Robust LQRs synthesis for structured uncertain systems: the WDLF and the CI approaches

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***Abstract*—This paper addresses the design of robust linear quadratic regulators (LQRs) for systems affected polynomially by uncertainty constrained in a semialgebraic set. The problem consists of determining a feedback controller that ensures a desired upper bound on the worst-case value of a quadratic cost. Two linear matrix inequality (LMI) approaches are proposed, the first one based on the construction of a Lyapunov function that weakly depends on the uncertainty, and the second one based on the construction of an index that quantifies the feasibility of different controllers. The proposed approaches have two main advantages with respect to the existing methods, namely, considering not only state-feedback design for polytopic systems but also output-feedback design for systems depending polynomially on the uncertainty, and providing conditions that are not only sufficient but also necessary under some assumptions. These advantages are illustrated through various examples, where it is shown that the existing methods may be more conservative or may be not applicable.**

***Index Terms*—LQR; Uncertain system; Polynomial dependence; Output feedback; Robustness; LMI.**

I. Introduction

An important problem in control systems consists of designing feedback controllers that ensure stability while minimizing a quadratic cost, which usually represents a weighted sum of the energies of the signals over an infinite horizon. For linear time-invariant (LTI) systems and in the case of state-feedback, such controllers are known as LQRs, and can be found by solving the algebraic Riccati equation (ARE), which is a quadratic matrix equation. See, e.g., [1]–[4].

There have been numerous studies about LQRs. In particular, the solution set of the ARE and the algebraic Riccati inequality have been investigated in [5], the case of singular control problems has been studied in [6], [7], the presence of time-varying components has been addressed in [8], [9], the case of nonlinear systems has been considered in [10], the use of random input gains has been investigated in [11], and the design of parametric LQRs for parametric systems has been proposed in [12].

A key problem that has been solved only partially consists of designing robust LQRs for uncertain systems, i.e., LQRs that ensure a desired upper bound on the worst-case value of the quadratic cost (where worst-case means supremum with respect to the admissible uncertainties). This problem is important because real systems are unavoidably affected by uncertainties, for instance because some physical quantities cannot be measured exactly or are allowed to change. Moreover, this problem is challenging for several reasons. Firstly, solving the ARE for an uncertain system would provide an LQR that depends on the uncertainties rather than an uncertainty-free controller, moreover, the ARE cannot be easily solved for uncertain systems. Secondly, exploiting the algebraic Riccati inequality with uncertainty-free Lyapunov functions would result in conservative solutions since such Lyapunov functions may be unable to verify robust stability of uncertain systems.

One of the pioneering works that have addressed this problem is [13], where an LMI method is proposed for optimal $\mathcal{H}_2$ norm control of polytopic systems (i.e., systems whose matrices are affine functions of an uncertain vector constrained in a convex bounded polytope). Another solution has been described in [14], where an LMI method is developed for polytopic systems with application to pulse width modulation (PWM) converters. Also, this problem has been addressed in [15], where output feedback LQR is designed for polytopic systems via a strategy based on the solution of nonlinear matrix inequalities through evolutionary algorithms and LMIs. While these methods have the nice feature to be mainly formulated in terms of LMIs, whose feasibility can be tested via convex optimization (see, e.g., [2]), they also have the drawbacks of considering only polytopic systems and providing conditions that are sufficient but not necessary.

Other related works include [16], [17] which propose the use of neural networks and reinforcement learning for adaptive optimal control, [18] which addresses robustness against disturbances based on tuned weighting matrices for the inverted pendulum, [19] which investigates the parameter dependence of a modified ARE for network synchronization, [20] which proposes the use of the total variation distance for designing a finite horizon LQR robust against disturbances, [21] which considers systems with norm bounded uncertainty and time delay and proposes a recursive solution for designing a finite horizon LQR based on an augmented Riccati equation, [22] which proposes the design of a finite horizon LQR for a sampled uncertain system based on the solution of two convex optimization problems, [23] which describes a procedure for robustifying an LQR designed for a nominal plant based on the use of suitable weighting matrices.

This paper addresses the design of robust LQRs for systems affected by structured uncertainties. The paper starts by considering continuous-time (CT) LTI systems where the system matrices are polynomial functions of an uncertain vector constrained in a semialgebraic set. The robust LQR problem consists of determining a feedback controller that ensures a desired upper bound on the worst-case value of a quadratic cost. Two approaches are proposed for dealing with this problem. The first one, weakly-dependent Lyapunov functions (WDLF) approach, is based on the construction of an uncertainty-dependent LQR obtained through Lyapunov functions that weakly depend on the uncertainties, followed

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by the extraction of an uncertainty-free controller. The second one, controller index (CI) approach, is based on the construction of an index that quantifies the feasibility of different controllers, followed by the extraction of a controller defined by such an index. A sufficient condition is derived for establishing whether the provided controller candidates solve the problem. Moreover, the necessity of this condition is investigated, providing assumptions for ensuring that the proposed approaches are asymptotically nonconservative. The extension of these approaches to the case of discrete-time (DT) LTI systems is hence presented. Both approaches are formulated through semidefinite programs (SDPs), which are convex optimization problems where a linear cost function is minimized subject to LMIs.

The proposed approaches have two main advantages with respect to the existing methods previously mentioned for addressing the robust LQR problem[1]. The first advantage is to consider a wider class of uncertain systems, where the matrices depend polynomially rather than affine on the uncertainty, the set of admissible uncertainties is a semialgebraic set rather than a convex bound polytope, and the output rather than the state is available for feedback. The second advantage is to provide conditions that are not only sufficient but also necessary under some assumptions. These advantages are illustrated through various examples in Section VII, where it is shown that the existing methods may be more conservative or may be not applicable.

The paper is organized as follows. Section II introduces the preliminaries. Section III describes the WDLF approach. Section IV describes the CI approach. Section V investigates the sufficiency and necessity of these approaches. Section VI addresses the extension to DT systems. Section VII presents the examples. Lastly, Section VIII reports the conclusions and future directions.

II. Preliminaries

This section introduces the preliminaries. In particular, Section II-A describes the problem formulation, and Section II-B reports some details about the Gram matrix.

A. Problem Formulation

The notation is as follows:

- "resp., s.t.": "respectively, subject to";
- $\mathbb{N}$ and $\mathbb{R}$: sets of nonnegative integers and real numbers;
- 0: null matrix of size specified by the context;
- I_n, I: identity matrices of size $n \times n$ and size specified by the context;
- $A \otimes B$: Kronecker's product of A, B;
- A^T: transpose of A;
- $\mathrm{he}(A)$: $A + A^T$;
- $\mathrm{tr}(A), \det(A)$: trace, determinant of A;
- $A \geq 0$ (resp., $A > 0$): symmetric positive semidefinite (resp., definite) A;
- $\|x\|_p$: p-norm of x;
- $\lceil x \rceil$: smallest integer greater than or equal to x;
- $\deg(A(x))$: degree of $A(x)$ in x;
- $\mathrm{conv}(A_1, A_2, \ldots)$: convex hull of $A_1, A_2, \ldots$;
- x_i: i-th entry of a vector x (unless specified otherwise).

[1]To the best knowledge of the author, [13]–[15] are the only existing methods that allow to solve the problem addressed in this paper.

Consider the CT LTI uncertain system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(p)x(t) + B(p)u(t) \\ y(t) & = & C(p)x(t) \\ x(0) & = & x_0 \\ p & \in & \mathcal{P} \end{array} \right. \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^r$ is the output, $x_0 \in \mathbb{R}^n$ is the initial condition, $p \in \mathbb{R}^q$ is the uncertainty, $A(p), B(p), C(p)$ are matrix polynomials, and $\mathcal{P}$ is the set of admissible uncertainties given by

$$\mathcal{P} = \{p \in \mathbb{R}^q : \ f_i(p) \geq 0, \ i = 1, \ldots, n_f\} \tag{2}$$

where $f(p) = (f_1(p), \ldots, f_{n_f}(p))^T$ is a vector polynomial.

Depending on $f(p)$, $\mathcal{P}$ can have various shapes of interest, such as:

- hypersphere, e.g.,

$$\mathcal{P} = \{p \in \mathbb{R}^q : \ \|p\|_2 \leq 1\}, \quad f(p) = 1 - \|p\|_2^2; \tag{3}$$

- hypercube, e.g.,

$$\mathcal{P} = \{p \in \mathbb{R}^q : \ \|p\|_\infty \leq 1\}, \quad f(p) = \begin{pmatrix} 1 - p_1^2 \\ \vdots \\ 1 - p_q^2 \end{pmatrix}; \tag{4}$$

- simplex, i.e.,

$$\mathcal{P} = \{p \in \mathbb{R}^q : \ p_1 + \ldots + p_q = 1, \ p_i \geq 0\}, \tag{5}$$

  which can be considered with a new variable $\tilde{p} \in \mathbb{R}^{\tilde{q}}$, $\tilde{q} = q - 1$, according to

$$p = \begin{pmatrix} \tilde{p} \\ 1 - \tilde{p}_1 - \ldots - \tilde{p}_{\tilde{q}} \end{pmatrix}, \quad f(\tilde{p}) = \begin{pmatrix} \tilde{p}_1 - \tilde{p}_1^2 \\ \vdots \\ \tilde{p}_{\tilde{q}} - \tilde{p}_{\tilde{q}}^2 \end{pmatrix}. \tag{6}$$

The LQR problem consists of solving

$$J_0(p) = \inf_{u(t)} \int_0^\infty \left(x^T(t)Qx(t) + u^T(t)Ru(t) \right) dt \tag{7}$$

where $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ are symmetric positive definite matrices. The control input that achieves $J_0(p)$ in the case of state-feedback (i.e., $C(p) = I_n$) is given by the control law

$$u(t) = K_0(p)x(t) \tag{8}$$

where $K_0(p) \in \mathbb{R}^{m \times n}$ is the controller

$$K_0(p) = -R^{-1}B(p)^T V_0(p) \tag{9}$$

and $V_0(p) \in \mathbb{R}^{n \times n}$ is the solution of the ARE

$$\begin{array}{rcl} 0 & = & Q + \mathrm{he}(V_0(p)A(p)) \\ & & -V_0(p)B(p)R^{-1}B(p)^T V_0(p), \end{array} \tag{10}$$

see, e.g., [2, Chapter 7], [4, Chapter 3].

Unfortunately, the controller obtained in this way, $K_0(p)$, is a function of the uncertainty. This means that, in order to implement such a controller, the real time measurement of the uncertainty is required. Clearly, this may be difficult or even impossible in real applications. Moreover, such a controller would require some computational capability in order to implement the expression (9). Last but not least, it is unclear if and how the ARE could be solved in order to obtain $V_0(p)$, since $A(p), B(p)$ depend on the uncertainty.

For these reasons, this paper addresses the problem of determining an uncertainty-free controller (if any) capable of ensuring a desired upper bound on the worst-case value of a quadratic cost, where worst-case means supremum with respect to the admissible uncertainties. Moreover, this controller is sought in the more general case of output-feedback rather than state-feedback.

Specifically, consider the system (1) controlled in closed-loop by

$$u(t) = Ky(t) \tag{11}$$

where $K \in \mathbb{R}^{m\times r}$ has to be determined, and introduce the worst-case cost

$$J^*(K) = \sup_{p\in\mathcal{P}} J(K,p) \tag{12}$$

where

$$\begin{array}{rcl} J(K,p) & = & \displaystyle\int_0^\infty \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt \\ & & \text{s.t.} \;\; u(t) = Ky(t). \end{array} \tag{13}$$

The problem addressed in this paper is formulated as follows.

Problem 1: (Robust LQR problem for CT systems) Consider the system (1) and worst-case cost (12). Given a scalar $\gamma > 0$, find K (if any) such that $J^*(K) < \gamma$. □

B. Gram Matrix

Here we briefly review the Gram matrix method for establishing whether a matrix polynomial is a sum of squares of matrix polynomials. The reader is referred to [24, Section 2], [25, Section III] and references therein for more details.

Let $E_i(p) \in \mathbb{R}^{n\times n}$, $i = 0,\ldots,m$, be symmetric matrix polynomials in $p \in \mathbb{R}^q$ of degree not greater than $2d$, $d \in \mathbb{N}$. Let $v \in \mathbb{R}^m$, and define

$$E(p) = E_0(p) + \sum_{i=1}^m v_i E_i(p). \tag{14}$$

Then, $E(p)$ is a symmetric matrix polynomial in p and affine in v, and can be expressed as

$$E(p) = \left(b(p)\otimes I_n\right)^T \left(F(v) + L(\alpha)\right)\left(b(p)\otimes I_n\right) \tag{15}$$

where $b(p) \in \mathbb{R}^\sigma$ is a vector polynomial whose entries form a basis for the polynomials in p of degree not greater than d, $F(v) \in \mathbb{R}^{\sigma\times\sigma}$ is a symmetric affine matrix function, $L(\alpha) \in \mathbb{R}^{\sigma\times\sigma}$ is a linear matrix function that parameterizes the set

$$\mathcal{L} = \left\{\tilde{L} = \tilde{L}^T : \;\; \left(b(p)\otimes I_n\right)^T \tilde{L}\left(b(p)\otimes I_n\right) = 0\right\}, \tag{16}$$

and $\alpha \in \mathbb{R}^\tau$ is a free vector. The quantities σ and τ are given by

$$\left\{\begin{array}{rcl} \sigma & = & \dfrac{(q+d)!}{q!d!} \\ \tau & = & \dfrac{1}{2}n\sigma(n\sigma+1) - \dfrac{1}{2}n(n+1)\dfrac{(q+2d)!}{q!(2d)!}. \end{array}\right. \tag{17}$$

For the case $n = 1$, $F(v)$ is said a Gram matrix of $E(p)$ with respect to $b(p)$, while the affine matrix function $F(v) + L(\alpha)$ parameterizes all such Gram matrices. For $n > 1$, these expressions generalize the Gram matrix method to the case of matrix polynomials.

The representation (15), also known as square matrix representation of matrix polynomials, is useful to establish if $E(p)$ is a sum of squares of matrix polynomials. Specifically, $E(p)$ is said a sum of squares of matrix polynomials for some v if there exist matrix polynomials $\tilde{E}_i(p) \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,k$, and v such that

$$E(p) = \sum_{i=1}^k \tilde{E}_i(p)^T \tilde{E}_i(p). \tag{18}$$

It turns out that $E(p)$ is a sum of squares of matrix polynomials for some v if and only if there exist α and v that satisfy the LMI

$$F(v) + L(\alpha) \geq 0. \tag{19}$$

In the sequel of this paper, the notation $E(p) \in \Sigma$ will be used to denote that $E(p)$ is a sum of squares of matrix polynomials. Moreover, the notation

$$E(p) \in \mathcal{C}(w(p), \delta), \tag{20}$$

where $w(p) \in \mathbb{R}^l$ is a vector polynomial and $\delta \in \mathbb{N}$, will be used to denote the existence of v and symmetric matrix polynomials $Y_i(p)$, $i = 1,\ldots,l$, of size equal to that of $E(p)$, such that

$$\left\{\begin{array}{l} Y_i(p), Z(p) \in \Sigma, \quad \forall i = 1,\ldots,l \\ \deg(Z(p)) \leq 2\left\lceil \dfrac{1}{2}\delta \right\rceil \end{array}\right. \tag{21}$$

where

$$Z(p) = E(p) - \sum_{i=1}^l w_i(p) Y_i(p). \tag{22}$$

III. WDLF Approach

The first approach proposed in this paper for solving Problem 1, and denoted as WDLF approach, consists of searching for an uncertainty-dependent controller that ensures the desired upper bound on the worst-case cost, followed by the extraction of an uncertainty-free controller from it. This is motivated by the fact that the search for such an uncertainty-dependent controller can be done through convex optimization in the case of state-feedback. Clearly, the obtained uncertainty-free controller might not ensure the desired upper bound on the worst-case cost. Hence, the idea is to search for an uncertainty-dependent controller that weakly depends on the uncertainty vector. Therefore, let us introduce the following assumption.

Assumption 1: The state and output of the system (1) coincide, i.e., $C(p) = I_n$. □

The WDLF approach is described hereafter through the following steps:

- definition of the Lyapunov-based matrix inequalities that provide an uncertainty-dependent controller;
- introduction of an uncertainty-dependence quantifier of such controller;
- definition of the SDP for determining a controller weakly dependent on the uncertainty;
- extraction of an uncertainty-free controller.

Let $\tilde{T}(p) \in \mathbb{R}^{m\times m}$, $\tilde{U}(p) \in \mathbb{R}^{m\times n}$ and $\tilde{V}(p) \in \mathbb{R}^{n\times n}$ be matrix functions to be determined, with $\tilde{T}(p), \tilde{V}(p)$ symmetric, and define

$$\begin{cases} \tilde{S}_1(p) = -\mathrm{he}\left(\tilde{V}(p)A(p)^T + B(p)\tilde{U}(p)\right) - x_0x_0^T \\ \tilde{S}_2(p) = \begin{pmatrix} \tilde{V}(p) \ , \ \tilde{U}(p)^T \\ \tilde{U}(p) \ , \ \tilde{T}(p) \end{pmatrix} \\ \tilde{S}_3(p) = \gamma - \mathrm{tr}\left(Q\tilde{V}(p)\right) - \mathrm{tr}\left(R\tilde{T}(p)\right). \end{cases} \quad (23)$$

From [2, Chapter 7], the existence of $\tilde{T}(p), \tilde{U}(p), \tilde{V}(p)$ such that $\tilde{S}_1(p), \tilde{S}_2(p), \tilde{S}_3(p)$ are positive definite over $\mathcal{P}$ implies that the uncertainty-dependent controller

$$\tilde{K}(p) = \tilde{U}(p)\tilde{V}(p)^{-1} \quad (24)$$

ensures the desired upper bound on the worst-case cost, i.e., $J^*(\tilde{K}(p)) < \gamma$.

Observe that the dependence of $\tilde{K}(p)$ on p is due to the product of $\tilde{U}(p)$ with $\tilde{V}(p)^{-1}$. While considering this product through convex optimization may be difficult or even impossible, a tractable way for reducing the dependence on p of $\tilde{K}(p)$ could be to reduce the dependence on p of $\tilde{U}(p)$ and $\tilde{V}(p)$. In particular, the latter means searching for a Lyapunov function that is weakly dependent on the uncertainty.

Hence, let $T(p) \in \mathbb{R}^{m\times m}$, $U(p) \in \mathbb{R}^{m\times n}$ and $V(p) \in \mathbb{R}^{n\times n}$ be matrix polynomials to be determined, with $T(p), V(p)$ symmetric, and define

$$\begin{cases} \Delta_1(p) = U(p) - U(p_0) \\ \Delta_2(p) = V(p) - V(p_0) \end{cases} \quad (25)$$

where $p_0 \in \mathcal{P}$ is arbitrarily chosen.

The idea is to reduce the magnitude of $\Delta_1(p)$ and $\Delta_2(p)$ over $\mathcal{P}$ in order to reduce the dependence of $\tilde{K}(p)$ on p. To this end, define

$$\begin{cases} S_i(p) = \tilde{S}_i(p)\Big|_{\substack{\tilde{T}(p)\to T(p) \\ \tilde{U}(p)\to U(p) \\ \tilde{V}(p)\to V(p)}} - \varepsilon I, \quad \forall i = 1,2,3 \\ S_4(p) = \begin{pmatrix} \zeta I_m \ , \ \Delta_1(p) \\ \Delta_1(p)^T \ , \ \zeta I_n \end{pmatrix} \\ S_5(p) = \zeta I_n - \Delta_2(p) \\ S_6(p) = \zeta I_n + \Delta_2(p) \end{cases} \quad (26)$$

where $\varepsilon, \zeta \in \mathbb{R}$ have to be determined. Define the quantities

$$\begin{cases} d_{S1} = \max\{\deg(A(p)), \deg(B(p))\} \\ d_{Si} = 0, \quad \forall i = 2, \ldots, 6 \end{cases} \quad (27)$$

and the optimization problem

$$\inf_{T(p),U(p),V(p),\varepsilon,\zeta} \zeta$$
$$\text{s.t.} \begin{cases} \varepsilon > 0 \\ \max\{\deg(T(p)), \deg(U(p)), \deg(V(p))\} \le d \\ S_i(p) \in \mathcal{C}(f(p), d + d_{Si}), \quad \forall i = 1, \ldots, 6 \end{cases} \quad (28)$$

where $d \in \mathbb{N}$ is introduced to bound the degree of the symmetric matrix polynomials $T(p), U(p), V(p)$. The optimization problem (28) is equivalent to an SDP since the cost function is linear in the decision variables and since the third constraint can be expressed as LMIs in the decision variables and auxiliary variables according to Section II-B.

Let $U^*(p), V^*(p)$ be $U(p), V(p)$ evaluated for the found optimal values of the decision variables of (28). The controller candidate for solving Problem 1 is defined as

$$K^* = U^*(p_0)V^*(p_0)^{-1}. \quad (29)$$

The numerical complexity of the WDLF approach can be measured in terms of the size of the SDP (28), i.e., the number of LMI scalar variables and the number of matrix rows. See, e.g., [2, Chapter 2] where the worst-case number of arithmetic operations required by interior-point and ellipsoid methods for solving an SDP is discussed. For conciseness, we report in Table I this size obtained by using the formulas in (17), for several values of n, d, q, in the case

$$\begin{cases} \mathcal{P} = \{p \in \mathbb{R}^q : \ \|p\|_2^2 \le 1\} \\ A(p) \text{ affine}, \quad B(p) \text{ constant}, \quad u \text{ scalar}. \end{cases} \quad (30)$$

This case is also considered in Example 1 in Section VII-A.

Case $q = 1$

n	$d=0$	$d=1$	$d=2$
2	[11,10]	[44,39]	[62,43]
3	[20,14]	[81,54]	[118,60]
4	[32,18]	[130,69]	[193,77]
5	[47,22]	[191,84]	[287,94]

Case $q = 2$

n	$d=0$	$d=1$	$d=2$
2	[13,12]	[68, 52]	[134, 62]
3	[26,17]	[133, 72]	[274, 87]
4	[44,22]	[221, 92]	[466,112]
5	[67,27]	[332,112]	[710,137]

TABLE I
SIZE OF THE SDP FOR THE WDLF APPROACH. EACH CELL SHOWS THE NUMBER OF LMI SCALAR VARIABLES AND THE NUMBER OF MATRIX ROWS IN THE SDP.

IV. CI APPROACH

This section describes the second approach proposed in this paper for solving Problem 1, denoted as CI approach, which consists of investigating the behaviour of different controllers by constructing a controller index over a search space of interest, and by retrieving the controller candidate by such an index. The CI approach is described hereafter through the following steps:

- definition of the Lyapunov-based matrix inequalities that certify the validity of a controller;
- introduction of a controller-dependent feasibility quantifier;
- construction of a simple region of interest in the controller space;
- definition of the SDP for determining a quantifier;
- extraction of a controller;
- construction of alternative regions of interest in the controller space.

Let us start by defining, for a symmetric matrix function $\tilde{W}(p) \in \mathbb{R}^{n\times n}$ to be determined,

$$\left\{\begin{array}{rcl} \tilde{X}_1(p) & = & -\mathrm{he}\left(\tilde{W}(p)\,(A(p)+B(p)KC(p))\right) \\ & & -Q - C(p)^T K^T RKC(p) \\ \tilde{X}_2(p) & = & \tilde{W}(p) \\ \tilde{X}_3(p) & = & \gamma - x_0^T \tilde{W}(p) x_0. \end{array}\right. \tag{31}$$

From [2, Chapter 7], the existence of $\tilde{W}(p)$ and K such that $\tilde{X}_1(p), \tilde{X}_2(p), \tilde{X}_3(p)$ are positive definite over $\mathcal{P}$ implies that K solves Problem 1.

Unfortunately, $\tilde{X}_1(p)$ is nonlinear in $\tilde{W}(p)$ and K, for instance due to a term with their product. This means that the set of pairs $(\tilde{W}(p), K)$ that make $\tilde{X}_1(p)$ positive semidefinite over $\mathcal{P}$ may be nonconvex.

In order to cope with this issue, the idea is to let K be a variable analogous to p rather than a decision variable. Specifically, define

$$\left\{\begin{array}{rcl} k & = & \mathrm{vec}(K) \\ \omega & = & \begin{pmatrix} k \\ p \end{pmatrix} \end{array}\right. \tag{32}$$

where $\mathrm{vec}(K) \in \mathbb{R}^{mr}$ is the vector obtained by stacking the columns of K in a consecutive order starting from the first one. Let $W(\omega) \in \mathbb{R}^{n\times n}$ be a symmetric matrix polynomial to be determined, and define

$$\left\{\begin{array}{rcl} X_1(\omega) & = & \tilde{X}_1(p)\Big|_{\tilde{W}(p)\to W(\omega)} - (\phi(k)+\psi)I_n \\ X_i(\omega) & = & \tilde{X}_i(p)\Big|_{\tilde{W}(p)\to W(\omega)} - \varepsilon I, \quad \forall i = 2,3 \end{array}\right. \tag{33}$$

where the polynomial $\phi(k)$ and the scalars $\psi, \varepsilon \in \mathbb{R}$ have to be determined. These quantities are introduced in order to quantify the feasibility of K for solving Problem 1. Indeed, the existence of $W(\omega), \phi(k), \psi, \varepsilon$ such that $X_1(\omega), X_2(\omega), X_3(\omega)$ are positive definite for all ω, implies that the following implication holds:

$$\phi(k) + \psi \geq 0 \quad \Rightarrow \quad J^*(K) < \gamma. \tag{34}$$

In order to search for $W(\omega), \phi(k), \psi$ with the properties just mentioned, we build a region of interest in the controller space. To this end, observe that a necessary condition for the worst-case cost to be bounded is that the matrix $A(p)+B(p)KC(p)$ is robustly Hurwitz over $\mathcal{P}$. Hence, define the set of controllers ensuring robust asymptotical stability over $\mathcal{P}$ as

$$\begin{array}{rcl} \mathcal{K}_{ras} & = & \{k \in \mathbb{R}^{mr} :\ \Re(\lambda) < 0, \\ & & \forall \lambda \in \mathrm{spec}(A(p)+B(p)KC(p)) \quad \forall p \in \mathcal{P}\}. \end{array} \tag{35}$$

Hence, let $\mathcal{K}_{oes} \subseteq \mathbb{R}^{mr}$ be any arbitrarily chosen outer estimate of $\mathcal{K}_{ras}$ that can be expressed as a semialgebraic set, i.e.,

$$\mathcal{K}_{oes} \supseteq \mathcal{K}_{ras} \tag{36}$$

and

$$\mathcal{K}_{oes} = \{k \in \mathbb{R}^{mr} :\ s_i(k) \geq 0, \quad \forall i = 1,\ldots,n_s\} \tag{37}$$

for some polynomials $s_i(k)$. The choice of the set $\mathcal{K}_{oes}$ will be discussed in the sequel of this section, where it will be also explained that this set is introduced in order to obtain a tradeoff between simplicity of implementation and velocity of convergence. Indeed, the sought controller can be searched for in the set $\mathcal{K}_{ras}$ only without loss of generality. In addition, since the CI approach requires that the set where the controller is searched for is bounded, we introduce the hypercube

$$\mathcal{K}_{box} = \{k \in \mathbb{R}^{mr} :\ \|k\|_\infty \leq \rho\} \tag{38}$$

where ρ is a chosen nonnegative scalar. The controller provided by the CI approach will be searched for in the set

$$\mathcal{K} = \mathcal{K}_{oes} \cap \mathcal{K}_{box}. \tag{39}$$

Next, define the vector polynomials

$$g(k) = \left(s_1(k), \ldots, s_{n_s}(k), \rho^2 - k_1^2, \ldots, \rho^2 - k_{mr}^2\right)^T \tag{40}$$

and

$$h(\omega) = \begin{pmatrix} f(p) \\ g(k) \end{pmatrix}. \tag{41}$$

Also, define

$$\mu = \int_{\mathcal{K}} (\phi(k) + \psi) dk \tag{42}$$

and the quantities

$$\left\{\begin{array}{rcl} d_{X1} & = & \deg_\omega(A(p)+B(p)KC(p)) \\ d_{Xi} & = & 0, \quad \forall i = 2,3 \end{array}\right. \tag{43}$$

where $\deg_\omega(\cdot)$ denotes the degree in ω.

Define the optimization problem

$$\begin{array}{ll} \sup\limits_{W(\omega),\phi(k),\psi,\varepsilon} & \mu - c\psi \\ \text{s.t.} & \left\{\begin{array}{l} \varepsilon > 0 \\ \deg(W(\omega)) \leq d \\ X_i(\omega) \in \mathcal{C}(h(\omega), d + d_{Xi}), \quad \forall i = 1,2,3 \\ -\phi(k) \in \mathcal{C}(g(k), d + d_{X1}) \\ \psi \leq 1 \end{array}\right. \end{array} \tag{44}$$

where $c \in \mathbb{R}$ is a positive scalar whose role will be clarified in the sequel of the paper, and $d \in \mathbb{N}$ is introduced to bound the degrees of the symmetric matrix polynomial $W(\omega)$ and polynomial $\phi(k)$. The optimization problem (44) is equivalent to an SDP since the cost function is linear in the decision variables and since the second and third constraints can be expressed as LMIs in the decision variables and auxiliary variables according to Section II-B.

Analogously to the WDLF approach, the numerical complexity of the CI approach can be measured in terms of the size of the SDP (44). For conciseness, we report in Table II this size obtained by using the formulas in (17), for several values of n, d, q, in the case

$$\begin{cases} \mathcal{P} = \{p \in \mathbb{R}^q : \ \|p\|_2^2 \le 1\} \\ A(p) \text{ affine}, \quad B(p) \text{ constant}, \quad u \text{ scalar}, \quad y = x. \end{cases} \tag{45}$$

This case is also considered in Example 1 in Section VII-A.

Case $q = 1$

n	$d = 0$	$d = 1$	$d = 2$
2	[27,23]	[54, 41]	[295, 78]
3	[74,39]	[156, 71]	[1329,164]
4	[170,59]	[365,109]	[4346,295]
5	[342,83]	[738,155]	[11563,480]

Case $q = 2$

n	$d = 0$	$d = 1$	$d = 2$
2	[31,25]	[65, 46]	[521, 97]
3	[89,42]	[192, 78]	[2229,198]
4	[206,63]	[447,118]	[6900,348]
5	[412,88]	[893,166]	[17513,556]

TABLE II

SIZE OF THE SDP FOR THE CI APPROACH. EACH CELL SHOWS THE NUMBER OF LMI SCALAR VARIABLES AND THE NUMBER OF MATRIX ROWS IN THE SDP.

Once (44) is solved, a controller candidate is defined as a zero of the polynomial $\phi^*(k)$, which is the polynomial $\phi(k)$ evaluated for the found optimal values of the decision variables of (44). Hence, define

$$\mathcal{Z} = \{k \in \mathcal{K} : \ \phi^*(k) = 0\}. \tag{46}$$

Any vector in the set $\mathcal{Z}$ defines a controller candidate for solving Problem 1. In order to single out one vector only from the set $\mathcal{Z}$, we define the controller candidate for solving Problem 1 provided by the CI approach as

$$K^* = \mathrm{mat}(k^*, n) \tag{47}$$

where $\mathrm{mat}(k, n) \in \mathbb{R}^{m\times n}$ is the matrix whose columns are stacked in consecutive order in the vector k starting from the first one, and k^* is defined by

$$k^* : \ k^* \lhd k, \quad \forall k \in \mathcal{Z} \tag{48}$$

where the symbol "$\lhd$" denotes the inequality operator defined as

$$k^* \lhd k \iff \begin{cases} |k^*|_l = |k|_l, \quad \forall l < l_0 \\ |k^*|_{l_0} < |k|_{l_0}. \end{cases} \tag{49}$$

The set $\mathcal{Z}$ can be found through linear algebra operations. Indeed, from Section II-B, the condition $-\phi(k) \in \mathcal{C}(g(k))$ can be written as $y_i(k), z(k) \in \Sigma$, $i = 1, \ldots, n_s + mr$, where $y_i(k)$ are polynomials and

$$z(k) = -\phi(k) - \sum_{i=1}^{n_s+mr} g_i(k) y_i(k). \tag{50}$$

Let $z^*(k)$ be $z(k)$ evaluated for the found optimal values of the decision variables of (44). It follows that

$$z^*(k) = 0, \quad \forall k \in \mathcal{Z}. \tag{51}$$

Define

$$\mathcal{Z}_0 = \{k \in \mathbb{R}^{mr} : \ z^*(k) = 0\}. \tag{52}$$

Since $z^*(k) \in \Sigma$, it follows from Section II-B that

$$z^*(k) = b(k)^T Z^* b(k) \tag{53}$$

where $b(k)$ is a vector polynomial and Z^* is a symmetric positive semidefinite matrix. Hence,

$$\mathcal{Z}_0 = \{k \in \mathbb{R}^{mr} : \ b(k) \in \ker(Z^*)\}. \tag{54}$$

As explained in [26, Chapter 1] and references therein, the vectors k such that $b(k) \in \ker(Z^*)$ can be found through pivoting operations and eigenvalue computations. Moreover, in the common case where $\ker(Z^*)$ has dimension equal to one, this process boils down to just reading k among the entries of a scaled eigenvector of Z^*. Once $\mathcal{Z}_0$ is found, the set $\mathcal{Z}$ is simply obtained by testing the conditions $k \in \mathcal{K}$ and $\phi^*(k) = 0$ for each vector $k \in \mathcal{Z}_0$.

We conclude this section by discussing the choice of the set $\mathcal{K}_{oes}$, which affects the set $\mathcal{K}$ where the controller candidate is searched for according to (39). As it will be explained in Theorem 3, the CI approach is guaranteed to be nonconservative by using a sufficiently large value of d for any arbitrarily chosen $\mathcal{K}_{oes}$ that satisfies (36). Nevertheless, the choice of $\mathcal{K}_{oes}$ can be useful to speed up the convergence of the CI approach by requiring a smaller value of d for providing a controller candidate that solves Problem 1, which is desirable in order to reduce the computational burden. This is due to the fact that the set $\mathcal{K}_{oes}$ reduces the search space for the controller candidate, in particular from $\mathcal{K}_{box}$ to $\mathcal{K}$, and this can be useful in order to reduce the degrees of $W(\omega), \phi(k)$ used to satisfy the conditions in the SDP (44).

Let us start by observing that a necessary condition for establishing that $k \in \mathcal{K}_{ras}$ can be obtained as follows. Let $p_0 \in \mathcal{P}$ be arbitrarily chosen, and define

$$A_{cl}(k) = A(p_0) + B(p_0)KC(p_0). \tag{55}$$

Clearly, $k \in \mathcal{K}_{ras}$ only if $A_{cl}(k)$ is Hurwitz. The condition that $A_{cl}(k)$ is Hurwitz can be expressed as a set of polynomial inequalities in k by using the Hurwitz table. Specifically, let $\theta_1(k), \ldots, \theta_{n+1}(k)$ be the entries in the first column of this table, with $\theta_1(k) = 1$, and write

$$\theta_i(k) = \frac{\tilde{\theta}_i(k)}{\bar{\theta}_i(k)} \tag{56}$$

where $\tilde{\theta}_i(k), \bar{\theta}_i(k)$ are coprime polynomials with $\tilde{\theta}_1(k) = \bar{\theta}_1(k) = 1$. Since the positivity of $\theta_1(k), \ldots, \theta_{i-1}(k)$ implies that $\bar{\theta}_i(k)$ is positive, it follows that $A_{cl}(k)$ is Hurwitz if and only if $\tilde{\theta}_i(k) > 0$ for all $i = 2, \ldots, n+1$. Hence, a choice for $\mathcal{K}_{oes}$ is

$$\mathcal{K}_1 = \left\{k \in \mathbb{R}^{mr} : \ \tilde{\theta}_i(k) \ge 0, \quad \forall i = 2, \ldots, n+1\right\} \tag{57}$$

This choice has the advantage of exactly characterizing the Hurwitz property of $A_{cl}(k)$, but also the disadvantage of providing a set $\mathcal{K}_{ras}$ (and, hence, also a set $\mathcal{K}$) with possibly non-simple shape, which could make difficult to calculate the integral in (42). Observe that $\mathcal{K}_1$ can be expressed as in (37) with $n_s \leq n$.

In order to cope with this issue, observe that a simpler necessary condition for establishing that $k \in \mathcal{K}_{ras}$ can be obtained as follows. Express the characteristic polynomial of $A_{cl}(k)$ as

$$\det(\lambda I_n - A_{cl}(k)) = \lambda^n + \sum_{i=0}^{n-1} a_i(k)\lambda^i \tag{58}$$

where each $a_i(k)$ is a polynomial. It follows that $A_{cl}(k)$ is Hurwitz only if $a_i(k) > 0$ for all $i = 0, \ldots, n-1$. Hence, another choice for $\mathcal{K}_{oes}$ is

$$\mathcal{K}_2 = \{k \in \mathbb{R}^{mr} : \ a_i(k) \geq 0, \quad \forall i = 0, \ldots, n-1\}. \tag{59}$$

This choice has the disadvantage of providing a possibly larger set $\mathcal{K}_{oes}$ than the previous choice (since the condition that the coefficients of the characteristic polynomial of $A_{cl}(k)$ are positive is only necessary for A_{cl} to be Hurwitz), but also the advantage of providing a set $\mathcal{K}_{oes}$ (and, hence, also a set $\mathcal{K}$) with possibly simpler shape than the previous choice. Indeed, it turns out that

$$\deg(a_i(k)) \leq \min\{n-i, \operatorname{rank}(B(p_0)), \operatorname{rank}(C(p_0))\}. \tag{60}$$

For instance, this means that all the polynomials $a_i(k)$ are affine functions whenever $\operatorname{rank}(B(p_0)) = 1$ or $\operatorname{rank}(C(p_0)) = 1$, which is the case of single-input or single-output systems. In such a case, the set $\mathcal{K}_{oes}$ (and, hence, also a set $\mathcal{K}$) is just a polytope, and the integral in (42) can be calculated more easily. Similarly to the previous choice, the set $\mathcal{K}_2$ can be expressed as in (37) with $n_s \leq n$.

Lastly, let us mention that a tighter set $\mathcal{K}_{oes}$ can be obtained by repeating the above choices for multiple values of p_0, denoted by $p_0^{(1)}, \ldots, p_0^{(l)}, \ldots \in \mathcal{P}$, and taking the intersection of the sets obtained. Specifically, such a set is

$$\mathcal{K}_3 = \bigcap_{i=1,\ldots,l} \mathcal{K}_{j(i)}\Big|_{p_0 = p_0^{(i)}}, \quad j(i) \in \{1, 2\} \tag{61}$$

which can be expressed as in (37) with $n_s \leq ln$.

V. Sufficiency and Necessity

This section investigates some key properties of the WDLF approach and CI approach. Specifically, Section V-A explains how a sufficient condition for solving Problem 1 can be obtained with these approaches. Then, Section V-B analyzes the necessity of this condition.

A. Sufficiency

The WDLF and CI approaches described in Sections III and IV provide a controller candidate K^* for solving Problem 1. Once such a controller has been obtained, one can test whether K^* solves this problem by establishing if $J^*(K^*) < \gamma$. This can be done through an SDP by using the methodology described in the previous sections and specialized to the case of known controller.

Specifically, let $\tilde{X}_i(p)$ be defined as in (31) for a fixed controller K, and introduce

$$\begin{cases} D_i(p) &= \tilde{X}_i(p)\Big|_{\tilde{W}(p)\to W(p)} - \varepsilon I, \quad \forall i = 1, 2 \\ D_3(p) &= \eta - x_0^T W(p) x_0 - \varepsilon \end{cases} \tag{62}$$

where the symmetric matrix polynomial $W(p)$ and the scalars η, ε have to be determined. Define

$$\begin{cases} d_{D1} &= \deg(A(p) + B(p)KC(p)) \\ d_{Di} &= 0, \quad \forall i = 2, 3 \end{cases} \tag{63}$$

and the optimization problem

$$\begin{array}{l} \hat{J}(K) = \displaystyle\inf_{W(p),\eta,\varepsilon} \eta \\ \text{s.t.} \quad \begin{cases} \varepsilon > 0 \\ \deg(W(p)) \leq d_\eta \\ D_i(p) \in \mathcal{C}(f(p), d_\eta + d_{Di}), \quad \forall i = 1, 2, 3 \end{cases} \end{array} \tag{64}$$

where $d_\eta \in \mathbb{N}$ is introduced to bound the degree of $W(p)$. Analogously to (28) and (44), the optimization problem (64) is equivalent to an SDP.

Definition 1: The set $\mathcal{P}$ is said strongly compact if it is compact and the highest degree forms of the polynomials in $f(p)$ have not common root except the origin. □

Definition 1 provides a stronger definition of compactness for $\mathcal{P}$ that will be exploited in the sequel of this paper. Observe that typical sets such as hypersphere, hypercube and simplex defined in (3)–(5) satisfy this definition.

The following theorem explains how one can establish if the controller candidates provided by the WDLF approach and the CI approach solve Problem 1.

Theorem 1: For all d_η one has

$$J^*(K) \leq \hat{J}(K). \tag{65}$$

Moreover, suppose that $J^*(K) < \infty$ and that $\mathcal{P}$ is strongly compact. Then, for all $\delta > 0$ there exists d_η such that

$$\hat{J}(K) - \delta \leq J^*(K). \tag{66}$$

Proof. Let us start by proving (65). To this end, suppose firstly that, for the chosen d_η, the constraints in (64) hold for some $W(p), \eta, \varepsilon$. Since $\varepsilon > 0$, from (20)–(21) it follows that $D_i(p) > 0$ for all $p \in \mathcal{P}$ for all $i = 1, 2, 3$, which implies that $J^*(K) < \eta$ and, in turn, $J^*(K) \leq \hat{J}(K)$. Also, suppose secondly that, for the chosen d_η, the constraints in (64) are infeasible. This implies that $\hat{J}(K) = \infty$, and (65) still holds.

Next, let us prove (66). Let $\delta > 0$ be arbitrarily chosen. It follows that there exists a symmetric matrix function $\tilde{W}(p)$ and a scalar $\tilde{\varepsilon} > 0$ such that

$$\tilde{X}_i(p)\Big|_{\gamma \to J^*(K)+\delta} - \tilde{\varepsilon} I \geq 0, \quad \forall p \in \mathcal{P} \quad \forall i = 1, 2, 3.$$

Since $\mathcal{P}$ is compact, it follows that there exists a symmetric matrix polynomial $W(p)$ that approximates arbitrarily well

$\tilde{W}(p)$ over $\mathcal{P}$. This implies that there exist a symmetric matrix polynomial $W(p)$ and a scalar $\varepsilon > 0$ such that

$$D_i(p) \geq 0, \quad \forall p \in \mathcal{P} \quad \forall i = 1,2,3.$$

Moreover, since $\mathcal{P}$ is also strongly compact, from [Section 1] [27] there exist symmetric matrix polynomials $Y_{i,j}(p) \in \Sigma$ such that $Z_i(p) \in \Sigma$ for all $i = 1,2,3$ for all $j = 1,\ldots,q$, where

$$Z_i(p) = D_i(p) - \sum_{j=1}^{n_f} f_j(p) Y_{i,j}(p).$$

Define

$$d^* = \min_{d \in \mathbb{N}} d$$
$$\text{s.t.} \quad \begin{cases} \deg(W(p)) \leq d \\ \deg(Z_i(p)) \leq 2\left\lceil \frac{1}{2}(d + d_{Di}) \right\rceil, \quad \forall i = 1,2,3. \end{cases}$$

Then, the second constraint in (64) already holds by choosing $d_\eta = d^*$. □

Theorem 1 states that $\hat{J}(K)$ is an upper bound on $J^*(K)$ for all d_η, moreover, the conservatism of this upper bound can be arbitrarily decreased by increasing d. Hence, a sufficient condition for the solution of Problem 1 is as follows.

Corollary 1: There exists a controller K that solves Problem 1 if, for some d_η, $\hat{J}(K^*) < \gamma$ where K^* is the controller candidate provided by the WDLF or CI approaches (in such a case, K^* solves Problem 1).

B. Necessity

Let us start by considering the WDLF approach in Section III.

Theorem 2: Suppose that there exists a controller $K = K_0$ that solves Problem 1. Moreover, suppose that there exist $T(p) = T_0$, $U(p) = U_0$ and $V(p) = V_0$ of degree 0 such that $U_0 = K_0 V_0$ and $\tilde{S}_1(p), \tilde{S}_2(p), \tilde{S}_3(p)$ are positive definite over $\mathcal{P}$, with $\mathcal{P}$ strongly compact. Then, there exists d such that the controller candidate K^* in (29) provided by the WDLF approach solves Problem 1.
Proof. Choose $T(p) = T_0$, $U(p) = U_0$ and $V(p) = V_0$. It follows that there exist $\varepsilon > 0$ and $\zeta = 0$ such that

$$S_i(p) > 0, \quad \forall p \in \mathcal{P} \quad \forall i = 1,\ldots,6.$$

Analogously to the proof of Theorem 1, there exist symmetric matrix polynomials $Y_{i,j}(p) \in \Sigma$ such that $Z_i(p) \in \Sigma$ for all $i = 1,\ldots,6$ for all $j = 1,\ldots,q$, where

$$Z_i(p) = S_i(p) - \sum_{j=1}^{q} f_j(p) Y_{i,j}(p).$$

Define

$$d^* = \min_{\tilde{d} \in \mathbb{N}} \tilde{d}$$
$$\text{s.t.} \quad \deg(Z_i(p)) \leq 2\left\lceil \frac{1}{2}(\tilde{d} + d_{Si}) \right\rceil, \quad \forall i = 1,\ldots,6.$$

The chosen $T(p), U(p), V(p), \varepsilon, \zeta$ satisfy the constraints in (28) for $d = d^*$, moreover, they are global minimizers of (28) since ζ cannot be negative. This proves that $T^*(p), U^*(p), V^*(p)$ (which are $T(p), U(p), V(p)$ evaluated for the found optimal values of the decision variables of (28)) have degree 0. Therefore, the controller K^* provided in (29) coincides with the controller $K(p)$ provided in (24), which solves Problem 1. □

Theorem 2 provides a condition that, if satisfied, ensures that the controller candidate provided by the WDLF approach solves Problem 1 for sufficiently large d. Such a condition boils down to the existence of a sought controller K_0 for which $J^*(K_0) < \gamma$ can be proved through a parameter-independent Lyapunov function. For instance, this is the case of systems weakly affected by the uncertainties.

Next, consider the CI approach in Section IV.

Theorem 3: Suppose that there exists $K = K_0$ that solves Problem 1, with $\|K_0\|_\infty \leq \rho$, and that $\mathcal{P}$ is strongly compact. Also, let c satisfy

$$0 < c < \frac{1}{d+1}. \tag{67}$$

Then, there exists d such that the controller candidate K^* in (47) provided by the CI approach solves Problem 1.
Proof. Define the optimization problem

$$\sup_{\tilde{W}(p), \tilde{\phi}, \tilde{\varepsilon}} \tilde{\phi}$$
$$\text{s.t.} \quad \begin{cases} \tilde{\varepsilon} > 0 \\ \tilde{\phi} \leq 1 \\ \tilde{X}_1(p) - \tilde{\phi} I \geq 0, \quad \forall p \in \mathcal{P} \\ \tilde{X}_i(p) - \tilde{\varepsilon} I \geq 0, \quad \forall p \in \mathcal{P} \quad \forall i = 2,3 \end{cases}$$

where $\tilde{W}(p) \in \mathbb{R}^{n \times n}$ is a symmetric matrix function, $\tilde{\phi}, \tilde{\varepsilon} \in \mathbb{R}$ are scalars, and $\tilde{X}_i(p)$ is defined as in (33) for a frozen controller K. It follows that this optimization problem is feasible and the supremum is achieved for any controller K. The global maximizers of $\tilde{W}(p), \tilde{\phi}$ are functions of the controller K. Denote any pair of such global maximizers as $\tilde{W}^*(\omega), \tilde{\phi}^*(k)$. Let k_0 be the vector corresponding to K_0. Since K_0 solves Problem 1, it follows that

$$\tilde{\phi}^*(k_0) > 0.$$

Since $\mathcal{P} \times \mathcal{K}$ is compact, it follows that $\tilde{W}^*(\omega)$ and $\tilde{\phi}^*(k)$ can be approximated arbitrarily well by polynomials over $\mathcal{P} \times \mathcal{K}$, in particular, there exist a symmetric matrix polynomial $W(\omega)$, a polynomial $\bar{\phi}(k)$ and a scalar ε such that $\bar{\phi}(k_0) > 0$ and

$$\begin{cases} \varepsilon > 0 \\ \bar{\phi}(k) \leq \bar{\psi} \\ X_1(\omega) - \bar{\phi}(k) I \geq 0, \quad \forall p \in \mathcal{P} \quad \forall k \in \mathcal{K} \\ X_i(\omega) - \varepsilon I \geq 0, \quad \forall i = 2,3 \quad \forall p \in \mathcal{P} \quad \forall k \in \mathcal{K} \end{cases} \tag{68}$$

where $\bar{\psi}$ is temporarily set to 1. Analogously to the proof of Theorem 1, there exist symmetric matrix polynomials $Y_{i,j}(\omega) \in \Sigma$ and polynomials $y_l(k) \in \Sigma$ such that

$Z_i(\omega), z(k) \in \Sigma$ for all $i = 1,2,3$ for all $j = 1,\ldots,n_s + mr + q$ for all $l = 1,\ldots,n_s+mr$, where

$$\begin{cases} Z_i(\omega) &= X_i(\omega) - \displaystyle\sum_{j=1}^{n_s+mr+q} h_j(\omega)Y_{i,j}(\omega) \\ z(k) &= \bar{\psi} - \bar{\phi}(k) - \displaystyle\sum_{l=1}^{n_s+mr} g_l(k)y_l(k). \end{cases}$$

This implies that $X_i(\omega) \in \mathcal{C}(h(\omega), d + d_{Xi})$ for all $i = 1,2,3$ and $\bar{\psi} - \bar{\phi}(k) \in \mathcal{C}(g(k), d + d_{X1})$ by choosing d such that

$$\begin{cases} \deg(W(\omega)) \le d \\ \deg(Z_i(\omega)) \le 2\left\lceil \dfrac{1}{2}(d + d_{Xi})\right\rceil, \quad \forall i = 1,2,3 \\ \deg(z(k)) \le 2\left\lceil \dfrac{1}{2}(d+d_{X1})\right\rceil. \end{cases}$$

Observe that

$$\bar{\phi}(k) \le \tilde{\phi}^*(k), \quad \forall k \in \mathcal{K}.$$

Hence, define

$$\bar{\mu} = \int_{\mathcal{K}} \bar{\phi}(k)dk$$

and

$$\begin{array}{cc} \sup\limits_{W(\omega),\bar{\phi}(k),\varepsilon} & \bar{\mu} \\ \text{s.t.} & (68). \end{array} \tag{69}$$

It follows that the maximizer of $\bar{\phi}(k)$ in (69), denoted by $\bar{\phi}^*(k)$, approximates arbitrarily well $\tilde{\phi}^*(k)$ over $\mathcal{K}$ by sufficiently increasing the degree of $\bar{\phi}(k)$ through d.

Next, define the set

$$\Xi = \left\{(\bar{\mu}, \bar{\psi}): \ \bar{\psi} \le 1, \ (68) \text{ holds}\right\}.$$

It follows that the set Ξ is bounded in the positive μ-direction, in particular,

$$\sup_{(\bar{\mu},\bar{\psi})\in\Xi} \bar{\mu} \le \int_{\mathcal{K}} \tilde{\phi}^*(k)dk.$$

Moreover, the set Ξ is bounded in the negative ψ-direction for finite values of μ, i.e.,

$$\forall \bar{\mu}\ \exists \bar{\psi}_- : \ \psi \ge \psi_-, \quad \forall (\bar{\mu},\bar{\psi}) \in \Xi.$$

This implies that the maximizer of $\bar{\phi}(k)$ in

$$\begin{array}{cc} \sup\limits_{W(\omega),\bar{\phi}(k),\bar{\psi},\varepsilon} & \bar{\mu} - c\bar{\psi} \\ \text{s.t.} & (68) \text{ holds}, \end{array} \tag{70}$$

denoted by $\bar{\phi}^{**}(k)$, is arbitrarily close to $\tilde{\phi}^*(k)$ by letting c be positive and sufficiently small as ensured by (67) when increasing d. Moreover, the maximizer of $\bar{\psi}$ in (70), denoted by $\bar{\psi}^{**}$, is the maximum of $\bar{\phi}^{**}(k)$ over $\mathcal{K}$. Hence, for d sufficiently large,

$$\begin{cases} \bar{\phi}^{**}(k) = \bar{\psi}^{**} \\ k \in \mathcal{K} \end{cases} \Rightarrow \quad J^*(K) < \gamma.$$

Observe that the controllers k satisfying the left hand side of the above condition are the controllers in the set $\mathcal{Z}$ in (46) by introducing the change of variables $\bar{\phi}(k) = \phi(k) + \psi$ and $\bar{\psi} = \psi$. □

Theorem 3 states that the controller candidate provided by the CI approach solves Problem 1 for sufficiently large d provided that there exists a sought controller in the set $\mathcal{K}$. In particular, this is guaranteed whenever the condition (67) is satisfied, which requires that the parameter c becomes arbitrarily small by increasing d.

VI. Extension To DT Systems

This section addresses the extension of the proposed approaches to DT systems. Specifically, Section VI-A introduces the problem formulation, while Sections VI-B and VI-C explain how the WDLF and CI approaches can be modified in order to deal with the problem.

A. Problem Formulation

Consider the DT LTI uncertain system

$$\begin{cases} x(t+1) &= A(p)x(t) + B(p)u(t) \\ y(t) &= C(p)x(t) \\ x(0) &= x_0 \\ p &\in \mathcal{P} \end{cases} \tag{71}$$

where $t \in \mathbb{N}$ is the time, and the other quantities are as in the system (1). The LQR problem for (71) consists of solving

$$J_0(p) = \inf_{u(t)} \sum_{t=0}^{\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) \tag{72}$$

where $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices. In the case of state-feedback (i.e., $C(p) = I_n$), the control input that achieves $J_0(p)$ is given by the control law (8)–(9) where $V_0(p) \in \mathbb{R}^{n\times n}$ is the solution of the DARE

$$\begin{array}{rcl} 0 &=& Q + A(p)^T V_0(p)A(p) - V_0(p) \\ && -(A(p)^T V_0(p)B(p))(R + B(p)^T V_0(p)B(p))^{-1} \\ && \cdot(B(p)^T V_0(p)A(p)), \end{array} \tag{73}$$

see, e.g., [3, Chapter 2], [4, Chapter 2].

The system (71) is controlled in closed-loop by (11) where $K \in \mathbb{R}^{m\times r}$ has to be determined in order to minimize the worst-case cost

$$J^*(K) = \sup_{p\in\mathcal{P}} J(K,p) \tag{74}$$

where

$$\begin{array}{rcl} J(K,p) &=& \displaystyle\sum_{t=0}^{\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) \\ && \text{s.t.} \ \ u(t) = Ky(t). \end{array} \tag{75}$$

Problem 2: (Robust LQR problem for DT systems) Consider the system (71) and worst-case cost (74). Given a scalar $\gamma > 0$, find K (if any) such that $J^*(K) < \gamma$. □

B. WDLF Approach

In the case of DT systems, the WDLF approach still supposes that Assumption 1 holds, and is modified by replacing $\tilde{S}_1(p)$ in (23) with

$$\tilde{S}_1(p) = V(p) - \left(\begin{array}{c} A(p)^T \\ B(p)^T \end{array} \right)^T \tilde{S}_2(p) \left(\begin{array}{c} A(p)^T \\ B(p)^T \end{array} \right) - x_0 x_0^T \tag{76}$$

and d_{S1} in (27) with

$$d_{S1} = 2 \max\{\deg(A(p)), \deg(B(p))\}. \tag{77}$$

C. CI Approach

In the case of DT systems, the CI approach is modified by replacing $\tilde{X}_1(p)$ in (31) with

$$\begin{array}{c} \tilde{X}_1(p) = W(p) - Q - C(p)^T K^T R K C(p) \\ - \left(A(p) + B(p)KC(p)\right)^T W(p) \left(A(p) + B(p)KC(p)\right) \end{array} \tag{78}$$

and d_{X1} in (43) with

$$d_{X1} = 2 \deg_\omega (A(p) + B(p)KC(p)). \tag{79}$$

Moreover, the set of controllers ensuring robust asymptotical stability in (35) is replaced by

$$\begin{array}{rcl} \mathcal{K}_{ras} & = & \{k \in \mathbb{R}^{mr} : \ |\lambda| < 1, \\ & & \forall \lambda \in \operatorname{spec}(A(p) + B(p)KC(p)) \quad \forall p \in \mathcal{P}\}. \end{array} \tag{80}$$

Next, let us discuss the construction of an outer estimate $\mathcal{K}_{oes}$ of the set $\mathcal{K}_{ras}$. With $A_{cl}(k)$ as in (55), the condition $k \in \mathcal{K}_{ras}$ holds only if $A_{cl}(k)$ is Schur. The condition that $A_{cl}(k)$ is Schur can be expressed as a set of polynomial inequalities in k by using the Jury table. Specifically, let $\theta_1(k), \ldots, \theta_{2(n+1)}(k)$ be the entries in the first column of this table, with $\theta_1(k) = 1$, and write

$$\theta_i(k) = \frac{\tilde{\theta}_i(k)}{\bar{\theta}_i(k)} \tag{81}$$

where $\tilde{\theta}_i(k), \bar{\theta}_i(k)$ are polynomials with $\tilde{\theta}_1(k) = \bar{\theta}_1(k) = 1$. Since the positivity of $\theta_1(k), \ldots, \theta_{2(i-1)+1}(k)$ implies that $\bar{\theta}_{2i+1}(k)$ is positive, it follows that $A_{cl}(k)$ is Schur if and only if $\tilde{\theta}_{2i+1}(k) > 0$ for all $i = 1, \ldots, n$. Hence, a choice for the set $\mathcal{K}_{oes}$ is

$$\mathcal{K}_1 = \left\{ k \in \mathbb{R}^{mr} : \ \tilde{\theta}_{2i+1}(k) \geq 0, \quad \forall i = 1, \ldots, n \right\}. \tag{82}$$

This choice has the advantage of exactly characterizing the Schur property of $A_{cl}(k)$, but also the disadvantage of providing a set $\mathcal{K}_{oes}$ (and, hence, also a set $\mathcal{K}$) with possibly non-simple shape, which could make difficult to calculate the integral in (42).

In order to cope with this issue, observe that a simpler necessary condition for establishing that $k \in \mathcal{K}_{ras}$ can be obtained as follows. Express the characteristic polynomial of $A_{cl}(k)$ as in (58). It follows that $A_{cl}(k)$ is Schur only if $\tilde{a}_j(k) > 0$ for all $i = 1, \ldots, 2n$, where

$$\left\{ \begin{array}{l} \tilde{a}_{2i+1} = c_i + a_i(k) \\ \tilde{a}_{2i+2} = c_i - a_i(k) \\ c_i = \dfrac{n!}{i!(n-i)!} \\ \forall i = 0, \ldots, n-1. \end{array} \right. \tag{83}$$

Hence, another choice for the set $\mathcal{K}_{oes}$ is

$$\mathcal{K}_2 = \{k \in \mathbb{R}^{mr} : \ \tilde{a}_i(k) \geq 0, \quad \forall i = 1, \ldots, 2n\}. \tag{84}$$

This choice has the disadvantage of providing a possibly larger set $\mathcal{K}_{oes}$ than the previous choice, but also the advantage of providing a set $\mathcal{K}_{oes}$ (and, hence, also a set $\mathcal{K}$) with possibly simpler shape than the previous choice. For instance, all the polynomials $\tilde{a}_i(k)$ are affine functions whenever $\operatorname{rank}(B(p_0)) = 1$ or $\operatorname{rank}(C(p_0)) = 1$, which is the case of single-input or single-output systems.

VII. Examples

This section presents four illustrative examples. The proposed approaches are compared with the following existing methods, which propose sufficient conditions for Problem 1:

- [13, Section 3], which considers state-feedback control design for CT and DT systems, and provides a controller for minimizing the worst-case cost through an SDP;
- [14, Section III], which considers state-feedback control design for CT systems, and provides a controller for minimizing the worst-case cost through an SDP;
- [15, Section II], which considers output-feedback control design for CT systems, and provides a controller for minimizing the worst-case cost through SDPs and evolutionary algorithms. In particular, evolutionary algorithms are used to search for two variables in order to solve a constrained optimization problem that, for frozen values of these two variables, boils down to an SDP. In this paper, these two variables are searched for by using the function fminsearch of Matlab, which solves the SDP obtained for frozen values of these two variables at each iteration.

The examples presented in this section aim to show a case where:

- the WDLF and CI approaches and the existing methods solve the problem;
- the CI approach solves the problem, while the WDLF approach and the existing methods do not (Example 2);
- the WDLF and CI approaches solve the problem, while the existing methods cannot be applied since the system is not polytopic (Example 3);
- the CI approach solves the problem, while the WDLF approach and the existing methods cannot be applied since the state is not available for feedback and since the dynamics are not CT (Example 4).

For conciseness, all examples consider Problem 1 with

$$\begin{cases} x_0 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathcal{P} = \{p \in \mathbb{R}^q : \ \|p\|_2 \leq 1\} \\ Q = I_n, \quad R = \frac{1}{2} I_m, \quad \gamma = 10. \end{cases} \tag{85}$$

Moreover, in all examples:

- the worst-case cost $J^*(K)$ is unbounded for the open loop system, i.e., $J^*(0) = \infty$;
- the WDLF approach is used with $p_0 = (1, 0, \ldots, 0)^T$;
- the CI approach is used with $\rho = 2$, $c = 10^{-3}$ and $p_0 = (1, 0, \ldots, 0)^T$.

For each found controller candidate K, provided either by the proposed approaches or by the existing methods, its worst-case cost $J^*(K)$ is calculated with the SDP (64) by using $d_\eta = 2$ (the upper bound $\hat{J}(K)$ provided by this SDP is tight in all examples).

The SDPs are solved with the toolbox SeDuMi [28] for Matlab on a standard computer with Windows 11, Intel Core i7, 3.2 GHz, 16 GB RAM. The reported SDP time is the time in seconds, rounded to the nearest not smaller integer, required for solving each SDP and extracting the controller.

A. Example 1

Consider the model of a DC motor (see, e.g., [29, Chapter 2])

$$\begin{cases} J_m \ddot{\psi}_m(t) + b_m \dot{\psi}_m(t) &= K_t i_a(t) \\ L_a \dot{i}_a(t) + R_a i_a(t) &= -K_e \dot{\psi}_m(t) + v_a(t) \end{cases}$$

where $\psi_m(t)$ is the angle, $i_a(t)$ is the current, $v_a(t)$ is the voltage, and $J_m, b_m, K_t, L_a, R_a, K_e$ are parameters. Define

$$\begin{cases} x(t) &= (\psi_m(t), \dot{\psi}_m(t), i_a(t))^T \\ u(t) &= v_a(t). \end{cases}$$

The model can be rewritten as

$$\dot{x}(t) = \begin{pmatrix} 0, & 1, & 0 \\ 0, & -\dfrac{b_m}{J_m}, & \dfrac{K_t}{J_m} \\ 0, & -\dfrac{K_e}{L_a}, & -\dfrac{R_a}{L_a} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{L_a} \end{pmatrix} u(t).$$

Choose the plausible values

$$\begin{cases} b_m = 0.5, \quad K_t = 2, \quad L_a = 0.5, \quad R_a = 1, \quad K_e = 3 \\ J_m \in [1, 2] \end{cases}$$

where J_m is the uncertainty. By defining

$$p = \frac{4}{J_m} - 3,$$

it follows that $p \in [-1, 1]$ and the model is described by the system (1) with

$$\begin{cases} A(p) = \begin{pmatrix} 0, & 1, & 0 \\ 0, & -0.125(p+3), & 0.5(p+3) \\ 0, & -6, & -2 \end{pmatrix} \\ B(p) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad C(p) = I_3. \end{cases}$$

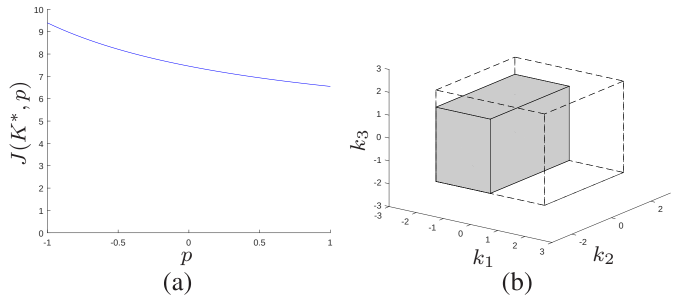

Fig. 1. Example 1: (a) Cost $J(K^*, p)$ for K^* found with the CI approach in Part 1 (blue solid line). The cost is unbounded for the open loop system for all values of p. (b) Set $\mathcal{K}$ in Part 1 (dashed line) and Part 2 (grey area).

The goal is to solve Problem 1 with the choices in (85).

Let us start by using the WDLF approach. We solve the SDP (28) for some values of d, and build each time the candidate controller K^* in (29). Table III shows K^*, its worst-case cost, and the SDP size.

d	K^*	$J^*(K^*)$	SDP size & time
0	N/A	N/A	$[20, 14]$, 1s
1	$(\mathbf{-1.329}, \mathbf{-0.877}, \mathbf{-0.922})$	**9.115**	$[\mathbf{81}, \mathbf{54}]$, **1s**

TABLE III
EXAMPLE 1: RESULTS OBTAINED WITH THE WDLF APPROACH. FOR $d = 0$, THE SDP IS INFEASIBLE.

Next, we use the CI approach:

- firstly (Part 1), with the simple choice $\mathcal{K}_{oes} = \mathbb{R}^{mr}$, which provides $\mathcal{K} = \mathcal{K}_{box}$. We solve the SDP (44) for some values of d, and build each time the candidate controller K^* in (47). Table IV (Part 1) shows K^*, its worst-case cost, and the SDP size. Figure 1a shows the cost $J(K^*, p)$ for $d = 2$;
- secondly (Part 2), with the choice $\mathcal{K}_{oes} = \mathcal{K}_2$, which provides

  $$\begin{aligned} \mathcal{K} \quad = \quad & \{k \in \mathbb{R}^3 : \ k_1 \in [-2, 0], \ k_2 \in [-2, 2], \\ & k_3 \in [-2, 5/4]\}. \end{aligned}$$

  Figure 1b shows $\mathcal{K}$, and Table VI (Part 2) shows the new results obtained. It can be observed that the CI approach in Part 2 solves Problem 1 already for $d = 0$ (rather than for $d = 2$ as in Part 1).

Part 1: $\mathcal{K}_{oes} = \mathbb{R}^{mr}$

d	K^*	$J^*(K^*)$	SDP size & time
0	$(0.003, -0.141, -0.167)$	∞	$[74, 39]$, 1s
1	$(-0.605, -0.667, -0.820)$	11.726	$[156, 71]$, 1s
2	$(\mathbf{-1.994}, \mathbf{-1.622}, \mathbf{-2.000})$	**9.396**	$[\mathbf{1329}, \mathbf{164}]$, **5s**

Part 2: $\mathcal{K}_{oes} = \mathcal{K}_2$

d	K^*	$J^*(K^*)$	SDP size & time
0	$(\mathbf{-1.025}, \mathbf{-0.410}, \mathbf{-0.750})$	**9.338**	$[\mathbf{74}, \mathbf{39}]$, **1s**

TABLE IV
EXAMPLE 1: RESULTS OBTAINED WITH THE CI APPROACH.

For comparison, we test the existing methods:

- [13, Section 3] and [14, Section III], which provide the controller $K = (-1.414, -0.966, -1.100)$ by solving an

SDP with size $[10, 10]$, time 1s. The worst-case cost is $J^*(K) = 9.121$;

- [15, Section II], which provides the controller $K = (-0.758, -0.475, -0.198)$ by solving 125 SDPs with size $[20, 25]$ each (the total time is 13s). The worst-case cost is $J^*(K) = 9.921$.

Concluding, the WDLF and CI approaches and the existing methods solve Problem 1 in this example.

B. Example 2

Consider the CT system (1) with

$$\begin{cases} A(p) = \begin{pmatrix} -1 + 1.6p \ , & 1 - 0.6p \\ -2.5 + 0.6p \ , & -0.5 - 1.6p \end{pmatrix} \\ B(p) = \begin{pmatrix} 0.6p \\ 0.6p + 0.5 \end{pmatrix}, \quad C(p) = I_2. \end{cases}$$

The goal is to solve Problem 1 with the choices in (85).

Let us start by using the WDLF approach. We solve the SDP (28) for some values of d, and build each time the candidate controller K^* in (29). Table V shows K^*, its worst-case cost, and the SDP size.

d	K^*	$J^*(K^*)$	SDP size & time
0	N/A	N/A	$[11, 10]$
1	$(-14.191, -9.975)$	∞	$[44, 39]$, 1s
2	$(-4.679, -0.039)$	∞	$[62, 43]$, 1s
3	$(-11.700, -1.066)$	∞	$[149, 65]$, 1s

TABLE V
EXAMPLE 2: RESULTS OBTAINED WITH THE WDLF APPROACH. FOR $d = 0$, THE SDP IS INFEASIBLE.

Next, we use the CI approach:

- firstly (Part 1), with $\mathcal{K}_{oes} = \mathbb{R}^{mr}$, which provides $\mathcal{K} = \mathcal{K}_{box}$. Table VI (Part 1) shows the results, and Figure 2a shows the cost $J(K^*, p)$ for $d = 2$;
- secondly (Part 2), with $\mathcal{K}_{oes} = \mathcal{K}_2$, which provides the set $\mathcal{K}$ shown in Figure 2b. Table VI (Part 2) shows the new results obtained. It can be observed that the CI approach in Part 2 solves Problem 1 already for $d = 0$ (rather than for $d = 2$ as in Part 1).

Part 1: $\mathcal{K}_{oes} = \mathbb{R}^{mr}$

d	K^*	$J^*(K^*)$	SDP size & time
0	$(0.000, 0.000)$	∞	$[27, 23]$, 1s
1	$(0.000, 0.000)$	∞	$[277, 78]$, 2s
2	$(\mathbf{-0.996, 0.052})$	$\mathbf{4.132}$	$[\mathbf{295, 78}]$, **2s**

Part 2: $\mathcal{K}_{oes} = \mathcal{K}_2$

d	K^*	$J^*(K^*)$	SDP size & time
0	$(\mathbf{-0.639, 0.273})$	$\mathbf{5.381}$	$[\mathbf{35, 29}]$, **1s**

TABLE VI
EXAMPLE 2: RESULTS OBTAINED WITH THE CI APPROACH.

For comparison, we test the existing methods:

- [13, Section 3] and [14, Section III], which do not provide any controller (the SDPs are infeasible);
- [15, Section II], which provides the controller $K = (-0.314, 0.017)$ by solving 400 SDPs with size $[11, 18]$ each (the total time is 33s). The worst-case cost is $J^*(K) = 64.302$.

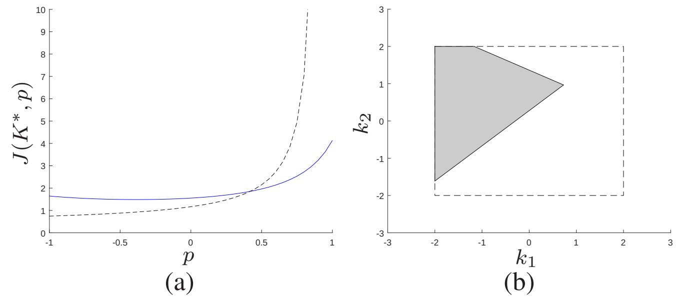

Fig. 2. Example 2: (a) Cost $J(K^*, p)$ for K^* found with the CI approach in Part 1 (blue solid line) and for the open loop system (black dashed line). (b) Set $\mathcal{K}$ in Part 1 (dashed line) and Part 2 (grey area).

Concluding, the CI approach solves Problem 1 in this example, while the WDLF approach and the existing methods do not (no controller provided or worst-case cost larger than γ).

C. Example 3

Consider the CT system (1) with

$$\begin{cases} A(p) = \begin{pmatrix} -1 \ , & p_1^2 \\ p_1 p_2 \ , & p_2 - 1 \end{pmatrix} \\ B(p) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad C(p) = I_2. \end{cases}$$

The goal is to solve Problem 1 with the choices in (85).

Let us start by using the WDLF approach. We solve the SDP (28) for some values of d, and build each time the candidate controller K^* in (29). Table VII shows K^*, its worst-case cost, and the SDP size.

d	K^*	$J^*(K^*)$	SDP size & time
0	$(\mathbf{0.181, 0.951})$	$\mathbf{4.914}$	$[\mathbf{13, 12}]$, **1s**

TABLE VII
EXAMPLE 3: RESULTS OBTAINED WITH THE WDLF APPROACH.

Next, we use the CI approach:

- firstly (Part 1), with $\mathcal{K}_{oes} = \mathbb{R}^{mr}$, which provides $\mathcal{K} = \mathcal{K}_{box}$. Table VIII (Part 1) shows the results, and Figure 3a shows the cost $J(K^*, p)$ for $d = 1$;
- secondly (Part 2), with $\mathcal{K}_{oes} = \mathcal{K}_2$, which provides the set $\mathcal{K}$ shown in Figure 3b. Table VIII (Part 2) shows the new results obtained. It can be observed that the CI approach in Part 2 solves Problem 1 already for $d = 0$ (rather than for $d = 2$ as in Part 1).

Concluding, the WDLF and CI approaches solve Problem 1 in this example. The existing methods cannot be applied since $A(p)$ is not affine and since $\mathcal{P}$ is not a convex bounded polytope.

D. Example 4

Consider the DT system (71) with

$$\begin{cases} A(p) = \begin{pmatrix} 0.5 - 0.3p \ , & -0.5 \\ 0.5p \ , & 0.3 \end{pmatrix} \\ B(p) = \begin{pmatrix} 1 \ , & 0 \\ -1 \ , & 1 \end{pmatrix}, \quad C(p) = \begin{pmatrix} 1 \ , & 0 \end{pmatrix}. \end{cases}$$

Part 1: $\mathcal{K}_{oes} = \mathbb{R}^{mr}$

d	K^*	$J^*(K^*)$	SDP size & time
0	$(-0.046, 0.031)$	31.779	$[31, 25]$, 1s
1	$(\mathbf{-0.528, 2.000})$	**5.014**	$[\mathbf{491, 97}]$, **2s**

Part 2: $\mathcal{K}_{oes} = \mathcal{K}_2$

d	K^*	$J^*(K^*)$	SDP size & time
0	$(\mathbf{-0.346, 1.243})$	**5.350**	$[\mathbf{39, 31}]$, **1s**

TABLE VIII
EXAMPLE 3: RESULTS OBTAINED WITH THE CI APPROACH.

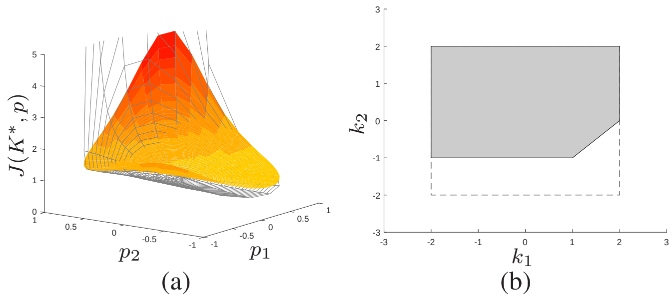

Fig. 3. Example 3: (a) Cost $J(K^*, p)$ for K^* found with the CI approach in Part 1 (filled colored surface) and for the open loop system (unfilled black surface). (b) Set $\mathcal{K}$ in Part 1 (dashed line) and Part 2 (grey area).

The goal is to solve Problem 2 with the choices in (85).

Since the state is not available for feedback in this example, the WDLF approach cannot be used. Hence, we use the CI approach with the changes mentioned in Section VI-C:

- firstly (Part 1), with $\mathcal{K}_{oes} = \mathbb{R}^{mr}$, which provides $\mathcal{K} = \mathcal{K}_{box}$. Table IX (Part 1) shows the results, and Figure 4a shows the cost $J(K^*, p)$ for $d = 2$;
- secondly (Part 2), with $\mathcal{K}_{oes} = \mathcal{K}_2$, which provides the set $\mathcal{K}$ shown in Figure 4b. Table IX (Part 2) shows the new results obtained. It can be observed that the CI approach in Part 2 solves Problem 1 already for $d = 1$ (rather than for $d = 2$ as in Part 1).

Concluding, the CI approach solves Problem 1 in this example. The WDLF approach and the existing methods cannot be applied since the state is not available for feedback and since the dynamics are not CT.

Part 1: $\mathcal{K}_{oes} = \mathbb{R}^{mr}$

d	K^*	$J^*(K^*)$	SDP size & time
0	$(0.000, 0.000)$	∞	$[27, 23]$, 1s
1	$(-0.034, 0.059)$	∞	$[277, 78]$, 2s
2	$(\mathbf{-0.418, -0.077})$	**4.517**	$[\mathbf{295, 78}]$, **2s**

Part 2: $\mathcal{K}_{oes} = \mathcal{K}_2$

d	K^*	$J^*(K^*)$	SDP size & time
0	$(0.000, 0.000)$	∞	$[31, 26]$, 1s
1	$(\mathbf{-0.256, -0.312})$	**3.131**	$[\mathbf{323, 92}]$, **2s**

TABLE IX
EXAMPLE 4: RESULTS OBTAINED WITH THE CI APPROACH.

VIII. CONCLUSIONS

This paper has proposed two approaches based on SDPs for designing robust LQRs for CT and DT LTI uncertain systems. The first approach, named WDLF, is based on the construction of an uncertainty-dependent LQR obtained through Lyapunov functions that weakly depend on the uncertainties. The second approach, named CI, is based on the construction of an index that quantifies the feasibility of different controllers. The proposed approaches have two main advantages with respect to the existing methods, namely, considering not only state-feedback design for polytopic systems but also output-feedback design for systems depending polynomially on the uncertainty, and providing conditions that are not only sufficient but also necessary under some assumptions. These advantages have been illustrated through various examples, where it has been shown that the existing methods may be more conservative or may be not applicable.

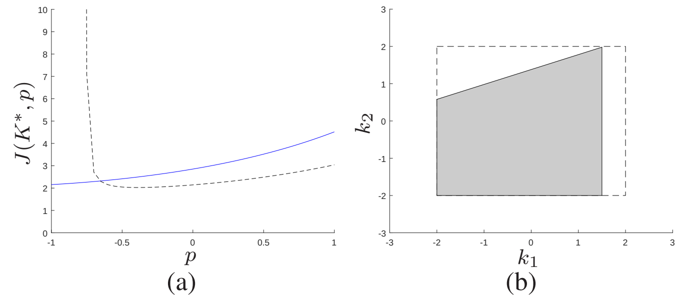

Fig. 4. Example 4: (a) Cost $J(K^*, p)$ for K^* found with the CI approach in Part 1 (blue solid line) and for the open loop system (black dashed line). (b) Set $\mathcal{K}$ in Part 1 (dashed line) and Part 2 (grey area).

Various directions can be explored in future work. One of these is the extension to the case where the system matrices are rational functions of uncertainties. Another direction can be the extension to the case where the uncertainties are time-varying. Last but not least, it would be interesting and useful to explore the possibility of reducing the numerical complexity in order for the proposed approaches to be applicable to large scale systems or in real time applications.

ACKNOWLEDGMENTS

The author thanks the Editors and the Reviewers for their useful comments that have significantly improved the quality of this paper.

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