Robust Data-Driven Predictive Control for Linear Time-Varying Systems

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***Abstract*— This paper presents a new robust data-driven predictive control scheme for linear time-varying (LTV) systems, in which the nominal system model is unknown. To tackle the challenges arising from the unknown nominal model and the time-varying nature of the system, a data-dependent optimization problem is formulated using input-state-output data. This optimization problem calculates an upper bound on the objective function and designs a state feedback controller to minimize this bound. Moreover, two significant concerns, namely the feasibility of the optimization problem and the stability of the closed-loop system under the designed controller, are thoroughly investigated. Compared with the existing data-enabled predictive control method for LTV systems, our proposed control scheme does not require the collected data to satisfy the persistently exciting (PE) condition and uniformly exponentially stabilizes the controlled system with guarantees. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed method.**

***Index Terms*— Data-Driven Control, Predictive Control, Linear Time-Varying Systems**

I. Introduction

MODEL predictive control (MPC) is an essential control scheme for dynamic systems that has gained immense popularity in scientific research and various industrial applications since its inception [1]. At each time step, MPC solves an optimization problem to determine the optimal control input by considering the predicted trajectory of the controlled system [2]. Over the years, numerous MPC methods have been developed for systems ranging from linear to nonlinear systems [2], [3]. However, most of these methods require a system model, which can be challenging to obtain, especially for complex systems [4], limiting their practical application.

In recent years, cutting-edge techniques such as big data and artificial intelligence have sparked considerable interest in data-driven control methods [5]. Unlike traditional model-based control methods that heavily depend on explicit model information, data-driven control methods focus on learning controllers directly from data [6]. The emergence of data-driven control methods offers a fresh perspective on solving the model issue of MPC methods. Several results have been reported in recent literature [7]–[12]. For linear time-invariant (LTI) systems, multiple data-driven predictive control methods are developed in [7]–[9] using Willems' fundamental lemma [13], which require the input sequence to satisfy the persistently exciting (PE) condition. To relax this requirement, a robust data-driven predictive control method is developed in [10] using the matrix Finsler's lemma [14]. For nonlinear systems, two data-driven predictive tracking control methods are designed in [11] and [12], respectively. The former approximates the local behavior of the nonlinear system using the online collected data, while the latter approximates the global behavior of nonlinear systems by applying the Koopman operator [15]. Nevertheless, the design of data-driven predictive controllers, particularly for nonlinear systems, is still relatively limited and deserves further research.

On the other hand, linear time-varying (LTV) systems represent a significant type of nonlinear system that arises in various practical problems, such as the linearization of nonlinear systems around a time-varying operating point [16]. As mentioned, model-based control methods for LTV systems also face the model issue, which leads to the development of data-driven control methods [17]–[19]. For example, a data-driven state-feedback controller is designed for LTV systems in [17]. In addition, a data-driven predictive controller, also called a data-enabled predictive controller, has been developed for LTV systems in [19]. This controller dynamically updates the data matrices online to approximate the system's current behavior, similar to the approach presented in [11]. Furthermore, this controller is developed based on Willems' fundamental lemma, which requires the input sequence to satisfy the PE condition of a sufficiently high order. However, collecting enough long data satisfying the PE condition may be challenging, particularly for unstable systems [20].

This paper proposes a new robust data-driven control scheme for LTV systems whose nominal model is unknown. Our approach is inspired by the robust model predictive control scheme in [2], which addresses the time-varying dynamics by calculating an upper bound on the objective function and designing a state feedback controller to minimize this bound at each time step. However, this approach relies on the known nominal model, rendering it inapplicable to situations where the nominal model is unknown. To address this issue, a data-based optimization problem is formulated using the pre-collected input-state-output data, which can achieve the two goals of the robust model predictive control scheme at each time step. Furthermore, the data-based optimization problem is feasible at each time step if it is feasible at the initial time

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step. Building upon this result, the proposed control scheme is constructed by solving the data-based optimization problem at each time step.

The rest of this paper is structured as follows. Section II states the problem formulation. Section III proposes a robust data-driven predictive control scheme for LTV systems. The effectiveness of the proposed method is illustrated by a numerical example in Section IV. Finally, Section V concludes this paper.

Notation: Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the set of natural numbers, integers, real numbers, and positive real numbers, respectively. I_n is the identity matrix with dimensions $n \times n$. $0_{n\times m}$ is a zero matrix with dimensions $n\times m$. The subscripts of I_n and $0_{n\times m}$ may be omitted if the dimensions are evident from the context. Given a matrix M, let $M^\top$ denote its transpose, M^{-1} denote its inverse if it is nonsingular, $\sigma_{\min}(M)$ and $\sigma_{\max}(M)$ denote its minimum and maximum eigenvalue if it is square, respectively, and $M < 0$ ($M \le 0$) means it is negative (semi)-definite. $\text{diag}(M_1, \dots, M_s)$ denotes a matrix where the square matrices $M_1, \dots, M_s$ are arranged on the diagonal. For a signal $z(i) : \mathbb{Z} \to \mathbb{R}^n$, define $\|z(i)\|_2 = \sqrt{z(i)^\top z(i)}$, and $z_{[i,j]} = [z(i)^\top, \dots, z(j)^\top]^\top$, where $i, j \in \mathbb{Z}$ and $i < j$.

II. Preliminaries and Problem Formulation

Consider the following discrete-time LTV system

$$
\begin{aligned}
x(k+1) &= A(k)x(k) + B(k)u(k), && \text{(1a)}\\
y(k) &= C(k)x(k) + D(k)u(k), && \text{(1b)}\\
&[A(k), B(k), C(k), D(k)] \in \Omega,
\end{aligned}
$$

where $u(k) \in \mathbb{R}^m$, $x(k) \in \mathbb{R}^n$, and $y(k) \in \mathbb{R}^p$ are the system input, state, and output, respectively; $A(k) \in \mathbb{R}^{n\times n}$, $B(k) \in \mathbb{R}^{n\times m}$, $C(k) \in \mathbb{R}^{p\times n}$, and $D(k) \in \mathbb{R}^{p\times m}$ are system matrices. In addition, Ω is a convex hull of a set of vertices, specifically, $Co\{[A_1, B_1, C_1, D_1], \dots, [A_s, B_s, C_s, D_s]\}$, where Co denotes the convex hull, and $[A_i, B_i, C_i, D_i]$, $\forall i \in \mathcal{N} = \{1, \dots, s\}$, are the vertices. Any $[A, B, C, D]$ in the convex set Ω is a linear combination of the vertices, i.e., $A = \sum_{i=1}^{s} \lambda_i A_i$, $B = \sum_{i=1}^{s} \lambda_i B_i$, $C = \sum_{i=1}^{s} \lambda_i C_i$, and $D = \sum_{i=1}^{s} \lambda_i D_i$ with $\sum_{i=1}^{s} \lambda_i = 1$, and $\lambda_i \ge 0$.

Without loss of generality, the following two assumptions are made for the system (1).

Assumption 1: The dimensions of the matrices A_i, B_i, C_i, and D_i, $\forall i \in \mathcal{N}$, are known, i.e., m, n, and p are known. However, the entries of these matrices are unknown.

Assumption 2: For each vertex i, $\forall i \in \mathcal{N}$, a length $T_i \ge n+m$ input-state-output trajectory can be pre-collected where the system (1) satisfies $A(k) = A_i$, $B(k) = B_i$, $C(k) = C_i$, and $D(k) = D_i$ for T_i time steps.

The first assumption is widely adopted in the data-driven control literature [6], [7]. The second assumption is similar to the requirement for both identifying the model of LTV systems [21] and designing data-driven controllers for LTV systems [17]. Furthermore, numerous practical systems meet this assumption, such as transport aircraft, which maintain an unchanged dynamic model before and after airdrop.

If the system matrices (A_i, B_i, C_i, D_i), $\forall i \in \mathcal{N}$ are known, a robust model predictive control scheme can be formulated for the LTV system (1), drawing inspiration from [2]. At each time k, a state feedback controller $u(k+\iota|k) = F(k)x(k+\iota|k)$, $\iota \in \mathbb{N}$, is designed to minimize an upper bound on the following robust performance objective

$$
\max_{[A(k+\iota), B(k+\iota), C(k+\iota), D(k+\iota)] \in \Omega, \iota \in \mathbb{N}} J(k), \tag{2}
$$

where

$$
J(k) = \sum\nolimits_{\iota=0}^{\infty} \left\|Q^{\frac{1}{2}} y(k+\iota|k)\right\|_2^2 + \left\|R^{\frac{1}{2}} u(k+\iota|k)\right\|_2^2; \tag{3}
$$

$Q \in \mathbb{R}^{p\times p}$ and $R \in \mathbb{R}^{m\times m}$ are weight matrices satisfying $Q > 0$ and $R > 0$, respectively; $F(k) \in \mathbb{R}^{m\times n}$ is the control gain matrix at time k; $x(k|k) = x(k)$ is the measured value of the system state at time k; $x(k+\iota|k)$, $\forall \iota \ge 1$ are the predicted value of the system state at time $k+\iota$ based on the measurement $x(k)$; similarly, $y(k+\iota|k)$, $\forall \iota \in \mathbb{N}$ are the predicted value of the system output at time $k+\iota$ based on $x(k)$; $u(k+\iota|k)$, $\forall \iota \in \mathbb{N}$ are the calculated control inputs based on $x(k)$ but only $u(k|k)$ is applied to the system (1).

Since the matrices (A_i, B_i, C_i, D_i), $\forall i \in \mathcal{N}$ are unknown, the robust model predictive control scheme is inapplicable. Traditionally, this problem can be addressed by using pre-collected data to identify the system model. However, this paper proposes a new approach to tackle this issue, which bypasses the identification step and directly learns the controller from the pre-collected data. This paper aims to develop a new robust data-driven predictive control scheme for the system (1). In some way, this scheme can be seen as a data-driven version of the robust model predictive control scheme, utilizing pre-collected data instead of the model information.

Before ending this section, a lemma is reviewed for future reference, which regards the criterion for uniform exponential stability of LTV systems [22], [23].

Lemma 1 ([22], [23]): The system (1) with $u(k) = 0$, $k \in \mathbb{N}$, is uniformly exponentially stable if and only if any one of the following statements is true

1) There exist positive definite matrices $P(k) \in \mathbb{R}^{n\times n}$, $k \in \mathbb{N}$, and positive scalars ν, η and ρ such that

$$
\begin{aligned}
A(k)^\top P(k+1)A(k) - P(k) &\le -\nu I_n,\\
\eta I_n \le P(k) &\le \rho I_n.
\end{aligned}
$$

2) There exist positive definite matrices $P(k) \in \mathbb{R}^{n\times n}$, $k \in \mathbb{N}$, uniformly exponentially stable function $\mu(k) : k \to \mathbb{R}^+$, and positive scalars η and ρ such that

$$
\begin{aligned}
A(k)^\top P(k+1)A(k) &\le \mu(k)^2 P(k),\\
\eta I_n \le P(k) &\le \rho I_n.
\end{aligned}
$$

III. Robust Data-Driven Predictive Control Scheme

This section presents a comprehensive approach to the development of a robust data-driven predictive control scheme for the system (1). Firstly, a data-based optimization problem is formulated to obtain an upper bound on the robust performance objective (2) and design a state feedback controller that minimizes this bound. Then, it is proved that the data-based optimization problem is feasible at each time step if it is feasible at the initial time. Building upon this result, the robust data-driven predictive control scheme is constructed by solving the data-based optimization problem at each time step. Finally, the stability of the closed-loop system under the designed controller is proved.

Under Assumption 2, for each vertex i, $\forall i \in \mathcal{N}$, an input-state-output trajectory of the system (1) is collected, denoted as, $(u_{i,[0,T_i-1]}, x_{i,[0,T_i]}, y_{i,[0,T_i-1]})$. Then, we define the following data matrices

$$\begin{aligned} X_i &= [x_i(0), \ldots, x_i(T_i-1)], X_{i,+} = [x_i(1), \ldots, x_i(T_i)], \\ U_i &= [u_i(0), \ldots, u_i(T_i-1)], Y_i = [y_i(0), \ldots, y_i(T_i-1)], \end{aligned}$$

which, together with (1), implies

$$X_{i,+} = A_i X_i + B_i U_i, \tag{4a}$$

$$Y_i = C_i X_i + D_i U_i. \tag{4b}$$

Then, the results of the upper bound and the state feedback controller are summarized in the following theorem.

Theorem 1 (Upper bound): Consider the system (1) under Assumptions 1-2. Given a scalar γ satisfying $0 < \gamma < 1$, if there exist a scalar $\eta_k = \eta(k) > 0$, matrices $S_{i,k} = S_i(k) \in \mathbb{R}^{T_i \times n}$, $\forall i \in \mathcal{N}$ at time k such that the following optimization problem (5) is feasible,

$$\min_{\eta_k, S_{i,k}, \forall i \in \mathcal{N}} \quad \eta_k \tag{5a}$$

$$\text{s.t.}$$

$$\begin{bmatrix} -\eta_k & x(k|k)^\top \\ x(k|k) & -\gamma^{-1} X_i S_{i,k} \end{bmatrix} \leq 0, \forall i \in \mathcal{N}, \tag{5b}$$

$$\begin{bmatrix} -\gamma X_i S_{i,k} & S_{i,k}^\top X_{i,+}^\top & S_{i,k}^\top Y_i^\top & S_{i,k}^\top U_i^\top \\ X_{i,+} S_{i,k} & -X_i S_{i,k} & 0 & 0 \\ Y_i S_{i,k} & 0 & -Q^{-1} & 0 \\ U_i S_{i,k} & 0 & 0 & -R^{-1} \end{bmatrix} \leq 0, \forall i \in \mathcal{N}, \tag{5c}$$

$$U_1 S_{1,k} = \cdots = U_s S_{s,k}, \tag{5d}$$

$$X_1 S_{1,k} = \cdots = X_s S_{s,k}, \tag{5e}$$

$$X_i S_{i,k} > 0, \forall i \in \mathcal{N}, \tag{5f}$$

then $x(k|k)^\top \gamma (X_i S_{i,k})^{-1} x(k|k)$ for any $i \in \mathcal{N}$ is an upper bound on the robust performance objective (2). Further, the upper bound is minimized by the state feedback controller $u(k+\iota|k) = F_k x(k+\iota|k)$, $\iota \in \mathbb{N}$ with the control gain matrix $F_k = U_i S_{i,k} (X_i S_{i,k})^{-1}$ for any $i \in \mathcal{N}$.

Proof: Under the state feedback controller $u(k+\iota|k) = F_k x(k+\iota|k)$, $\iota \in \mathbb{N}$, the closed-loop system is given by

$$x(k+\iota+1|k) = \bar{A}_{k+\iota|k} x(k+\iota|k), \tag{6a}$$

$$y(k+\iota|k) = \bar{C}_{k+\iota|k} x(k+\iota|k), \tag{6b}$$

where $\bar{A}_{k+\iota|k} = A_{k+\iota|k} + B_{k+\iota|k} F_k$ and $\bar{C}_{k+\iota|k} = C_{k+\iota|k} + D_{k+\iota|k} F_k$ with $[A_{k+\iota|k}, B_{k+\iota|k}, C_{k+\iota|k}, D_{k+\iota|k}]$ being the predicted matrices of $[A(k+\iota), B(k+\iota), C(k+\iota), D(k+\iota)]$ at time $k+\iota$.

According to (5e), we define $P_k = (X_1 S_{1,k})^{-1} = \cdots = (X_s S_{s,k})^{-1}$, where $P_k \in \mathbb{R}^{n \times n}$ is positive definite due to (5f). From (5d) and (5e), the control gain matrix F_k satisfies

$$F_k = U_1 S_{1,k} (X_1 S_{1,k})^{-1} = \cdots = U_s S_{s,k} (X_s S_{s,k})^{-1}. \tag{7}$$

Substituting $P_k = (X_i S_{i,k})^{-1}$, $\forall i \in \mathcal{N}$, into (7) gives

$$F_k = U_i S_{i,k} P_k, \forall i \in \mathcal{N}. \tag{8}$$

Define $\bar{A}_{i,k} \in \mathbb{R}^{n \times n}$ and $\bar{C}_{i,k} \in \mathbb{R}^{p \times n}$, $\forall i \in \mathcal{N}$ as

$$\bar{A}_{i,k} = A_i + B_i F_k, \tag{9a}$$

$$\bar{C}_{i,k} = C_i + D_i F_k. \tag{9b}$$

Substituting (8) into (9a) gives

$$\bar{A}_{i,k} = [A_i, B_i] \begin{bmatrix} I_n \\ U_i S_{i,k} P_k \end{bmatrix} = [A_i, B_i] \begin{bmatrix} P_k^{-1} P_k \\ U_i S_{i,k} P_k \end{bmatrix}. \tag{10}$$

Further, substituting $P_k^{-1} = X_i S_{i,k}$, $\forall i \in \mathcal{N}$, into (10) gives

$$\bar{A}_{i,k} = [A_i, B_i] \begin{bmatrix} X_i \\ U_i \end{bmatrix} S_{i,k} P_k, \forall i \in \mathcal{N}, \tag{11}$$

which, together with (4a), gives

$$\bar{A}_{i,k} = X_{i,+} S_{i,k} P_k, \forall i \in \mathcal{N}. \tag{12}$$

Similarly, substituting (8), $P_k^{-1} = X_i S_{i,k}$, $\forall i \in \mathcal{N}$, and (4b) into (9b) yields

$$\bar{C}_{i,k} = Y_i S_{i,k} P_k, \forall i \in \mathcal{N}. \tag{13}$$

In addition, pre- and post-multiplying (5c) with diag(P_k, I_n, I_p, I_m), and substituting $P_k^{-1} = X_i S_{i,k}$, $\forall i \in \mathcal{N}$ into the resulting expression yield

$$\begin{bmatrix} -\gamma P_k & P_k S_{i,k}^\top X_{i,+}^\top & P_k S_{i,k}^\top Y_i^\top & P_k S_{i,k}^\top U_i^\top \\ X_{i,+} S_{i,k} P_k & -P_k^{-1} & 0 & 0 \\ Y_i S_{i,k} P_k & 0 & -Q^{-1} & 0 \\ U_i S_{i,k} P_k & 0 & 0 & -R^{-1} \end{bmatrix} \leq 0 \tag{14}$$

for all $i \in \mathcal{N}$. Substituting (8), (12) and (13) into (14) gives

$$M_{i,k} \leq 0, \forall i \in \mathcal{N}, \tag{15}$$

where $M_{i,k} = \begin{bmatrix} -\gamma P_k & \bar{A}_{i,k}^\top & \bar{C}_{i,k}^\top & F_k^\top \\ \bar{A}_{i,k} & -P_k^{-1} & 0 & 0 \\ \bar{C}_{i,k} & 0 & -Q^{-1} & 0 \\ F_k & 0 & 0 & -R^{-1} \end{bmatrix}$.

Since, at each time k, the system matrices (A_k, B_k, C_k, D_k) is a linear combination of the vertices (A_i, B_i, C_i, D_i), $\forall i \in \mathcal{N}$, we have $A_k = A(k) = \sum_{i=1}^s \lambda_{i,k} A_i$, $B_k = B(k) = \sum_{i=1}^s \lambda_{i,k} B_i$, $C_k = C(k) = \sum_{i=1}^s \lambda_{i,k} C_i$, and $D_k = D(k) = \sum_{i=1}^s \lambda_{i,k} D_i$, where $\lambda_{i,k} \in \mathbb{R}$, $i = 1, \ldots, s$ could be any values satisfying $\sum_{i=1}^s \lambda_{i,k} = 1$ and $\lambda_{i,k} \geq 0$, $\forall i \in \mathcal{N}$.

Combining $M_{i,k} \leq 0$ in (15) and $\lambda_{i,k} \geq 0$, $\forall i \in \mathcal{N}$, yields

$$\sum\nolimits_{i=1}^s \lambda_{i,k} M_{i,k} \leq 0. \tag{16}$$

Substituting $M_{i,k}$ right defined in (15) into (16), and further substituting (9a)-(9b), $\bar{A}_k = A_k + B_k F_k$, and $\bar{C}_k = C_k + D_k F_k$ into the resulting equation yield

$$\begin{bmatrix} -\gamma P_k & \bar{A}_k^\top & \bar{C}_k^\top & F_k^\top \\ \bar{A}_k & -P_k^{-1} & 0 & 0 \\ \bar{C}_k & 0 & -Q^{-1} & 0 \\ F_k & 0 & 0 & -R^{-1} \end{bmatrix} \leq 0. \tag{17}$$

Applying Schur complement [24] to (17) gives

$$\bar{A}_k^\top P_k \bar{A}_k - \gamma P_k \leq -\bar{C}_k^\top Q \bar{C}_k - F_k^\top R F_k. \tag{18}$$

Since (16) holds for all $\lambda_{i,k}$, $i \in \mathcal{N}$ satisfying $\sum_{i=1}^{s} \lambda_{i,k} = 1$ and $\lambda_{i,k} \geq 0$, $\forall i \in \mathcal{N}$, both (17) and (18) holds for all $[A_k, B_k, C_k, D_k] \in \Omega$. Therefore, (18) holds for all $[A_{k+\iota|k}, B_{k+\iota|k}, C_{k+\iota|k}, D_{k+\iota|k}] \in \Omega$, $\iota \in \mathbb{N}$, i.e.,

$$\bar{A}_{k+\iota|k}^{\top} P_k \bar{A}_{k+\iota|k} - \gamma P_k \leq -\bar{C}_{k+\iota|k}^{\top} Q \bar{C}_{k+\iota|k} - F_k^{\top} R F_k, \iota \in \mathbb{N},$$

which is equivalent to

$$\begin{aligned} &x(k+\iota|k)^{\top}(\bar{A}_{k+\iota|k}^{\top} P_k \bar{A}_{k+\iota|k} - \gamma P_k) x(k+\iota|k) \\ \leq &- x(k+\iota|k)^{\top}(\bar{C}_{k+\iota|k}^{\top} Q \bar{C}_{k+\iota|k} + F_k^{\top} R F_k) x(k+\iota|k) \end{aligned} \tag{19}$$

for all $\iota \in \mathbb{N}$. Substituting (6a), (6b) and $u(k+\iota|k) = F_k x(k+\iota|k)$, $\iota \in \mathbb{N}$, into (19) gives

$$\begin{aligned} &x(k+\iota+1|k)^{\top} P_k x(k+\iota+1|k) - x(k+\iota|k)^{\top} \gamma P_k x(k+\iota|k) \\ \leq &- y(k+\iota|k)^{\top} Q y(k+\iota|k) - u(k+\iota|k)^{\top} R u(k+\iota|k) \end{aligned} \tag{20}$$

for all $\iota \in \mathbb{N}$. Summing (20) from $\iota = 0$ to $\iota = \infty$ yields

$$J(k) \leq x(k|k)^{\top} \gamma P_k x(k|k),$$

which holds for all $[A_{k+\iota|k}, B_{k+\iota|k}, C_{k+\iota|k}, D_{k+\iota|k}] \in \Omega$, $\iota \in \mathbb{N}$. It follows that

$$\max_{[A_{k+\iota}, B_{k+\iota}, C_{k+\iota}, D_{k+\iota}] \in \Omega, \iota \in \mathbb{N}} J(k) \leq x(k|k)^{\top} \gamma P_k x(k|k).$$

Therefore, $x(k|k)^{\top} \gamma P_k x(k|k)$ is an upper bound on the robust performance objective (2).

Further, applying Schur complement to (5b) and substituting $P_k = (X_i S_{i,k})^{-1}$, $\forall i \in \mathcal{N}$ into the result give

$$x(k|k)^{\top} \gamma P_k x(k|k) \leq \eta_k. \tag{21}$$

In addition, since the objective function (5a) is convex, and the constraints (5b)-(5f) are LMIs, the optimization problem (5) is convex. Hence, if (5) is feasible, the objective function η_k is minimized. Based on (21), the upper bound $x(k|k)^{\top} \gamma P_k x(k|k)$ is also minimized by the designed state feedback controller. ■

Remark 1: In the problem (5), if only the constraints (5d)-(5f) are considered, a sufficient condition to ensure their feasibility is that the matrices U_i and X_i satisfy

$$\operatorname{rank}\big([U_i^{\top}, X_i^{\top}]^{\top}\big) = n + m, \forall i \in \mathcal{N}. \tag{22}$$

Remark 2: The condition (22) can be theoretically guaranteed by using Willems' fundamental lemma [13]. According to this lemma, if the input sequence $u_{i,[0,T_i-1]}$ is PE of order $n+1$, and the pair (A_i, B_i) is controllable, then (22) is satisfied. The minimum length of pre-collected data required to satisfy the PE condition is $(n+1)m + n$ [6]. However, in practice, it is possible to satisfy (22) using pre-collected data of length $m+n$, i.e., $T_i \geq n+m$, as assumed in Assumption 2. Therefore, the condition (22) can be checked directly, for which $T_i \geq n+m$ is its necessary condition.

In order to establish a robust data-driven predictive control scheme, it is necessary to solve the problem (5) at each time step. Consequently, it becomes crucial to investigate whether the problem (5) is feasible at each time $k \in \mathbb{N}$. The answer to this question is provided in the following theorem.

Theorem 2 (Feasibility): Under the same conditions as Theorem 1, if there exist a scalar $\eta_0 > 0$, matrices $S_{i,0} \in \mathbb{R}^{T_i \times n}$, $\forall i \in \mathcal{N}$ such that the problem (5) is feasible at time $k = 0$, then the problem (5) is feasible at any time $k \in \mathbb{N}$.

Proof: The feasibility of the problem (5) at time $k = 0$ implies that LMIs (5b)-(5f) are feasible at time $k = 0$. Applying Schur complement to (5b) with $k = 0$ gives

$$x(0|0)^{\top} \gamma (X_i S_{i,0})^{-1} x(0|0) \leq \eta_0, \forall i \in \mathcal{N}. \tag{23}$$

From (19) with $k = \iota = 0$, and $P_0 = (X_i S_{i,0})^{-1}$, we get

$$\begin{aligned} &x(0|0)^{\top}(\bar{A}_{0|0}^{\top}(X_i S_{i,0})^{-1} \bar{A}_{0|0} - \gamma (X_i S_{i,0})^{-1}) x(0|0) \\ \leq &- x(0|0)^{\top}(\bar{C}_{0|0}^{\top} Q \bar{C}_{0|0} + F_0^{\top} R F_0) x(0|0), \forall i \in \mathcal{N}, \end{aligned} \tag{24}$$

where $\bar{A}_{0|0} = A_{0|0} + B_{0|0} F_0$ and $\bar{C}_{0|0} = C_{0|0} + D_{0|0} F_0$, $\forall [A_{0|0}, B_{0|0}, C_{0|0}, D_{0|0}] \in \Omega$. Since $[A_0, B_0, C_0, D_0] \in \Omega$, (24) holds for $[A_0, B_0, C_0, D_0]$, which implies

$$\begin{aligned} &x(1|1)^{\top}(X_i S_{i,0})^{-1} x(1|1) - x(0|0)^{\top} \gamma (X_i S_{i,0})^{-1} x(0|0) \\ \leq &- y(0|0)^{\top} Q y(0|0) - u(0|0)^{\top} R u(0|0), \forall i \in \mathcal{N}, \end{aligned} \tag{25}$$

where $x(1|1) = (A_0 + B_0 F_0) x(0|0)$, $y(0|0) = (C_0 + D_0 F_0) x(0|0)$, and $u(0|0) = F_0 x(0|0)$.

Combining (23), (25) and $0 < \gamma < 1$ yields

$$x(1|1)^{\top} \gamma (X_i S_{i,0})^{-1} x(1|1) \leq \gamma \eta_0 \leq \eta_0, \forall i \in \mathcal{N}. \tag{26}$$

Therefore, $(\eta_0, S_{i,0}, \forall i \in \mathcal{N})$ is a solution to LMI (5b) at time $k = 1$. In addition, if only LMIs (5c)-(5f) are considered, $(\eta_0, S_{i,0}, \forall i \in \mathcal{N})$ is also a solution to them at time $k = 1$. It follows that $(\eta_0, S_{i,0}, \forall i \in \mathcal{N})$ is a solution to LMIs (5b)-(5f) at time $k = 1$. As a result, the problem (5) is feasible at time $k = 1$. Repeating the above process shows that the problem (5) is feasible at any time $k \in \mathbb{N}$. ■

Utilizing Theorems 1-2, a robust data-driven predictive control scheme can be formulated for the system (1). The scheme is outlined in Algorithm 1.

Algorithm 1: Robust data-driven predictive control scheme

1 Pre-collect Input-state-output sequence and set $k = 0$;
2 Let $x(k|k) = x(k)$ and solve (5);
3 Set $F(k) = U_i S_{i,k} (X_i S_{i,k})^{-1}$ for any $i \in \mathcal{N}$;
4 Apply the input $u(k|k) = F(k) x(k|k)$;
5 Set $k = k + 1$ and return to 2;

Before applying Algorithm 1, it is necessary to address another crucial question: whether the designed controller stabilizes the system (1). To answer this question and facilitate the subsequent stability analysis, a special case of Lemma 1 is introduced as follows:

Lemma 2: The system (1) with $u(k) = 0$ is uniformly exponentially stable if and only if there exist positive definite matrices $P(k) \in \mathbb{R}^{n \times n}$, a scalar $0 < \gamma < 1$, and positive scalars η and ρ such that $A(k)^{\top} P(k+1) A(k) \leq \gamma P(k)$ and $\eta I_n \leq P(k) \leq \rho I_n$.

Proof: The proof is straightforward by applying Lemma 1, and thus is omitted. ■

With the help of Lemma 2, the answer to the stability issue is summarized in the following theorem.

Theorem 3 (Stability): Under the same conditions as Theorem 1, if there exist a scalar $\eta_k > 0$, matrices $S_{i,k} \in \mathbb{R}^{T_i \times n}$, $\forall i \in \mathcal{N}$ such that the optimization problem (5) is feasible at each time $k \in \mathbb{N}$, then the state feedback controller $u(k) = F_k x(k)$ with $F_k = U_i S_{i,k}(X_i S_{i,k})^{-1}$ for any $i \in \mathcal{N}$ uniformly exponentially stabilizes the system (1).

Proof: Since $S_{i,1}$, $i \in \mathcal{N}$, is the optimal solution to the problem (5) at time $k = 1$, from (5b), we have

$$x(1)^\top \gamma (X_i S_{i,1})^{-1} x(1) \leq x(1)^\top \gamma (X_i S_{i,0})^{-1} x(1), \quad (27)$$

for all $i \in \mathcal{N}$ where $x(1) = x(1|1)$.

Combining (25) with (27) gives

$$\begin{aligned} &x(1)^\top (X_i S_{i,1})^{-1} x(1) - x(0)^\top \gamma (X_i S_{i,0})^{-1} x(0) \\ &\leq -y(0)^\top Q y(0) - u(0)^\top R u(0), \forall i \in \mathcal{N}, \end{aligned} \quad (28)$$

where $x(0) = x(0|0)$, $u(0) = u(0|0)$, and $y(0) = y(0|0)$.

Generalizing (28) to any time k and substituting $P_k = (X_i S_{i,k})^{-1}$, $i \in \mathcal{N}$ into the resulting expression yield

$$\begin{aligned} &x(k+1)^\top P_{k+1} x(k+1) - x(k)^\top \gamma P_k x(k) \\ &\leq -y(k)^\top Q y(k) - u(k)^\top R u(k). \end{aligned} \quad (29)$$

Substituting $x(k+1) = \bar{A}_k x(k)$, $y(k) = \bar{C}_k x(k)$, and $u(k) = F_k x(k)$ into (29) gives

$$\bar{A}_k^\top P_{k+1} \bar{A}_k - \gamma P_k \leq -\bar{C}_k^\top Q \bar{C}_k - F_k^\top R F_k, \quad (30)$$

which, together with $\bar{C}_k^\top Q \bar{C}_k + F_k^\top R F_k \geq 0$, implies

$$\bar{A}_k^\top P_{k+1} \bar{A}_k \leq \gamma P_k. \quad (31)$$

Define $\xi_{\min} = \min_{k \in \mathbb{N}} \sigma_{\min}(P_k)$ and $\xi_{\max} = \max_{k \in \mathbb{N}} \sigma_{\max}(P_k)$. It follows that

$$\xi_{\min} I_n \leq P_k \leq \xi_{\max} I_n, \forall k \in \mathbb{N}. \quad (32)$$

According to Lemma 2, since there exist positive definite matrices P_k, $k \in \mathbb{N}$, a scalar $0 < \gamma < 1$, and positive scalars $\xi_{\min}$ and $\xi_{\max}$ such that inequalities (31) and (32), the system (1) is uniformly exponentially stabilized by the designed state feedback controller. ■

Remark 3: The proposed method can be extended to incorporate typical input-output constraints, such as norm constraints on the input and output and peak constraints on each entry of the input and output [2]. The basic idea involves converting the input and output constraints into LMIs and incorporating them into the optimization problem (5). This aspect will be further investigated in future work.

Remark 4: Compared to the robust model predictive control scheme presented in [2], which relies on a known nominal model, our method removes the need for the model information and instead relies solely on pre-collected data.

Remark 5: The control gain matrix F_k in the state feedback controller $u(k) = F_k x(k)$ may take any value in $\mathbb{R}^{m \times n}$. In the case where the problem (5) is feasible, the data matrix X_i, $\forall i \in \mathcal{N}$, must have full row rank while U_i, $\forall i \in \mathcal{N}$, may or may not have full row rank. If U_i has full row rank, the term $U_i S_{i,k}(X_i S_{i,k})^{-1}$ can represent any value in $\mathbb{R}^{m \times n}$. Hence, the structure imposed on F_k is not conservative in this case. If U_i does not have full row rank, the term $U_i S_{i,k}(X_i S_{i,k})^{-1}$ can only describe a subset of $\mathbb{R}^{m \times n}$. Consequently, the structure imposed on F_k introduces some conservativeness in this case.

IV. Numerical Example

This section demonstrates the effectiveness of the proposed method through an LTV system that has the same form as (1), where $x(k) \in \mathbb{R}^4$, $u(k) \in \mathbb{R}^2$, and $y(k) \in \mathbb{R}^2$. The number of vertices is $s = 3$, and the corresponding matrices are

$$A_1 = \begin{bmatrix} 0.30 & -0.35 & 0.71 & 0.04 \\ -0.15 & 0.42 & 0.14 & 0.03 \\ 0.56 & 0.11 & -0.22 & 0.47 \\ 0.01 & -0.09 & 0.52 & 0.81 \end{bmatrix}, B_1 = \begin{bmatrix} -1.07 & 0.33 \\ -0.81 & -0.75 \\ -2.94 & 1.37 \\ 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -0.10 & 0.32 & 0 & -0.16 \\ -0.24 & 0 & -0.03 & 0.63 \end{bmatrix}, D_1 = 0,$$

$$A_2 = \begin{bmatrix} 0.19 & 0.44 & -0.42 & 0.39 \\ 0.20 & 0.31 & 0.53 & -0.22 \\ 0.59 & -0.30 & 0.07 & 0.32 \\ -0.01 & 0.47 & 0.24 & 0.33 \end{bmatrix}, B_2 = \begin{bmatrix} 2.91 & -0.47 \\ 0.83 & -0.27 \\ 1.38 & 1.10 \\ -1.06 & -0.28 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.70 & 0 & -1.58 & 0 \\ 0 & -0.82 & 0.51 & 0.03 \end{bmatrix}, D_2 = 0,$$

$$A_3 = \begin{bmatrix} 0.21 & 0.37 & 0.33 & 0.05 \\ 0.34 & 0.30 & 0.04 & -0.18 \\ 0.11 & 0.14 & 0.15 & 0.04 \\ -0.05 & -0.10 & 0.24 & 0.37 \end{bmatrix}, B_3 = \begin{bmatrix} -0.16 & -0.88 \\ -0.15 & -0.48 \\ -0.53 & -0.71 \\ 1.68 & -1.17 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} -0.19 & 1.53 & -1.06 & 1.23 \\ -0.27 & 0 & 0 & -0.23 \end{bmatrix}, D_3 = 0.$$

Since the eigenvalues of A_1 are not all within the unit circle, the investigated LTV system is potentially open-loop unstable. The matrices A_i, B_i, C_i, and D_i, $i = 1, 2, 3$, are presumed to be unknown and only used to build the unknown system in the simulation. The parameters λ_i, $i = 1, 2, 3$ are randomly generated at each time step satisfying $\sum_{i=0}^{3} \lambda_i = 1$ and $\lambda_i \geq 0$. Additionally, the measurement data is assumed to be noise-free. We choose $Q = 10 I_2$ and $R = I_2$ in the objective function $J(k)$, $\gamma = 0.8$ in the problem (5), and the initial state $x_0 = [0.68, 0.14, 0.72, 0.11]^\top$ in the simulation.

To implement our robust data-driven predictive (RDPC) method, we collect an input-state-output trajectory ($u_{i,[0,5]}$, $x_{i,[0,6]}$, $y_{i,[0,5]}$) for each vertex i, $i = 1, 2, 3$, where the input signals $u_{[0,5]}$ take random values in the interval $[-0.1, 0.1]$. We verify that the data matrices $U_i = [u_i(0), \ldots, u_i(5)]$ and $X_i = [x_i(0), \ldots, x_i(5)]$, $i = 1, 2, 3$, satisfy the condition (22). In addition, $T_i = 6$ is the minimum length we can find to make the problem (5) feasible. The collected inputs do not meet the PE condition of order $8 + 2L$, where L is the prediction horizon, which is essential for implementing the data-enabled predictive controller in [19].

Furthermore, we compare the proposed method with the robust model predictive control (RMPC) scheme presented in Section 4 of [2] using the same objective function. For a fair comparison, we also present simulation results of the RMPC scheme incorporating the parameter γ. The latter can also be regarded as an improved version of the controller with the minimum delay rate in Remark 7 of [2].

The input and output outcomes of the closed-loop system for both the proposed RDPC and RMPC with/without γ schemes are shown in Fig. 1. The simulation results for the RDPC are shown in blue, while those for the RMPC with and without γ are shown in dashed green and red, respectively. It can be observed that all control methods achieve the desired control targets. Compared to the RMPC with γ, the proposed RDPC exhibits the same control performance. Compared to the RMPC without γ, the RDPC has both advantages and disadvantages. In terms of inputs, u_1 of the RDPC requires less control effort than that of the RMPC, but conversely, u_2 of the RDPC requires more. As for the outputs, both y_1 and y_2 of the RDPC have a faster convergence rate than that of the RMPC, but y_1 of the RDPC has a larger overshoot. We also consider the objective function values over the simulation time, denoted as $\hat{J} = \sum_{k=0}^{25} \|Q^{\frac{1}{2}} y(k)\|_2^2 + \|R^{\frac{1}{2}} u(k)\|_2^2$, which are shown in Table I. It demonstrates that the objective function value of the RDPC is equivalent to that of the RMPC with γ while being smaller than that of the RMPC without γ. Finally, it is worth noting that the RDPC and RMPC with γ having the same behavior may be attributed to two factors. Firstly, the pre-collected data can uniquely determine the system model. Secondly, both control schemes are designed based on the same Lyapunov function.

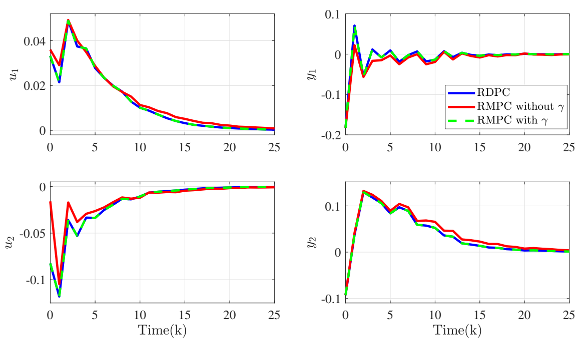

Fig. 1: The input and output outcomes of the closed-loop system under the RDPC and RMPC schemes.

TABLE I: The objective function values of the RDPC and RMPC schemes.

Methods	RMPC without γ	RMPC with γ	RDPC
$\hat{J}$	1.4693	1.3603	1.3603

V. Conclusion

This paper has proposed a new RDPC scheme for LTV systems, where the nominal system model is unknown. To tackle the challenges arising from the unknown nominal model and the time-varying characteristics of the system, a data-dependent optimization problem has been formulated. This optimization problem calculates an upper bound on the robust performance objective and designs a state feedback controller to minimize this bound. Furthermore, two significant concerns, namely the feasibility of the optimization problem and the stability of the closed-loop system under the designed controller, have been thoroughly investigated. Finally, a numerical example has been provided to demonstrate the effectiveness of the proposed method. Overall, the proposed method has two main advantages. Firstly, it does not rely on the explicit model information compared to the RMPC scheme. Secondly, it relies on less data than the data-enabled predictive control scheme based on Willems' fundamental lemma. On the other hand, the proposed method may exhibit conservatism as it designs a common controller for all vertices. However, this could be improved by designing distinct controllers for different vertices, which will be studied in future work. Moreover, future work will also examine extending the proposed method to incorporate the measurement noise.

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