Neural Networks 180 (2024) 106695

Contents lists available at ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Full Length Article

State transition learning with limited data for safe control of switched nonlinear systems[☆]

Chenchen Fan[a,1], Kai-Fung Chu[b,1,*], Xiaomei Wang[c], Ka-Wai Kwok[c,d], Fumiya Iida[b]

[a] Department of Rehabilitation Sciences, The Hong Kong Polytechnic University, Hung Hom, Hong Kong, China
[b] Department of Engineering, University of Cambridge, Trumpington Street, Cambridge, CB2 1PZ, UK
[c] Department of Mechanical Engineering, The University of Hong Kong, Pokfulam, Hong Kong, China
[d] Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Shatin, Hong Kong, China

ARTICLE INFO

Keywords:
Switched system
Safe control
Machine learning
Limited data

ABSTRACT

Switching dynamics are prevalent in real-world systems, arising from either intrinsic changes or responses to external influences, which can be appropriately modeled by switched systems. Control synthesis for switched systems, especially integrating safety constraints, is recognized as a significant and challenging topic. This study focuses on devising a learning-based control strategy for switched nonlinear systems operating under arbitrary switching law. It aims to maintain stability and uphold safety constraints despite limited system data. To achieve these goals, we employ the control barrier function method and Lyapunov theory to synthesize a controller that delivers both safety and stability performance. To overcome the difficulties associated with constructing the specific control barrier and Lyapunov function and take advantage of switching characteristics, we create a neural control barrier function and a neural Lyapunov function separately for control policies through a state transition learning approach. These neural barrier and Lyapunov functions facilitate the design of the safe controller. The corresponding control policy is governed by learning from two components: policy loss and forward state estimation. The effectiveness of the developing scheme is verified through simulation examples.

1. Introduction

In recent decades, the fields of robotics and control have witnessed a notable surge in interest and research focused on safety (Dawson, Gao, & Fan, 2023; Guiochet, Machin, & Waeselynck, 2017). Additionally, stability is a fundamental requirement in system control to achieve reliable and desired performance. Consequently, control problems involving both safety and stability requirements are worthy of study, where the controller is not only to stabilize the system but also to drive it to operate safely within safe constraints while avoiding unsafe regions. Such safe control problems are a typical class of constrained control (Chen, Zhang, Liu, Wang, & Wang, 2023; de Jesús Rubio et al., 2024), and widely applied in various domains, such as autonomous vehicles (Zhang et al., 2022), bipedal robots (Peng, Donca, Castillo, & Hereid, 2023), multi-robot systems (Lafmejani, Berman, & Fainekos, 2022), human–robot interaction (Landi, Ferraguti, Costi, Bonfè, & Secchi, 2019). From the perspective of system control, the concept of safety is often linked to system reachability and invariant sets (Weng, Capito, Ozguner, & Redmill, 2021; Xiang, Tran, & Johnson, 2017). To be more specific, the safety constraints imposed on the evolution of system states or outputs can be translated into the idea that reachable system states or outputs will be confined in the invariant safe sets defined by these constraints (Fan et al., 2024; Fan, Lam, Xie, & Song, 2021). In the research of robotics and control, a highly effective and commonly employed method for ensuring set invariance in safe control systems is the control barrier function approach (Ames et al., 2019; Rauscher, Kimmel, & Hirche, 2016). The mainstream of this approach is to synthesize a safe-critical controller by incorporating secure constraints as set invariance based on the control barrier function, while also addressing control performance concerns. The Lyapunov function

☆ This work was partially supported by PolyU Start-up Fund for RAPs under the Strategic Hiring Scheme with project ID P0051252, by the Research Grants Council, Hong Kong under Grants 17210023, 17209021, C4026-21G, and STG1/E-401/23-N, by Innovation and Technology Commission, Hong Kong under Grant MRP/029/20X, by Multi-Scale Medical Robotics Center Limited and Centre for Garment Production Limited, InnoHK, and also by European Union's Horizon 2020 research innovation programme under the Marie Skłodowska-Curie grant agreement No 101034337.

* Corresponding author.
E-mail address: kfc35@cam.ac.uk (K.-F. Chu).

[1] Contributed equally.

https://doi.org/10.1016/j.neunet.2024.106695
Received 20 May 2024; Received in revised form 19 August 2024; Accepted 2 September 2024
Available online 3 September 2024

method is widely recognized as an effective and fundamental way to achieve performance-based control, like stability, and robustness (Fan, Lam, Xie, & Li, 2022; Xie, Lam, & Fan, 2018). Thus, it is natural that safe control problems often involve both control barrier functions and Lyapunov functions (Dawson et al., 2023; Romdlony & Jayawardhana, 2016). Furthermore, quadratic programming techniques have been commonly utilized to formulate safety-critical problems (Ames, Xu, Grizzle, & Tabuada, 2016; Bena, Hossain, Chen, Wu, & Nguyen, 2023), with integrating both control barrier function and Lyapunov function induced constraints for safe controller design. However, it can be found that the construction of control barrier functions and Lyapunov functions are crucial but also challenging when designing safe controllers, especially for complex dynamic systems.

Switching dynamics are frequently encountered in real-world systems, emerging due to either intrinsic changes or reactions to external perturbations. Switched dynamics, a typical model of hybrid systems with multiple subsystem modes and a rule governing their switching (Liberzon & Morse, 1999), have drawn increasing attention owing to their strong ability to model many complex practical processes. They find applications in a wide range of domains, such as switching converters in power grids, failure and recovery in production processes, varying loads or operations of robot manipulations (Lin & Antsaklis, 2009), and switching communication/neural networks (Bao, Zhang, & Zhang, 2023; Fan, Lam, Chu, Lu, & Kwok, 2024). Furthermore, subsystems may exhibit complex nonlinear dynamics under switching in practical scenarios, which results in a switched nonlinear system model. For switched nonlinear systems, the main efforts are devoted to constructing different types of Lyapunov functions to adapt to various kinds of switching laws thus addressing stability and stabilization problems (Liu, Zhou, Liang, & Wang, 2017), as well as robust analysis and control problems (Peng, Yang, & Jiang, 2024). Notably, the integration of switching laws and nonlinear dynamics introduces complexity and poses additional challenges in terms of analysis and control. For safe control of switched nonlinear systems which needs to fulfill safety and stability simultaneously, explicitly developing controller design is hard to implement.

A promising research direction to address the complex nonlinear dynamics problem is via learning-based neural network control (Fei, Li, Li, & Li, 2023). A multiagent reinforcement learning algorithm to handle nonlinear dynamics in the policy learner has been developed in Abdallah and Lesser (2008). Lusch et al. introduce a method to use deep learning for identifying and representing Koopman eigenfunctions from data, enabling the approximation of strongly nonlinear dynamics as linear. The framework yields parsimonious and interpretable embeddings of dynamics onto low-dimensional manifolds, facilitating global linearization. Furthermore, it extends Koopman representations to continuous spectra systems, utilizing an auxiliary network for efficient embedding while preserving physical interpretability (Lusch, Kutz, & Brunton, 2018). Koopman eigenfunctions have been used to truncate extensive datasets, enabling the development of a data-efficient, model-free reinforcement learning algorithm (Donge, Lian, Lewis, & Davoudi, 2023). This approach reduces the requisite data for optimal control learning while retaining the dynamical intricacies of the original nonlinear system, eliminating the need for an exact model. Recently, neural network-based adaptive control problems for switched nonlinear systems have been investigated (Li, Ahn, Guo, & Xiang, 2020), using only sampled system output. The approach conserves communication resources by intermittently monitoring an event-triggering mechanism. It ensures semiglobal uniform ultimate boundedness of closed-loop system states under arbitrary switchings and is validated through application to a continuous stirred tank reactor system. However, few studies concern the safe control of switched nonlinear systems via safe certification. Only Kıvılcım, Karabacak, and Wisniewski (2019) reported on barrier functions for safety verification of switched nonlinear systems, but it lacks practical instructions for constructing control barrier functions while simultaneously synthesizing controllers.

One even more challenging problem is that learning-based approaches usually require a large amount and diversified data to train the universal function approximator: neural networks. However, data may be problematic in many cases, including instances where the data used for model training is influenced by disturbances and noise (Chu, Lam, Fan, & Li, 2021). Furthermore, there are situations where data is either inaccessible or unavailable, such as the task of generating images with limited available data (Karras et al., 2020), and learning control policies in partially observable environments (Littman, Cassandra, & Kaelbling, 1995). The situation could be even worse in switched systems as there are multiple subsystems in the switched system and not all subsystem dynamics can be involved in the data collection process. In other words, a subset of subsystem data may not be included in the dataset for different reasons, such as unmeasurable data or the need to conserve resources, causing a potential data distribution mismatch issue between model training and execution. Therefore, we have to be prepared for the safe control of switched nonlinear systems with limited available data.

In this paper, we address the safe control problem of switched nonlinear systems where data of certain subsystems may not be available. Inspired by AI-based control and the necessity of safe certificate, we propose a state transition learning approach to train a neural Lyapunov function, a neural barrier function, and a neural control policy for the switched nonlinear systems in the scenario where data of some subsystems is not available to the model training. The proposed approach trains a neural control Lyapunov and barrier function, and the neural policy is jointly trained with the neural safe certificate to ensure the safety requirement of the policies. The training of policy with limited data is achieved by the policy loss from available subsystem data and forward state estimation for unavailable subsystems, which learns to reflect the subsystem transition effect of the switched systems to the control signal. We evaluate the proposed approach with a numerical example and a switched robotic manipulator setting. The experimental results indicate that the proposed approach can safely control the switched system with a satisfactory controlling performance.

The primary contributions can be summarized as follows:

(1) We establish a safe control scheme for switched nonlinear systems by developing control design conditions using control barrier functions and Lyapunov functions to ensure both safety and stability performance.
(2) We propose a state transition learning approach to handle switching dynamics, constructing a loss function that complements missing/unavailable subsystem data with forward state estimation, which relaxes the knowledge requirement of model dynamics compared to the existing approaches. This approach avoids the data distribution mismatch issue appearing in the existing supervised learning methods in switched nonlinear systems with limited data circumstances.
(3) We compare various control approaches with different data availability scenarios. The experiment results suggest that the proposed approach can safely control the switched nonlinear with limited training data, comparable to another control method with a complete dataset.

The remainder of this work is structured in the following manner. The problem formulation is outlined in Section 2, followed by the presentation of the developed methodology in Section 3. The validation of the proposed techniques is demonstrated through experiments in Section 4, and finally, the paper concludes in Section 5.

Notation: The notations used in this paper adhere to standard conventions. $\mathbb{R}^n$ and $\mathbb{R}^n_+$ represent the set of n-dimensional vectors with real and nonnegative real components, respectively. The Lie derivative is introduced as $L_f W(x) = \frac{\partial W(x)}{\partial x} f(x)$, $L_g W(x) = \frac{\partial W(x)}{\partial x} g(x)$. A continuous function β: $[0, c) \rightarrow [0, \infty)$ with constant $c > 0$ belongs to $\mathcal{K}$-class function if β is strictly increasing and $\beta(0) = 0$. A continuous function α: $(-c_1, c_2) \rightarrow (-\infty, \infty)$ with $c_1, c_2 > 0$ belongs to extended $\mathcal{K}$-class function if α is strictly increasing and $\alpha(0) = 0$. The symbol $\mathbb{E}$ stands for the expectation operator. The superscript T indicates the matrix transpose.

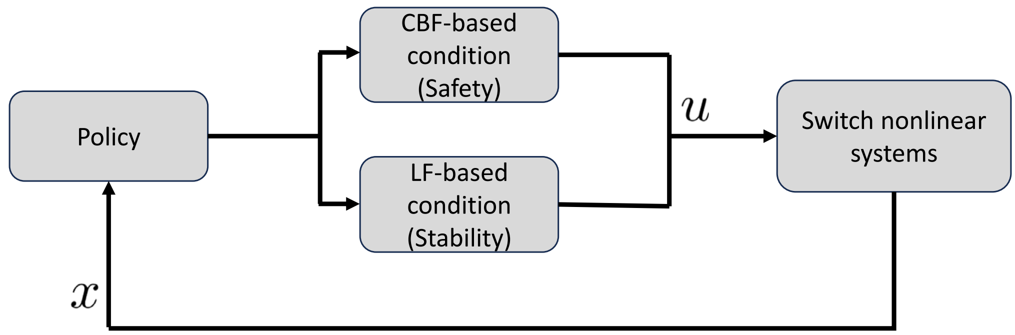

Fig. 1. Block diagram of the controller design principle.

2. Problem formulation

The paper focuses on considering a class of switched nonlinear dynamic systems in the following form:

$$\dot{x} = f_{\sigma(t)}(x) + g_{\sigma(t)}(x)u, \quad x(t_0) = x_0, \tag{1}$$

where $x \in \mathcal{X} \in \mathbb{R}^x$ and $u \in \mathcal{U} \in \mathbb{R}^u$ represent the system state and control input, respectively. The function $\sigma(t)$ represents the switching signal, which is defined as $\sigma(t) : [0, \infty) \to \mathcal{N} = \{1, 2, \ldots, N\}$. It is a piecewise constant function that is continuous from the right. Denoting a strictly increasing switching sequence as $\{t_k\}$, $k = 0, 1, 2, \ldots$, for any $t \in [t_k, t_{k+1})$, $\sigma(t) = i \in \mathcal{N}$ indicates that the ith subsystem is activated. $f_\sigma(t)$ and $g_\sigma(t)$, can be rewritten as f_i and g_i, $i = 1, 2, \ldots, N$, which are assumed to be locally Lipschitz continuous regarding x. The switching law considered in this work is arbitrary switching.

The nonlinearity of subsystems ($f_i(x)$ and $g_i(x)$, $i \in \mathcal{N}$, are nonlinear items from dynamic system (1)) introduced challenges for analytical control design methods compared to linear items. Our objective in this work is to devise a state-feedback controller u that ensures both the stability and safety of the switched nonlinear system (1) under arbitrary switching. The block diagram for illustrating the design principle of controller that guarantees both stability and safety is shown in Fig. 1.

In light of addressing the safety concerns associated with the switched nonlinear system (1), we establish a safe set, denoted by $\mathcal{S} \subset \mathcal{X}$. Inspired by the set invariance of safe set (Athanasopoulos, Smpoukis, & Jungers, 2016; Gurriet, Mote, Singletary, Feron, & Ames, 2019), we can define safety for system (1) as follows:

Definition 1 (*safety*)**.** Given the initial conditions within the safe set, the switched system (1) under arbitrary switching is safe if the system state trajectory consistently remains within the safe set, i.e., $x(t) \in \mathcal{S}$, for all $t \geq 0$ for all initial conditions $x_0 \in \mathcal{S}$.

Control Barrier Function (CBF): The control barrier function is an effective tool for investigating safety scenarios. It is assumed that the safe set $\mathcal{S}$ of system state is related to a superlevel set of a continuously differentiable function $h : \mathcal{X} \to \mathbb{R}$, i.e.,

$$h(x) \geq 0, \quad \forall x \in \mathcal{S}. \tag{2}$$

Based on the property of control barrier functions, for switched systems (1) subjected to arbitrary switching, h is qualified as a control barrier function if the following condition holds:

$$\sup_{u \in \mathcal{U}} [L_{f_{\sigma(t)}} h(x) + L_{g_{\sigma(t)}} h(x)u + \alpha(h(x))] \geq 0, \tag{3}$$

where α is an extended $\mathcal{K}$-class function, and $\alpha(h(x))$ can be often chosen as a linear form $\lambda h(x)$, $\lambda > 0$. Since $\sigma(t)$ takes the value from $\mathcal{N} = \{1, 2, \ldots, N\}$, $L_{f_{\sigma(t)}} h(x) = \frac{\partial h(x)}{\partial x} f_i(x)$ and $L_{g_{\sigma(t)}} h(x) = \frac{\partial h(x)}{\partial x} g_i(x)$, under arbitrary switching law, $h(x)$ can be regarded as a common control barrier function for all subsystem i, $i = 1, 2, \ldots, N$.

Besides safety, we need to employ Lyapunov functions for investigating the stability of the controlled switched system (1). This is also called a stabilization problem of system (1) to find control u to stabilize the system to an equilibrium point $x^* = 0$.

Lyapunov Function (LF): Consider a continuously differentiable function $V : \mathcal{X} \to \mathbb{R}_+$, if (i) it is positive definite, i.e., $V(x^*) = 0$ and $V(x) > 0$, $\forall x \in \mathcal{X} \setminus \{0\}$, (ii) it can satisfy the following condition:

$$\inf_{u \in \mathcal{U}} [L_{f_{\sigma(t)}} V(x) + L_{g_{\sigma(t)}} V(x)u + \beta(V(x))] \leq 0, \tag{4}$$

where $\beta(\cdot)$ is a class of $\mathcal{K}$ function, and can be chosen as $\gamma h(x)$, $\gamma > 0$. With $\sigma(t) = i \in \mathcal{N}$, $V(x)$ can also be called a common control Lyapunov function for all subsystems.

The safe control problem of switched nonlinear system (1) can be established by the following theorem based on the above-proposed common control barrier function and Lyapunov function.

Theorem 1. *If switched nonlinear system (1) under arbitrary switching has a control barrier function $h(x) \geq 0$, $\forall x \in \mathcal{S}$ and a Lyapunov function $V(x)$, and scalars $\lambda > 0$, $\gamma > 0$, satisfying*

$$L_{f_i} h(x) + L_{g_i} h(x)u + \lambda h(x) \geq 0, \tag{5}$$

$$L_{f_i} V(x) + L_{g_i} V(x)u + \gamma V(x) \leq 0, \tag{6}$$

where $i \in \mathcal{N}$. Then, the controller u satisfying (5) and (6) can guarantee that the switched system (1) is safe and stable.

Proof. It is evident that $\sigma(t)$ can be replaced by i, $i = 1, 2, \ldots, N$. Then, from any initial condition $x_0 \in \mathcal{S}$, an arbitrary subsystem i, $i \in \mathcal{N}$, operates, resulting in the $h(x) \leq 0$ in terms of (5). When switching, x in the switching instant satisfies $h(x) \leq 0$ and will be the initial state of the next arbitrary subsystem j, $j \neq i \in \mathcal{N}$. Therefore, the subsequent subsystem also operates with an initial condition belonging to $\mathcal{S}$, and ensures $h(x) \leq 0$ based on (5). And so on, u satisfying constraint (5) can guarantee the safety of switched system (1) under arbitrary switching in terms of Definition 1.

Meanwhile, u should also satisfy (6), that is $\dot{V}(x) = L_{f_i} V(x) + L_{g_i} V(x)u \leq -\gamma V(x)$, $i \in \mathcal{N}$. Since $\gamma > 0$, it is obtained that $\dot{V}(x) \leq -\gamma V(x) \leq 0$ for all subsystems arbitrary switching. Thus the stability of system (1) is held with u also satisfying constraint (6). The proof is done. □

Theorem 1 develops the conditions that a safe and stable controller of switched nonlinear systems (1) under arbitrary switching should meet. In this work, we consider the state-feedback controller $u(x)$. In general, the controller is designed by solving a quadratic programming problem in terms of Theorem 1 as

$$\begin{aligned} u^*(x) &= \arg\min_u \frac{1}{2} u^T u, \\ \text{subject to} \quad & L_{f_i} h(x) + L_{g_i} h(x)u + \lambda h(x) \geq 0, \\ & L_{f_i} V(x) + L_{g_i} V(x)u + \gamma V(x) \leq 0. \end{aligned} \tag{7}$$

However, it is hard to compute the controller u directly by solving the above quadratic programming problem without the specific form

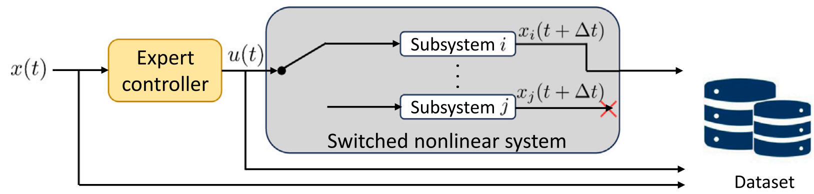

Fig. 2. Data collection phase.

of $h(x)$ and $V(x)$, especially when dealing with complex switching nonlinear dynamics. Learning-based methods are powerful to address such problems effectively. Additionally, when dealing with switched systems comprising a number of subsystems, we encounter situations where the state of certain subsystems is inaccessible for use. Designing controllers with limited subsystem data requires the exploration of specific techniques.

3. Methodology

We introduce the state transition learning approach called Complementary neural Lyapunov and Barrier function with Forward State Estimation (CLBFSE) to address the problem discussed in Section 2. There are three main components parameterized using a neural network: neural Lyapunov function $V_{\theta_V}: \mathcal{X} \to \mathbb{R}_+$, neural control barrier function $h_{\theta_h}: \mathcal{X} \to \mathbb{R}$, and neural control policy $\pi_{\theta_\pi}: \mathcal{X} \to \mathbb{R}^u$. All these three networks can be trained either offline or online. To simplify the algorithm, we discuss the offline version throughout this paper. The algorithm can be extended to an online version if the sequential order of the data is a concern.

3.1. Operation

The proposed CLBFSE is depicted in Figs. 2–4. There are three phases: data collection, training, and testing. In the data collection phase (Fig. 2), random states $x(t)$ are sampled and fed to an expert controller to generate the control signal $u(t)$. The states $x(t)$, control signal $u(t)$, and next states $x(t+\Delta t)$ are stored in the dataset for training. In our problem, a subset of subsystems may be inaccessible and thus the dataset may not contain the data of inaccessible system j. In the training phase (Fig. 3), the three neural functions are trained based on the dataset and loss functions $\mathcal{L}_V$, $\mathcal{L}_\pi$, and $\mathcal{L}_h$ to be discussed in later subsections. The purpose of this training is to update the parameters θ_V, θ_h, and θ_π such that they can perform the tasks we designed. In the testing phase (Fig. 4), we may deploy the model and use the functions with trained parameters to control a switched nonlinear system. During testing, any subsystems can be switched which challenges the training because of the unseen control pair data $(x(t), u(t), x(t+\Delta t))$ in the dataset. The training algorithm of the proposed approach is given in Algorithm 1.

3.2. Neural Lyapunov function

The Lyapunov function is positive definite and satisfies (4) as discussed in Section 2. Therefore, the neural Lyapunov function should also structured and trained based on the conditions.

Using the Rectified Linear Unit (ReLU) activation function is a straightforward way to guarantee that a neural network is nonnegative. However, it just forces the negative outputs to be zero without reflecting the mathematical definition of positive definiteness. To ensure positive definiteness, we use an uncommon neural network composed of a bilinear form of fully connected layers. To be specific, we define the neural Lyapunov function $V_{\theta_V}(x) = w(x)^T w(x)$, where w is a multilayer perceptron. With this neural network structure, we ensure that the neural Lyapunov function V_{θ_V} is positive definite.

For condition (4), we define a Lyapunov loss function so that it satisfies constraint (6) in the case of a switched nonlinear system. The Lyapunov loss function $\mathcal{L}_{Lya}$ is defined as

$$\begin{aligned}\mathcal{L}_{Lya} =& \lambda_{Lya1} V_{\theta_V}(x^*)^2 + \lambda_{Lya2} ReLU(L_{f_i} V_{\theta_V}(x) \\ &+ L_{g_i} V_{\theta_V}(x)\pi(x) + \gamma V_{\theta_V}(x) + \epsilon),\end{aligned} \tag{8}$$

where $ReLU$ is the rectified linear unit activation function, ϵ is a small positive constant to serve the function of incentivizing the rigorous adherence to inequalities, and λ_{Lya1} and λ_{Lya2} are the hyperparameters of loss terms.

3.3. Neural control barrier function

A control barrier function can ensure the safety of a control system if it is properly defined, which includes conditions (2) and (3). Therefore, we must ensure the neural barrier function has the same properties during the design and training phase.

Different from the Lyapunov function which is a positive definite, the control barrier function does not have to be positive definite if condition (2) is satisfied. To mitigate unnecessary restrictions on the output of the control barrier function, we use a fully-connected layer without a ReLU activation function as the last output layer.

To train the neural network to behave as a control barrier function according to conditions (2) and (3), we define the training loss function $\mathcal{L}_h$ to be

$$\begin{aligned}\mathcal{L}_h =& \lambda_{h1} \mathbb{E}_{x\in S}(ReLU(\epsilon - h_{\theta_h}(x))) \\ &+ \lambda_{h2} \mathbb{E}_{x\notin S}(ReLU(\epsilon + h_{\theta_h}(x))) \\ &+ \lambda_{h3} ReLU(-L_{f_{\sigma(t)}} h_{\theta_h}(x) - L_{g_{\sigma(t)}} h_{\theta_h}(x)\pi(x) - \lambda h_{\theta_h}(x)),\end{aligned} \tag{9}$$

where ϵ is a small positive constant similar to the one in the Lyapunov loss function, and λ_{h1}, λ_{h2} and λ_{h3} are the hyperparameters of loss terms. With this loss function, we can obtain a neural control barrier that satisfies the conditions.

3.4. Neural control policy

A neural control policy network π can be defined to control the switched nonlinear system such that $u = \pi_{\theta_\pi}(x)$. One way to train the network is to formulate the problem as a supervised learning problem, which mimics the control signals of an expert controller. We can update the network parameters using a supervised loss composed of the control signal samples from a pre-defined expert controller, such as Linear Quadratic Regulator (LQR), which is given by

$$\mathcal{L}_{\text{expert}} = \mathbb{E}_i[(\pi_{\theta_\pi}(x_i) - \pi_{\text{expert}}(x_i))^2]. \tag{10}$$

However, as we have discussed in Section 2, the optimal control policy of subsystems may not be available in the dataset due to the nonlinearity. We have to consider the scenario where the available dataset cannot capture the distribution of control policy of all the subsystems, leading to sub-optimal control. Similar issues were reported in behavior cloning (Codevilla, Santana, López, & Gaidon, 2019). Therefore, to avoid the aforementioned data distribution mismatch issue, we regulate the supervised policy loss function with an additional forward state estimation which is given by

$$\mathcal{L}_{\text{FSE}} = \mathbb{E}_{ij}[(F_j(x_i, \pi_{\theta_\pi}(x_i)) - x_i')^2], \tag{11}$$

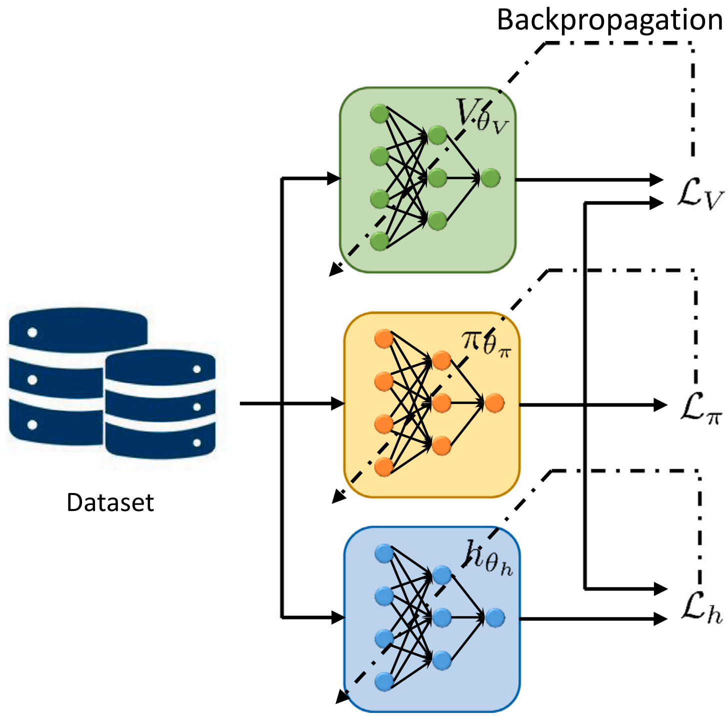

Fig. 3. Training phase.

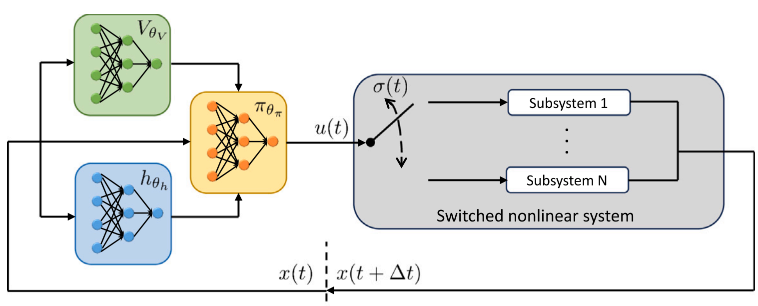

Fig. 4. Testing phase.

where data of subsystem i are given while subsystem j is not accessible, F_j is an estimated function to approximate the forward dynamics of subsystem j, and x_i' is the next state of x_i controlled by an expert controller (e.g., LQR). By minimizing the forward state estimation, we are guiding the policy π_{θ_π} to control the subsystem j (data of subsystem j are not given) with a similar next state as if it was controlled by an expert controller. Furthermore, we need to ensure the safety and stability of the policy by imposing Eqs. (5) and (6) as the constraints. To facilitate the training of the neural control policy network, we integrate the control policy loss functions (10) and (11) with the constraints (5) and (6) by applying Lagrangian relaxation, resulting in a single loss function as follows.

$$\begin{aligned}\mathcal{L}_\pi =& \lambda_{\pi 1}\mathcal{L}_{\text{expert}} + \lambda_{\pi 2}\mathcal{L}_{\text{FSE}} \\ &+ \lambda_{\pi 3} ReLU(-L_{f_{\sigma(t)}}h(x) - L_{g_{\sigma(t)}}h(x)\pi_{\theta_\pi}(x) - \lambda h(x)) \\ &+ \lambda_{\pi 4} ReLU(L_{f_i}V(x) + L_{g_i}V(x)\pi_{\theta_\pi}(x) + \gamma V(x) + \epsilon),\end{aligned} \tag{12}$$

where $\lambda_{\pi 1}$, $\lambda_{\pi 2}$, $\lambda_{\pi 3}$, and $\lambda_{\pi 4}$ are the weights (Lagrange multipliers) of the corresponding terms.

Remark 1 (*Convergence analysis*). The Eq. (12) represents the loss function used to obtain the neural control policy. It is worth noting that $\frac{\partial^2 \mathcal{L}_{\text{expert}}}{\partial^2 \pi_{\theta_\pi}} > 0$, and $\frac{\partial^2 \mathcal{L}_{\text{FSE}}}{\partial^2 \pi_{\theta_\pi}} = 2\mathbb{E}_{ij}[g_j(x_i)^2] \geq 0$. Additionally, ReLU is a convex function. As a result, we can conclude that the loss function $\mathcal{L}_\pi$ is convex since it is a nonnegative weighted sum of four convex functions. Moreover, the training of the neural control policy will guarantee convergence in terms of the direction provided by the gradient-descent-based approach to minimize the loss function $\mathcal{L}_\pi$.

4. Experiments

4.1. Experimental systems

We evaluate our proposed method on a numerical system and a single-link robotic manipulator system. The details are given as follows:

Example 1. Consider a numerical example of a switched nonlinear system derived from Noghreian and Koofigar (2018), which consists of two subsystems with the following parameters:

$$f_1(x) = \begin{pmatrix} x_2 - x_1^2 \\ x_1^2 x_2 + x_3 + x_1^2 \\ x_1 e^{x_1} \end{pmatrix}, \quad f_2(x) = \begin{pmatrix} x_2 \\ x_1^2 x_2 + x_3 \\ \sin x_1 \end{pmatrix} \tag{13}$$

$$g_1(x) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad g_2(x) = \begin{pmatrix} 0 \\ 0 \\ 0.5 \end{pmatrix}. \tag{14}$$

Algorithm 1 Complementary Neural Lyapunov and Barrier Policy

Input: Dataset D_i of subsystem i
Output: Parameters θ_V, θ_h, θ_π
Initialization : neural Lyapunov function parameters θ_V, neural control barrier function parameters θ_h, and policy parameters θ_π

for epoch = 1 to N **do**
  Shuffle dataset D_i
  Divide dataset D_i into batches
  for each batch in D_i **do**
    Update θ_V according to the loss (8)
    Update θ_h according to the loss (9)
    Update θ_π according to the loss (12)
  end for
end for
return θ_V, θ_h, θ_π

Example 2. The single-link robotic manipulator serves as a foundational and versatile building block for the development of more intricate robotic systems. We evaluate our proposed approach in a single-link manipulator composed of a rigid link connected via a gear train to a DC motor (Lian & Li, 2020; Spong, Hutchinson, & Vidyasagar, 2020).

Consider the robotic manipulator moves with varying loads, resulting in a switching behavior among several subsystems with different parameters. The switching dynamics are described by

$$J_{\sigma(t)}\ddot{q} + B\dot{q} + M_{\sigma(t)} g L_{\sigma(t)} \sin(q) = K_\tau u. \tag{15}$$

By letting $x = \begin{bmatrix} q & \dot{q} \end{bmatrix}^T$, (15) can be rewritten as a switched system (1) with

$$f_{\sigma(t)}(x) = \begin{pmatrix} \dot{q} \\ -(\frac{1}{J_{\sigma(t)}})(B\dot{q} + M_{\sigma(t)} g L_{\sigma(t)} \sin q) \end{pmatrix}, \tag{16}$$

$$g_{\sigma(t)}(x) = \begin{pmatrix} 0 \\ \frac{K_\tau}{J_{\sigma(t)}} \end{pmatrix}. \tag{17}$$

We borrow the parameters from Lian and Li (2020), which are $\sigma(t) = i$, $i \in \{1, 2\}$, and $K_\tau = 2$, $B = 1$ kg m^2 s^{-1}, $J_1 = 1$ kg m^2, $J_2 = 2$ kg m^2, $M_1 g L_1 = 10$ kg m^2 s^{-2}, and $M_2 g L_2 = 30$ kg m^2 s^{-2}.

To show the arbitrary switching law, two switching signals are evaluated: dwell time switching and random switching. The first means that each subsystem maintains for a dwell time before switching to another subsystem. In random switching, we randomly generate a switching signal and thus there is no constraint on the dwell time. The dwell time and random switching signal are plotted in Figs. 5 and 6, respectively. The training data was collected by executing the nominal control on a random initial state. I.e., (x^t, u^t, x^{t+1}) is a data sample in the dataset. 100 samples were recorded for each parameter setting and subsystem.

4.2. Compared methods

Several methods are compared in the experiments:

- No control: No control signal is fed to the system. In other words, u is always zero in this case.
- Linear Quadratic Regulator (LQR) (Bemporad, Morari, Dua, & Pistikopoulos, 2002): The system dynamics is linearized and the control policy is computed by an LQR controller.
- Supervised learning (Bain & Sammut, 1995): The π_{θ_π} is trained with data from all subsystems based on a mean square error supervised loss (first term in (12)).

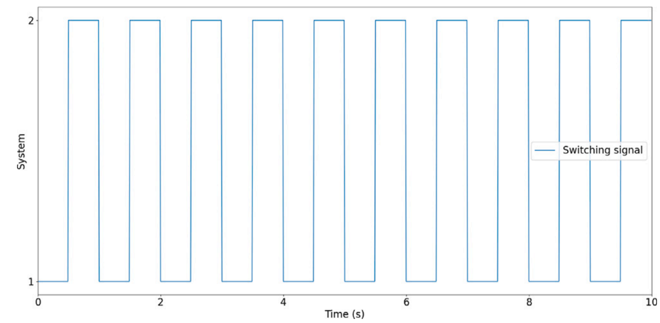

Fig. 5. Dell time switching signal.

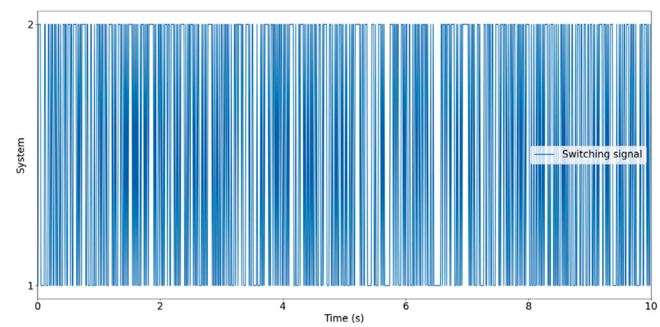

Fig. 6. Random switching signal.

- Supervised learning with limited data (Wang et al., 2023): The π_{θ_π} is trained with limited data of subsystem i only based on a mean square error supervised loss (first term in (12)).
- CLBFSE: Proposed approach presented in Section 3. It is trained with limited data of subsystem i only.

4.3. Experiment results

4.3.1. Test 1: Numerical example system

Example 1 represents a category of switched nonlinear systems characterized by exponential, square, and sine functions. Furthermore, it presents an underactuated system, as evidenced by the function $g_i(x)$, thereby exacerbating control challenges. Initial conditions are set to $x_1 = 0.5$, $x_2 = 0.5$, and $x_3 = 0.5$, with the imposition of reference safety constraints: $|x_1| \leq 1.5$, $|x_2| \leq 1.5$, and $|x_3| \leq 1.5$. The objective of the controllers is to regulate the states towards a reference value of zero during experimentation.

Figs. 7–9 and Figs. 10–12 illustrate the behavior of x_1, x_2, and x_3 under various control policies, considering random and dwell time switching, respectively. Notably, the system exhibits instability, as evidenced by the observed values without control, which tend to diverge exponentially. In general, all compared control policies effectively maintain the states within the safe boundary of ± 1.5 and converge towards a value proximal to the reference value of zero. Moreover, from Figs. 7–12, it can be observed that our proposed method shows great potential in faster convergence speed. Differences among these control policies manifest in their respective convergent values and rates, quantifiable through $x_i(T)$ and MAE$x_i(t)$, as presented in Table 1. We use state value at the final timestep and the corresponding MAE as the metrics to evaluate the proposed method. This table encapsulates the numerical states at $t = T$ and the average mean absolute error relative to the reference for all time t. Given the reference's zero value, smaller magnitudes denote superior control performance. Notably, despite possessing additional information such as model dynamics and subsystem data, traditional control methods such as LQR and supervised control are surpassed by the proposed approach in efficacy.

Table 1
Error of each control method for Example 1.

Methods	Random switching					
	$x_1(T)$	$x_2(T)$	$x_3(T)$	MAE$x_1(t)$	MAE$x_2(t)$	MAE$x_3(t)$
Expert control (LQR)	0.0063	−0.0174	0.0233	0.5254	0.2109	0.3746
No control	–	–	–	–	–	–
Supervised control	−0.0910	0.0216	−0.0276	0.3416	0.1599	0.2340
Supervised (limited data)	**0.0203**	0.0198	−0.0083	0.3789	0.1614	0.2734
Proposed approach	−0.0564	**−0.0041**	**0.0037**	**0.1770**	**0.1055**	**0.1609**
Methods	Switching with dwell time: 0.5 s					
	$x_1(T)$	$x_2(T)$	$x_3(T)$	MAE$x_1(t)$	MAE$x_2(t)$	MAE$x_3(t)$
Expert control (LQR)	−0.0023	0.0013	0.0016	0.3494	0.1450	0.2450
No control	–	–	–	–	–	–
Supervised control	−0.0862	0.0063	−0.0362	0.2800	0.1343	0.1934
Supervised (limited data)	**0.0363**	0.0090	−0.0052	0.2733	0.1223	0.1988
Proposed approach	−0.0580	**−0.0048**	**−0.0033**	**0.1670**	**0.1002**	**0.1559**

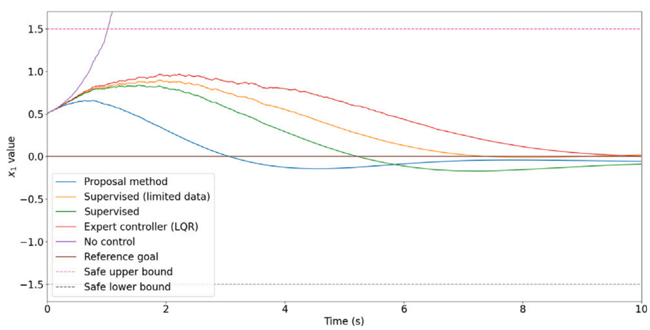

Fig. 7. Value of x_1 under random switching for Example 1.

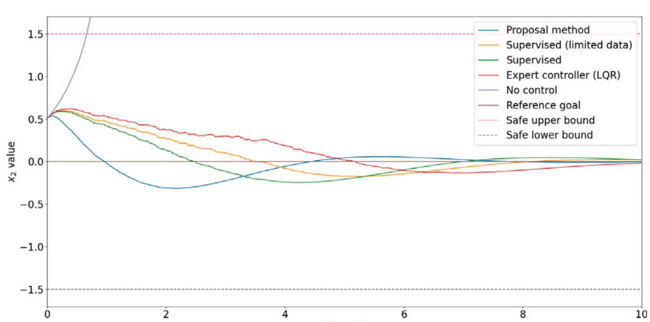

Fig. 8. Value of x_2 under random switching for Example 1.

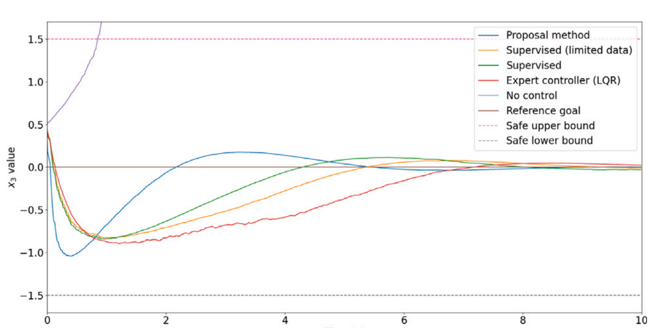

Fig. 9. Value of x_3 under random switching for Example 1.

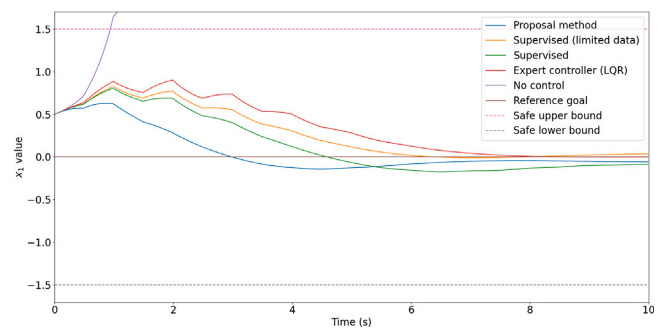

Fig. 10. Value of x_1 under $0.5s$ dwell time switching for Example 1.

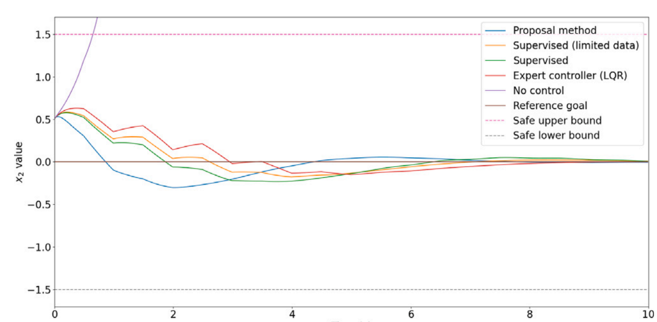

Fig. 11. Value of x_2 under $0.5s$ dwell time switching for Example 1.

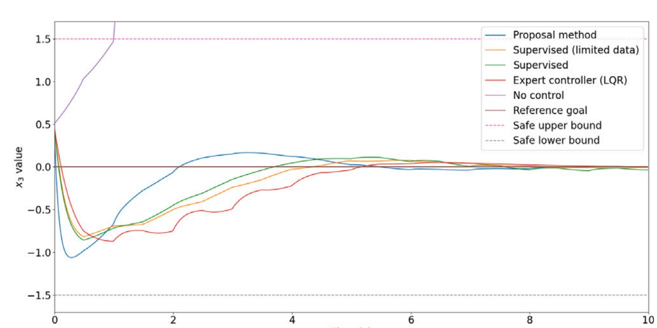

Fig. 12. Value of x_3 under $0.5s$ dwell time switching for Example 1.

4.3.2. *Test 2: Single-link robotic manipulator*

By setting a reference safe constraint as $|q| \leq 4.5$, $|\dot{q}| \leq 4.5$ and given the initial conditions as $q_0 = 2.5$ and $\dot{q} = 2.5$, the safe control results via different compared methods under random switching and dwell time switching are depicted in Figs. 13 and 14, and Figs. 15 and 16, respectively. For the values of q and $\dot{q}$ (Figs. 13–16), all control methods can be generally converged (stable) and operate within the safe region except the "no control". It can be observed that our controller has similar controlling performance when compared to the supervised learning trained with a complete dataset. Recall that our controller

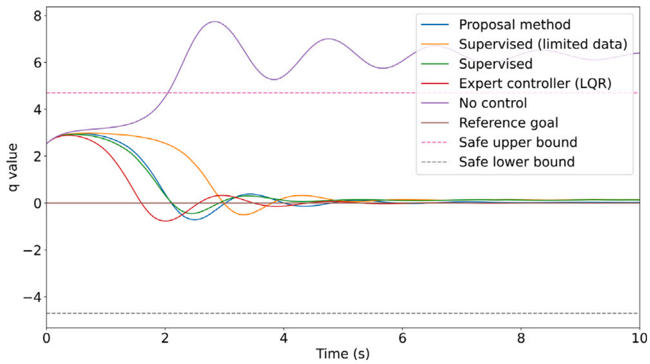

Fig. 13. Value of q under random switching for Example 2.

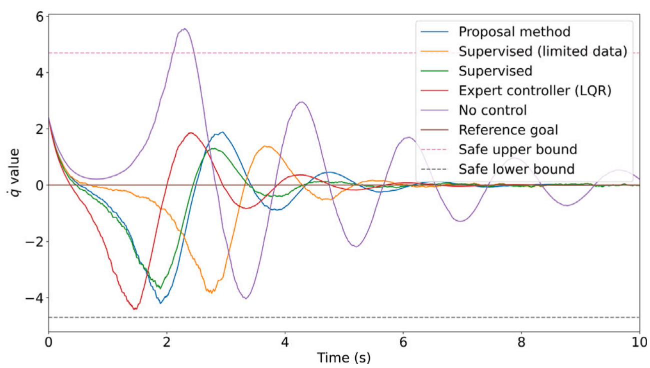

Fig. 14. Value of $\dot{q}$ under random switching for Example 2.

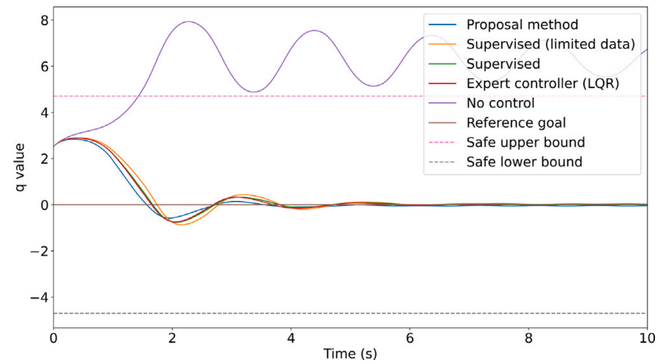

Fig. 15. Value of q under $0.5s$ dwell time switching for Example 2.

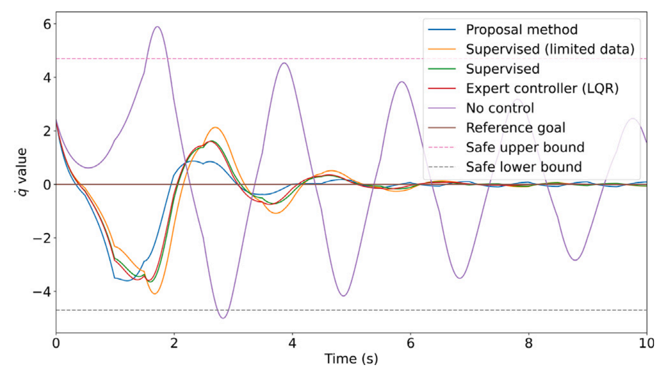

Fig. 16. Value of $\dot{q}$ under $0.5s$ dwell time switching for Example 2.

was trained with only one subsystem. Therefore, it is satisfactory to see the superior learning ability of our training approach. Moreover, it is obvious that our proposed method with limited data outperforms the supervised learning with limited data in terms of convergence speed, which verifies the effectiveness of the complementary idea with forward state estimation. To examine the convergence in more detail, we record the last timestep error difference and mean absolute error (MAE) of q and $\dot{q}$ to the goal in Table 2. We use state value at the final timestep and the corresponding MAE as the metrics to evaluate the proposed method. From the table, we can find that the system controlled by our proposed approach has the lowest error in most of the cases if we exclude the expert control (optimal control). Similar convergence properties can be observed from Figs. 13–16. Regarding the peak value of $\dot{q}$, it can be seen from Figs. 14 and 16 that our proposed method has the potential for a safer performance when close to the boundary. Therefore, it can be concluded that the switched nonlinear system under arbitrary switching is safe and stable with our designed control policy.

Table 2
Error of each control method for Example 2.

Methods	Random switching			
	$q(T)$	$\dot{q}(T)$	MAE$q(t)$	MAE$\dot{q}(t)$
Expert control (LQR)	0.0004	0.0023	0.4263	0.5978
No control	6.3964	0.2227	5.7101	1.3600
Supervised control	0.1307	−0.0200	0.5840	**0.4993**
Supervised (limited data)	0.1409	−0.0224	0.8397	0.5227
Proposed approach	**0.0284**	**−0.0005**	**0.5766**	0.6245
Methods	Switching with dwell time: 0.5 s			
	$q(T)$	$\dot{q}(T)$	MAE$q(t)$	MAE$\dot{q}(t)$
Expert control (LQR)	−0.0017	−0.0023	0.4345	0.5977
No control	6.7144	1.5798	5.8956	2.2320
Supervised control	**0.0206**	**−0.0556**	0.4458	0.5960
Supervised (limited data)	0.0213	−0.0611	0.4919	0.6693
Proposed approach	−0.0391	−0.0909	**0.3941**	**0.5030**

4.4. Discussions

4.4.1. Discuss the insights of the proposed approach

First, our controller via a state transition learning approach possesses a good convergence and safe performance. Many existing works use a unified neural Lyapunov-barrier function, but we trained separate neural Lyapunov function and neural control barrier function. The unified neural Lyapunov-barrier function will impose an additional constraint that the control barrier function is positive definite. However, the safe constraints can be given or characterized with various kinds of functions not limited to positive definite functions. Thus compared with the unified neural Lyapunov-barrier function, separate neural Lyapunov function and neural control barrier function in our work may be less conservative.

On another aspect, the highlight of the proposed state transition learning for switched systems can reduce the dependence on the dataset. It should be noted that in switched systems, many efforts are devoted to mitigating the impact of switching behavior by studying each subsystem individually. The intuitive idea behind our proposed approach is that one can control one subsystem with the desired performance, and then collect the data only from this subsystem to drive other systems among switching to follow a similar trajectory. This will make a switched system operate like a single system, and thus mitigate the impact of switching behavior, just as our proposed approach does. Therefore, our state transition learning is suitable for dealing with the control of switched systems.

4.4.2. Discuss the limitations of the proposed approach

The limitation of using the neural control barrier function is that safety is guaranteed by a neural network, which is an approximator of the safety certificate. Thus, it may not rigorously guarantee safety as in the conventional safe control. During our experimental trials, we observed that insufficient training iterations resulted in the potential violation of the safe bound. One of the unaddressed technical challenges is to theoretically analyze the relationship between the system state boundedness and parameters of neural networks. In this case, by adjusting the parameters or adding relaxed compensators, the certainty of ensuring safety may be further enhanced. Moreover, all the considered scenarios are that the safe region is not in conflict with stability. Because we focus on the effectiveness of state transition learning methods in handling the limited data of switched systems, our method no longer focuses on making trade-offs when safe constraints conflict with stability.

5. Conclusion

This study addresses safe control problems of switched nonlinear systems with limited data by a state transition learning approach. We have synthesized a controller to ensure both safety and stability performance using control barrier functions and Lyapunov functions. To avoid difficulties in constructing these functions, We have developed neural control barrier functions and neural Lyapunov functions to find the safe control policy via a machine learning approach. In view of the limited data and data distribution mismatch issues during the training of control policy, we constructed a complementary loss function that estimates the next state to complement the missing subsystem data. Simulation experiments have been conducted to verify the effectiveness of the developed safe control scheme for switched nonlinear systems. In particular, the control policy trained by the proposed approach can achieve a similar control performance to the one with a complete dataset. There are several possible directions for future work. One potential area is to explore robust and safe control strategies to deal with the disturbances and uncertainties in the system dynamics. Additionally, in the context of multi-robot systems, where communication networks may be subject to switching dynamics, there is an opportunity for an extension of our work that explores safe cooperative control strategies under switching networks. Furthermore, We considered the supervised learning setting where the dataset is available for offline training. If the real-world switched system is available for numerous training and testing iterations such as a digital twin system, the problem can be solved by reinforcement learning-based approach.

CRediT authorship contribution statement

Chenchen Fan: Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Kai-Fung Chu:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Xiaomei Wang:** Writing – review & editing, Writing – original draft, Validation, Investigation. **Ka-Wai Kwok:** Writing – review & editing, Validation, Supervision, Funding acquisition. **Fumiya Iida:** Writing – review & editing, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Ka-Wai Kwok reports financial support was provided by Research Grants Council. Ka-Wai Kwok reports financial support was provided by Innovation and Technology Commission. Xiaomei Wang reports financial support was provided by Multi-Scale Medical Robotics Center Limited InnoHK. Chenchen Fan reports financial support was provided by Centre for Garment Production Limited InnoHK. Chenchen Fan reports financial support was provided by The Hong Kong Polytechnic University. Kai-Fung Chu reports financial support was provided by European Union's Horizon 2020. If there are other authors they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

Abdallah, S., & Lesser, V. (2008). A multiagent reinforcement learning algorithm with non-linear dynamics. *Journal of Artificial Intelligence Research*, *33*, 521–549.

Ames, A. D., Coogan, S., Egerstedt, M., Notomista, G., Sreenath, K., & Tabuada, P. (2019). Control barrier functions: Theory and applications. In *2019 18th European control conference* (pp. 3420–3431). IEEE.

Ames, A. D., Xu, X., Grizzle, J. W., & Tabuada, P. (2016). Control barrier function based quadratic programs for safety critical systems. *IEEE Transactions on Automatic Control*, *62*(8), 3861–3876.

Athanasopoulos, N., Smpoukis, K., & Jungers, R. M. (2016). Safety and invariance for constrained switching systems. In *2016 IEEE 55th conference on decision and control* (pp. 6362–6367). IEEE.

Bain, M., & Sammut, C. (1995). A framework for behavioural cloning. In *Machine intelligence*: *vol. 15*, (pp. 103–129).

Bao, Y., Zhang, Y., & Zhang, B. (2023). Resilient fixed-time stabilization of switched neural networks subjected to impulsive deception attacks. *Neural Networks*, *163*, 312–326.

Bemporad, A., Morari, M., Dua, V., & Pistikopoulos, E. N. (2002). The explicit linear quadratic regulator for constrained systems. *Automatica*, *38*(1), 3–20.

Bena, R. M., Hossain, S., Chen, B., Wu, W., & Nguyen, Q. (2023). A hybrid quadratic programming framework for real-time embedded safety-critical control. In *2023 IEEE international conference on robotics and automation* (pp. 3418–3424). IEEE.

Chen, Z., Zhang, H., Liu, J., Wang, Q., & Wang, J. (2023). Adaptive prescribed settling time periodic event-triggered control for uncertain robotic manipulators with state constraints. *Neural Networks*, *166*, 1–10.

Chu, K.-F., Lam, A. Y., Fan, C., & Li, V. O. (2021). Disturbance-aware neuro-optimal system control using generative adversarial control networks. *IEEE Transactions on Neural Networks and Learning Systems*, *32*(10), 4565–4576.

Codevilla, F., Santana, E., López, A. M., & Gaidon, A. (2019). Exploring the limitations of behavior cloning for autonomous driving. In *Proceedings of the IEEE/CVF international conference on computer vision* (pp. 9329–9338).

Dawson, C., Gao, S., & Fan, C. (2023). Safe control with learned certificates: A survey of neural Lyapunov, barrier, and contraction methods for robotics and control. *IEEE Transactions on Robotics*, *39*(3), 1749–1767.

de Jesús Rubio, J., Orozco, E., Cordova, D. A., Hernandez, M. A., Rosas, F. J., & Pacheco, J. (2024). Observer-based differential evolution constrained control for safe reference tracking in robots. *Neural Networks*, *175*, Article 106273.

Donge, V. S., Lian, B., Lewis, F. L., & Davoudi, A. (2023). Data-efficient reinforcement learning for complex nonlinear systems. *IEEE Transactions on Cybernetics*.

Fan, C., Lam, J., Chu, K.-F., Lu, X., & Kwok, K.-W. (2024). Output reachable set-based leader-following consensus of positive agents over switching networks. *IEEE Transactions on Cybernetics*, *54*(7), 3918–3930.

Fan, C., Lam, J., Lu, X., Liu, J. J., Wang, X., & Kwok, K.-W. (2024). Reachable set-based consensus of positive multiagent systems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, *54*(4), 2320–2330.

Fan, C., Lam, J., Xie, X., & Li, P. (2022). Stability and stabilization of almost periodic piecewise linear systems with dwell time uncertainty. *IEEE Transactions on Automatic Control*, *68*(2), 1130–1137.

Fan, C., Lam, J., Xie, X., & Song, X. (2021). Observer-based output reachable set synthesis for periodic piecewise time-varying systems. *Information Sciences*, *571*, 246–261.

Fei, Y., Li, D., Li, Y., & Li, J. (2023). Deterministic learning-based neural network control with adaptive phase compensation. *Neural Networks*, *160*, 175–191.

Guiochet, J., Machin, M., & Waeselynck, H. (2017). Safety-critical advanced robots: A survey. *Robotics and Autonomous Systems*, *94*, 43–52.

Gurriet, T., Mote, M., Singletary, A., Feron, E., & Ames, A. D. (2019). A scalable controlled set invariance framework with practical safety guarantees. In *2019 IEEE 58th conference on decision and control* (pp. 2046–2053). IEEE.

Karras, T., Aittala, M., Hellsten, J., Laine, S., Lehtinen, J., & Aila, T. (2020). Training generative adversarial networks with limited data. In *Advances in neural information processing systems*: *vol. 33*, (pp. 12104–12114).

Kıvılcım, A., Karabacak, Ö., & Wisniewski, R. (2019). Safety verification of nonlinear switched systems via barrier functions and barrier densities. In *2019 18th European control conference* (pp. 776–780). IEEE.

Lafmejani, A. S., Berman, S., & Fainekos, G. (2022). NMPC-LBF: Nonlinear MPC with learned barrier function for decentralized safe navigation of multiple robots in unknown environments. In *2022 IEEE/RSJ international conference on intelligent robots and systems* (pp. 10297–10303). IEEE.

Landi, C. T., Ferraguti, F., Costi, S., Bonfè, M., & Secchi, C. (2019). Safety barrier functions for human-robot interaction with industrial manipulators. In *2019 18th European control conference* (pp. 2565–2570). IEEE.

Li, S., Ahn, C. K., Guo, J., & Xiang, Z. (2020). Neural-network approximation-based adaptive periodic event-triggered output-feedback control of switched nonlinear systems. *IEEE Transactions on Cybernetics*, *51*(8), 4011–4020.

Lian, J., & Li, C. (2020). Event-triggered control for a class of switched uncertain nonlinear systems. *Systems & Control Letters*, *135*, Article 104592.

Liberzon, D., & Morse, A. S. (1999). Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, *19*(5), 59–70.

Lin, H., & Antsaklis, P. J. (2009). Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic control*, *54*(2), 308–322.

Littman, M. L., Cassandra, A. R., & Kaelbling, L. P. (1995). Learning policies for partially observable environments: Scaling up. In *Machine learning proceedings 1995* (pp. 362–370). Elsevier.

Liu, L., Zhou, Q., Liang, H., & Wang, L. (2017). Stability and stabilization of nonlinear switched systems under average dwell time. *Applied Mathematics and Computation*, *298*, 77–94.

Lusch, B., Kutz, J. N., & Brunton, S. L. (2018). Deep learning for universal linear embeddings of nonlinear dynamics. *Nature Communications*, *9*(1), 4950.

Noghreian, E., & Koofigar, H. R. (2018). Adaptive output feedback tracking control for a class of uncertain switched nonlinear systems under arbitrary switching. *International Journal of Systems Science*, *49*(3), 486–495.

Peng, C., Donca, O., Castillo, G., & Hereid, A. (2023). Safe bipedal path planning via control barrier functions for polynomial shape obstacles estimated using logistic regression. In *2023 IEEE international conference on robotics and automation* (pp. 3649–3655). IEEE.

Peng, D., Yang, H., & Jiang, B. (2024). Robust switching time optimization for networked switched systems via model predictive control. *IEEE Transactions on Neural Networks and Learning Systems*, *35*(8), 10961–10972.

Rauscher, M., Kimmel, M., & Hirche, S. (2016). Constrained robot control using control barrier functions. In *2016 IEEE/RSJ international conference on intelligent robots and systems* (pp. 279–285). IEEE.

Romdlony, M. Z., & Jayawardhana, B. (2016). Stabilization with guaranteed safety using control Lyapunov–barrier function. *Automatica*, *66*, 39–47.

Spong, M. W., Hutchinson, S., & Vidyasagar, M. (2020). *Robot modeling and control*. John Wiley & Sons.

Wang, Q., McCarthy, R., Bulens, D. C., McGuinness, K., O'Connor, N. E., Sanchez, F. R., et al. (2023). Improving behavioural cloning with positive unlabeled learning. In *Conference on robot learning* (pp. 3851–3869). PMLR.

Weng, B., Capito, L., Ozguner, U., & Redmill, K. (2021). Towards guaranteed safety assurance of automated driving systems with scenario sampling: An invariant set perspective. *IEEE Transactions on Intelligent Vehicles*, *7*(3), 638–651.

Xiang, W., Tran, H.-D., & Johnson, T. T. (2017). Output reachable set estimation for switched linear systems and its application in safety verification. *IEEE Transactions on Automatic Control*, *62*(10), 5380–5387.

Xie, X., Lam, J., & Fan, C. (2018). Robust time-weighted guaranteed cost control of uncertain periodic piecewise linear systems. *Information Sciences*, *460*, 238–253.

Zhang, Y., Liang, X., Li, D., Ge, S. S., Gao, B., Chen, H., et al. (2022). Barrier Lyapunov function-based safe reinforcement learning for autonomous vehicles with optimized backstepping. *IEEE Transactions on Neural Networks and Learning Systems*, *35*(2), 2066–2080.