{C})$ by a finite group.

G-structures with $G = \Theta(GL(q,\mathbb{C})) \times GL(p,\mathbb{C})$; $p, q \geq 2$; will be referred to as Grassmann structures. A complex manifold M then admits a Grassmann structure, if and only if the holomorphic tangent bundle $T(M)$ admits a non-trivial decomposition as the tensor product $A \otimes B$, where $rank(A) = p$, $rank(B) = q$. In [HM1] we proved a general theorem about flatness of G-structures on uniruled projective manifolds (*cf.* §3 for the definition of "uniruled"), for G reductive and connected. In the case of Grassmann structures it says that a uniruled projective manifold admitting a Grassmann structure is biholomorphic to some $G(p,q)$.

(1.3) In the description of Grassmann structures, there is an associated linear representation of G on $T_x(M)$, and an associated rational homogeneous submanifold $\mathcal{W}_x \subset T_x(M)$ consisting of projectivizations of highest weight vectors. Fixing a $\mathcal{G}$-frame $\varphi \in Isom(M(q,p;\mathbb{C}), T_x(M))$, $\mathcal{W}_x$ is the set of all $[\varphi(X)]$, where $X \in M(q;p;\mathbb{C})$ is a matrix of *rank* 1. $\mathcal{W}_x \subset \mathbb{P}T_x(M)$ is well-defined, independent of the choice of φ, since $rank(QXP) = rank(X)$, whenever P and Q are invertible. We call $\mathcal{W}_x$ varieties of highest weight tangents. On each Euclidean cell Ω of the Grassmannian $G(p,q)$, the bundle $\mathcal{W}|_\Omega \to \Omega$ is a constant family with respect to Euclidean coordinates on Ω.

For any irreducible Hermitian symmetric space S of the compact type, there is the notion of Harish-Chandra coordinates, which generalizes the charts on Euclidean cells Ω, as described. For the general theory we refer the reader to Wolf [Wo] and Mok [Mk2]. Fix a canonical metric g on S. In what follows the notation G, as customary, will mean the identity component of the group of isometries of S. Fix $o \in S$ and let $K \subset G$ be the isotropy subgroup at o. The isotropy representation of K on $T_o(S)$ is faithful, so that K can be identified as a linear subgroup of $GL(T_o(S))$. K is reductive, with a one-dimensional centre. The complexification $K^{\mathbb{C}}$ then acts irreducibly on $T_o(S)$ and there is an associated variety of highest weight tangents $\mathcal{W}_o$. Here by a highest weight vector we mean a highest weight vector of the semisimple part of $K^{\mathbb{C}}$. A $K^{\mathbb{C}}$-structure will also be called an S-structure. We have

Proposition 1.3.1. *On S, Euclidean translations on a Harish-Chandra coordinate chart Ω extend to a biholomorphic automorphism of S. As a consequence in terms of trivializations given by Harish-Chandra coordinates the bundle $\mathcal{W} \to S$ of varieties of highest weight tangents is a constant family when restricted to Ω, and the S-structure on S is flat.*

We list in the following table all irreducible Hermitian symmetric spaces $S = G/K$ of the compact type and their varieties of highest weight orbits $\mathcal{W}_o$ at a reference point $o \in S$. It turns out that $\mathcal{W}_o \subset \mathbb{P}T_o(S)$ is itself biholomorphic to a Hermitian symmetric space of the compact type of rank 1 or 2, and is irreducible except in the case of Grassmannians $G(p,q)$; $p,q \geq 2$. There the rank-1 matrices correspond to decomposable tensors $u \otimes v$, and the highest weight orbit is the image of $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$ in $\mathbb{P}^{pq-1}$ by the Segre embedding given by $\sigma([u],[v]) = [u \otimes v]$. In the table $\mathbb{O}$ will stand for the octonions (Cayley numbers).

Table of irreducible Hermitian symmetric spaces S
of the compact type and their varieties of highest weight tangents $\mathcal{W}_o$

Type	G	K	$G/K = S$	$\mathcal{W}_o$	Embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2, n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$\mathrm{Spin}(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

(1.4) An irreducible Hermitian symmetric space S of the compact type is a Fano manifold, *i.e.*, a projective manifold with ample anti-canonical line bundle. Any Fano manifold is covered by rational curves, by Mori [Mo]. For the general notion of minimal rational curves, we refer the reader to [HM2]. In the case of S, the Picard group of holomorphic line bundles is infinite cyclic, and we have the notion of degree with respect to the positive generator $\mathcal{O}(1)$ of $Pic(S)$. It turns out that the vector space $\Gamma(S, \mathcal{O}(1))$ defines a holomorphic embedding $\tau : S \to \mathbb{P}^N$ of S into some $\mathbb{P}^N$, in such a way that there exists lines of $\mathbb{P}^N$ lying on $\tau(S)$. Such lines C are degree-1 rational curves on S with respect to $\mathcal{O}(1)$ and they are the only degree-1 curves. For S minimal rational curves as defined in [HM2] are precisely the degree-1 rational curves.

The S-structures on S, determined by the bundle $\mathcal{W} \to S$ of highest weight orbits, is intimately related to the space of minimal rational curves on S, which we will illustrate here by the case of Grassmannians.

For $S = G(p,q)$, $\tau : G(p,q) \to \mathbb{P}^N$ is given by the Plücker embedding, as follows. For a p-plane E in $W \cong \mathbb{C}^{p+q}$ with basis $\{e_1, \dots, e_p\}$ we define $\tau([E]) = [e_1 \wedge \dots \wedge e_p] \in \mathbb{P}\Lambda^p W$, which is clearly defined independent of the choice of basis. Choose a system of Harish-Chandra coordinates on a Euclidean cell Ω such that $[E]$ corresponds to the origin. Let $X \in M(q,p;\mathbb{C})$ be a tangent vector at $[E]$ and consider the Euclidean line L on Ω passing through $[E]$ such that X is tangent to L. For each $t \in \mathbb{C}$ we have a point $[E_t] \in L$ where E_t is spanned by $\{e_1 + t\eta_1, \dots, e_p + t\eta_p\}$ corresponding to the column vectors of $\begin{bmatrix} tX \\ I_p \end{bmatrix}$. If X is a matrix of rank-1, then $\eta_1, \dots, \eta_p$ are proportional to each other, say $\eta_j = c_j \eta$ for some $\eta \in Span\{e_{p+1}, \dots, e_n\}$, and some $c_j \in \mathbb{C}$. Then $(e_1 + t\eta_1) \wedge \dots \wedge (e_p + t\eta_p) = (e_1 + tc_1\eta) \wedge \dots \wedge (e_p + tc_p\eta) = e_1 \wedge \dots \wedge e_p + t\eta \wedge \omega$ for some $\omega \in \Lambda^{p-1}E$. In other words, $\tau(\overline{L})$ is a line on $\mathbb{P}\Lambda^p W$. We have shown that for $x = [E] \in G(p,q)$, each $[\alpha] \in \mathcal{W}_x$ is tangent to a minimal rational curve $C = \overline{L}$. As $\tau(C')$ is a line on $\mathbb{P}\Lambda^p W$ for any minimal rational curve C', there is only one C with $T_x C = \mathbb{C}\alpha$. Furthermore, if C is any minimal rational curve on $G(p,q)$ passing through $[E]$, and $\gamma(t) = tY + O(t^2)$ is a local parametrization of C at $[E]$, then all column vectors of Y are proportional, *i.e.* Y is of rank 1, by an expansion of $\tau(\gamma(t))$ as in the above. We have thus demonstrated the following proposition in the special case of $S = G(p,q)$.

Proposition 1.4.1. *Let S be an irreducible Hermitian symmetric space of the compact type, $x \in S$ and $\mathcal{W}_x \subset \mathbb{P}T_x(S)$ be the variety of highest weight orbits at x. Then $\mathcal{W}_x$ is precisely the variety $\mathcal{C}_x$ of all $[\alpha] \in \mathbb{P}T_x(S)$ tangent to a minimal rational curve C. Furthermore, for each $[\alpha] \in \mathcal{W}_x$ the minimal rational curve C is uniquely determined by $[\alpha]$.*

$\mathcal{C}_x$ as defined above will be called the variety of minimal rational tangents. For a conceptual and uniform proof of $\mathcal{W}_x = \mathcal{C}_x$ using Grothendieck's splitting theorem [Gr] for principal G-bundles over rational curves with G reductive, we refer the reader to [HM1].

For a local holomorphic curve Γ on a complex manifold X we have a tautological lifting $\hat{\Gamma}$ of Γ to $\mathbb{P}T(X)$ by lifting each $x \in \Gamma$ to its tangent line $[T_x(\Gamma)] \in \mathbb{P}T_x(X)$. In terms of Harish-Chandra coordinates we have the following description of minimal rational curves which in the case of $G(p,q)$ follows from the preceding discussion. Here we use the term Euclidean cell for the image of a Harish-Chandra coordinate chart.

Proposition 1.4.2. *Let S be an irreducible Hermitian symmetric space of the compact type. Non-empty intersections of minimal rational curves with Euclidean cells Ω are precisely affine lines $L = C \cap \Omega$ whose tangents lie on $\mathcal{W}$. In other words, the tautological lifting $\hat{L}$ of L to $\mathbb{P}T(S)$ is a constant section of $\mathcal{W}$ over L, in terms of Harish-Chandra coordinates.*

(1.5) In what follows G-structures and fiber bundles are understood to be holomorphic. The notion of flatness for a holomorphic G-structure on a complex manifold M can be expressed as a system of differential equations on the G-structure. From this description it is easy to deduce that flatness is a closed condition. More precisely, we have

Definition-Proposition 1.5.1. *Let V be an n-dimensional vector space, $G \subset GL(V)$ be a Lie subgroup. Let $\pi : \Delta^n \times \Delta \to \Delta$ be the projection to the second factor, T^π be the relative tangent bundle of the regular family $\pi : \Delta^n \times \Delta \to \Delta$, and $\mathcal{F}^\pi$ be the relative frame bundle with $\mathcal{F}^\pi_x = Isom(V, T^\pi_x)$. Let $\mathcal{G}^\pi \subset \mathcal{F}^\pi$ be a principal G-bundle. We call $\mathcal{G}^\pi$ a holomorphic family of G-structures on Δ^n. Suppose the G-structure $\mathcal{G}^\pi|_{\Delta^n \times \{t\}}$ is flat for $t \neq 0$. Then, the G-structure $\mathcal{G}^\pi|_{\Delta^n \times \{0\}}$ is also flat.*

By going to prolongation bundles in the theory of G-structures, the proof can be reduced to the Frobenius conditions, *cf.* Singer-Sternberg [SS]. We are contented with an illustration in the case of S-structures, as follows.

Example. Consider the case of S-structures with $S = Q^n$, $n \geq 3$. In what follows by a holomorphic metric we mean a holomorphic non-degenerate covariant symmetric tensor θ, given locally by $\Sigma \theta_{ij} dz^i \otimes dz^j$ with $det(\theta_{ij}) \neq 0$. A germ of holomorphic Q^n-structure at $x \in X$ gives a germ of holomorphic metric θ at x, unique up to conformal factors (*i.e.*, scalar multiples). Flatness of the Q^n-structure amounts to saying that θ is conformally equivalent to the flat holomorphic metric. We say that θ is conformally flat if and only if it is conformally equivalent to the flat holomophic metric. As in Riemannian geometry, θ is conformally flat if and only if the (conformally invariant) Bochner-Weyl tensor $W_{ijk\ell}$ vanishes. Obviously the vanishing of $W_{ijk\ell}$ is a closed condition. For the interpretion of flatness of S-structures in general in terms of "curvature tensors", which measure obstructions to prolongation of $K^{\mathbb{C}}$-structures, *cf.* [HM3].

(1.6) On the projective space any two distinct points can be joined by a minimal rational curve. While this is obviously not the case for irreducible Hermitian symmetric spaces S of *rank* ≥ 2, as $\mathcal{C}_o(S) \neq \mathbb{P}T_o(S)$, it is always possible to join any two distinct points by a chain of minimal rational curves. We will say that a chain $K = C_1 + \cdots + C_m$ of minimal rational curves is non-overlapping, if and only if $C_j \cap C_{j+1}$ is a single point, and $C_j \cap C_k = \emptyset$ whenever $|k-j| \geq 2$. We have more precisely

Proposition 1.6.1. *For S of rank r any two distinct points $x, y \in S$ can be joined by a non-overlapping chain $K = C_1 + \cdots + C_m$ of minimal rational curves for some m, $1 \leq m \leq r$.*

Proposition 1.6.1 follows readily from the following result, which is consequence of the Polysphere Theorem (*cf.* Wolf [Wo] or Mok [Mk2]).

Proposition 1.6.2. *For a polysphere $(\mathbb{P}^1)^s$ denote by L_k, $1 \leq k \leq s$, the holomorphic line bundle obtained by pulling back the positive generator $\mathcal{O}(1)$ on the k-th direct fact $\mathbb{P}^1_k$ by the canonical projection. Let S be an irreducible Hermitian symmetric space of the compact type of rank r and denote by L the positive generator of its Picard group. Then, there exists a holomorphic embedding $\sigma : (\mathbb{P}^1)^r \to S$ such that $\sigma^* L \cong L_1 \otimes \cdots \otimes L_r$.*

Since the analogue of Proposition 1.6.1 holds obviously for the polysphere, Proposition 1.6.2 implies readily the validity of Proposition 1.6.1 for S. We will illustrate Proposition 1.6.2 by the example of Grassmannians $G(p,q)$. Without loss of generality we assume that $p \leq q$. Then $G(p,q)$ is of rank $r = p$.

Example. For $x, y \in G(p,q)$ distinct choose a Euclidean cell $\Omega \subset G(p,q)$ so that $x, y \in \Omega$ and x is the origin and y is represented by $Z \in M(q,p;\mathbb{C})$. Let $Q \in GL(q,\mathbb{C})$ and $P \in GL(p,\mathbb{C})$ be arbitrary. Then, $\Phi(Z) = QZP$ extends to an automorphism of $G(p,q)$. Let $D \subset M(q,p;\mathbb{C})$ be the vector subspace $D \cong \mathbb{C}^p$, consisting of matrices of the form

$$diag(\zeta_1, \dots, \zeta_p) := \begin{bmatrix} \zeta_1 & & \\ & \ddots & 0 \\ & \zeta_p & \end{bmatrix}.$$

Then, P, Q can be chosen such that $QZP \in D$. Thus, to join x to y without loss of generality we may assume that $Z \in D$. We describe $\overline{D} \subset G(p,q)$ as a polysphere, as follows. Let $(e_{p+1}, \dots, e_n; e_1, \dots, e_p)$ be the ordered basis of $W \cong \mathbb{C}^{p+q}$ as in (1.2). Then, $Z = diag(\zeta_1, \dots, \zeta_p)$ represents the p-plane $E_Z = \mathrm{Span}\{e_1 + \zeta_1 e_{p+1}, \dots, e_p + \zeta_p e_{2p}\}$. For $1 \leq k \leq p$ write $V_k = \mathbb{C}e_k + \mathbb{C}e_{p+k}$. Then, $\overline{D} \subset G(p,q)$ consists of all p-planes spanned by some $\{v_1, \dots, v_p\}$, $v_k \in V_k$, from which it follows that $\overline{D} \cong (\mathbb{P}^1)^p$. Let $\tau : G(p,q) \to \mathbb{P}\Lambda^p W$ be the Plücker embedding and write C_k for any rational curve on $\overline{D}$ corresponding to some $(a_1, \dots, a_{k-1}) \times \mathbb{P}^1 \times (a_{k+1}, \dots, a_p)$ under the biholomorphism $\overline{D} \cong (\mathbb{P}^1)^p$. Each a_j, $j \neq k$, corresponds to a line in V_k, generated by some $u_k \in V_k$. Then $\tau(C_k) = \{[u_1 \wedge \cdots \wedge u_{k-1} \wedge v_k \wedge u_{k+1} \wedge \cdots \wedge u_p] : v_k \in V_k\}$, which shows readily that $\tau(C_k)$ is a line C on $\mathbb{P}\Lambda^p W$. In fact, if

$\omega = u_1 \wedge \cdots \wedge u_{k-1} \wedge e_k \wedge u_{k+1} \wedge \cdots \wedge u_p$ and $\omega' = u_1 \wedge \cdots \wedge u_{k-1} \wedge e_{p+k} \wedge u_{k+1} \wedge \cdots \wedge u_p$, then $C = \mathbb{P}(\mathbb{C}\omega + \mathbb{C}\omega')$. Thus, each C_k is a minimal rational curve, as desired.

§2 Analytic continuation of local isomorphisms of flat S-structures

(2.1) To each irreducible Hermitian symmetric space S of the compact type and of *rank* ≥ 2 one can associate a flat S-structure by means of Harish-Chandra coordinates, as we have illustrated in §1. For the question on deformation rigidity, it is essential to be able to recover S from the associated S-structure, as given in

Theorem 2.1.1 (Ochiai [Oc]). *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Let X be a compact simply-connected complex manifold with a flat S-structure. Then, X is biholomorphic to S.*

Given an S-structure on X we have an associated bundle $\mathcal{W} \subset \mathbb{P}T_X$ of varieties of highest weight orbits. The assumption that X admits a flat S-structure means that given any $x \in X$, some neighborhood U_x of x can be identified with an open set U on S in such a way that $\mathcal{W}|_{U_x}$ agrees with the bundle $\mathcal{C}|_U$ over U of varieties of minimal rational tangents. The proof of Theorem 2.1.1 in [Oc] is algebraic in nature. In its place, we will present in this section a complex-analytic and geometric proof. First of all, Theorem 2.1.1 is a consequence of the following result on extending local isomorphisms of holomorphic S-structure.

Theorem 2.1.2 (Ochiai [Oc]). *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Denote by $\pi : \mathcal{C} \to S$ the bundle of varieties of minimal rational tangents. Let U, $V \subset S$ be two connected open sets and $f : U \to V$ be a biholomorphism such that $f_*\mathcal{C}|_U = \mathcal{C}|_V$. Then, f extends to a biholomorphic automorphism of S.*

Theorem 2.1.2 implies Theorem 2.1.1, as follows. For X as in Theorem 2.1.1 admitting a flat S-structure, choose a connected open neighborhood U_x of x and a biholomorphism $f : U_x \cong U \subset X$ onto some open subset U of X such that $f_*(\mathcal{W}|_{U_x}) = \mathcal{C}|_U$. Starting with one choice of x and f, Theorem 2.1.2 allows us to continue f holomorphically along any continuous curve, by matching different f_y on U_y on intersecting regions using global automorphisms of S. This leads to a developing map, which is well-defined on X since X is simply connected. The resulting unramified holomorphic map $F : X \to S$ is necessarily a biholomorphism, since S is simply connected.

We will give an alternate proof of Theorem 2.1.2 by means of analytic continuation, along the lines of Mok-Tsai [MT]. Let $C \subset S$ be a minimal rational curve. Recall that the tautological lifting $\hat{C}$ of C lies on $\mathcal{C}$. Such liftings define a holomorphic 1-dimensional foliation $\mathcal{F}$ of $\mathcal{C}$. The proof of Theorem 2.1.1 will be divided into 3 steps. First we will show that any S-structure-preserving local biholomorphism is necessarily $\mathcal{F}$-preserving. Then, we will show that any $\mathcal{F}$-preserving local biholomorphism on S extends to a birational self-map F by means of anlaytic continuation. Finally, we argue that F is unramified, and conclude that F is a biholomorphic automorphism by Hartogs extension on anti-canonical sections.

As preparation for our proof of Theorem 2.1.2 we discuss here some preliminaries in projective geometry involving the second fundamental form of projective submanifolds, *i.e.*, differentials of their Gauss map. For this will need the ensuing standard lemma on rulings of local complex submanifolds of Euclidean spaces and Zak's theorem on tangencies. For a complex manifold B and a complex submanifold $A \subset B$ we denote by $N_{A|B}$ the holomorphic normal bundle of A in B. When B is endowed a Kähler metric for $x \in A$ we will identify the vector space $N_{A|B,x}$ with the orthogonal complement $T_x^\perp(A)$ of $T_x(A)$ in $T_x(B)$ with respect to g.

Lemma 2.1.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain and $Z \subset \Omega$ be a closed complex submanifold. At $z \in Z$ denote by $\sigma_z : T_z(Z) \times T_z(Z) \to N_{Z|\Omega,z}$ the second fundamental form with respect to the Euclidean flat connection ∇ on Ω. Denote by $Ker(\sigma_z) \subset T_z(Z)$ the complex vector subspace of all η such that $\sigma_z(\tau, \eta) = 0$ for any $\tau \in T_z(Z)$. Suppose $Ker(\sigma_z)$ is of the same positive rank d on Z. Then, the distribution $z \to Ker(\sigma_z)$ is integrable and the integral submanifolds are open subsets of d-dimensional affine linear subspaces.*

Proof. At $z \in Z$, let η, ξ, τ be germs of holomorphic vector fields on Z such that η, ξ are $Ker(\sigma)$-valued. We proceed to prove that $\nabla_\eta \xi$ is also $Ker(\sigma)$-valued. Since ∇ is torsion-free for any germ of holomorphic vector field χ at $z \in Z$ we have $[\chi, \tau] = \nabla_\chi \tau - \nabla_\tau \chi$, and χ is $Ker(\sigma)$-valued if and only if for any choice of τ, $\nabla_\chi \tau$ is tangent to Z, or equivalently $\nabla_\tau \chi$ is tangent to Z. Since ∇ is flat, we have

$$\nabla_\tau(\nabla_\eta \xi) = \nabla_\eta(\nabla_\tau \xi) + \nabla_{[\eta,\tau]}\xi \ ,$$

which implies that $\nabla_\tau(\nabla_\eta \xi)$ is tangent to Z and hence that $\nabla_\eta \xi$ is $Ker(\sigma)$-valued. Together with $[\eta, \xi] = \nabla_\eta \xi - \nabla_\xi \eta$ it follows that $[Ker(\sigma), Ker(\sigma)] \subset Ker(\sigma)$. The distribution $Ker(\sigma)$ is hence integrable, and on an integral submanifold Σ, the tangent bundle $T(\Sigma)$ of Σ is invariant under parallel transport with respect to ∇. In other words, Σ is an open subset of some affine-linear subspace of $\mathbb{C}^n$, as desired.

□

Theorem 2.1.4 (Special case of Zak's Theorem on tangencies, Zak [Za]). *Let $W \subset \mathbb{P}^N$ be a k-dimensional complex submanifold other than a projective linear subspace and $\mathbb{P}E \subset \mathbb{P}^N$ be a $k-$dimensional projective subspace. Then, the set of points on Z at which $\mathbb{P}E$ is tangent to Z is finite.*

From Zak's Theorem and Lemma 2.1.3 we conclude

Proposition 2.1.5. *Let $W \subset \mathbb{P}^N$ be a k-dimensional projective submanifold other than a projective linear subspace. For $w \in W$ denote by $\sigma_w : T_w(W) \times T_w(W) \to N_{W|\mathbb{P}^n,w}$ the second fundamental form in the sense of projective geometry. Then, $Ker(\sigma_w) = 0$ for a generic point $w \in W$.*

Proof. For the canonical projection $\pi : \mathbb{C}^{N+1} - \{o\} \to \mathbb{P}^N$ write $Z = \pi^{-1}(W)$. The Proposition is equivalent to stating that for a generic $\alpha \in Z$, $Ker(\sigma_\alpha) = \mathbb{C}\alpha$ for the second fundamental form $\sigma_\alpha : T_\alpha(Z) \times T_\alpha(Z) \to N_{Z|\mathbb{C}^{N+1},\alpha}$ in the sense of Euclidean geometry. Suppose otherwise, $Ker(\sigma_\alpha)$ would define a d-dimensional distribution on some non-empty open subset D of Z for some $d \geq 2$, defining by Lemma 2.1.3 a ruling of D by open subsets Σ of affine linear spaces. Clearly we can choose $D \subset Z$ to be invariant under scaling. Then, the closure $\overline{\Sigma}$ contains o, so that Σ is actually contained in a d-dimensional complex vector subspace. Furthermore, since $T_\alpha(\Sigma) = Ker(\sigma_\alpha)$, $T(Z)$ is parallel along Σ, and there is a $(k+1)$-dimensional vector subspace E of $\mathbb{C}^{N+1}$ such that E is tangent to Z along Σ at any non-zero

$\alpha \in \Sigma$. $\mathbb{P}E$ is tangent to $W \subset \mathbb{P}^N$ along $\mathbb{P}\Sigma$, which is of dimension $d-1 \geq 1$. As this contradicts Zak's Theorem on tangencies, Proposition 2.1.5 is established. □

Remark. Proposition 2.1.5 is equivalent to saying that the Gauss map on W is generically finite-to-one. For a more self-contained proof of the latter fact, which precedes Zak [Za], we refer the reader to Ein [Ei].

(2.2) For the proof of Theorem 2.1.2 of Ochiai's, as explained we start with

Proposition 2.2.1. *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2, and $\pi : \mathcal{C} \to S$ be the bundle of varieties of minimal rational tangents. Let $U, V \subset S$ be connected open subsets of S and $f : U \to V$ be a biholomorphic map such that $f_*(\mathcal{C}|_U) = \mathcal{C}|_V$. Then, for the 1-dimensional holomorphic foliation $\mathcal{F}$ on $\mathcal{C}$ defined by liftings $\hat{C}$ of minimal rational curves C, we have $f^*(\mathcal{F}|_{\pi^{-1}(V)}) = \mathcal{F}|_{\pi^{-1}(U)}$.*

The foliation $\mathcal{F}$ on $\mathcal{C}$ has the following crucial property: for a non-zero vector η tangent to $\mathcal{F}$ at $[\alpha]$, $d\pi(\eta)$ must be proportional to α. Thus, at $[\alpha], \mathcal{F}$ is defined by a lifting of $\mathbb{C}\alpha$ to $T_{[\alpha]}(\mathcal{C})$. On $\mathcal{C}|_U = \pi^{-1}(U)$, $\mathcal{F}$ and $f^*\mathcal{F}$ define two liftings of the tautological line bundle L to $T(\mathcal{C})$, and the difference of these liftings define at $x \in U$ a twisted vertical holomorphic vector field in $\Gamma(\mathcal{C}_x, Hom(L, T(\mathcal{C}_x))$. As the latter space is in fact non-zero, this does not yield the Proposition. Consistent with the emphasis of these lectures we will give a proof of Proposition 2.2.1 basing on the existence of Harish-Chandra coordinates. We use the fact that the two foliations are obtained from each other by a local map f on S and exploit the symmetry of the Hessian $D^2 f$.

The proof given here is elementary in nature, avoids the use of isotropy representations as in [HM3, (2.5), Lemma 4], and shows how the Gauss map on $\mathcal{C}_x$ enters into the picture. A result stronger than Proposition 2.2.1 holds in a general setting on uniruled projective manifolds. For the proof, which relies on the deformation theory of rational curves, we refer the reader to [HM5].

Proof of Proposition 2.2.1. Denote by ∇ the Euclidean flat connections on both U and V and write ∇' for the pulled-back connection $f^*\nabla$ on U. Denote by $(z_1, \ldots, z_n)$ *resp.* $(w_1, \ldots, w_n)$ Harish-Chandra coordinates on U *resp.* V. Fix a base point $x \in U$. Without loss of generality we may assume that $df(x)$ is the identity map with respect to (z_i) and (w_k). For a non-zero tangent vector $\alpha = \sum \alpha^i \frac{\partial}{\partial z_i}$ at x by abuse of notations we will write $\frac{\partial}{\partial z_\alpha}$ for the constant vector field on U which is equal to α at x. We have

$$\nabla'_{\frac{\partial}{\partial z_\alpha}} \frac{\partial}{\partial z_\beta} = f^*\Big(\nabla_{f_* \frac{\partial}{\partial z_\alpha}} f_* \frac{\partial}{\partial z_\beta} \Big)$$

At the point x we have

$$\nabla'_{\frac{\partial}{\partial z_\alpha}} \frac{\partial}{\partial z_\beta}(x) = \sum_k \alpha^i \beta^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \frac{\partial}{\partial z_k}(x),$$

where $f^* \frac{\partial}{\partial w_k}(x)$ is identified with $\frac{\partial}{\partial z_k}(x)$ since $df(x) = id$.

Denote by $\tilde{\mathcal{C}}_x \subset T_x(S)$ the cone of all minimal rational tangent vectors at x, so that $\mathcal{C}_x$ is the projectivization of $\tilde{\mathcal{C}}_x - \{o\}$. For $\alpha \in \tilde{\mathcal{C}}_x$, $\alpha \neq 0$, we will write $P_\alpha \subset T_x(X)$ to consist of all vectors tangent to $\tilde{\mathcal{C}}_x \subset T_x(X)$ at α. Thus, $T_{[\alpha]}(\mathcal{C}_x) \cong P_\alpha/\mathbb{C}_\alpha$. Write $(u_1, \ldots, u_n)$ *resp.* $(v_1, \ldots, v_n)$ for the standard fiber coordinates for the tangent bundles $T(U)$ *resp.* $T(V)$ with respect to the Harish-Chandra coordinates (z_i) *resp.* (w_j).

Consider now at x two non-zero minimal rational tangent vectors α and β. α and β will also be considered as points on $\tilde{\mathcal{C}}_x$ or as points on $\tilde{\mathcal{C}}_y$, $y = f(x)$, when we identify $T_x(U)$ with $T_y(V)$ via df, Let C be the minimal rational curve on S passing through x with $T_x(C) = \mathbb{C}\alpha$. Write $C \cap U = L$ and let L' be the graph of the constant section of $\tilde{\mathcal{C}}|_U$ over L containing β. Then $f_* L'$ is a section of $\tilde{\mathcal{C}}|_V$ over $f(L)$ containing $df(\beta) = \beta$. Suppose μ and ν are two vectors tangent to $\tilde{\mathcal{C}}|_V$ at the point $\beta \in \tilde{\mathcal{C}}_y$ such that for the canonical projection $\pi : T(V) \to V$ we have $\pi(\mu) = \pi(\nu)$. Then the difference $\mu - \nu$ projects to zero, and is hence a vertical tangent vector, *i.e.*, belonging to $T_\beta(\tilde{\mathcal{C}}_x)$. Although $T_\beta(\tilde{\mathcal{C}}_x)$ and P_β correspond to each other they are different vector spaces with $T_\beta(\tilde{\mathcal{C}}_x) \subset T_\beta(T_x(X))$ and $P_\beta \subset T_x(X)$. At the point β by Proposition 1.4.2 we may take ν to be the horizontal tangent vector α, and μ to be the pull-back of the horizontal vector α by f, i.e.,

$$\mu = \alpha + \sum_{i,j,k} \alpha^i \beta^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \frac{\partial}{\partial u_k}.$$

It follows that the difference

$$\mu - \nu = \sum_{i,j,k} \alpha^i \beta^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \frac{\partial}{\partial u_k} \in T_\beta(\tilde{\mathcal{C}}_x).$$

Equivalently, that

$$\sum_{i,j,k} \alpha^i \beta^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \frac{\partial}{\partial z_k} \in P_\beta,$$

where we identify $T_x(U)$ with $T_y(V)$ via df. Since the left hand side is symmetric in α and β we conclude that

$$D^2 f(\alpha, \beta) \in P_\alpha \cap P_\beta,$$

where $D^2 f$ denotes the Hessian. Endow V with the standard Euclidean inner product with respect to the Harish-Chandra coordinates (w_k). Now fix α and let $\beta = \alpha(t)$, $\alpha(0) = \alpha$, be a smooth real one-parameter family of minimal rational tangent vectors defined for small t such that $\alpha(t) = \alpha + t\xi + t^2\zeta_t$, where $\xi \in P_\alpha$ is tangent to $\tilde{\mathcal{C}}$ at α and ζ_t is orthogonal to P_α. Then,

$$D^2 f(\alpha, \alpha + t\xi + t^2\zeta_t) \in P_\alpha \cap P_{\alpha(t)}. \qquad (*)$$

For a complex vector subspace B of a finite-dimensional Hermitian vector space A and for η a vector in A we denote by $pr(\eta, B)$ the orthogonal projection of η into B. We denote by $B^\perp$ the orthogonal complement of B in A. Observe that

$$D^2 f(\alpha, \alpha + t\xi + t^2\zeta_t) \in P_\alpha \implies D^2 f(\alpha, \alpha), \quad D^2 f(\alpha, \xi) \in P_\alpha,$$
$$\text{so that } pr\left(D^2 f(\alpha, \xi), P_{\alpha(t)}^\perp\right) = O(t).$$

Using this and the second half of $(*)$ we have

$$D^2 f(\alpha, \alpha(t)) = D^2 f(\alpha, \alpha + t\xi + t^2\zeta_t) \in P_{\alpha(t)} \Longrightarrow pr\Big(D^2 f(\alpha,\alpha), P^{\perp}_{\alpha(t)}\Big) = O(t^2).$$

We are going to deduce from this that $D^2 f(\alpha,\alpha)$ is proportional to α. Towards that end we need

Lemma 2.2.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain and $Z \subset \Omega$ be a closed complex submanifold. At $x \in Z$ denote by $\sigma_z : T_z(Z) \times T_z(Z) \to N_{S|\Omega,z}$ the second fundamental form with respect to the Euclidean flat connection on Ω. Let τ be a vector tangent to Z at z and $\gamma : (-\epsilon,\epsilon) \to Z, \gamma(0) = z$, be a smooth curve such that $\gamma'(0) = 2Re(\eta)$. Identify vectors at different points of Ω by the standard trivialization $T_\Omega \cong \Omega \times \mathbb{C}^n$. Then, $pr(\tau, T^{\perp}_{\gamma(t)}(Z)) = O(t^2)$ if and only if $\sigma_z(\tau,\eta) = 0$.*

Proof. Let $\tilde\tau(t)$ be a smooth vector field of (1,0)-tangent vectors along γ such that $\tilde\tau(0) = \tau$ and $\tilde\tau(t)$ is tangent to Z at $\gamma(t)$. With respect to the Euclidean flat connection ∇, we have $\tilde\tau'(0) = \nabla_\eta \tilde\tau(0)$. In what follows for $z \in Z$ we write $T^{\perp}_z$ for $T^{\perp}_z(Z)$. Since

$$\sigma_z(\tau,\eta) = pr\Big(\nabla_\eta\, \tilde\tau(0), T^{\perp}_z\Big),$$

we have

$$\tilde\tau'(0) \in T_z \quad \Longleftrightarrow \quad \sigma_z(\tau,\eta) = 0.$$

Consider now the vector field $\tau - \tilde\tau(t)$ along γ, which vanishes at $t = 0$. Then,

$$\begin{aligned} \tilde\tau'(0) \in T_z(Z) \quad &\Longleftrightarrow \quad (\tau - \tilde\tau)'(0) \in T_z(Z)\ , \\ i.e., \quad &\tau = \tilde\tau + t\mu + O(t^2) \end{aligned}$$

for some $\mu = -\tilde\tau'(0) \in T_z(Z)$. Finally,

$$\begin{aligned} pr\Big(\tau, T^{\perp}_{\gamma(t)}\Big) &= pr\Big((\tau - \tilde\tau) + \tilde\tau, T^{\perp}_{\gamma(t)}\Big) = pr\Big(\tau - \tilde\tau, T^{\perp}_{\gamma(t)}\Big) \\ &= t \cdot pr\Big(\mu, T^{\perp}_{\gamma(t)}\Big) + O(t^2), \end{aligned}$$

so that

$$pr\Big(\tau, T^{\perp}_{\gamma(t)}\Big) = O(t^2) \Longleftrightarrow \mu \in T_z(Z) \Longleftrightarrow \sigma_z(\tau,\eta) = 0,$$

as desired. □

End of proof of Proposition 2.2.1. We have proven that for each non-zero $\alpha \in \tilde{\mathcal{C}}_x$, $D^2 f(\alpha,\alpha) \in P_\alpha$, and it remains to show that this forces the stronger property that $D^2 f(\alpha,\alpha)$ is proportional to α. Note that $T_\beta(\tilde{\mathcal{C}}_x)$ is identified with P_β. On the smooth curve $\alpha(t)$, $|t| < \epsilon$, $\alpha(0) = \alpha$, $\alpha'(0) = 2Re(\eta)$, $\eta \in T_\alpha(\tilde{\mathcal{C}}_x) = P_\alpha$, we already know that

$$pr\Big(D^2 f(\alpha,\alpha), P^{\perp}_{\alpha(t)}\Big) = O(t^2).$$

By Lemma 2.2.2, for τ tangent to $\tilde{\mathcal{C}}_x$ at α we have

$$pr\Big(\tau, P^{\perp}_{\alpha(t)}\Big) = O(t^2) \Longleftrightarrow \sigma_\alpha(\tau,\eta) = 0.$$

Since $\alpha'(0)$ is an arbitrary $(1,0)$-vector tangent to $\tilde{\mathcal{C}}_x$ at α we conclude that

$$\sigma_\alpha\Big(D^2f(\alpha,\alpha),\eta\Big) = 0$$

for any $\eta \in P_\alpha$. In other words, $D^2f(\alpha,\alpha)$ lies in $Ker(\sigma_\alpha)$. By Proposition 2.1.5, for the projective submanifold $\tilde{\mathcal{C}}_x \subset \mathbb{P}T_x(S)$, $Ker(\sigma_\alpha) = \mathbb{C}\alpha$ for a generic $\alpha \in \tilde{\mathcal{C}}_x$, so that $D^2f(\alpha,\alpha) \in \mathbb{C}\alpha$ for a generic $\alpha \in \tilde{\mathcal{C}}_x$ and hence for every $\alpha \in \tilde{\mathcal{C}}_x$. Clearly this implies that $f : U \to V$ preserves the 1-dimensional foliation $\mathcal{F}$, *i.e.* $f^*(\mathcal{F}|_V) = \mathcal{F}|_U$. The proof of Proposition 2.2.1 is complete. □

(2.3) We are now led to consider an $\mathcal{F}$-preserving biholomorphism $f : U \to V$; $U, V \subset S$ connected open sets. We proceed to extend f to a biholomorphic automorphism of S, by the method of analytic continuation.

To illustrate our approach consider the case of projective spaces $\mathbb{P}^n$, $n \geq 2$. Although $\mathcal{C}_x = \mathbb{P}T_x(\mathbb{P}^n)$ in this case, $\mathbb{P}T(\mathbb{P}^n)$ is still equipped with a 1-dimensional holomorphic foliation whose leaves are liftings $\hat{L}$ of lines L on $\mathbb{P}^n$. The problem of extending $\mathcal{F}$-preserving biholomorphisms still makes sense on $\mathbb{P}^n$. Restrict now to the case of $n = 2$ and take $U = B^2 \subset \mathbb{C}^2 \subset \mathbb{P}^2$ to be the unit ball. We have

Lemma (for illustration). *Let $f : B^2 \to V$ be a biholomorphism onto $V \subset \mathbb{P}^2$ such that f transforms any non-empty open subset $L \cap B^2$ of a line L into an open subset of some line $L' \subset \mathbb{P}^2$. Then, f extends to a biholomorphic automorphism $F : \mathbb{P}^2 \to \mathbb{P}^2$.*

Proof. The dual space $(\mathbb{P}^2)^*$ of lines in $\mathbb{P}^2$ is itself biholomorphic to $\mathbb{P}^2$. Let $D \subset (\mathbb{P}^2)^*$ be the open subset consisting of all lines L such that $L \cap B^2 \neq \emptyset$. We proceed to determine D. Let $[w_0, w_1, w_2]$ be homogeneous coordinates on $\mathbb{P}^2$ and $\mathbb{C}^2 \subset \mathbb{P}^2$ be the affine part corresponding to $z_0 \neq 0$. A line $L \subset \mathbb{P}^2$ is defined by a homogeneous linear equation $\eta_0 w_0 + \eta_1 w_0 + \eta_2 w_2 = 0$, $(\eta_0, \eta_1, \eta_2) \neq 0$. In the affine part $w_0 \neq 0$ we write $(z_1, z_2) = (\frac{w_1}{w_0}, \frac{w_2}{w_0})$ for inhomogeneous coordinates. For a line L not passing through $[1,0,0]$, $\eta_0 \neq 0$, and L is given by $\xi_1 z_1 + \xi_2 z_2 + 1 = 0$ where $(\xi_1, \xi_2) = (\frac{\eta_1}{\eta_0}, \frac{\eta_2}{\eta_0})$ are inhomogeneous coordinates for the affine part of $(\mathbb{P}^2)^*$ corresponding to $\eta_0 \neq 0$. The distance of L to the origin $o \in B^2$ in terms of the Euclidean metric on $\mathbb{C}^2$ is given by $dist(o; L) = \frac{1}{\sqrt{|\xi_1|^2 + |\xi_2|^2}}$, which is < 1 if and only if $|\xi_1|^2 + |\xi_2|^2 > 1$. In other words, $D \subset (\mathbb{P}^2)^*$ consists precisely of points $[\eta_0, \eta_1, \eta_2] \in (\mathbb{P}^2)^*$ such that either $\eta_0 = 0$ (and L is a line through the origin o in B^2), or $\eta_0 \neq 0$ and $(\xi_1, \xi_2) = (\frac{\eta_1}{\eta_0}, \frac{\eta_2}{\eta_0})$ is exterior to the unit ball $(B^2)'$ in $\mathbb{C}^2 \subset (\mathbb{P}^2)^*$, $\mathbb{C}^2$ corresponding to $\eta_0 \neq 0$. Whenever $L \cap B^2 \neq \emptyset$ it is a disk on L and hence connected. The assumption that $f : B^2 \cong V \subset \mathbb{P}^2$ is $\mathcal{F}$-preserving implies that $f(L \cap B^2)$ is a connected open subset of some line. Thus, f induces a holomorphic map $f^\# : D \to \mathbb{P}^2$. As $D \cap \mathbb{C}^2 = \{(\xi_1, \xi_2) : |\xi_1|^2 + |\xi_2|^2 > 1\}$, D is strongly pseudoconcave, and by Hartogs extension theorem $f^\# : D \to \mathbb{P}^2$ extends to a meromorphic map $f^\# : (\mathbb{P}^2)^* \to (\mathbb{P}^2)^*$. Applying the same argument to the restriction of $f^{-1} : V \cong B^2$ to some Euclidean ball we conclude that $f^\#$ is a birational map. ¿From the fact that $f : B^2 \cong V$ is a biholomorphism it follows that there exists some Euclidean ball $(B^2)'' \subset (\mathbb{P}^2)^*$ such that $f^\#|_{(\mathbb{P}^2)^* - (B^2)'}$ is a biholomorphism onto $(\mathbb{P}^2)^* - G$ for some open subset $G \subset (B^2)''$. The extension $f^\# : \mathbb{P}^2 \to \mathbb{P}^2$ must therefore be holomorphic and is hence a biholomorphic automorphism of $(\mathbb{P}^2)^*$. From this one readily recovers a biholomorphic automorphism $F : \mathbb{P}^2 \to \mathbb{P}^2$ extending $f : B^2 \to V$. □

The same line of argument when applied to $\mathbb{P}^n$, $n \geq 3$, is already not so straightforward. The moduli space of all lines on $\mathbb{P}^n$ is $G(2, n-1)$, the Grassmannian of 2-planes in $\mathbb{C}^{n+1}$. One can define a holomorphic map on some open neighborhood D of a projective $(n-1)$-plane $\mathbb{P}^{n-1} \subset G(2, n-1)$ corresponding to all lines passing through a fixed base point $x_o \in \mathbb{P}^n$. To be able to extend $f^{\#}$ meromorphically in one step to $G(2, n-1)$, of dimension $2n-2$, one would have to find a tubular neighborhood G of $\mathbb{P}^{n-1}$ with smooth boundary, $G \subset\subset D$, such that ∂G is $(2n-3)$-pseudoconcave, by Andreotti-Grauert [AG]. In practice, this amounts to representing G as $\{\varphi < 0\}$, φ smooth in a neighborhood of $\overline{G}$, such that $d\varphi|_{\partial G} \neq 0$ and $\sqrt{-1}\partial\overline{\partial}\varphi|_{T_z^{\mathbb{C}}(G)}$ has at least one negative eigenvalue for every $z \in \partial G$. In place of implementing this we adopt a more direct way of analytically continuing f along lines, as follows.

Lemma (provisional). *For $n \geq 3$ let $f : B^n \to V$ be a biholomorphism onto $V \subset \mathbb{P}^n$ such that f transforms open subsets of lines onto open subsets of lines. Then, f extends to a birational map $F : \mathbb{P}^n \to \mathbb{P}^n$.*

Proof. Let $L_o \subset \mathbb{P}^n$ be a line such that $L_o \cap B^n$ is non-empty. $L_o \cap B^n$ is a disk on the line L_o. By assumption $f(L_o \cap B^n)$ is an open subset of some line $L'_o \subset \mathbb{P}^n$. The same applies to a line $L \subset \mathbb{P}^n$ such that $[L]$ is sufficiently close to $[L_o]$, with $f(L \cap B^n) \subset L' \subset \mathbb{P}^n$, so that the assignment $f^{\#}([L]) = L'$ defines a holomorphic map in a neighborhood of $[L_o]$ in $G(2, n-1)$. Let $\rho : \mathbb{P}T(\mathbb{P}^n) \to G(2, n-1)$ be the universal family of lines on $\mathbb{P}^n$. Then $f^{\#} \circ \rho$ is defined in a neighborhood of the lifting $\hat{L}$ of L to $\mathbb{P}T(\mathbb{P}^n)$. We proceed next to define a meromorphic map $\hat{f}$ in a neighborhood of L. In what follows we will more generally be dealing with $\mathcal{F}$-preserving meromorphic maps $f : \Omega \dashrightarrow \mathbb{P}^n$ on a domain $\Omega \subset \mathbb{P}^n$. By this we mean that at a generic point of Ω, f is a local biholomorphism and $\mathcal{F}$-preserving. By the identity theorem on holomorphic functions if the germ of f is a local biholomorphism and $\mathcal{F}$-preserving at some point $x_o \in \Omega$, then f is $\mathcal{F}$-preserving on Ω.

Let now D be an open neighborhood of $L_o \subset \mathbb{P}T(\mathbb{P}^n)$ on which $f^{\#} \circ \rho$ is defined, $x_o \in L_o$ a point at which f is defined, $\sigma_1, \sigma_2 : W \to \mathbb{P}T(\mathbb{P}^n)$ be two distinct holomorphic sections over some neighborhood of x_o such that $\sigma_i(W) \subset D$; $i = 1, 2$. Let $\Sigma_o \subset W \times \mathbb{P}^n$ be the intersection $Graph(f^{\#} \circ \rho \circ \sigma_1) \cap Graph(f^{\#} \circ \rho \circ \sigma_2)$. Here by the graph of the meromorphic map $f^{\#} \circ \rho \circ \sigma_i$; $i = 1, 2$; we mean the topological closure of the graph of $f^{\#} \circ \rho \circ \sigma_i$ over the open subset $W_i \subset W$ where $f^{\#} \circ \rho \circ \sigma_i$ is holomorphic. ($W - W_i$ is a complex-analytic subvariety of W.) There is a unique irreducible component Σ of Σ_o such that the canonical projection $\Sigma \to W$ is surjective. Σ is the topological closure of $Graph(f^{\#} \circ \rho \circ \sigma'_1) \cap Graph(f^{\#} \circ \rho \circ \sigma'_2)$, where σ'_1, σ'_2 are the restriction of σ_1, σ_2 to $W_1 \cap W_2$. If f is defined on a non-empty connected open subset W_o of W, then, denoting $g = f|_{W_o}$, Σ is the graph of some meromorphic $\hat{g}$ over W such that $\hat{g}|_{W_o} \equiv g$. This observation, together with the following obvious lemma, implies readily that the germ of f at $o \in B^n$ admits an extension $\hat{f} : U \to \mathbb{P}^n$ to some tubular neighborhood U of L.

Lemma 2.3.1. *Let $W_o \subset W \subset \mathbb{P}^n$ be non-empty connected open subsets of $\mathbb{P}^n$. Let $g : W_o \to \mathbb{P}^n$ be an $\mathcal{F}$-preserving meromorphic map. Suppose $g^{\#} \circ \rho$ is defined on the graph of two distinct holomorphic sections $\sigma_1, \sigma_2 : W \to \mathbb{P}T(\mathbb{P}^n)$ over W.*

Define now $\Sigma \subset W \times \mathbb{P}^n$ to be the unique irreducible component of $Graph(g^{\#} \circ \rho \circ \sigma_1) \cap Graph(g^{\#} \circ \rho \circ \sigma_2)$ which projects onto W. Then, Σ is the graph of an $\mathcal{F}$-preserving meromorphic map $\hat{g} : W \to \mathbb{P}^n$ such that $\hat{g}|_{W_o} \equiv g$.

Proof of Lemma (provisional) continued. We note that in the application of Lemma 2.3.1 the important thing is to have some holomorphic section of $\mathbb{P}T(\mathbb{P}^n)$ over W. There is no difficulty with finding such local sections on tubular neighborhoods of pieces of the line L since the lift $\hat{L}$ of L to $\mathbb{P}T(\mathbb{P}^n)$ already lies in the domain of definition of $f^{\#} \circ \rho$.

Recall that $f : B^n \cong V \subset \mathbb{P}^n$ is an $\mathcal{F}$-preserving biholomorphism, *i.e.*, f transforms intersections of B^n with lines to open subsets of lines. The preceding argument shows that, for any line L passing through o, the germ of f at o can be analytically continued to some open neighborhood U of L. Since every point $y \in \mathbb{P}^n$, $y \neq o$, can be joined to o by a unique line L, implementing the procedure of anlaytic continuation simultaneously on all such lines we obtain a well-defined $\mathcal{F}$-preserving meromorphic map $\hat{f} : \mathbb{P}^n \to \mathbb{P}^n$ extending $f : B^n \cong V$. Arguing with f^{-1} we conclude that $f : \mathbb{P}^n \to \mathbb{P}^n$ is birational, as desired. □

The proof that $\hat{f}$ is a biholomorphism requires a little work, which will be taken up in (2.4) in a more general context. For irreducible Hermitian symmetric spaces S of *rank* ≥ 2 as in Theorem 2.1.2, 2 distinct points $x, y \in S$ may or may not be joined by a line. However, as explained in (1.6) any 2 distinct points can be joined by a non-overlapping chain of lines, by the Polysphere Theorem. The same argument as for $\mathbb{P}^n$, replacing $\mathbb{P}T(\mathbb{P}^n)$ by the bundle $\pi : \mathcal{C} \to S$ of varieties of minimal rational tangents, on which the holomorphic foliation $\mathcal{F}$ is defined, leads to the following result on analytic continuation.

Lemma 2.3.2. *Let $K = C_1 + C_2 + \cdots + C_m$ be a non-overlapping chain of lines on S, $o \in C_1$ and f be a germ of $\mathcal{F}$-preserving meromorphic map at o. Then, there exists a tubular neighborhood U of K and an $\mathcal{F}$-preserving meromorphic map $\hat{f} : U \to S$ such that $\hat{f}$ extends the germ f.*

For an open ball $\Omega \subset \mathbb{C}^n \subset S$ consider the space $\mathcal{M} = \mathcal{M}_\Omega$ of all $\mathcal{F}$-preserving meromorphic maps $f : \Omega \dashrightarrow S$. Denote by $\pi : \hat{\Omega} \to S$ the hull of maximal existence of $\mathcal{M}$, which is a Riemann domain over S, *i.e.*, π is a local biholomorphism. (Such a hull can be constructed for any family of meromorphic maps, *cf.* Narasimhan [Na]). ¿From the universality of the hull, if $\pi' : \Omega' \to S$ is any Riemann domain over S such that any $f \in \Omega$ admits a (unique) extension to Ω', then there is a canonical map $\tau : \Omega' \to \hat{\Omega}$ extending the identity map on the univalent domain $\Omega \subset \Omega'$. By Lemma 2.3.2, any $f \in \mathcal{M}$ can be analytically continued along tubular neighborhoods of non-overlapping chains of lines. Since S is rationally connected by a non-overlapping chain of lines, it follows that $\pi : \hat{\Omega} \to S$ is surjective. However it is not clear that π is injective. We will now prove that it is in fact the case by making use of $\mathbb{C}^*$-actions on S. This will yield

Proposition 2.3.3. *In the notations of Theorem 2.2.1, $f : U \to V$ extends to a birational map $F : S \dashrightarrow S$.*

Proof. We fix a system of Harish-Chandra coordinates on a Euclidean cell $\cong \mathbb{C}^n$ on S, and take Ω to be the unit ball on $\mathbb{C}^n$, $o \in \Omega$. For $\lambda \in \mathbb{C}^*$ we denote by $\lambda : \mathbb{C}^n \to \mathbb{C}^n$ the mapping $\lambda(z) = (\lambda z_1, \ldots, \lambda z_n)$. λ extends to a biholomorphism of S, defining a $\mathbb{C}^*$-action on S. Let now $\lambda \in \mathbb{C}^*$, $|\lambda| < 1$. Write $\Omega_{1/\lambda} = \frac{1}{\lambda}\Omega$. The biholomorphism $\lambda : \Omega_{1/\lambda} \to \Omega$ is $\mathcal{F}$-preserving, inducing a bijection between $\mathcal{M}_\Omega$ and $\mathcal{M}_{\Omega_{1/\lambda}}$. It extends therefore to a biholomorphism $\hat{\Omega}_{1/\lambda} \to \hat{\Omega}$. Denote also by $\lambda : \hat{\Omega}_{1/\lambda} \to \hat{\Omega}$ the extended map. $\hat{\Omega}_{1/\lambda}$ is equivalently the hull of Ω with respect to the family $\{f|_\Omega : f \in \mathcal{M}_{\Omega_{1/\lambda}}\} \subset \mathcal{M}_\Omega$. By universality there is a local biholomorphism $\tau : \hat{\Omega} \to \hat{\Omega}_{1/\lambda}$ extending the identity map on Ω, and $\lambda \circ \tau : \hat{\Omega} \to \hat{\Omega}$ extends the self-map $\lambda : \Omega \to \Omega$. From now on we will write λ for $\lambda \circ \tau$.

We proceed to prove that $\pi : \hat{\Omega} \to S$ is injective and hence a biholomorphism. To start with, we show that $\pi^{-1}(o)$ is a single point. Suppose otherwise. Let ζ_1, $\zeta_2 \in \pi^{-1}(o)$, $\zeta_1 \neq \zeta_2$. There exists some connected open neighborhood U of o such that $\pi^{-1}(U)$ is a disjoint union of open sets U_α, $\alpha \in A$, with $\zeta_1 \in U_1$, $\zeta_2 \in U_2$. Let γ be a continuous curve on $\hat{\Omega}$ joining ζ_1 to ζ_2.

For λ sufficiently small $\lambda(\pi(\gamma)) \subset U$, so that $\lambda(\gamma) \subset \pi^{-1}U$, and for all $\zeta \in \gamma$, $\lambda\zeta \in U_\alpha$ for a unique α, contradicting with $\lambda\zeta_1 \in U_1$, $\lambda\zeta_2 \in U_2$. Now the same argument works with o replaced by any $x \in \Omega$, using $\mathbb{C}^*$-actions λ_x with centres x. (In this case $\lambda_x : \Omega \to \Omega$ for λ sufficiently small.) We conclude that $\pi : \hat{\Omega} \to S$ is univalent over Ω. ¿From this we proceed to prove that π is univalent over S.

Suppose $y \in S$ is such that $\pi^{-1}(y)$ contains at least 2 distinct points ζ_1, ζ_2. Consider now the simultaneous extension of $\lambda : \Omega \to \Omega$; $0 < |\lambda| < 1$; to $\lambda : \hat{\Omega} \to \hat{\Omega}$. Given any continuous curve γ on $\hat{\Omega}$, some λ_o; $|\lambda_o| < 1$; and the knowledge of $\lambda_o\xi$ in a vicinity of γ, $\lambda\xi$ is uniquely determined for λ sufficiently close to λ_o and ξ sufficiently close to γ, by lifting $\lambda\pi(\xi)$. From this it follows that $\lambda\xi$ depends holomorphically and jointly on (λ, ξ). For a similar reason for λ sufficiently close to 1, $\lambda\xi_1 \neq \lambda\xi_2$. On the other hand, since

$$\pi(\lambda\xi_1) = \lambda(\pi(\xi_1)) = \lambda(y) = \lambda(\pi(\xi_2)) = \pi(\lambda\xi_2) ,$$

we see that if λ is chosen small enough such that $\pi(\lambda y) \in U$, we must have $\lambda\xi_1 = \lambda\xi_2$. This contradicts with the identity theorem on holomorphic functions, and we conclude that $\pi : \hat{\Omega} \to S$ is univalent, hence a biholomorphism. We can now identify the hull $\pi : \hat{\Omega} \to S$ for $\mathcal{M}_\Omega$ as S itself. Returning to Proposition 2.3.3, we have proven that $f : U \to V$ extends to a rational map F on S. Applying the same arguments to the inverse mapping $f^{-1} : V \to U$ we conclude that $F : S \dashrightarrow S$ is a birational map. The proof of Proposition 2.3.3 is complete. □

(2.4) For the proof of Theorem 2.1.2 (and hence Theorem 2.1.1) it remains to establish

Proposition 2.4.1. *Let S be an irreducible Hermitian symmetric space of the compact type and $F : S \dashrightarrow S$ be a birational self-map. Suppose for a generic line L of S, $F|_L$ is a biholomorphism of L onto a line L'. Then, F is a biholomorphic automorphism of S.*

Proof. We denote by $Z \subset S$ the set of indeterminacies of F and by R the ramification divisor of $F|_{S-Z}$. The union $B = R \cup Z \subset S$ is a subvariety. To prove Proposition 2.4.1 we claim that it suffices to show that $R = \emptyset$. Assuming $R = \emptyset$, then for any holomorphic anti-canonical section $s \in \Gamma(S, K_S^{-1})$, F^*s is well-defined on $S - Z$ and extends to a holomorphic section over S by Hartogs extension, since

Z is of codimension ≥ 2. Applying the same argument to the inverse map F^{-1}, we conclude that F induces a linear isomorphism $\theta : \Gamma(S, K_S^{-1}) \to \Gamma(S, K_S^{-1})$, which induces a projective-linear isomorphism: $[\theta^*] : \mathbb{P}\Gamma(S, K_S^{-1})^* \to \mathbb{P}\Gamma(S, K_S^{-1})^*$. Identifying S as a complex submanifold of $\mathbb{P}\Gamma(S, K_S^{-1})^*$, F is nothing other than the restriction of $[\theta^*]$ to S, thus a biholomorphism.

It remains to show that F is unramified on $S-Z$. Suppose otherwise. Let $x \in R$ be a smooth point. Since $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is not contained in any hyperplane there exists some line L passing through x, $L \not\subset B = R \cup Z$, such that $F|_L$ maps L biholomorphically onto a line L', so that $dF(x) \not\equiv 0$. For $y = F(x)$, $dF(x) : T_x(S) \to T_y(S)$ has a non-trivial kernel. The projective linear map $[dF(x)] : \mathbb{P}T_x(S) \dashrightarrow \mathbb{P}T_y(S)$ induces a rational map $[dF(x)]|_{\mathcal{C}_x} : \mathcal{C}_x \dashrightarrow \mathcal{C}_y$ with image lying in a linear section Σ of $\mathcal{C}_y$, $\Sigma \subsetneq \mathcal{C}_y$, as $\mathcal{C}_y$ is not contained in any hyperplane section. It follows that $[dF(x)]|_{\mathcal{C}_x} : \mathcal{C}_x \dashrightarrow \mathcal{C}_y$ must have positive-dimensional generic fibers. In particular, there exist a holomorphic one-parameter family of lines $\{L_t : t \in \triangle\}$ passing through x such that $L_o = L$ and such that each L_t is mapped biholomorphically onto the same L', so that F fails to be locally biholomorphic at a generic point of L, contradicting the choice of L. With this we have proven that F is unramified on $S - Z$. Consequently, $F : S \to S$ is actually a biholomorphic automorphism, as explained. □

§3 Minimal rational curves on the central fiber and the distribution spanned by their tangents

(3.1) A projective manifold is said to be uniruled if deformations of some rational curve cover the whole manifold. By Mori's theory any Fano manifold is uniruled. In what follows we will give a very informal introduction on the deformation theory of rational curves to the extent necessary for the understanding of [HM2]. We refer the reader to Kollár [Ko] for the general theory of rational curves, and to [HM5] for a more systematic introduction.

To start with we need some preliminaries on holomorphic vector bundles on the Riemann sphere $\mathbb{P}^1$. On $\mathbb{P}^1$ the isomorphism class of a holomorphic line bundle is determined by its degree. A holomorphic line bundle of degree k is denoted by $\mathcal{O}(k)$. Thus, the tangent bundle $T(\mathbb{P}^1)$ is isomorphic to $\mathcal{O}(2)$, because there exists a holomorphic vector field with two isolated zeros, *e.g.*, the Euler vector field given by $z\frac{\partial}{\partial z}$ on $\mathbb{C} \subset \mathbb{P}^1$. In general holomorphic vector bundles on compact Riemann surfaces are complicated objects, but in the case of $\mathbb{P}^1$ we have Grothendieck's splitting theorem, which says that any rank-r holomorphic vector bundle V over $\mathbb{P}^1$ splits into a direct sum of holomorphic line bundles $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ such that the degrees a_i, counting multiplicities, are uniquely determined by V. (*cf.* Grauert-Remmert [GR] for a proof). For example, the tangent bundle $T(\mathbb{P}^n)$, when restricted to a projective line (*i.e.*, a degree-1 rational curve) $\mathbb{P}^1 \subset \mathbb{P}^n$, splits into $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-1}$. A holomorphic vector bundle V over $\mathbb{P}^1$ is said to be semipositive whenever each $a_i \geq 0$. Suppose $X \subset \mathbb{P}^n$ is a projective submanifold, and $C \subset \mathbb{P}^n$ is a projective line already contained in X, such that $T(X)|_C$ is semipositive, then it must be of the form $\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$. $T(X)|_C$ is semipositive whenever $T(X)$ is spanned by global sections, *i.e.*, by holomorphic vector fields on X. This is the case for Hermitian symmetric spaces S of the compact type, as S is homogeneous. For

S irreducible, $H_2(S,\mathbb{Z}) \cong \mathbb{Z}$, and there exists a holomorphic embedding τ of S into some $\mathbb{P}^N$, defined by the positive generator $\mathcal{O}(1)$ of the Picard group of S, such that τ induces an isomorphism $\tau_* : H_2(S,\mathbb{Z}) \cong H_2(\mathbb{P}^N,\mathbb{Z})$, as exemplified by the Plücker embedding on Grassmannians. Rational curves C on S of degree 1 with respect to $\mathcal{O}(1)$ are then projective lines on $\mathbb{P}^N$. Since $T(S)|_C \subset T(\mathbb{P}^N)|_C$ and $T(S)|_C$ is semipositive, it follows that $T(S)|_C \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ with $K_S^{-1} \cdot C = p+2$, for every degree-1 (rational) curve $C \subset S$.

Let X be a projective manifold. By a parametrized rational curve we mean a non-constant holomorphic map $f : \mathbb{P}^1 \to X$. The image of f is called a rational curve. Given a holomorphic family $f_t : \mathbb{P}^1 \to X$, $t \in \Delta$, of parametrized rational curves, the derivative $\frac{d}{dt}\Big|_o f_t$ defines a holomorphic section of $f_o^*T(X)$. However, given a member f_o of the space $Hol(\mathbb{P}^1, X)$ of parametrized rational curves in X, and $\sigma \in \Gamma(\mathbb{P}^1, f_o^*T(X))$, it is not always possible to fit f_o into a holomorphic family of $f_t \in Hol(\mathbb{P}^1, X)$, such that $\frac{d}{dt}|_o f_t = \sigma$. Setting in power series $f_t = f + \sigma t + g_2 t^2 + \cdots$ locally, the obstruction of lifting to higher coefficients lies in $H^1(\mathbb{P}^1, f_o^*T(X))$. In case the latter vanishes, $Hol(\mathbb{P}^1, X)$ is smooth in a neighborhood U of $[f_o]$, and the tangent space at $[f] \in U$ can be identified with $\Gamma(\mathbb{P}^1, f^*T(X))$. This is in particular the case whenever $f^*T(X)$ is semipositive on X. In this case deformations of f sweep out some open neighborhood of $C = f(\mathbb{P}^1)$. We call f a free rational curve, and X is said to be uniruled.

Each irreducible component $\mathcal{H}$ of $Hol(\mathbb{P}^1, X)$ can be given the structure of a quasi-projective variety. $Aut(\mathbb{P}^1)$ acts on $\mathcal{H}$ and the quotient space $\mathcal{H}/Aut(\mathbb{P}^1)$ can be endowed the structure of a quasi-projective variety such that $\mathcal{H} \to \mathcal{H}/Aut(\mathbb{P}^1)$ realizes $\mathcal{H}$ as a principal G-bundle with $G = Aut(\mathbb{P}^1) \cong \mathbb{P}SL(2,\mathbb{C})$. When each member $[f]$ of $\mathcal{H}$ is free, $\mathcal{H}/Aut(\mathbb{P}^1)$ is non-singular.

We denote by $Hol((\mathbb{P}^1,0);(X,x))$ the space of all parametrized rational curves $f : \mathbb{P}^1 \to X$, $f(0) = x \in X$. In analogy with the deformation of parametrized rational curves we can also consider deformations of parametrized rational curves $f : \mathbb{P}^1 \to X$ fixing one point, $f(0) = x$. Candidates for infinitesimal deformation are then given by $\Gamma(\mathbb{P}^1, f^*T(X) \otimes \mathfrak{m}_0)$, where $\mathfrak{m}_0$ denotes the maximal ideal sheaf of 0 on $\mathbb{P}^1$. For $f^*T(X)$ semipositive each Grothendieck direct summand of $f^*T(X)\otimes\mathfrak{m}_0 \cong f^*T(X)\otimes\mathcal{O}(-1)$ is of degree ≥ -1. The obstruction to deforming f fixing 0 given by $H^1(\mathbb{P}^1, f^*T(X) \otimes \mathfrak{m}_0)$ must then vanish, since $H^1(\mathbb{P}^1, \mathcal{O}(a)) = 0$ whenever $a \geq -1$.

(3.2) Let now S be an irreducible Hermitian symmetric space of the compact type and of *rank* ≥ 2 and consider a regular family $\pi : \mathcal{X} \to \Delta$ of compact Kähler manifolds over the unit disk such that the fiber $X_t := \pi^{-1}(t)$ is biholomorphic to S for $t \neq 0$. There is a holomorphic line bundle L on X such that the restriction of L to each X_t, $t \neq 0$, is isomorphic to $\mathcal{O}(1)$. Taking positive powers of L, by the Direct Image Theorem $\Gamma(X_o, L^k)$ must grow like k^n, $n = dim\, S$, so that X_o is Moišezon. Since X_o is Kähler and Moišezon, it must be projective-algberaic, by Moišezon's Theorem. We write X for X_o and also denote $L|_X$ by $\mathcal{O}(1)$.

We consider some degree-1 rational curves $C_1 \subset X_t$ for some $t \neq 0$. Deformations of C_1 as a subvariety in $\mathcal{X}$ fill up each X_t for $t \neq 0$ and must therefore fill up $X = X_o$. Since C_1 is of degree-1 and $X = X_o$ is also Kähler, C_1 cannot decompose under deformation, *viz.*, every deformation C of C_1 lying on X must also be of degree 1 with respect to $\mathcal{O}(1)$ and is hence irreducible and reduced. Thus, X is uniruled by such degree-1 rational curves C.

Consider now one choice of such a rational curve $C \subset X$, represented by $f_o : \mathbb{P}^1 \to X$, such that $f_o^*T(X)$ is semipositive, and consider $\mathcal{H} \subset Hol(\mathbb{P}^1, X)$ the irreducible component containing $[f_o]$ as a member. For every $[f] \in \mathcal{H}$ we call $C = f(\mathbb{P}^1)$ a minimal rational curve. Collect the set of all $[g] \in \mathcal{H}$ such that $g^*T(X)$ fails to be semipositive. The set $\mathcal{E}$ of all $C' = g(\mathbb{P}^1)$ cannot fill up an open subset of X, otherwise infinitesimal deformations of some choice of such g must generate all tangent directions at some point of C', *i.e.*, $g^*T(X)$ is semipositive, a contradiction. By our choice of $\mathcal{H}$, $\mathcal{D} := \mathcal{H}/Aut(\mathbb{P}^1)$ is compact (since C does not decompose under deformation) and hence projective. The subset $\mathcal{E} \subset \mathcal{D}$ is a subvariety, and they fill up a proper subvariety $E \subset X$. For $x \notin E$ consider $\mathcal{H}_x = Hol((\mathbb{P}^1, 0); (X, x)) \cap \mathcal{H}$. Then, $\mathcal{H}_x$ is non-singular. Let $Aut(\mathbb{P}^1, 0)$ be the subgroup of $Aut(\mathbb{P}^1)$ fixing 0. $Aut(\mathbb{P}^1, 0)$ acts on $\mathcal{H}_x$ and $\mathcal{M}_x := \mathcal{H}_x/Aut(\mathbb{P}^1, 0)$ can be endowed the structure of a projective manifold. We call $\mathcal{M}_x$ the normalized Chow space of minimal rational curves marked at x. For $y \in X_t$, $t \neq 0$, $\mathcal{M}_y$ similarly defined is irreducible, and is isomorphic to the variety $\mathcal{C}_o(S)$ of minimal rational tangents as defined in (1.2). By deforming $[C] \in \mathcal{M}_x$ to X_t, $t \neq 0$, we see readily that $\mathcal{M}_x$ is connected.

(3.3) For $x \in X - E$, $\mathcal{M}_x$ parametrizes free minimal rational curves C marked at x. Representing C by $f : \mathbb{P}^1 \to X$, we say that C is standard if $f^*T(X) \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ for some $p, q \geq 0$. We argue briefly that for a generic $x \in X - E$ and a generic choice of $[C] \in \mathcal{M}_x$, C is standard. (For details we refer the reader to [Mk1] or [HM3,5].) Otherwise one can obtain a one-parameter family of $[C]$ on $\mathcal{D}$ containing a pair of distinct points $x_1 \neq x_2$, and hence construct a ruled surface $\tau : R \to B$ over a non-singular algebraic curve B. There is an associated tautological holomorphic map $F : R \to X$ of maximal rank, together with two disjoint holomorphic sections B_1, B_2 of $\tau : R \to B$, corresponding to x_1 and x_2, so that B_1 *resp.* $B_2 \subset R$ are exceptional divisors blown down to x_1 *resp.* x_2. As $R - B_1 - B_2$ is a holomorphic $\mathbb{C}^*$-bundle, the compactifying divisors B_1 and B_2 must have opposite self-intersection numbers, contradicting with Grauert's blowing-down criterion (that $B_i \cdot B_i < 0$ for $i = 1, 2$). The preceding argument is a special case of Mori's break-up trick in [Mo].

Let now $x \in X - E$ be generic and $[C] \in \mathcal{M}_x$ be a standard minimal rational curve, represented by $f : \mathbb{P}^1 \to X$, $f(0) = x$. Then, $df : T(\mathbb{P}^1) \to f^*T(X) \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ must be injective, as $T(\mathbb{P}^1) \cong \mathcal{O}(2)$. We associate to $[C]$ the tangent space $df(T_0(\mathbb{P}^1)) \subset T_x(X)$. This way we obtain a map defined for $[C] \in \mathcal{M}_x$ standard (and more generally for $[C]$ immersed at 0). The extension of this map to a rational map $\Phi_x : \mathcal{M}_x \dashrightarrow \mathbb{P}T_x(X)$ will be called the tangent map. ¿From the description of infinitesimal deformations, the tangent map Φ_x is generically finite-to-one. (More details will be given in (4.1).) Let $\mathcal{M}_x^o \subset \mathcal{M}_x$ be the Zariski-dense open subset on which Φ_x is holomorphic. The strict transform $\overline{\Phi(\mathcal{M}_x^o)}$ of $\mathcal{M}_x$ under Φ_x, to be denoted by $\mathcal{C}_x$, is called the variety of minimal rational tangents at $x \in X$. For the natural projection $\pi : T_x(X) - \{o\} \to \mathbb{P}T_x(X)$, we denote by $\tilde{\mathcal{C}}_x \subset T_x(X)$ the cone of minimal rational tangent vectors defined by $\tilde{\mathcal{C}}_x = \pi^{-1}(\mathcal{C}_x) \cup \{o\}$. We summarize the relevant facts in

Proposition 3.3.1. *Let S be an irreducible symmetric space of the compact type and of rank ≥ 2. Let $\pi : \mathcal{X} \to \Delta$ be a regular family of compact Kähler manifolds such that for $t \neq 0$, $X_t = \pi^{-1}(t)$ is biholomorphic to S. Then, for the central fiber $X = X_o$, and for a generic point $x \in X$, the normalized Chow space $\mathcal{M}_x$ of minimal* *rational curves marked at x is a projective manifold, a generic member $[C] \in \mathcal{M}_x$ is a standard minimal rational curve, and the tangent map $\Phi_x : \mathcal{M}_x \dashrightarrow \mathcal{C}_x$ is a generically finite-to-one rational map.*

(3.4) To recover an S-structure on the central fiber X, one major difficulty is the possibility that generic varieties of minimal rational tangents are linearly degenerate, *i.e.*, contained in hyperplanes. In this case we would have a proper meromorphic distribution W on X such that at a generic point $x \in X$, W_x is spanned by the cones $\tilde{\mathcal{C}}_x \subset T_x(X)$. Regarding such distributions we have

Proposition 3.4.1. *Suppose $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is linearly degenerate at a generic point, and write $W \subsetneq T(X)$ for the proper meromorphic distribution spanned by $\tilde{\mathcal{C}}_x \subset T_x(X)$. Then, W is not integrable.*

Proof. We argue by contradiction. There is a subvariety $A = Sing(W) \subset X$ of codimension ≥ 2 such that W is holomorphic on $X - A$. Suppose W is integrable. We assert that the leaves L of the foliation on $X - A$ defined by W are closed subvarieties of $X - A$ such that $\overline{L} \subset X$ are subvarieties. Recall that for $x \in X - E$, every minimal rational curve passing through x is free and $\mathcal{M}_x$ is irreducible. To prove the assertion take $x \in X - A - E$, write $\mathcal{V}_1$ for the subvariety swept out by minimal rational curves emanating from x. Then, $\mathcal{V}_1$ is irreducible and tangent to W at a generic point. Inductively let $\mathcal{V}^o_{k+1} \subset X - A$ for the subvariety swept out by minimal rational curves emanating from $\mathcal{V}_k - E$, $\mathcal{V}_{k+1} = \overline{\mathcal{V}^o_{k+1}}$. The process must stop in a finite number of steps to give a chain $\mathcal{V}_1 \subsetneq \cdots \subsetneq \mathcal{V}_m = \mathcal{V}_{m+1}$ such that $\mathcal{V}_m$ is irreducible, $T_x(\mathcal{V}_m) = W_x$ at generic points x, by the integrability of W. For the leaf L_x of W passing through x, we have $L_x = \mathcal{V}_m \cap (X - A)$ and $\overline{L}_x = \mathcal{V}_m$.

There is a projective variety $\mathcal{L}$ parametrizing subvarieties of X, such that a generic member of $\mathcal{L}$ represents the closure $\overline{L}$ of some leaf L of W on $X - A$, and each $\overline{L}$ is represented by a member of $\mathcal{L}$. Let $\mathcal{N} \subset \mathcal{L}$ be a hypersurface and denote by $H \subset X$ the divisor swept out by $\mathcal{N}$. For a minimal rational curve C_o lying on some $\overline{L}$ such that $[\overline{L}] \notin \mathcal{N}$, the intersection $C_o \cap H$ can only lie on $A = Sing(W)$. For C_o a free rational curve some deformation C of C_o will avoid A since $A \subset X$ is of codimension ≥ 2, so that $C \cap H = \emptyset$. However, since X is of Picard number 1, $H \subset X$ is an ample divisor, and every curve on X must intersect H, a contradiction. In other words, W cannot be integrable. □

§4 Rigidity of irreducible Hermitian symmetric spaces under Kähler deformation

(4.1) For S an irreducible Hermitian symmetric space of the compact type, $\pi : \mathcal{X} \to \Delta$ a regular family with $X_t := \pi^{-1}(t) \cong S$ for $t \neq 0$, assuming that varieties of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$; $X = X_o$; are linearly degenerate, we have proven in §3 the non-integrability of the meromorphic distribution $W \subsetneq T(X)$ spanned at generic points by $\tilde{\mathcal{C}}_x$. We proceed now to produce a sufficient condition for the integrability of W, which will eventually contradict Proposition 3.4.1, proving that $W = T(X)$.

We start with a more precise description of the tangent map. Let $f : \mathbb{P}^1 \to X$ be a parametrized free minimal rational curve, $f(0) = x$ and $df(0) \neq 0$, *i.e.*, f is immersed at 0. Let z be the Euclidean coordinate on $\mathbb{C} \subset \mathbb{P}^1$ and $(z_1, \ldots, z_n)$ be local holomorphic coordinates at $x \in X$ such that $(z_1(x), \ldots, z_n(x)) = (0, \ldots, 0)$. Write $C = f(\mathbb{P}^1)$. Let $F(t,z) = z(\alpha + t\xi + O(t^2)) + O(z^2)$; t, z sufficiently small; be a local parametrization of minimal rational curves such that $f_t(z) = F(t,z)$ parametrizes C_t, $C_o = C$, $f_o = f$. Here the implicitly understood constants in $O(t^2)$ *resp.* $O(z^2)$ are independent of z *resp.* t. The tangent vector $\frac{d}{dt}|_{t=0} f_t \in T_{[f]}(\mathcal{H}_x)$ is given by $\sigma(z) = z\xi + O(z^2)$. ξ depends on the parametrization of C_t. Once $\frac{\partial F}{\partial z}(0,0) = \alpha$ is fixed, ξ is uniquely determined modulo $\mathbb{C}\alpha$. We have $T_{[f]}(\mathcal{H}_x) = \Gamma(\mathbb{P}^1, f^*T(X))$, and $T_{[C]}(\mathcal{M}_x) = T_{[f]}(\mathcal{H}_x)/df(\Gamma(\mathbb{P}^1, T(\mathbb{P}^1) \otimes \mathfrak{m}_0)$. For the tangent map $\Phi_x : \mathcal{M}_x \dashrightarrow \mathcal{C}_x$ we have $\Phi_x([C_t]) = [\alpha + t\xi + O(t^2)] \in \mathbb{P}T_x(X)$. For $\sigma \in T_{[f]}(\mathcal{H}_x)$ let $\overline{\sigma} \in T_{[C]}(\mathcal{M}_x)$ be its residue class. Then $d\Phi_x(\overline{\sigma}) \in T_{[\alpha]}(\mathcal{C}_x)$ is given by $d\Phi_x(\overline{\sigma}) = \xi \, mod \, \mathbb{C}\alpha$ under the identification $T_{[\alpha]}(\mathbb{P}T_x(X))$ with $T_x(X)/\mathbb{C}\alpha$. We note that the latter identification depends on the choice of α (*i.e.*, changes when α is replaced by a proportional vector). From the preceding description $d\Phi_x(\overline{\sigma}) = 0$ if and only if, composing with an automorphism of $\mathbb{P}^1$ fixing 0 if necessary, σ can be chosen to vanish to second order at 0. In case C is standard, we have $f^*T(X) \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$, and any $\sigma \in \Gamma(\mathbb{P}^1, f^*T(X) \otimes \mathfrak{m}_0)$ vanishing to the second order must be tangent to C. In other words, $d\Phi_x([C]) : T_{[C]}(\mathcal{M}_x) \to T_{[\alpha]}(\mathcal{C}_x)$ is an isomorphism and $\Phi_x : \mathcal{M}_x \dashrightarrow \mathcal{C}_x$ is generically finite-to-one, as stated in (3.2). Letting $df(0)(T_0(\mathbb{P}^1)) = \mathbb{C}\alpha$ by abuse of notations we write P_α for the image of $(\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)_0$ under $df(0)$. Then $T_{[\alpha]}(\mathcal{C}_x) = P_\alpha/\mathbb{C}\alpha$.

For notational simplicity alone we will argue as if standard minimal rational curves were always embedded. We note that although the Grothendieck direct summands are not uniquely determined, the positive part P_α is *a priori* independent of the choice of Grothendieck decomposition, as follows. Let y be a point on C distinct from x. Then, $P_\alpha \subset T_x(X)$ is the vector subspace spanned at x by $\Gamma(C, T(X) \otimes \mathfrak{m}_y)$.

(4.2) We return now to the question of integrability of the distribution W on X spanned by $\tilde{\mathcal{C}}_x$ at generic points, assuming $W \subsetneq T(X)$. First of all we look for a sufficient condition in terms of $\tilde{\mathcal{C}}_x$. By the Frobenius condition, W is integrable if and only if for any two W-valued holomorphic vector fields η, η' on some open set $U \subset X - A$, $A = Sing(W)$, the Lie bracket $[\eta, \eta']$ is also W-valued. Clearly, for $x \in U$, $[\eta, \eta'](x) \in W_x$ if either $\eta(x) = 0$ or $\eta'(x) = 0$, from the local formula of the Lie bracket. It follows that for η, η' in general, $[\eta, \eta'](x) \mod W$ depends only on $\eta(x)$ and $\eta'(x)$. This defines the Frobenius form $\varphi_x : \Lambda^2 W_x \to T_x(X)/W_x$ such that W is integrable if and only $\varphi \equiv 0$. To show that W is integrable it is therefore sufficient to verify that over some non-empty open subset $U \subset X - A$, for any $x \in U$ there exists pairs of germs of W-valued holomorphic vector fields (η, η') at x such that

(a) $\varphi(\eta, \eta')(x) = 0$;

(b) the set of all $(\eta \wedge \eta')(x)$ for such pairs (η, η') span $\Lambda^2 W_x$ as a vector space.

Let now $E_x \subset \Lambda^2 W_x$ be the subspace $Span\{\alpha \wedge \xi : \alpha \in \tilde{\mathcal{C}}_x, \xi \in P_\alpha\}$. We prove

Proposition 4.2.1. *Suppose at generic points $x \in X$ we have $E_x = \Lambda^2 W_x$. Then, W is integrable.*

Proof. Let $C \subset X$ be a standard minimal rational curve marked at x, $T_x(C) = \mathbb{C}\alpha$. Let $y \in C$ be a point distinct from x. Consider a holomorphic one-parameter family $\{f_t : t \in \Delta_\varepsilon\}$ of parametrized standard minimal rational curves marked at y, $\varepsilon > 0$, $f_t(0) = y$, $f_o(\mathbb{P}^1) = C$. Write $f_t(\mathbb{P}^1) = C_t$; $C_o = C$. Then, $\frac{df_t}{dt} \in \Gamma(\mathbb{P}^1, f_t^*T(X) \otimes \mathfrak{m}_0)$, corresponding to $\sigma_t \in \Gamma(C_t, T(X) \otimes \mathfrak{m}_y)$. Assume $\sigma_o(x) \neq 0$. For any $t \in \Delta_\varepsilon$, $\zeta \in \mathbb{P}^1$, $z = f_t(\zeta)$ and $0 \neq \beta \in T_z(C_t)$, as observed $\sigma_t(\zeta) = \chi \in P_\beta$, so that the surface swept out by $\{C_t : t \in \Delta_\varepsilon\}$ gives a germ of complex-analytic surface Σ at x such that $T_z(\Sigma) = Span\{\beta, \chi\} \subset W_z$ for any $z \in \Sigma$. In other words, Σ is an integral surface of W. Thus, $\frac{df_t}{dz}$ and $\frac{df_t}{dt}$ correspond to W-valued holomorphic vector fields η *resp.* η' on Σ such that $[\eta, \eta'] \equiv 0$ on Σ, so that $\varphi(\eta, \eta')(x) = 0$. Since $[\alpha] \in \mathcal{C}_x$ is arbitrary, $\sigma_o(0) = \xi \in P_\alpha$ can be chosen arbitrarily, and by assumption $E_x := Span\{\alpha \wedge \xi : \alpha \in \tilde{\mathcal{C}}_x, \xi \in P_\alpha\} = \Lambda^2 W_x$, both conditions (a) and (b) preceding the Proposition are satisfied, and W is integrable, as desired. □

(4.3) To study the distribution W on the central fiber, we need to understand generic varieties of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ as subvarieties, in order to verify hypotheses in Proposition 4.2.1. Consider a holomorphic section $\sigma : \Delta_\varepsilon \to \mathcal{X}$ defined for some $\varepsilon > 0$ such that $\sigma(0) = x$. For $t \neq 0$; $\mathcal{C}_{\sigma(t)} \subset \mathbb{P}T_{\sigma(t)}(X_t)$ is projectively equivalent to $\mathcal{C}_o(S) \subset \mathbb{P}T_o(S)$ on the model manifold S. It is difficult to study the projective geometry of $\mathcal{C}_x$ directly as limits of $\mathcal{C}_{\sigma(t)}$. On the other hand, we have the Chow spaces $\mathcal{M}_{\sigma(t)}$ of minimal rational curves marked at $\sigma(t)$ and tangent maps $\Phi_{\sigma(t)} : \mathcal{M}_{\sigma(t)} \dashrightarrow \mathcal{C}_{\sigma(t)}$, which is an isomorphism for $t \neq 0$. It is easier to study $\{\mathcal{M}_{\sigma(t)}\}$ as a regular family of projective manifolds. Recall from (1.3) that $\mathcal{C}_o(S)$ is itself biholomorphic to a Hermitian symmetric space of the compact type, and is irreducible except in the case of $G(p,q)$ of *rank* ≥ 2, where $\mathcal{C}_o(G(p,q))$ is isomorphic to $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$, embedded in $\mathbb{P}^{pq-1}$ by the Segre embedding.

Considering the regular family $\mathcal{M}_{\sigma(t)} \to \Delta_\varepsilon$, $\sigma(o) = x$, we argue in [HM2,§3] by induction that $\mathcal{M}_x \cong \mathcal{C}_o(S)$. This can be achieved by induction on dimension, except for the case of $S = G(p,q)$, $p, q \geq 2$, where there is the difficulty that as abstract manifolds, $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$ are not rigid under Kähler deformation, as exemplified by the deformation of $\mathbb{P}^1 \times \mathbb{P}^1$ to any Hirzebruch surface of even genus (cf. Siu [Si,3.1]). In the latter case the key phenomenon is that limits of direct factors may decompose in the central fiber. In our situation we need the extra interpretation of individual direct factors $\mathbb{P}^{p-1} \times \{b\}$ *resp.* $\{a\} \times \mathbb{P}^{q-1}$ as parameter spaces for families of lines through a fixed base point of projective subspaces $\cong \mathbb{P}^p$ *resp.* $\mathbb{P}^q$ in $G(p,q)$. These projective subspaces are of degree 1 with respect to $\mathcal{O}(1)$, so that their limits under deformation cannot decompose in the central fiber. Basing on this and on the study of intersections of limiting projective subspaces on the central fiber, in the case of $S = G(p,q)$ we can still prove in [HM2] that $\mathcal{M}_x \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$ for a generic point x on the central fiber X.

At a generic point $x \in X$ on the central fiber X of $\pi : \mathcal{X} \to \Delta$, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is the image of the tangent map $\Phi_x : \mathcal{M}_x \dashrightarrow \mathcal{C}_x$. As explained $\mathcal{M}_x$ is biholomorphic to $\mathcal{C}_o(S)$ for the model S. Let $\sigma : \Delta \to \mathcal{X}$ be a holomorphic section, $\sigma(o) = x$. The family $\sigma^*\mathcal{M}$ consisting of $\{\mathcal{M}_{\sigma(t)} : t \in \Delta\}$ is holomorphically trivial. Identifying $\sigma^*\mathcal{M} \to \Delta$ with $\mathcal{C}_o(S) \times \Delta \to \Delta$, $\mathcal{C}_o(S) \subset \mathbb{P}T_o(S)$, there is a holomorphic family of linear maps $\varphi_t : T_o(S) \to T_{\sigma(t)}(X_t)$ such that $\Phi_{\sigma(t)} : \mathcal{M}_{\sigma(t)} \dashrightarrow \mathcal{C}_{\sigma(t)}$ is induced by $[\varphi_t] : \mathbb{P}T_o(S) \to \mathbb{P}T_{\sigma(t)}(X_t)$, $\Phi_{\sigma(t)} = [\varphi(t)]|_{\mathcal{M}_{\sigma(t)}}$. Suppose the distribution

$W \subsetneq T(X)$. Let $E_x = Span\{\alpha \wedge \xi : \alpha \in \tilde{\mathcal{C}}_x, \alpha \neq 0, \xi \in P_\alpha\}$. By Proposition 4.2.1 W is integrable if $E_x = \Lambda^2 W_x$. We are going to establish

Proposition 4.3.1. *For the model variety of minimal rational tangents $\mathcal{C}_o(S)$ and $E_o \subset \Lambda^2 T_o(S)$ defined by $E_o = Span\{\alpha \wedge \xi : \alpha \in \tilde{\mathcal{C}}_o, \alpha \neq 0, \xi \in P_\alpha\}$, we have $E_o = \Lambda^2 T_o(S)$.*

E_x is the image of E_o under the canonical linear projection $\Lambda^2 T_o(S) \to \Lambda^2 W_x$ induced by $\varphi_o : T_o(S) \to W$. Given Proposition 4.3.1, at a generic point x on the central fiber X, we have $E_x = \Lambda^2 W_x$, so that $W \subsetneq T(X)$ is integrable, contradicting Proposition 3.4.1. In other words, given Proposition 4.3.1, we have proven by contradiction that $W = T(X)$, so that $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_o(S) \subset \mathbb{P}T_o(S)$ for a generic $x \in X$. By the latter we mean that there exists a projective linear isomorphism $\nu : \mathbb{P}T_o(S) \to \mathbb{P}T_x(X)$ such that $\nu(\mathcal{C}_o(S)) = \mathcal{C}_x$.

For the proof of Proposition 4.3.1 we will make use of the following description of $\mathcal{C}_o(S)$ obtained from $\mathbb{C}^*$-actions on the latter derived from Grothendieck's splitting theorem (*cf.* [HM2,§5]).

Fact. At $[\alpha] \in \mathcal{C}_o(S)$, $dim\, \mathcal{C}_o(S) = p$, one can choose inhomogeneous coordinates such that the affine part of $\mathcal{C}_o(S)$ is the graph of a vector-valued quadratic polynomial in p variables.

Example.

(1) For $S = Q^n$, $n \geq 3$, $\mathcal{C}_o(S) \cong Q^{n-2} \mapsto \mathbb{P}^{n-1}$. Since any two non-degenerate complex bilinear forms are conjugate to each other, we may choose Euclidean coordinates $(w_0, \ldots, w_{n-1})$ on $T_o(Q^n) \cong \mathbb{C}^n$ such that $\mathcal{C}_o(S)$ is defined by $w_0 w_{n-1} - w_1^2 - w_2^2 - \cdots - w_{n-2}^2 = 0$. We have $[1, 0, \ldots, 0] \in \mathcal{C}_o(S)$. In terms of inhomogeneous coordinates $z_k = \frac{w_k}{w_0}$, $1 \leq k \leq n$, the affine part of $\mathcal{C}_o(S)$ is given by the graph of the quadratic polynomial $z_{n-1} = z_1^2 + z_2^2 + \cdots + z_{n-2}^2$.

(2) For $S = G(p,q)$, $p, q \geq 2$, $\mathcal{C}_o(S) \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \mapsto \mathbb{P}^{pq-1}$, defined by the Segre embedding, which arises from $\varphi : \mathbb{C}^p \times \mathbb{C}^q \to \mathbb{C}^p \otimes \mathbb{C}^q \cong \mathbb{C}^{pq}$, $\varphi(u, v) = u \otimes v$. Write $\{e_i\}_{0 \leq i \leq p-1}$ *resp.* $\{\varepsilon_j\}_{0 \leq j \leq q-1}$ for bases of $\mathbb{C}^p$ *resp.* $\mathbb{C}^q$ and take $\alpha = \varphi(e_0, \varepsilon_0)$. Then the affine part of $\mathcal{C}_o(S)$ is given by the graph of $F : \mathbb{C}^{p+q-2} \to \mathbb{C}^{(p-1)(q-1)}$, where

$$F(z_1, \ldots, z_{p-1}; \zeta_1, \ldots, \zeta_{q-1}) = (z_i \zeta_j)_{1 \leq i \leq p-1, 1 \leq j \leq q-1} \, .$$

Proof of Proposition 4.3.1. We give here a proof of the Proposition assuming the preceding description of $\mathcal{C}_o = \mathcal{C}_o(S)$ in inhomogeneous coordinates. In a neighborhood of $[\alpha] \in \mathcal{C}_o, [\beta] \in \mathcal{C}_o$ is described by $\beta = \alpha + z_1 e_1 + \cdots + z_p e_p + Q(z_1, \cdots, z_p)$, where $(z_1, \cdots, z_{n-1})$ are inhomogeneous coordinates, with corresponding Euclidean basis $(e_1, \cdots, e_{n-1})$; and, writing $A = Span\{e_1, \cdots, e_p\}$ and $B = Span\{e_{p+1}, \cdots, e_{n-1}\}$, $Q : A \to B$ is a vector-valued quadratic polynomial. Let now $\{\alpha(t) : t \in \triangle\}$, $\alpha(0) = \alpha$, be a holomorphic curve on $\tilde{\mathcal{C}}_o$. Then, $\alpha'(t) \in P_{\alpha(t)}$ and $\alpha(t) \wedge \alpha'(t) \in E_o$. Furthermore $(\alpha(t) \wedge \alpha'(t))' = \alpha(t) \wedge \alpha''(t) \in E_o$. It follows that $\alpha \wedge \eta \in E_o$ whenever $\eta \in A$ or $\eta = D^2 Q(\xi, \xi)$ for some $\xi \in A$, where $D^2 Q$ is the vector-valued Hession of Q, which can be identified with the second fundamental form σ_α of $\tilde{\mathcal{C}}_o$ in $T_x(X)$ at α with respect to the flat connection on $T_x(X)$. By polarization $\alpha \wedge \eta \in E_o$ also for $\eta = \sigma_\alpha(\xi, \xi')$ whenever $\xi, \xi' \in A$. But since Q is quadratic, the linear span of $\tilde{\mathcal{C}}_o$ is the same as the linear span of all derivatives of β up to the second order. By the linear non-degeneracy of $\tilde{C}_o$, $Span\{\sigma_\alpha(\xi,\xi')\} = Span\{\partial_i\partial_j Q(0)\} = B$. Consequently, $\alpha \wedge T_o(S) \subset E_o$. Varying α we conclude that $\Lambda^2 T_o(S) = E_o$, as desired. □

(4.4) Over the total space $\mathcal{X}$ of the regular family $\pi : \mathcal{X} \to \Delta$ consider the fibered space $\mathcal{C} \to \mathcal{X}$ of varieties of minimal rational tangents. For each $x \in X_t$, $t \neq 0$, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_o(S) \subset \mathbb{P}T_o(S)$. From (4.3) we see that this holds also for a generic point x on the central fiber $X = X_o$. Using Hartogs extension we strengthen this to the following result.

Proposition 4.4.1. *Let $\pi : \mathcal{X} \to \Delta$ be a regular family of compact Kähler manifolds such that the fibers $X_t = \pi^{-1}(t)$ are biholomorphic to S. Then, for any $x \in X = X_o$, the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is projectively equivalent to $\mathcal{C}_o(S) \subset \mathbb{P}T_o(S)$ on the model manifold S.*

For the proof of Proposition 4.4.1 we will make use of

Proposition 4.4.2 (Matsushima-Morimoto [MM]). *Let V be a finite-dimensional complex vector space and $H \subset GL(V)$ be a linear reductive subgroup. Then, $GL(V)/H$ is a Stein manifold.*

Example. Consider the case of $S = Q^n$, $n \geq 3$, $V = T_o(S)$, $H = K^{\mathbb{C}}$ is the group of linear transformations preserving a fixed non-degenerate complex bilinear form up to a multiplicative constant. Then, $GL(n,\mathbb{C})/K^{\mathbb{C}}$ parametrizes the set of all proportionality classes of non-degenerate complex bilinear forms. Consider the vector space E of complex bilinear forms on $\mathbb{C}^n$. Let $\Sigma \subset E$ be the subvariety of all complex bilinear forms, represented by $(a_{ij})_{1\leq i,j\leq n}$ with respect to a fixed basis of $\mathbb{C}^n$, such that $det(a_{ij}) = 1$. Then Σ is affine-algebraic and $GL(n,\mathbb{C})/K^{\mathbb{C}}$ is a quotient of Σ by a finite group, hence afffine-algebraic and *a fortiori* Stein.

Proof of Proposition 4.4.1. In the case of S-structures with $H = K^{\mathbb{C}}$, $o \in S$, the choice of a linear isomorphism $\varphi : T_o(S) \cong \mathbb{C}^n$ determines a projective submanifold $Z_\varphi \subset \mathbb{P}^{n-1}$ with $Z_\varphi = [\varphi](\mathcal{C}_o(S))$. For $\gamma \in K^{\mathbb{C}}$, $Z_\varphi = Z_{\varphi\circ\gamma}$, so that $GL(n,\mathbb{C})/K^{\mathbb{C}}$ parametrizes the set of all submanifolds $Z \subset \mathbb{P}^{n-1}$ projectively equivalent to $\mathcal{C}_o(S)$. For the regular family $\pi : \mathcal{X} \to \Delta$ there is a proper subvariety $E \subset X$ such that $\mathcal{C}_x$ is projectively equivalent to $\mathcal{C}_o(S)$ for all $x \in \mathcal{X} - E$. For any $y \in X$ and U an open neighborhood of y such that $T(U)$ is holomorphically trivial, we obtain from the varieties $\mathcal{C}_x$ a holomorphic map $\psi : \mathcal{X} - E \to GL(n,\mathbb{C})/K^{\mathbb{C}}$. Since $GL(n,\mathbb{C})/K^{\mathbb{C}}$ is Stein by Proposition 4.4.1, it can be holomorphically embedded as a closed analytic subvariety of some Euclidean space, by the embedding theorem of Bishop-Narasimhan-Remmert. It follows that ψ extends holomorphically across E by Hartogs extension. ¿From this we conclude that all varieties of minimal rational tangents $\mathcal{C}_x$ are projectively equivalent to the model $\mathcal{C}_o(S)$, as desired. The proof of Proposition 4.4.1 is complete. □

End of Proof of the Main Theorem. By Hirzebruch-Kodaira [HK] we may assume that $S \neq \mathbb{P}^n$, *i.e.*, $rank(S) \geq 2$. Let $\mathcal{F} \to X$ be the holomorphic frame bundle of X. For $x \in X$ let $\mathcal{H}_x \subset \mathcal{F}_x$ be the subset of all linear isomorphisms $\varphi : T_o(S) \to T_x(X)$ such that $[\varphi](\mathcal{C}_o(S)) = \mathcal{C}_x$. Then the holomorphic subbundle of frames $\mathcal{H} \to X$ defines a $K^{\mathbb{C}}$-structure. For each $t \neq 0$ we have a holomorphic $K^{\mathbb{C}}$-structure $\mathcal{H}(t)$ on X_t such that $\mathcal{H}(t)$ converges to $\mathcal{H}$ as t tends to 0, in the obvious sense. Each $\mathcal{H}(t)$ is flat on X_t for $t \neq 0$. By Definition-Proposition 1.5.1, $\mathcal{H}$ is also flat on X.

Since X is simply-connected, it is biholomorphic to S by Theorem 2.1.1. The proof of the Main Theorem is complete. □

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