Input-output Gain Analysis of Linear Discrete-time Systems with Cone Invariance

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***Abstract*—This paper investigates the input-output gain of linear discrete-time cone-preserving systems. The cone linear absolute-norm and cone max-norm, which are applied to describe input-output gains of cone-preserving systems, are introduced. Subsequently, by utilizing the property of cone-preserving systems, several necessary and sufficient conditions to characterize input-output gains of the system in terms of cone-induced norms are provided. The results indicate that input-output gains of cone-preserving systems can be characterized by the static gain matrix. The duality between the two cone-induced gains is also unveiled.**

***Index Terms*—Cone invariance, input-output gain, positive systems, static gain.**

I. Introduction

The study of positive systems, characterized by trajectories that remain within the nonnegative orthant under any nonnegative initial conditions, has garnered significant interest recently. This is attributed to their extensive applications in real-world physical processes that involve nonnegative variables, such as viral infections [8], disease transmission [1], and electrical networks [14]. Different from positive systems, the state of systems with cone-invariance resides in a proper cone rather than the nonnegative orthant. This property enables systems with cone invariance to not only generalize positive systems, but also to possess distinct applications involving cooperation and comparison, such as the rendezvous in multi-agent systems [4] and chemical reaction networks [2].

Stability and input-output gain, as fundamental properties in analyzing dynamic systems, have drawn significant attention in the study of positive systems in recent years. The research revealed some unique properties for positive systems. As stated in Theorem 13 of [9], the relationship between the stability of positive systems and property of the system matrix has been established. The result also showed that the equivalent stability condition for linear positive systems is characterized by linear inequalities. Furthermore, the stability conditions of different types of positive systems, e.g., time-delayed positive systems [11], [13], [15], [16], switched positive systems [23], positive periodic systems [24], have been extensively analyzed. The foundation of performance analysis for positive systems is stability. Along this line, when the system is stable, the input-output performance of positive systems which is mainly referred to the L_1- and L_∞-gains has been investigated. In [5], the analytical formula of L_1- and L_∞-gain for linear continuous-time positive systems were given, and the result showed that the L_1-gain of a positive system equals the L_∞-gain of its dual positive system. The discrete-time case was taken into consideration in [6], [18]. It showed the DC-dominance property of positive systems. In other words, the input-output gains of positive systems are characterized by the static gain matrix.

As mentioned before, cone-preserving systems can be viewed as the generalization of positive systems. The question naturally arises whether cone-preserving systems also exhibit these unique properties characteristic of positive systems. Recently, there are some studies on cone-preserving systems [7], [10], [21]. The research showed that some unique properties of positive systems are due to cone-preserving properties rather than nonnegativity. The stability of time-delayed cone-preserving systems was analyzed in [19], [22], [25]. These studies established equivalent asymptotic stability conditions for continuous-time/discrete-time systems, featuring cone invariance and various types of time delays. The results indicated that the stability of cone-preserving systems with time delays remains unaffected by the magnitude and variation of these delays. Since the stability is a prerequisite condition for input-output gain analysis, upon examining the stability conditions, investigating the input-output performance of cone-preserving systems is a logical step. In [20], Shen and Lam applied the cone linear absolute-norm and cone max-norm instead of the L_1- and L_∞-norm in positive systems to describe the input-output performance for linear continuous-time cone-preserving systems. The result showed that the input-output gain of the cone-preserving systems is exactly characterized by the static gain matrix.

Motivated by above works, criteria for analyzing input-output gains of linear discrete-time cone-preserving systems are investigated in this paper. The definitions of cone linear absolute-norm and cone max-norm introduced in [20] are recalled first. Equivalent conditions to guarantee the cone-preserving property and the asymptotic stability of the system are introduced. Then, by utilizing the two cone-induced

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norms, the characterizations of input-output gains of linear discrete-time cone-preserving systems are given. Furthermore, equivalent conditions that apply partial order relationship to describe the input-output gain are also investigated.

II. Preliminaries

Mathematical notions and cone related definitions are first introduced. To begin, we present the following mathematical notions. $\mathbb{N}=\{1,2,\dots\}$ and $\mathbb{N}_0=\{0,1,\dots\}$. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the sets of n-dimensional real vector spaces and real matrices with dimensions $n\times m$, respectively. A^{T} represents the transpose of A. I represents an identity matrix of a suitable dimension.

Preliminary definitions about cones from Chapter 1 of [3] are revisited. Let set $S\subseteq\mathbb{R}^n$, S^G represents the set containing all nonnegative linear combinations of elements within the set S. ∂S and $\mathrm{int}S$ represent the boundary and interior of set S, respectively. A cone K is said to be closed if it satisfies the condition $K=K^G$, solid if $\mathrm{int}K\neq\emptyset$, and pointed if $K\cap(-K)=\{0\}$. If the cone is closed, solid, and pointed, it is said to be a proper cone. The dual of cone K is denoted by the set $K^*=\left\{y\in\mathbb{R}^n \mid y^{\mathrm{T}}x\geq 0,\forall x\in K\right\}$. With the definition of a proper cone, several partial order relations are introduced. $x\prec_K y$ means that $y-x\in\mathrm{int}K$, while $x\preceq_K y$ means that $y-x\in K$. Moreover, matrix A is referred to as a K-nonnegative matrix, if $Ax\in K,\ \forall x\in K$.

Definitions related to cone linear absolute-norm, cone max-norm, and cone-induced matrix norm from [17] are recalled.

Definition 1. [17] *Given a proper cone $K\subset\mathbb{R}^n$ and a vector $\eta\in\mathrm{int}K^*$, the cone linear absolute norm of the vector $x\in\mathbb{R}^n$ is defined as*

$$\|x\|_{\eta,1}=\inf\left\{\eta^{\mathrm{T}}u:-u\preceq_K x\preceq_K u\right\}.$$

Remark 1. *When $x\in K$, the set $V_1=\{u\mid -u\preceq_K x\preceq_K u\}$ equals set $V_2=\{u\mid u=x+y,\forall y\in K\}$, and the cone linear absolute-norm satisfies*

$$\|x\|_{\eta,1}=\inf\left\{\eta^{\mathrm{T}}u:u=x+y,\forall y\in K\right\}.$$

Since $\eta\in\mathrm{int}K^*$, one can conclude that $\eta^{\mathrm{T}}u\geq\eta^{\mathrm{T}}x$. By letting $y=0$, the infimum of $\eta^{\mathrm{T}}u$ is obtained and the cone linear absolute-norm is calculated by $\|x\|_{\eta,1}=\eta^{\mathrm{T}}x$, where $x\in K$.

Definition 2. [20] *Given a proper cone $K\subset\mathbb{R}^n$ and a vector $u\in\mathrm{int}K$, an order interval B_u is given as*

$$B_u=\{x\in\mathbb{R}^n\mid -u\preceq_K x\preceq_K u\}.$$

And the cone max-norm of the vector $y\in\mathbb{R}^n$ is defined as

$$\|y\|_{u,\infty}=\inf\{t\geq 0\mid y\in tB_u\}.$$

It could be found that the cone max-norm $\|x\|_{u,\infty}$ exhibits monotonic behavior with respect to cone K (i.e. $\|x\|_{u,\infty}\geq\|y\|_{u,\infty}$ for all $x\succeq_K y$.).

Definition 3. [3] *Let $u\in\mathrm{int}K$, the cone induced operator norm for the matrix is defined as*

$$\|A\|_{u,\infty}=\sup_{\|x\|_{u,\infty}=1}\|Ax\|_{u,\infty}.$$

Moreover, if A is K-nonnegative, then $\|A\|_{u,\infty}=\|Au\|_{u,\infty}$ holds.

III. Main Results

The definition of system with cone invariance is given, and several input-output gains of discrete-time cone-preserving system characterized by two cone-induced norms are discussed in this section. A linear discrete-time system is first taken into consideration,

$$\begin{aligned}x(k+1)&=Ax(k)+B_w w(k),\\ y(k)&=Cx(k)+D_w w(k),\end{aligned}\tag{1}$$

where $x(k)\in\mathbb{R}^{n_x}$, $w(k)\in\mathbb{R}^{n_w}$ and $y(k)\in\mathbb{R}^{n_y}$ are state vector, disturbance vector, and output vector accordingly. The system matrices $A\in\mathbb{R}^{n_x\times n_x}$, $B_w\in\mathbb{R}^{n_x\times n_w}$, $C\in\mathbb{R}^{n_y\times n_x}$, and $D_w\in\mathbb{R}^{n_y\times n_w}$. The definition of a cone-preserving system and an equivalent condition for this is given.

Definition 4. *Given three proper cones $K_x\subset\mathbb{R}^{n_x}$, $K_w\subset\mathbb{R}^{n_w}$ and $K_y\subset\mathbb{R}^{n_y}$, system (1) is said to be monotone with respect to (K_x,K_w,K_y) if, for any disturbance $w(k)\in K_w$ and initial condition $x(0)\in K_x$, system (1) satisfies that $x(k)\in K_x$ and $y(k)\in K_y$ for all $k\in\mathbb{N}_0$.*

Lemma 1. [19], [25] *Given three proper cones $K_x\subset\mathbb{R}^{n_x}$, $K_w\subset\mathbb{R}^{n_w}$ and $K_y\subset\mathbb{R}^{n_y}$, system (1) is monotone with respect to (K_x,K_w,K_y) if and only if A is K_x-nonnegative, $B_wK_w\subset K_x$, $CK_x\subset K_y$ and $D_wK_w\subset K_y$.*

Now, the asymptotic stability of system (1) that is monotone with respect to (K_x,K_w,K_y) is considered. As a cone-preserving system with time delay to be zero, the asymptotic stability criterion in [19] can be applied to the system (1). Furthermore, an alternative asymptotic stability condition is proposed, which is derived from the equivalence between Schur stability and asymptotic stability as demonstrated in [25].

Lemma 2. [19], [25] *For a linear discrete-time cone-preserving system (1) with disturbance $w(k)=0,\ \forall k\in\mathbb{N}_0$, the statements below are equivalent:*
i) System (1) is asymptotically stable;
ii) Matrix A is a Schur matrix;
iii) There exists a vector $\lambda\in\mathrm{int}K_x$ satisfying $(A-I)\lambda\prec_{K_x}0$.

Remark 2. *Note that, if there exists a vector $\lambda\in K_x$ satisfying condition $(A-I)\lambda\prec_{K_x}0$, one can find a positive scalar l satisfying $(A-I)(\lambda+lv')\prec_{K_x}0$ for any vector $v'\succ_{K_x}0$. In other words, the condition that there exists a vector $\lambda\in K_x$ satisfying $(A-I)\lambda\prec_{K_x}0$, is also an equivalent stability condition for system (1). Similar results for positive systems can be found in [6] and Chapter 2 of [12].*

Lemma 2 gives an equivalent condition to determine whether system (1) is asymptotically stable, and it also shows an equivalent condition to determine whether a K_x-nonnegative matrix is a Schur matrix.

A. Cone Linear Absolute-Norm Induced Gain

In this subsection, cone linear absolute-norm induced gain is investigated. According to Definition 1, the value of cone linear absolute-norm can be calculated directly. Combining the asymptotic stability condition and the definition of cone linear absolute-norm, we can use the cone linear absolute-norm to describe the input-output performance of system (1), and theorems are given.

Theorem 1. *Given proper cones K_x, K_w, K_y and vectors $v_1 \in \text{int}K_w^*$, $v_2 \in \text{int}K_y^*$. Suppose system (1) is monotone with respect to (K_x, K_w, K_y) and asymptotically stable. Then, for $v \in K_w$, under zero initial conditions, there exists a scalar γ such that $\sum_{k=0}^{\infty} \|y(k)\|_{v_2,1} \leq \gamma \sum_{k=0}^{\infty} \|w(k)\|_{v_1,1}$ holds for all $w(k) \in K_w$ and the infimum of γ is*

$$\gamma_0 = \inf\{\gamma\} = \sup_{\|v\|_{v_1,1}=1} \left\| \left[C(I-A)^{-1}B_w + D_w\right] v \right\|_{v_2,1}. \quad (2)$$

Proof. According to state space equation of system (1), when the initial conditions $x(0) = 0$, one has

$$\begin{aligned} x(1) &= B_w w(0), \\ x(2) &= AB_w w(0) + B_w w(1), \\ &\vdots \\ x(n) &= A^{n-1}B_w w(0) + A^{n-2}B_w w(1) + \cdots \\ &\quad + AB_w w(n-2) + B_w w(n-1). \end{aligned} \quad (3)$$

By state space equation of system (1) and (3), the output vector $y(k)$ satisfies

$$\begin{aligned} y(0) &= D_w w(0), \\ y(1) &= CB_w w(0) + D_w w(1), \\ y(2) &= CAB_w w(0) + CB_w w(1) + D_w w(2), \\ &\vdots \\ y(n) &= CA^{n-1}B_w w(0) + \cdots + CAB_w w(n-2) \\ &\quad + CB_w w(n-1) + D_w w(n). \end{aligned} \quad (4)$$

By Definition 1 and (4), we have

$$\begin{aligned} \sum_{k=0}^{0} \|y(k)\|_{v_2,1} &= v_2^{\mathrm{T}} D_w w(0), \\ \sum_{k=0}^{1} \|y(k)\|_{v_2,1} &= v_2^{\mathrm{T}} \left(D_w w(0) + CB_w w(0)\right) + v_2^{\mathrm{T}} D_w w(1), \\ &\vdots \\ \sum_{k=0}^{n} \|y(k)\|_{v_2,1} &= \sum_{l=0}^{n-1} \left[v_2^{\mathrm{T}} \left(D_w + C \left(\sum_{m=0}^{n-1-l} A^m \right) B_w \right) w(l) \right] \\ &\quad + v_2^{\mathrm{T}} D_w w(n). \end{aligned} \quad (5)$$

Since $v_2^{\mathrm{T}} CA^{k_1} B_w w(k_2) \geq 0$ for all $k_1 \in \mathbb{N}_0$ and $k_2 \in \mathbb{N}_0$, (5) gives the inequality

$$\sum_{k=0}^{n} \|y(k)\|_{v_2,1} \leq \sum_{l=0}^{n-1} \left[v_2^{\mathrm{T}} C \left(\sum_{m=0}^{n-1} A^m \right) B_w w(l) \right] + \sum_{l=0}^{n} v_2^{\mathrm{T}} D_w w(l). \quad (6)$$

When $n \to \infty$, inequality (6) becomes

$$\sum_{k=0}^{\infty} \|y(k)\|_{v_2,1} \leq \sum_{l=0}^{\infty} v_2^{\mathrm{T}} D_w w(l) + \sum_{l=0}^{\infty} \left[v_2^{\mathrm{T}} C \left(\sum_{m=0}^{\infty} A^m \right) B_w w(l) \right]. \quad (7)$$

Since system (1) is asymptotically stable, inequality (7) implies

$$\begin{aligned} \sum_{k=0}^{\infty} \|y(k)\|_{v_2,1} &\leq v_2^{\mathrm{T}} D_w \sum_{l=0}^{\infty} w(l) + v_2^{\mathrm{T}} C(I-A)^{-1} B_w \sum_{l=0}^{\infty} w(l) \\ &\leq \gamma_0 v_1^{\mathrm{T}} \sum_{l=0}^{\infty} w(l) = \gamma_0 \sum_{l=0}^{\infty} \|w(l)\|_{v_1,1} \end{aligned} \quad (8)$$

holds for all $w(k) \in K_w$. By inequality (8), one can conclude that γ_0 is one of the scalar γ letting inequality $\sum_{k=0}^{\infty} \|y(k)\|_{v_2,1} \leq \gamma \sum_{k=0}^{\infty} \|w(k)\|_{v_1,1}$ holds.

Then our goal is to check whether γ_0 is infimum of γ, in other words, whether γ can reach the value of γ_0. Suppose $w(0) = v_1'$, where $v_1' \in K_w$ with $v_1^{\mathrm{T}} v_1' = 1$, and $w(k) = 0$ for all $k \geq 1$. Then according to (5), the equation

$$\sum_{k=0}^{n} \|y(k)\|_{v_2,1} = v_2^{\mathrm{T}} D_w w(0) + \left[v_2^{\mathrm{T}} C \left(\sum_{m=0}^{n-1} A^m \right) B_w w(0) \right] \quad (9)$$

holds. When $n \to \infty$, (9) gives

$$\lim_{n\to\infty} \sum_{k=0}^{n} \|y(k)\|_{v_2,1} = v_2^{\mathrm{T}} \left[D_w + C(I-A)^{-1}B_w\right] v_1'. \quad (10)$$

By choosing vector v_1' to maximize the right hand side of (10), one has

$$\lim_{n\to\infty} \sum_{k=0}^{n} \|y(k)\|_{v_2,1} = \sup_{\|v\|_{v_1,1}=1} \left\| \left[C(I-A)^{-1}B_w + D_w\right] v \right\|_{v_2,1} = \gamma_0. \quad (11)$$

The γ can reach the value of γ_0 and Theorem 1 is proved. □

Theorem 1 gives a way of using cone linear absolute norm to describe the input-output performance of system (1). In our works, cones K_x, K_y, K_w and vectors v_1, v_2 are arbitrarily chosen proper cones and vectors, respectively. When the proper cones K_x, K_y, K_w are positive orthants and vectors v_1, v_2 have all entries equal to 1, equation (2) turns to the characterization of ℓ_1-gain of positive systems. For a positive system, the ℓ_1-gain cannot only be calculated by the 1-norm of the transition matrix but also be calculated via linear inequalities. It arouses our interest whether, for system (1), we can find several partial order inequalities to characterize the cone linear absolute-norm induced gain of system (1). The conditions are given in Theorem 2 below.

Theorem 2. *Given proper cones K_x, K_w, K_y and vectors $v_1 \in \text{int}K_w^*$, $v_2 \in \text{int}K_y^*$. Suppose system (1) is monotone with respect to (K_x, K_w, K_y), system (1) is asymptotically stable and satisfies $\sum_{k=0}^{\infty} \|y(k)\|_{v_2,1} < \gamma \sum_{k=0}^{\infty} \|w(k)\|_{v_1,1}$ for all nonzero $w(k) \in K_w$, if and only if there exist a scalar $\gamma > 0$ and a vector $p \succ_{K_x^*} 0$ satisfying*

$$(A-I)^{\mathrm{T}} p + C^{\mathrm{T}} v_2 \prec_{K_x^*} 0, \quad (12)$$

$$B_w^{\mathrm{T}} p + D_w^{\mathrm{T}} v_2 \prec_{K_w^*} \gamma v_1. \quad (13)$$

Proof. Sufficiency: The asymptotic stability is proved first. According to inequalities (12), the inequality holds as follows:

$$(A-I)^{\mathrm{T}} p \prec_{K_x^*} -C^{\mathrm{T}} v_2. \quad (14)$$

Since $v_2 \in \text{int}K_y^*$ and $Cx' \in K_y$, for any $x' \in K_x$, inequality $v_2^{\mathrm{T}} Cx' \geq 0$ holds. Based on the definition of dual of a cone,

$C^{\mathrm{T}}v_2 \in K_x^*$ holds. Therefore, when inequality (14) holds, there exists a vector $p \succ_{K_x^*} 0$ such that $(A-I)^{\mathrm{T}} p \prec_{K_x^*} 0$. Based on Lemma 2, system (1) is asymptotically stable.

Since $y(k) \in K_y$ and $w(k) \in K_w$, according to Remark 1, the cone linear absolute-norm of $y(k)$ and $w(k)$ are written as

$$\|y(k)\|_{v_2,1} = v_2^{\mathrm{T}} y(k), \tag{15}$$

$$\|w(k)\|_{v_1,1} = v_1^{\mathrm{T}} w(k). \tag{16}$$

Then the function $\sum_{k=0}^{\infty} \|y(k)\|_{v_2,1} - \sum_{k=0}^{\infty} \gamma \|w(k)\|_{v_1,1}$ is written as

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left(\|y(k)\|_{v_2,1} - \gamma \|w(k)\|_{v_1,1}\right) \\
&= \sum_{k=0}^{\infty} \left(v_2^{\mathrm{T}} y(k) - \gamma v_1^{\mathrm{T}} w(k)\right) \\
&= \sum_{k=0}^{\infty} \left[v_2^{\mathrm{T}} \left(Cx(k) + D_w w(k)\right) - \gamma v_1^{\mathrm{T}} w(k)\right].
\end{aligned} \tag{17}$$

According to inequality (13), (17) satisfies the following inequality:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left[v_2^{\mathrm{T}} \left(Cx(k) + D_w w(k)\right) - \gamma v_1^{\mathrm{T}} w(k)\right] \\
&\leq \sum_{k=0}^{\infty} \left(v_2^{\mathrm{T}} Cx(k) - p^{\mathrm{T}} B_w w(k)\right).
\end{aligned} \tag{18}$$

Based on inequality (12), inequality (18) can be further simplified to

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left[v_2^{\mathrm{T}} \left(Cx(k) + D_w w(k)\right) - \gamma v_1^{\mathrm{T}} w(k)\right] \\
&\leq \sum_{k=0}^{\infty} \left[p^{\mathrm{T}}(I-A)x(k) - p^{\mathrm{T}} B_w w(k)\right] \\
&= \sum_{k=0}^{\infty} \left[p^{\mathrm{T}} \left(x(k) - x(k+1)\right)\right] \\
&= \lim_{k\to\infty} \left[p^{\mathrm{T}} \left(x(0) - x(k)\right)\right] = p^{\mathrm{T}} x(0) = 0.
\end{aligned} \tag{19}$$

The sufficiency of conditions (12) and (13) in Theorem 2 is proved.

Necessity: According to Theorem 1, inequality

$$\gamma > \left\| \left[C(I-A)^{-1} B_w + D_w\right] v \right\|_{v_2,1}, \tag{20}$$

holds for all $\|v\|_{v_1,1} = 1$, where $v \in K_w$. It can be rewritten as

$$v_2^{\mathrm{T}} \left[C(I-A)^{-1} B_w + D_w\right] v < \gamma, \tag{21}$$

for all $\|v\|_{v_1,1} = 1$, where $v \in K_w$. For inequality (21), we first define a vector $p = (I-A)^{-\mathrm{T}} \left(C^{\mathrm{T}} v_2 + \varepsilon \xi\right)$, where $\varepsilon > 0$ and $\xi \in \mathrm{int}K_x^*$. One can find that $p \in \mathrm{int}K_x^*$. Based on inequality (21), it is always the case that a sufficiently small scalar $\varepsilon > 0$ exists, so that

$$v^{\mathrm{T}} \left(B_w^{\mathrm{T}} p + D_w^{\mathrm{T}} v_2\right) < \gamma v^{\mathrm{T}} v_1. \tag{22}$$

Notice that inequality (22) holds for all $v \in K_w$, and $\gamma v_1 - \left(B_w^{\mathrm{T}} p + D_w^{\mathrm{T}} v_2\right) \in \mathrm{int}K_w^*$. Then we can claim that the vector p satisfies inequality (13). Furthermore, inequality

$$\begin{aligned}
(A-I)^{\mathrm{T}} p + C^{\mathrm{T}} v_2 &= (A-I)^{\mathrm{T}} (I-A)^{-\mathrm{T}} \left(C^{\mathrm{T}} v_2 + \varepsilon \xi\right) + C^{\mathrm{T}} v_2 \\
&= -C^{\mathrm{T}} v_2 - \varepsilon \xi + C^{\mathrm{T}} v_2 \prec_{K_x^*} 0
\end{aligned} \tag{23}$$

holds. The proof of necessity has been established.. □

B. Cone Max-Norm Induced Gain

The cone max-norm induced gain of system (1) is analyzed in this subsection. Similar to section 3.1, two theorems will be given to characterize the cone max-norm induced gain of system (1).

Theorem 3. *Given proper cones K_x, K_w, K_y and vectors $v_3 \in \mathrm{int}K_w$, $v_4 \in \mathrm{int}K_y$. Suppose system (1) is monotone with respect to (K_x, K_w, K_y) and asymptotically stable, then there exists a scalar η letting $\sup \|y(k)\|_{v_4,\infty} \leq \eta \sup \|w(k)\|_{v_3,\infty}$ for all $w(k) \in K_w$ and the infimum of η is*

$$\eta_0 = \inf\{\eta\} = \left\| \left[C(I-A)^{-1} B_w + D_w\right] v_3 \right\|_{v_4,\infty}. \tag{24}$$

Proof. Without loss of generality, we assume $\sup \|w(k)\|_{v_3,\infty} = 1$. A system with constant disturbance $w(k) \equiv v_3$ is given as follows:

$$\begin{aligned}
\bar{x}(k+1) &= A\bar{x}(k) + B_w v_3, \\
\bar{y}(k) &= C\bar{x}(k) + D_w v_3.
\end{aligned} \tag{25}$$

Letting $e_x(k) = \bar{x}(k) - x(k)$ and $e_y(k) = \bar{y}(k) - y(k)$, a system for $e_x(k)$ and $e_y(k)$ is given as follows:

$$\begin{aligned}
e_x(k+1) &= Ae_x(k) + B_w \left(v_3 - w(k)\right), \\
e_y(k) &= Ce_x(k) + D_w \left(v_3 - w(k)\right).
\end{aligned} \tag{26}$$

According to Lemma 1, the inequalities $\bar{x}(k) \succeq_{K_x} x(k)$ and $\bar{y}(k) \succeq_{K_y} y(k)$ hold for all $\|w(k)\|_{v_3,\infty} = 1$, and $\sup \|\bar{y}(k)\|_{v_4,\infty}$ is the infimum of η. By the state equation of system (25), $\bar{y}(k)$ is denoted as

$$\bar{y}(k) = C \left(\sum_{i=0}^{k-1} A^i\right) B_w v_3 + D_w v_3. \tag{27}$$

According to (27), inequality

$$\bar{y}(k+1) - \bar{y}(k) = CA^k B_w v_3 \succeq_{K_y} 0 \tag{28}$$

holds for all $k \in \mathbb{N}_0$, and value of $\|\bar{y}(k)\|_{v_4,\infty}$ increases with the increase of k. When $k \to \infty$, (27) leads to

$$\lim_{k\to\infty} \bar{y}(k) = C(I-A)^{-1} B_w v_3 + D_w v_3. \tag{29}$$

According to (29), the infimum of η is

$$\left\| \left[C(I-A)^{-1} B_w + D_w\right] v_3 \right\|_{v_4,\infty}, \tag{30}$$

and Theorem 3 is proved. □

Remark 3. *The infimum of η in Theorem 3 can also be written as the supremum of a cone max-norm which is similar to (2) in Theorem 1 as follows:*

$$\sup_{\|v\|_{v_3,\infty}=1} \left\| \left[C(I-A)^{-1} B_w + D_w\right] v \right\|_{v_4,\infty}. \tag{31}$$

Since the expression in (31) reaches the maximum value when $v = v_3$, we use (24) instead of (31) in Theorem 3.

Similar to Theorem 2, the cone max-norm induced gain can be characterized by partial order inequalities, and the results are given in Theorem 4 below.

Theorem 4. *Given proper cones K_x, K_w, K_y and vectors $v_3 \in \mathrm{int}K_w$, $v_4 \in \mathrm{int}K_y$. Suppose system (1) is monotone with respect to (K_x, K_w, K_y), system (1) is asymptotically stable and satisfies*

$\sup\|y(k)\|_{v_4,\infty} < \eta \sup\|w(k)\|_{v_3,\infty}$ *for all* $w(k)\in K_w$*, if and only if there exist a scalar* $\eta>0$ *and a vector* $p\succ_{K_x} 0$ *satisfying*

$$(A-I)p+B_w v_3 \prec_{K_x} 0, \tag{32}$$
$$Cp+D_w v_3 \prec_{K_y} \eta v_4. \tag{33}$$

Proof. Sufficiency: First, the asymptotic stability is proved. Based on inequality (32), an inequality

$$(A-I)\,p \prec_{K_x} -B_w v_3 \preceq_{K_x} 0 \tag{34}$$

hold. According to Lemma 2, the systems is asymptotically stable. Then one can assume that $\sup\|w(k)\|_{v_3,\infty}=f$, where $f>0$. By applying mathematical induction, the fact that there exists a vector $p\in \mathrm{int}K_x$ such that $x(k)\preceq_{K_x} fp$ for all $k\in\mathbb{N}_0$ is first proved below. When $k=0$, we have $x(0)=0\prec_{K_x} fp$. Then we assume that $x(i)\preceq_{K_x} fp$ for all $i\le n$. Let $k=n$, we have

$$\begin{aligned} x(n+1) &= Ax(n)+B_w w(n)\\ &\preceq_{K_x} fAp+B_w w(n)\\ &\preceq_{K_x} f(Ap+B_w v_3). \end{aligned} \tag{35}$$

According to inequality (32), inequality $x(n+1)\preceq_{K_x} fp$ holds. By the above induction, the inequality $x(k)\preceq_{K_x} fp,\ \forall k\in\mathbb{N}_0$ holds. According to inequality (33), the following inequality

$$\begin{aligned} y(k) &= Cx(k)+D_w w(k),\\ &\preceq_{K_y} f(Cp+D_w v_3),\\ &\prec_{K_y} f\eta v_4, \end{aligned} \tag{36}$$

holds for all $k\in\mathbb{N}_0$. Since system (1) is monotone with respect to (K_x,K_w,K_y), the inequality $y(k)+f\eta v_4\succeq_{K_y} 0$ holds. Combining with inequality (36), the cone max-norm of $y(k)$ satisfies the inequality $\|y(k)\|_{v_3,\infty}<f\eta$. The proof of the sufficiency has been established.

Necessity: Since system (1) is asymptotically stable and satisfies $\sup\|y(k)\|_{v_4,\infty}<\eta\sup\|w(k)\|_{v_3,\infty}$, matrix A is a Schur matrix and the inequality

$$\left\|\left[C(I-A)^{-1}B_w+D_w\right]v_3\right\|_{v_4,\infty}<\eta \tag{37}$$

holds. Inequality (37) also indicates that there exists a positive scalar $\varepsilon>0$ such that

$$\left[C(I-A)^{-1}B_w+D_w\right]v_3+\varepsilon q\prec_{K_y}\eta v_4, \tag{38}$$

holds, for all $\|q\|_2\le 1$. By letting $p=(I-A)^{-1}(B_w v_3+\varepsilon v')\in \mathrm{int}K_x$, where $v'\in\mathrm{int}K_x$ and $\left\|(I-A)^{-1}v'\right\|_2\le 1$, the following two inequalities

$$(A-I)p+B_w v_3\prec_{K_x} 0, \tag{39}$$
$$Cp+D_w v_3\prec_{K_y}\eta v_4, \tag{40}$$

hold. The necessity of Theorem 4 is proved. □

Remark 4. *Theorems 1–4 provide the characterization of the cone-induced gains using cone-induced norms and partial order inequalities. When the given cones are n-dimensional polyhedral cones with n edges, one can employ the affine transformation to convert the characterization of the cone-induced gains into the weighted* ℓ_1*- or* ℓ_∞*-gain characterization. However, for a broader range of proper cones, such as second-order cones, determining how to utilize linear inequalities or linear matrix inequalities to characterize the conditions in Theorems 1–4 remains a potentially significant future research direction.*

C. Duality of Cone-Induced Gain

For discrete-time positive systems, there is duality property that the ℓ_1-gain of a positive system is equal to the ℓ_∞-gain of the dual of the system. For cone-preserving systems, this property is not intuitive. In the following, we will discuss whether such a property holds for cone-preserving systems. First, a dual system of the discrete-time cone-preserving system (1) is given as follows:

$$\begin{aligned} \bar{x}(k+1) &= A^{\mathrm{T}}\bar{x}(k)+C^{\mathrm{T}}\bar{w}(k),\\ \bar{y}(k) &= B_w^{\mathrm{T}}\bar{x}(k)+D_w^{\mathrm{T}}\bar{w}(k). \end{aligned} \tag{41}$$

It should be pointed out that the proper cones of $\bar{x}(k)$, $\bar{w}(k)$ and $\bar{y}(k)$ are changed into K_x^*, K_y^* and K_w^* accordingly. The reason why the cones of $\bar{x}(k)$, $\bar{w}(k)$ and $\bar{y}(k)$ are different from the ones in system (1) is given in what follows. Take disturbance $\bar{w}(k)$ as an example. For system (1), $Cv\in K_y$ holds for all $v\in K_x$. For a vector $v'\in K_y^*$, inequality $v'^{\mathrm{T}}Cv\ge 0$ always holds. In other words, $v^{\mathrm{T}}C^{\mathrm{T}}v'\ge 0$ holds for all $v\in K_x$, and $C^{\mathrm{T}}v'\in K_x^*$ for all $v'\in K_y^*$. Therefore, $\bar{w}(k)\in K_y^*$ and so do matrices A, B_w and D_w and $\bar{x}(k)\in K_x^*$, $\bar{y}(k)\in K_w^*$. Based on Theorem 2 and Theorem 4 for the cone linear absolute-norm induced gain and the cone max-norm induced gain, the duality property of cone-induced gains is given as follows.

Theorem 5. *The cone linear absolute-norm of the cone-preserving system (1) is equal to the cone max-norm of the dual of the system (41) for all* $v_1=v_4\in K_w^*$ *and* $v_2=v_3\in K_y^*$*, where* v_1 *and* v_2 *are the vectors of cone linear absolute-norm induced gain in system (1) and* v_3 *and* v_4 *are the vectors of cone-max norm induced gain in dual system (41).*

Proof. Theorem 5 is proved by contradiction. We assume that the cone linear absolute-norm induced gain of system (1) is γ_1 and the cone max-norm induced gain of system (41) is γ_2, where $\gamma_1\neq\gamma_2$. Let $\bar{\gamma}=0.5(\gamma_1+\gamma_2)$. First, we assume that $\gamma_2>\gamma_1$, which indicates the inequality $\gamma_2>\bar{\gamma}>\gamma_1$. Since γ_1 is the infimum of γ satisfying (12)–(13), one can find a vector $p'\succ_{K_x^*} 0$ such that

$$(A-I)^{\mathrm{T}}p'+C^{\mathrm{T}}v_2\prec_{K_x^*} 0, \tag{42}$$
$$B_w^{\mathrm{T}}p'+D_w^{\mathrm{T}}v_2\prec_{K_w^*}\bar{\gamma}v_1, \tag{43}$$

based on Theorem 2. According to Theorem 4, when inequalities (42)–(43) hold, the cone-max norm induced gain of system (41) satisfies $\gamma_2\le\bar{\gamma}$, which contradicts the assumption that $\gamma_2>\bar{\gamma}$. Then, one has shown that the assumption $\gamma_2<\gamma_1$ is false in a similar way, which is omitted here. Therefore, $\gamma_1=\gamma_2$, and the Theorem 5 is proved. □

Theorem 5 also indicates that the expression

$$\begin{aligned} &\sup_{\|v\|_{v_1,1}=1}\left\|\left[C(I-A)^{-1}B_w+D_w\right]v\right\|_{v_2,1}\\ &=\sup_{\|v\|_{v_2,\infty}=1}\left\|\left[B_w^{\mathrm{T}}(I-A)^{-\mathrm{T}}C^{\mathrm{T}}+D_w^{\mathrm{T}}\right]v\right\|_{v_1,\infty} \end{aligned}$$

holds.

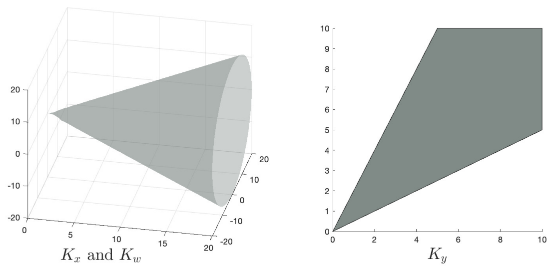

Fig. 1. Cones K_x, K_w and K_y

IV. Illustrative Examples

Example 1. *The following system matrices of system (1) are used in the first example.*

$$A=\begin{bmatrix}0.37 & -0.06 & 0.10\\ 0.08 & 0.18 & -0.16\\ 0.10 & -0.12 & 0.09\end{bmatrix},\ B_w=\begin{bmatrix}2.07 & -0.35 & -0.45\\ 0.36 & 1.35 & -0.42\\ 0.72 & 0.59 & -0.75\end{bmatrix},$$
$$C=\begin{bmatrix}0.27 & -0.09 & 0.35\\ 0.29 & -0.13 & -0.41\end{bmatrix},\ D_w=\begin{bmatrix}0.51 & 0.03 & 0.05\\ 0.50 & 0.07 & 0.01\end{bmatrix}.$$

The K_x and K_w are three-dimensional second-order cones $\mathscr{I}$ satisfying

$$\mathscr{I}=\left\{x\in\mathbb{R}^3 : x^{\mathrm{T}}Qx\geq 0, x^{\mathrm{T}}e\geq 0\right\},$$

where $e=[1,\ 0,\ 0]^{\mathrm{T}}$ and $Q_3=2ee^{\mathrm{T}}-I$, and K_y is a polyhedral cone given as

$$K_y=\text{Cone}\left(\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\right).$$

The cones K_x, K_w and K_y are depicted in Fig. 1.

Since the eigenvalues of matrix A are 0.0332, 0.3259 and 0.2809, the system is asymptotically stable. Let $v_1=[3,\ 1,\ 1]^{\mathrm{T}}$ and $v_2=[4,\ 1]^{\mathrm{T}}$. Based on Theorem 1, the cone linear absolute-norm induced gain of system (1) is maximized with the value to be 6.3576, when $v=[0.629,\ -0.477,\ -0.410]^{\mathrm{T}}$. Disturbances $w_1(k)$ and $w_2(k)$ are given below:

$$w_1(k)=\left[4/(k+1)^2 \quad 1/(k+1)^2 \quad 1/(k+1)^2\right]^{\mathrm{T}},\qquad k>0,$$
$$w_2(k)=\begin{cases}\left[0.629 \quad -0.477 \quad -0.410\right]^{\mathrm{T}}, & k=0\\ 0, & k>0\end{cases}.$$

The variations of the performance ratio

$$\frac{\sum_{i=0}^{k}\|y(i)\|_{v_2,1}}{\sum_{i=0}^{k}\|w(i)\|_{v_1,1}} \tag{44}$$

with disturbances $w_1(k)$ and $w_2(k)$ are depicted in Fig. 2. One can find that the ratio (44) for both of the two disturbances is less than the cone linear absolute-norm induced gain, and the trajectory of ratio (44) with the disturbance w_2 converges to the upper bound of the cone linear absolute-norm induced gain.

To analyze the cone max-norm induced gain performance, two vectors, $v_3\in\int K_w$ and $v_4\in\int K_y$, for cone max-norm are chosen as follows:

$$v_3=\left[1.2 \quad 0.1 \quad 0.4\right]^{\mathrm{T}},\qquad v_4=\left[2 \quad 3\right]^{\mathrm{T}}.$$

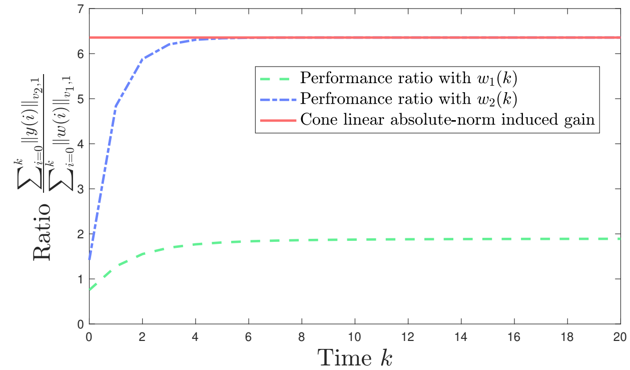

Fig. 2. Variation of the ratio (44) with disturbances $w_1(k)$ and $w_2(k)$

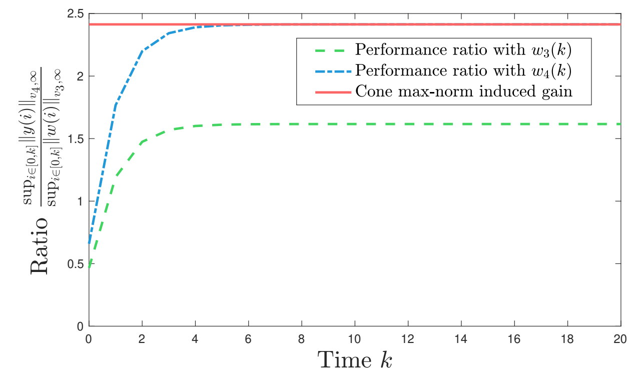

Fig. 3. Variation of the performance ratio (46) with $w_3(k)$ and $w_4(k)$

Based on Theorem 3, the cone max-norm induced gain is

$$\left\|\left[C(I-A)^{-1}B_w+D_w\right]v_3\right\|_{v_4,\infty}=2.4133. \tag{45}$$

To demonstrate the effectiveness of the calculated upper bound and the necessity of Theorem 3, two disturbances are introduced as follows:

$$w_3(k)=\left[6 \quad 1 \quad 2\right]^{\mathrm{T}},\quad w_4(k)=\left[1.2 \quad 0.1 \quad 0.4\right]^{\mathrm{T}}.$$

Fig. 3 shows the variation of the performance ratio

$$\frac{\sup_{i\in[0,k]}\|y(i)\|_{v_4,\infty}}{\sup_{i\in[0,k]}\|w(i)\|_{v_3,\infty}}. \tag{46}$$

It shows that the performance ratio (46) monotonically increases under the given disturbances $w_3(k)$ and $w_4(k)$. When the constant disturbance $w_4(k)$ equals to the vector v_3, the ratio monotonically increases and converges to the cone max-norm induced gain, which verifies Remark 3.

Example 2. *To illustrate the duality of the two gains, the following example is given. Assume $K_w^*\subset\mathbb{R}^2$ has two edges $\left[3 \quad 1\right]^{\mathrm{T}}$ and $\left[0 \quad 1\right]^{\mathrm{T}}$, and let*

$$v_1=v_4=\left[1 \quad 2\right]^{\mathrm{T}},$$

and

$$\left[C(I-A)^{-1}B_w+D_w\right]^{\mathrm{T}}v_2=\left[C(I-A)^{-1}B_w+D_w\right]^{\mathrm{T}}v_3=[2\ 1]^{\mathrm{T}},$$

where v_1 and v_2 are the vectors of cone linear absolute-norm induced gain in system (1) and v_3 and v_4 are the vectors of cone-max norm induced gain in dual system (41). Then the cone linear absolute-norm of the original system (1) is

$$\begin{aligned}&\sup_{\|v\|_{v_1,1}=1}\left\|\left[C(I-A)^{-1}B_w+D_w\right]v\right\|_{v_2,1}\\&=\sup_{\|v\|_{v_1,1}=1}v^{\mathrm{T}}\left[C(I-A)^{-1}B_w+D_w\right]^{\mathrm{T}}v_2.\end{aligned}\tag{47}$$

When $v=\begin{bmatrix}1 & 0\end{bmatrix}^{\mathrm{T}}$, $v^{\mathrm{T}}\left[C(I-A)^{-1}B_w+D_w\right]^{\mathrm{T}}v_2$ gets the maximum value 2. The cone max-norm of the dual of the system (41) is

$$\begin{aligned}&\sup_{\|v\|_{v_3,\infty}=1}\left\|\left[B_w^{\mathrm{T}}(I-A)^{-\mathrm{T}}C^{\mathrm{T}}+D_w^{\mathrm{T}}\right]v\right\|_{v_4,\infty}\\&=\left\|\left[B_w^{\mathrm{T}}(I-A)^{-\mathrm{T}}C^{\mathrm{T}}+D_w^{\mathrm{T}}\right]v_3\right\|_{v_4,\infty}=2.\end{aligned}\tag{48}$$

One can find that cone linear absolute-norm of the original system (1) equals to the cone max-norm of its corresponding dual system (41).

V. CONCLUSION

In this paper, the cone linear absolute-norm and cone max-norm are introduced to characterize the input-output gain of linear discrete-time cone-preserving systems. Several equivalent conditions describing the input-output performance of cone-preserving systems are proposed. The conditions have shown that the two cone-induced gains can be calculated via the linear programming. As generalizations of ℓ_1- and ℓ_∞-gain of positive systems, the theoretical results have also indicated the duality of two cone-induced gains.

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