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Felix Schremmer

Newton strata in Levi subgroups

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Abstract. Certain Iwahori double cosets in the loop group of a reductive group, known under the names of P-alcoves or (J, w, δ)-alcoves, play an important role in the study of affine Deligne–Lusztig varieties. For such an Iwahori double coset, its Newton stratification is related to the Newton stratification of an Iwahori double coset in a Levi subgroup. We study this relationship further, providing in particular a bijection between the occurring Newton strata. As an application, we prove a conjecture of Dong-Gyu Lim, giving a non-emptiness criterion for basic affine Deligne–Lusztig varieties.

1. Introduction

Let G be a reductive group over the local field F, whose completion of the maximal unramified extension we denote by $\breve{F}$. In the context of reduction of Shimura varieties, one would choose F to be a finite extension of p-adic rationals, whereas in the context of moduli spaces of shtukas, F would be the field of formal Laurent series over a finite field. In any case, the Galois group $\mathrm{Gal}(\breve{F}/F)$ is (topologically) generated by the Frobenius σ. We moreover pick a σ-stable Iwahori subgroup $I \subseteq G(\breve{F})$. Then the affine Deligne–Lusztig variety associated to two elements $x, b \in G(\breve{F})$ is defined as

$$X_x(b) = \{g \in G(\breve{F})/I \mid g^{-1}b\sigma(g) \in IxI\}.$$

Evidently, the isomorphism type of $X_x(b)$ only depends on the σ-conjugacy class

$$[b] = \{g^{-1}b\sigma(g) \mid g \in G(\breve{F})\} \subseteq G(\breve{F})$$

and the Iwahori double coset IxI. The latter Iwahori double cosets are typically indexed using the extended affine Weyl group $\widetilde{W}$, so that each coset is given by IxI for a uniquely determined element $x \in \widetilde{W}$. The set $B(G)$ of σ-conjugacy classes has an important parametrization due to Kottwitz [9,10], characterizing each $[b] \in B(G)$ by its Newton point $\nu(b)$ and Kottwitz point $\kappa(b)$.

F. Schremmer (✉): University of Hong Kong, Hong Kong, China.
e-mail: schremmer@hku.hk

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Following [3, Section 2], we assume without loss of generality that the group G is quasi-split over F. We choose a maximal torus T whose unique parahoric subgroup $T_0(\breve{F})$ is contained in I and a σ-stable Borel subgroup $T \subset B$ such that, in the corresponding appartment of the Bruhat–Tits building of $G_{\breve{F}}$, the alcove fixed by I is opposite to the dominant cone defined by B. With this notation, the Kottwitz point $\kappa(b)$ for $b \in G(\breve{F})$ lies in $(X_*(T)/\mathbb{Z}\Phi^\vee)_\Gamma$, where $\Phi^\vee$ is the set of coroots and Γ the absolute Galois group of F. The dominant Newton point $\nu(b)$ is an element of $X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$, where $\Gamma_0 \subset \Gamma$ is the absolute Galois group of $\breve{F}$. We identify the extended affine Weyl group $\widetilde{W}$ as the semidirect product of the finite Weyl group $W = N_G(T)/T$ with $X_*(T)_{\Gamma_0}$.

Geometric properties of the affine Deligne–Lusztig variety $X_x(b)$ are closely related to those of the corresponding Newton stratum $[b] \cap IxI \subset G(\breve{F})$. Obviously, one is empty if and only if the other is empty. Further properties can be related following [12, Section 3.1].

2. Reduction to Levi subgroups

It is an important question to study which of these affine Deligne–Lusztig varieties are non-empty, i.e. to determine the set

$$B(G)_x = \{[b] \in B(G) \mid [b] \cap IxI \neq \emptyset\}.$$

An important breakthrough result of Görtz–He–Nie [3] is a characterization of all elements $x \in \widetilde{W}$ where $B(G)_x$ contains the basic σ-conjugacy class. For this purpose, they introduce the notion of a (J, w, σ)-alcove, generalizing the previous notion of a P-alcove known for split groups. Write $\Delta \subseteq \Phi$ for the set of simple roots.

Definition 1. Let $x \in \widetilde{W}$, $w \in W$ and $J \subseteq \Delta$ such that $J = \sigma(J)$. Then we say that x is a (J, w, σ)-*alcove element* if the following conditions are both satisfied:

(a) The element $\tilde{x} = w^{-1}x\sigma(w)$ lies in the extended affine Weyl group $\widetilde{W}_M$ of the standard Levi subgroup $M = M_J \supseteq T$ defined by J.
(b) For all positive roots $\alpha \in \Phi^+$ that are not in the root system Φ_J generated by J, the corresponding root subgroup $U_\alpha \subseteq G(\breve{F})$ satisfies

$$U_{w\alpha} \cap {}^xI \subseteq U_{w\alpha} \cap I.$$

Observe that x is a (J, w, σ)-alcove element if it is a (J, w', σ)-alcove element for any $w' \in wW_J$, where W_J is the subgroup of W generated by the simple reflections coming from J (or equivalently the finite Weyl group of M_J). Following Viehmann [13, Section 4], we say that x is a *normalized* (J, w, σ)-alcove element if w has minimal length in its coset wW_J. This extra assumption is independent of x, i.e. any (J, w, σ)-alcove element will also be a normalized (J, w', σ)-alcove element for a more refined choice of w'.

Assume that x is a (J, w, σ)-alcove element and write $\tilde{x} = w^{-1}x\sigma(w) \in \widetilde{W}_M$. If $[b] \in B(G)_x$, it is a result of Görtz–He–Nie [3, Theorem 3.3.1] that each element in Newton stratum $IxI \cap [b]$ is of the form $i^{-1}wm\sigma(w^{-1}i)$ for some $i \in I$ and

$$m \in \bigcup_{[b']}\Big((I \cap M(\breve{F}))\tilde{x}(I \cap M(\breve{F})) \cap [b']\Big) \subseteq M(\breve{F}),$$

with the union taken over all $[b'] \in B(M)$ contained in $[b]$. Our main result states that, whenever x is a *normalized* (J, w, σ)-alcove element, this union is spurious.

Theorem 2. *Let x be a normalized (J, w, σ)-alcove element and $\tilde{x}=w^{-1}x\sigma(w)\in \widetilde{W}_M$. Then we get a bijective map*

$$B(M)_{\tilde{x}} \to B(G)_x,$$

sending a σ-conjugacy class $[b]_M \in B(M)_{\tilde{x}}$ to the unique σ-conjugacy class $[b]_G \in B(G)$ with $[b]_M \subseteq [b]_G$. In this case, the dominant Newton points $\nu_M(b)$ and $\nu_G(b)$ agree as elements of $X_*(T)_{\Gamma_0} \otimes \mathbb{Q}$.

Unfortunately, the relationship discussed here has been a source of confusion in the past. While surjectivity of the map $B(M)_{\tilde{x}} \to B(G)_x$ follows from [3, Theorem 3.3.1], it should be noted that the natural map $B(M) \to B(G)$ is neither injective nor surjective. In particular, for $[b]_G \in B(G)_x$, the intersection $[b]_G \cap M(\breve{F})$ may consist of more than one σ-conjugacy class of M. We would like to make the following remarks regarding previous literature:

- The reader will notice that the published version of [3, Proposition 3.5.1] does not hold true in the claimed generality. It is actually only proved for basic σ-conjugacy classes, as explained in the erratum[1].
- In [13, Example 5.4], Viehmann claims to provide a counterexample to the injectivity statement in Theorem 2. However, the situation considered there does not give rise to a *normalized* (J, w, σ)-alcove. Her example does explain how crucial this assumption is for our present work.

We remark that for split G, the bijection of Theorem 2 is known from [1, Corollary 2.1.3].

We can use the correspondence in Theorem 2 to study affine Deligne–Lusztig varieties. In the setting of Theorem 2, [3, Theorem 3.3.1] gives us a closed immersion of the affine Deligne–Lusztig variety $X_{\tilde{x}}(b)$ (for M) into $X_x(b)$:

$$X_{\tilde{x}}(b) \to X_x(b), \qquad g(I \cap M) \mapsto gw^{-1}I.$$

This map is usually not surjective. However, we can make it surjective as follows: For $b \in G(\breve{F})$, denote its σ-centralizer by

$$\mathrm{J}_b(F) := \{g \in G(\breve{F}) \mid g^{-1}b\sigma(g) = b\}.$$

Observe that $\mathrm{J}_b(F)$ acts on $X_x(b)$ by left multiplication.

[1] Cf. https://www.esaga.uni-due.de/f/ulrich.goertz/pdf/Erratum-GHN.pdf.

Corollary 3. *The affine Deligne–Lusztig variety $X_x(b)$ is a disjoint union of closed subsets*

$$X_x(b) = \bigsqcup_{j\in \mathrm{J}_b(F)/(\mathrm{J}_b(F)\cap M(\breve F))} (jM(\breve F)w^{-1}I/I \cap X_x(b)).$$

For each $j \in \mathrm{J}_b(F)$, we get an isomorphism

$$X_{\tilde x}(b) \to jM(\breve F)w^{-1}I/I \cap X_x(b), \qquad g(I\cap M) \mapsto jgw^{-1}I.$$

Proof. Similar to the proof of [1, Theorem 2.1.4], using Theorem 2 instead of [1, Corollary 2.1.3]. □

The geometric correspondence between the affine Deligne–Lusztig varieties $X_{\tilde x}(b)$ and $X_x(b)$ is mirrored by a corresponding representation-theoretic result of He–Nie [7, Theorem C], comparing class polynomials of x and $\tilde x$. These are certain structure constants describing the cocenter of the Iwahori–Hecke algebra of $\widetilde W$ resp. $\widetilde W_M$. If one knows all class polynomials for a given element $x \in \widetilde W$, one can use these to determine many geometric properties the affine Deligne–Lusztig varieties $X_x(b)$ for $[b] \in B(G)$, cf. [5, Theorem 6.1]. These properties include dimension as well as the number of top dimensional irreducible components up to the action of the σ-centralizer of $b \in G(L)$. Moreover, in a certain sense, the number of rational points of the Newton stratum $IxI \cap [b]$ can be expressed using these class polynomials [8, Proposition 3.7]. In this sense, it already follows from [7] that these numerical invariants agree for $X_{\tilde x}(b)$ and $X_x(b)$.

Proof of Theorem 2. We follow the reduction method of Deligne–Lusztig, adapted to the affine case by Görtz–He [2]. I.e. we do an induction on $\ell(x)$.

If x is of minimal length in its σ-conjugacy class, then $B(G)_x$ contains only one element, being the σ-conjugacy class defined by x. Moreover, He–Nie [7, Proposition 4.5] prove that in this case $\tilde x$ is of minimal length in its σ-conjugacy class in $\widetilde W_M$. Hence we only have to show that $\nu_M(\tilde x)$ agrees with $\nu_G(x)$.

Following the definition of Newton points, we consider σ-twisted powers

$$x^{\sigma,n} = x\sigma(x)\cdots\sigma^{n-1}(x) \in \widetilde W.$$

Observe that each $x^{\sigma,n}$ is a (J, w, σ^n)-alcove element. Let n be sufficiently large such that $x^{\sigma,n}$ is a pure translation element, i.e. equal to the image of some $\mu \in X_*(T)_{\Gamma_0}$ in $\widetilde W$, and such that σ^n is the identity map on $\widetilde W$. Then the Newton point $\nu_G(x)$ is the unique dominant element in the W-orbit of μ/n. Similarly, the Newton point $\nu_M(\tilde x)$ is the unique dominant (with respect to $B\cap M$) element in the W_J-orbit of $w^{-1}\mu/n$.

The fact that $x^{\sigma,n}$ is a $(J, w, 1)$-alcove element implies that $\langle w^{-1}\mu, \alpha\rangle \geq 0$ for all $\alpha \in \Phi^+\setminus\Phi_J$. Hence $\nu_M(\tilde x)$ is already dominant with respect to B, and thus $\nu_G(x) = \nu_M(\tilde x)$. This finishes the proof in case x has minimal length in its σ-conjugacy class.

If x is not of minimal length in its σ-conjugacy class, we can use [6, Theorem A] to obtain a sequence

$$x = x_1 \xrightarrow{s_1} \cdots \xrightarrow{s_n} x_{n+1},$$

for simple affine reflections $s_i \in \widetilde{W}$ and elements $x_{i+1} = s_i x_i \sigma(s_i)$ such that $\ell(x_1) = \cdots = \ell(x_n) > \ell(x_{n+1})$. From [7, Lemma 7.1], we find elements $w_1, \ldots, w_n \in W$ such that each x_i is a normalized (J, w_i, σ)-alcove element. Denote the corresponding elements by $\tilde{x}_i = w_i^{-1} x_i \sigma(w_i) \in \widetilde{W}_M$, so that the proof of [7, Corollary 4.4] shows

$$\ell_{\widetilde{W}_M}(\tilde{x}_1) = \cdots = \ell_{\widetilde{W}_M}(\tilde{x}_n) > \ell_{\widetilde{W}_M}(\tilde{x}_{n+1}).$$

Said proof moreover reveals that each $\tilde{x}_{i+1}$ is conjugate to $\tilde{x}_i$ either by a simple affine reflection in $\widetilde{W}_M$ or a length zero element in $\widetilde{W}_M$.

The Deligne–Lusztig reduction method of Görtz–He [2] yields

$$B(G)_x = \cdots = B(G)_{x_n} = B(G)_{x_{n+1}} \cup B(G)_{s_n x_n},$$
$$B(M)_{\tilde{x}} = \cdots = B(M)_{\tilde{x}_n}.$$

We moreover know from the aforementioned article of He–Nie that $\tilde{x}_{n+1} = \tilde{s}\tilde{x}_n\sigma(\tilde{s})$ for some simple affine reflection $\tilde{s} = w_n s_n w_n^{-1} \in \widetilde{W}_M$ of M. Hence

$$B(M)_{\tilde{x}_n} = B(M)_{\tilde{x}_{n+1}} \cup B(M)_{\tilde{s}\tilde{x}_n}.$$

By induction, we get bijective and Newton-point preserving maps

$$B(M)_{\tilde{x}_{n+1}} \to B(G)_{x_{n+1}}, \qquad B(M)_{\tilde{s}\tilde{x}_n} \to B(G)_{s_n x_n}.$$

We conclude that the map $B(M)_{\tilde{x}} \to B(G)_x$ is well-defined, surjective and Newton-point preserving. If $[b_1]_M, [b_2]_M \in B(M)_{\tilde{x}}$ have the same image $[b]_G \in B(G)_x$ under this map, then $\nu_M(b_1) = \nu_G(b) = \nu_M(b_2)$ and $\kappa_M(b_1) = \kappa_M(\tilde{x}) = \kappa_M(b_2)$, hence $[b_1]_M = [b_2]_M$. This finishes the induction and the proof. □

Let us note the following consequence of Theorem 2.

Corollary 4. *If x is a (J, w, σ)-alcove element and $[b_1], [b_2] \in B(G)_x$, then*

$$\nu_G(b_1) \equiv \nu_G(b_2) \pmod{\Phi_J^\vee}.$$ □

3. Lim's conjecture on the nonemptiness of the basic locus

As an application of Corollary 4, we prove a conjecture of Dong-Gyu Lim [11, Conjecture 1], yielding an alternative criterion to the one from [3] for the nonemptiness of the basic Newton stratum in IxI.

Definition 5. Let $x \in \widetilde{W}$ be written as $x = \omega s_1 \cdots s_{\ell(x)}$ for a length zero element ω and simple affine reflections $s_1, \ldots, s_{\ell(x)} \in \widetilde{W}$. We define the *$\sigma$-support* of x to be the smallest subset $J \subseteq \widetilde{W}$ containing $s_1, \ldots, s_{\ell(x)}$ and being closed under the action of the composite automorphism $\sigma \circ \omega$. Denote it by $\mathrm{supp}_\sigma(x)$. We say that x is *spherically σ-supported* if the subgroup of $\widetilde{W}$ generated by $\mathrm{supp}_\sigma(x)$ is finite.

It follows from [4, Proposition 5.6] that x has spherical σ-support if and only if $B(G)_x = \{[b]\}$ for a basic σ-conjugacy class $[b]$.

Proposition 6. *Assume that the Dynkin diagram of Φ is σ-connected, i.e. that the Frobenius σ acts transitively on the set of irreducible components of the root system Φ.*

Let $x \in \widetilde{W}$, and denote by $[b] \in B(G)$ the unique basic σ-conjugacy class with $\kappa(b) = \kappa(x)$. Then $X_x(b) = \emptyset$ if and only if the following two conditions are both satisfied:

(a) The element x does not *have spherical σ-support, i.e. $B(G)_x$ contains a non-basic σ-conjugacy class.*
(b) There exists $J \subsetneq \Delta$ and $w \in W$ such that x is a (J, w, σ)-alcove element.

Proof. If x has spherical σ-support, we get $IxI \subseteq [b]$, so that indeed $X_x(b) \neq \emptyset$. In the case that x is not a (J, w, σ)-alcove element for any $J \subsetneq \Delta$, we easily obtain $X_x(b) \neq \emptyset$ by [3, Theorem A].

Assume now conversely that (a) and (b) both hold true, so we have to show $X_x(b) = \emptyset$. Let (J, w) be as in (b) such that moreover x is a normalized (J, w, σ)-alcove element. Let $[b_x] \in B(G)_x$ denote the generic σ-conjugacy class.

Assume that $[b] \in B(G)_x$. From (b) together with Corollary 4, we see

$$\nu(b) \equiv \nu(b_x) \pmod{\Phi_J^\vee}.$$

In particular

$$\langle \nu_G(b_x) - \nu_G(b), 2\rho - 2\rho_J \rangle = 0.$$

Since $\nu_G(b_x)$ is dominant and b is basic, we conclude $\langle \nu_G(b_x), \alpha \rangle = 0$ for all $\alpha \in \Phi^+ \backslash \Phi_J$. Thus $\Phi = \Phi_J \cup \Phi_{J'}$ where $J' \subseteq \Delta$ is the stabilizer of $\nu(b_x)$.

Each irreducible component of Φ contains a unique longest root, and by σ-irreducibility, these longest roots form a single σ-orbit. Since Φ_J and $\Phi_{J'}$ are two σ-stable subsets of Φ covering the entire root system, one of these two sets must contain all longest roots. This is only possible if $J = \Delta$ or $J' = \Delta$.

We assumed $J \neq \Delta$ in (b), so we conclude that $\nu_G(b_x)$ must be central. Thus $[b_x]$ is basic itself. This contradicts (a). □

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Declarations

Data Availability Statement None

Conflict of interest None

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