ORIGINAL ARTICLE

Journal of Risk and Insurance

Pareto-efficient risk sharing in centralized insurance markets with application to flood risk

Tim J. Boonen[1] | Wing Fung Chong[2] | Mario Ghossoub[3]

[1]Department of Statistics and Actuarial Science, University of Hong Kong, Pok Fu Lam, Hong Kong

[2]Maxwell Institute for Mathematical Sciences and Department of Actuarial Mathematics and Statistics, Heriot-Watt University, Edinburgh, UK

[3]Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, USA

Correspondence
Mario Ghossoub, Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, USA.
Email: mario.ghossoub@uwaterloo.ca

Abstract
Centralized insurance can be found in both the private and public sectors. This paper provides a micro-economic study of the risk-sharing mechanisms in these markets, where multiple policyholders interact with a centralized monopolistic insurer. With minimal assumptions on the risk preferences of the market participants, we characterize Pareto optimality in terms of the agents' risk positions and their assessment of the likelihoods associated with their loss tail events. We relate Pareto efficiency in this market to a naturally associated cooperative game. Based on our theoretical results, we then consider a model of flood insurance coverage via an illustrative example. The lessons drawn from our theoretical results and this example lead to important policy implications for the existing National Flood Insurance Program in the United States.

1 | INTRODUCTION

Centralized insurance markets have long existed in the private sector. An insurance company typically provides policies to multiple policyholders. Each policy holder pays an ex ante premium in exchange for ceding a part of their insurable risk to the company. An ex post indemnification is then paid out by the insurer to each policy holder. Hence, the insurer plays

the role of a centralized entity that pools the contracted parts of the insurable risks from all policyholders on its portfolio, in return for funding through the collected premia.

The same architecture of centralization is also observed in the public sector, particularly in areas dedicated to sharing various forms of risks through social planning. Insurance contracting therein is often circumscribed to a monopolistic market, in which the centralized insurer is a benevolent social planner. For instance, in many countries, the national social security system is often funded through tax collections from the employed labor force; and the government de facto acts as a centralized insurer for its citizens, providing coverage with a focus on social welfare rather than maximizing expected profits. Another relevant example is that of the government of Japan, which acts as a benevolent monopolistic reinsurer providing reinsurance coverage for earthquake insurance written in the private sector. When a massive earthquake occurs with a significant impact on the risk bearing capacity of the primary insurance market, the Japanese government steps in and absorbs losses, using the collected reinsurance premia as a reserve. The focus is not on expected profit maximization, but rather on social welfare. A third example is that of the National Flood Insurance Program (NFIP)[1] in the United States, which provides flood insurance coverage for homeowners, business owners, and renters. NFIP was established through the National Flood Insurance Act of 1968 to reduce the burden on the nation's resources, rather than with any expectation of profitability. NFIP is managed by the Federal Emergency Management Agency (FEMA). Through FEMA, and via either Direct Servicing Agents (DSA)[2] or the Write-Your-Own (WYO)[3] Program as intermediary, the United States' federal government plays the role of a benevolent monopolistic centralized insurer. NFIP's flood insurance coverage is financed through premium collection from its policyholders, which are deposited via FEMA to the National Flood Insurance Fund (NFIF). If a policy holder is serviced by DSA, they will be indemnified directly by FEMA in case of a claim. If a policy holder is serviced by a company in the WYO program, they will be indemnified by the company, which will then be subsequently reimbursed by FEMA. At the end, FEMA withdraws funds from NFIF for the indemnification or reimbursement.

1.1 | This paper's contributions

Our main interest in this paper is to provide a microeconomic study of the risk sharing mechanism in these centralized insurance markets with multiple policyholders and a single insurer. We consider a market model in which several policyholders, each with a given initial risky endowment, wish to transfer their risk to a centralized insurance provider, in exchange for a premium payment. A contract in this market consists of a vector of premium payments (one from each policy holder) and a vector of indemnity functions (one for each policy holder). While the policyholders evaluate contracts only by considering the risk exposure that results from their own premium disbursement and their own indemnification, the centralized insurer

---

[1]For a comprehensive overview of NFIP, we refer to https://sgp.fas.org/crs/homesec/R44593.pdf, https://uscode.house.gov/view.xhtml?path=/prelim@title42/chapter50&edition=prelim, and https://www.fema.gov/flood-insurance/work-with-nfip/manuals/current. All URL links in this paper were last accessed on March 1, 2024. See also Kousky and Shabman (2014) regarding the pricing of NFIP.

[2]DSA operate as private contractors selling flood insurance on behalf of FEMA.

[3]Through WYO Program, private insurance companies are paid to directly write and service the policies in their own names. In 2022, there are over 50 private insurance companies participating in NFIP.

evaluates contracts at the aggregate level, that is, by considering the risk exposure resulting from the aggregate premium collection as well as the aggregate promised indemnification. The economic problem of interest is thus how to design a good, if not the best or most efficient, risk sharing mechanism in such centralized markets. A classical notion of efficiency in exchange economies and risk-sharing markets is that of Pareto optimality. A contract is Pareto optimal (or Pareto efficient) if no agent (policyholders and/or insurer) can be made better off without making someone else worse off.

The main contributions of this paper are as follows. First, we provide a characterization of the set of Pareto-optimal contracts (Theorem 2.1). At a Pareto optimum, indemnification functions are those that minimize the total of all agents' risk exposure (see Equation 3), while the premia are those that ensure that all agents are rationally participating in the market (see Equations 1 and 2). Second, we provide a characterization of the indemnity functions at a Pareto optimum in terms of the likelihood associated to the loss tail events (see Equation 13). Third, we relate this risk-sharing market to a cooperative game (Section 3), which assigns to each coalition its maximum welfare gain from risk sharing within the coalition. This gain is given as the largest possible reduction (from the status quo) in the total risk exposure of all agents in a given coalition, resulting from risk sharing within that coalition (see Equations 10 and 11). We show that the core[4] of this game is associated with a particular premium at a Pareto optimum (Theorems 3.1 and 3.2). All of our results hold regardless of the dependence structure among the risky endowments of the multiple policyholders.

We then apply our results to an illustrative example that considers an alternative model for flood insurance in the United States, based on a modified version of NFIP (Sections 4 and 6). We choose not to apply our results directly to NFIP due to the sheer size of the set of model inputs required for such an analysis. In fact, as of September 2022, NFIP has more than five million policyholders nationwide. Our simplified model considers a market design in which each state is able to aggregate all of the flood risk exposures originating from its eligible entities in a given period. For each state, this aggregate is seen as an insurable risk against which the state wishes to obtain insurance coverage from the federal government, such as via FEMA. The federal government thus plays the role of a monopolistic insurer, collecting premia from the states at the beginning of the period, and offering flood indemnities to each state at the end of the period. Our theoretical results on the characterization of Pareto optimal contracts in centralized insurance markets with multiple policyholders and a single insurance provider can then be applied to characterize the set of Pareto efficient contracts in this model of centralized flood insurance market. We apply our results to a publicly available data set in OpenFEMA, which contains historical claim events for NFIP.

In the aforementioned example, we consider a market setting in which a centralized insurance provider, such as a representative insurer procured by FEMA, proposes flood risk coverage to two different sets of states: one set is California (CAL), New York (NY), and Texas (TX), which are geographically distant states, while the other set is Alabama (AL), Louisiana (LA), and Mississippi (MS), which are adjacent. The centralized insurance provider covers the flood losses for each tuple of states over a 1-month period. Several findings are drawn from the example (Section 6), and we mention the most notable here.

---

[4]The core is a well-established solution concept in cooperative game theory, in which an allocation prevents a subgroup of agents from forming an alliance among themselves and excluding the remaining agents; see, for instance, Osborne and Rubinstein (1994).

First, the example shows a clear benefit of risk aggregation. Indeed, within each of the two sets of states concerned, more coverage, leading to more maximum welfare gain, is obtained when the contracting with the centralized insurer is done at the aggregate level compared to contracting done at the standalone state level (Section 6.2.1). Second, there is a clear advantage of geographical diversification. The set CAL-NY-TX, comprising relatively geographically distant states, benefits from more insurance coverage, as well as a larger total premium discount, compared to the less geographically diversified set AL-LA-MS (Sections 6.2.2 and 6.3). A larger maximum welfare gain is achieved for the tuple CAL-NY-TX than for the tuple AL-LA-MS (Section 6.2.2). This can be attributed to the fact that the multivariate flood risk distribution of the states in these two tuples have noticeable differences in their spatial dependence due to their geographical locations, although they have marginals in more or less the same order (Tables 1 and 2). Through diversification, the geographically distant tuple CAL-NY-TX gains more than the geographically adjacent tuple AL-LA-MS by sharing their risk exposure with the centralized insurer. While the benefits of diversification in managing spatially dependent catastrophic risks have been discussed in the literature (e.g., Ermoliev et al., 2000; Kousky & Cooke, 2012), we provide in our illustrative example a quantification of the benefits of geographical diversification. Third, as one would expect, the risk appetite of one state within a given tuple affects the coverage of other states in a given Pareto-optimal contract (Section 6.2.3). This effect certainly does not exist when contracting is done directly at the standalone state level.

The theoretical results in this paper and the lessons learned from this example have some important policy implications regarding the design of a sustainable national flood insurance program. Currently, the individual policyholders enrolled in NFIP share their flood risk exposure directly with the federal government through individual bilateral contracts, which do not necessarily consider the benefit of risk aggregation. Based on our findings, leveraging the benefits of risk aggregation in NFIP could allow for a larger welfare gain nationwide, which translates into a larger sum of premium discounts in the flood insurance policies. We thus suggest that flood insurance contracts in NFIP be written in batch processing, and periodically, to be compatible with a multilateral risk sharing setting that would be able to exploit any potential diversification benefits. Any eligible entity can join NFIP or renew their existing contract only at the beginning of each period. Such a period could be a month, a quarter, a half-year, a year, and so on. By doing so, the federal government could design flood insurance coverage schemes that take into consideration the benefits of risk aggregation from the existing pool of policyholders at the design stage. The policyholders in NFIP would then be indemnified at the end of the period.

1.2 | Related literature

Our work is related to the literature on Pareto-optimal risk sharing in centralized insurance markets. Within this literature, efficiency of contracts in markets with multiple policyholders and one insurer has hitherto not been well-studied. However, a few papers have dealt with the optimality of contract design in markets with one insurer and one policy holder who is subject to a multivariate risk. For instance, Denuit and Vermandele (1998) focus on risk minimization in the stop-loss order when the multivariate loss variable is exchangeable. Cai and Wei (2012) extend the work of Denuit and Vermandele (1998) to the case of positive dependence through stochastic ordering. Cheung et al. (2014) study the

minimization of a risk measure, but their focus is on the worst-case dependence structure for the multivariate risk vector. Recently, Guerra and de Moura (2021) study optimal insurance design with multivariate risk in an expected-utility framework, in which they allow for contract randomization; and they derive conditions for optimality. Another relevant recent paper is Asimit et al. (2021), in which indemnities are written for multiple mutually exclusive environments, and thus the aggregate risk is essentially composed of several risks emanating from the same policy holder, but in different environments with an extremely negative dependence. Ghossoub and Zhu (2024) examine a sequential-move centralized insurance market with multiple policyholders. They characterize Stackelberg equilibria and Pareto-efficient contracts.

The technical setting of this paper is different from the papers above, as the focus is not on a single policy holder subject to a multivariate risk vector, but on multiple policyholders, each holding an individual risk, hence leading to important considerations of risk aggregation from the single insurer's point of view. Additionally, the setting of this paper is different from Asimit et al. (2021), since we allow for multiple policyholders with generic translation-invariant risk measures, without any distributional or dependence assumption among the multiple risks. It is also different from Ghossoub and Zhu (2024), as in our setting the insurer does not have a first-move advantage.

It should also be emphasized that the proposed risk-sharing framework herein is for centralized insurance markets. This is in contrast to the recent developments in decentralized and peer-to-peer insurance, such as Abdikerimova et al. (2024), Denuit and Dhaene (2012), Denuit et al. (2023, 2022), Denuit and Robert (2022), Feng, Liu, and Taylor (2022), and Feng, Liu, and Zhang (2022).

As this paper provides some policy implications and potential improvements to the existing NFIP structure, it is also related to the recent literature on flood insurance and natural disaster risk management more broadly. For instance, Collier and Ragin (2020) provide an empirical analysis of the incentives of insurers in NFIP to influence overinsurance of participating policyholders. They show that the distribution systems and commission rates of the intermediaries in NFIP play a large role in whether policyholders overinsure. Collier et al. (2022) find that policyholders in NFIP typically fully insure their homes, with premia that are significantly higher than the contract's expected value. They show that this contrasts with the theoretical prediction of purchasing low coverage limits, thereby retaining most of their flood exposures, assuming expected-utility preferences. They contend that probability distortions in decision making could offer a normative explanation. Hinck (2024) studies the design of optimal insurance contracts together with the possibility of government disaster relief payments. Finally, Li and Su (2024) investigate the use of catastrophe bonds as a risk management tool for wildfires.

1.3 | Outline

The rest of this paper is set out as follows. Section 2 introduces our market model and assumptions, and characterizes Pareto optimality. Section 3 discusses the relationship between the set of Pareto optima and the core of a naturally associated cooperative game. Section 4 provides an illustrative example for flood risk. Section 5 shows how our results can be used to obtain more explicit descriptions of the set of Pareto optima when the agents are endowed with more specific risk

measures. Section 6 revisits the example, and Section 7 outlines some policy implications to the existing NFIP. Finally, Section 8 concludes. All proofs are relegated to Appendix A.

2 | THE INSURANCE MARKET

2.1 | Market model

Consider a collection $\Omega := \{\omega_1, \omega_2, ..., \omega_N\}$ of future states of the world, equip Ω with its power set $\mathcal{F} = 2^\Omega$, and assume given an objective probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$. We take the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as representing future uncertainty in a given time horizon.

There are $n \in \mathbb{N}$ policyholders wishing to share their endowed risks with a centralized insurer. For each $i \in \mathcal{N} := \{1, ..., n\}$, let X_i denote the nonnegative loss of the ith policy holder, which we model as a given nonnegative random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. For each $i \in \mathcal{N}$ and $j \in \mathcal{N}' := \{1, ..., N\}$, let $x_{j,i}$ denote the realization of the loss X_i in state ω_j. Hence, for $j \in \mathcal{N}'$,

$$(X_1, X_2, ..., X_n)(\omega_j) = (x_{j,1}, x_{j,2}, ..., x_{j,n}).$$

We consider a one-period, pure risk endowment economy, where all risks are realized at the end of the period. The risk-sharing mechanism in this market is as follows. For each $i \in \mathcal{N}$, policy holder i pays ex ante the amount $\pi_i \in \mathbb{R}$ to the centralized insurer. For $j \in \mathcal{N}'$, if state ω_j realizes, then the centralized insurer will indemnify $I_i(x_{j,i})$ to policy holder i. Herein, for each $i \in \mathcal{N}$, the indemnity function I_i lies in the ex ante specified admissible set

$$\mathcal{I} := \{I : \mathbb{R}_+ \to \mathbb{R}_+ : I(0) = 0, 0 \le I(x) - I(y) \le x - y \text{ for all } x > y \ge 0\}.$$

The conditions in $\mathcal{I}$ can be interpreted as follows. No indemnity is provided if a policy holder does not experience any losses. Moreover, a larger indemnity payment is provided, and a larger retention is expected, when a policy holder experiences a larger loss. These conditions rule out any *ex post* moral hazard that might otherwise occur from a misreporting of the true loss by a policy holder. For any $M \in \mathbb{N}$, denote by $\mathcal{I}^M$ the M-fold Cartesian product of $\mathcal{I}$ by itself.

For each $i \in \mathcal{N}$, the end-of-period, posttransfer risk exposure of policy holder i is thus given by

$$R_i(X_i) + \pi_i,$$

where $R_i(x) := x - I_i(x)$ is the retained loss of policy holder i. Similarly, the end-of-period, posttransfer risk exposure of the centralized insurer is given by

$$\sum_{i=1}^{n} (I_i(X_i) - \pi_i).$$

We assume that the preferences of each policy holder i and that of the centralized insurer are represented by given risk measures ρ_i and ρ, respectively. Next, we recall some properties of risk measures.

Definition 2.1. Given a collection $\mathcal{X}$ of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a risk measure $\eta : \mathcal{X} \to \mathbb{R}$ is said to be:

- *Translation invariant* if $\eta(X + c) = \eta(X) + c$, for all $(X, c) \in \mathcal{X} \times \mathbb{R}$.
- *Monotone* if $\eta(X) \leq \eta(Y)$, for all $X, Y \in \mathcal{X}$ such that $X \leq Y$.
- *Positively homogeneous* if $\eta(\alpha X) = \alpha \eta(X)$, for all $(X, \alpha) \in \mathcal{X} \times \mathbb{R}^+$.
- *Subadditive* if $\eta(X + Y) \leq \eta(X) + \eta(Y)$, for all $X, Y \in \mathcal{X}$.
- *Convex* if $\eta(\alpha X + (1 - \alpha)Y) \leq \alpha \eta(X) + (1 - \alpha)\eta(Y)$, for all $X, Y \in \mathcal{X}$ and all $\alpha \in [0, 1]$.
- *Coherent* if it is translation invariant, monotone, positively homogeneous, and subadditive.
- *Comonotonic additive* if $\eta(X + Y) = \eta(X) + \eta(Y)$, for all $X, Y \in \mathcal{X}$ such that they are comonotonic.[5]
- *With nonnegative loading* if $\eta(X) \geq \mathbb{E}[X]$, for all $X \in \mathcal{X}$.

Under positive homogeneity, the properties of subadditivity and convexity are equivalent. Hence, a coherent risk measure is convex. As the first requirement, we assume that the risk measures ρ_i and ρ are translation invariant. Consequently, each policy holder i measures their risk exposure by

$$\rho_i(R_i(X_i) + \pi_i) = \rho_i(R_i(X_i)) + \pi_i,$$

and the centralized insurer measures its risk exposure by

$$\rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right) = \rho\left(\sum_{i=1}^n I_i(X_i)\right) - \sum_{i=1}^n \pi_i.$$

2.2 | Individual rationality and Pareto optimality

A risk-sharing mechanism, or contract, in this market is a pair

$$\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\right) \in \mathcal{I}^n \times \mathbb{R}^n.$$

The aim of the risk-sharing mechanism is to achieve a Pareto-efficient ex ante redistribution of risk endowments, among all possible contracts that incentivize the agents (policyholders and insurer) to participate in the market. We define these notions below.

Definition 2.2. A contract $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{I}^n \times \mathbb{R}^n$ is said to be:

- *Individually rational* (IR) if it incentivizes the agents to participate in the market, that is,

---

[5]$X, Y \in \mathcal{X}$ are said to be comonotonic if (X, Y) has the same joint distribution as $(f_X(Z), f_Y(Z))$, for some nondecreasing functions f_X, f_Y and some $Z \in \mathcal{X}$; that is, X, Y are most positively dependent on each other.

$$\rho_i\left(R_i^*(X_i) + \pi_i^*\right) \le \rho_i(X_i), \forall\, i \in \mathcal{N},$$

$$\rho\left(\sum_{i=1}^n \left(I_i^*(X_i) - \pi_i^*\right)\right) \le \rho(0) = 0,$$

where $R_i^*(X_i) := X_i - I_i^*(X_i)$, for $i \in \mathcal{N}$.

- *Pareto optimal* (PO) if it is IR and there does not exist any other IR contract $\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\right)$ such that

$$\rho_i(R_i(X_i) + \pi_i) \le \rho_i\left(R_i^*(X_i) + \pi_i^*\right), \forall i \in \mathcal{N},$$

$$\rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right) \le \rho\left(\sum_{i=1}^n \left(I_i^*(X_i) - \pi_i^*\right)\right),$$

with at least one strict inequality.

Let $\mathcal{IR}$ denote the set of all IR contracts. Then, in particular, $\mathcal{IR} \ne \varnothing$ since it contains the status-quo, that is, the no-risk-sharing contract consisting of no indemnification and no premium payments.

Remark 2.1. Note that by translation invariance of the risk measures $\rho_i, i \in \mathcal{N}$, and $\rho, (\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{IR}$ if and only if

$$\pi_i^* \le \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right), \forall i \in \mathcal{N}, \text{ and,} \tag{1}$$

$$\rho\left(\sum_{i=1}^n I_i^*(X_i)\right) \le \sum_{i=1}^n \pi_i^*. \tag{2}$$

Let $\mathcal{P} \subseteq \mathcal{IR}$ denote the set of all PO contracts, and let $\mathcal{S}'$ be the set of all minimizers for the following sum-minimization problem:

$$\inf_{\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\right) \in \mathcal{IR}} \sum_{i=1}^n \rho_i(R_i(X_i) + \pi_i) + \rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right).$$

Theorem 2.1. *The following hold:*

(i) $\mathcal{P} = \mathcal{S}'$.
(ii) $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{S}'$ *if and only if*

(a) $\{I_i^*\}_{i=1}^n$ *solves*

$$\inf_{\{I_i\}_{i=1}^n \in \mathcal{I}^n} \sum_{i=1}^n \rho_i(R_i(X_i)) + \rho\left(\sum_{i=1}^n I_i(X_i)\right), \text{ and} \tag{3}$$

(b) $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ *satisfies* (1) *and* (2).

(iii) *For any* $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ *that solves* (3), *there exist* $\{\pi_i^*\}_{i=1}^n \in \mathbb{R}^n$ *such that* $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ *satisfies* (1) *and* (2).

The first result of Theorem 2.1 shows that the set of all Pareto optima coincides with the set $\mathcal{S}'$, thereby extending Asimit and Boonen (2018, Theorem 3.1) to the case of multiple policyholders. In addition, the second and third results of Theorem 2.1 suggest that for a contract to be PO, it is both necessary and sufficient that the collection of indemnities $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ be optimal for (3), regardless of the premium payments $\{\pi_i^*\}_{i=1}^n \in \mathbb{R}^n$ given by

$$\pi_i^* = \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right), \forall i \in \mathcal{N}, \tag{4}$$

is one such possibility. However, such a choice does not provide an improvement in any posttransfer risk position of the n policyholders from the status-quo, and thus (4) refers to the indifference premia. Rather, it allocates all the welfare gains from the risk-sharing mechanism to the centralized insurer. We expand on this in Section 3.

Note also that Theorem 2.1 does not guarantee that the premium payments $\{\pi_i^*\}_{i=1}^n \in \mathbb{R}^n$ in a given PO contract $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ can be chosen to be nonnegative. Section 3 also addresses this issue. On a similar note, the following discusses the conditions that ensure that the premium payments $\{\pi_i^*\}_{i=1}^n \in \mathbb{R}^n$ in a given PO contract $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ can be chosen with nonnegative loading.

Proposition 2.2.   *Assume that the risk measures* ρ_i, $i \in \mathcal{N}$, *are comonotonic additive and with nonnegative loading. Let* $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ *be a solution of* (3), *and let, for* $i \in \mathcal{N}$,

$$\pi_i^* \in \left[\mathbb{E}\left[I_i^*(X_i)\right], \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right)\right],$$

if $\rho\left(\sum_{i=1}^n I_i^*(X_i)\right) \leq \mathbb{E}\left[\sum_{i=1}^n I_i^*(X_i)\right]$, *while*

$$\pi_i^* \in \left[\max\left\{\rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - \frac{1}{n}\left(\sum_{j=1}^n \left(\rho_j(X_j) - \rho_j\left(R_j^*(X_j)\right)\right) - \rho\left(\sum_{j=1}^n I_j^*(X_j)\right)\right),\right.\right.$$
$$\left.\left.\mathbb{E}\left[I_i^*(X_i)\right]\right\}, \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right)\right], \tag{5}$$

if $\mathbb{E}\left[\sum_{i=1}^n I_i^*(X_i)\right] \leq \rho\left(\sum_{i=1}^n I_i^*(X_i)\right)$ *(e.g., when* ρ *is further assumed to be with nonnegative loading). Then* $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$, *and* $\{\pi_i^*\}_{i=1}^n$ *are with nonnegative loading, that is,* $\pi_i^* \geq \mathbb{E}[I_i^*(X_i)]$, *for* $i \in \mathcal{N}$.

Finally, we provide another characterization of PO contracts that is useful in elucidating the relationship between indemnities and premia at an optimum.

Theorem 2.3. *The following are equivalent:*

(i) $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$.

(ii) *There exist $\{\lambda_i\}_{i=1}^n \in \mathbb{R}^n$ such that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is optimal for the problem*

$$\inf_{\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\right) \in \mathcal{IR}} \left\{ \rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i) \right) : \rho_i(R_i(X_i) + \pi_i) = \lambda_i, \forall i \in \mathcal{N} \right\}. \tag{6}$$

From Theorem 2.3, it follows that if $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$, then there exist some $\{\lambda_i\}_{i=1}^n \in \mathbb{R}^n$ such that

$$\pi_i^* = \lambda_i - \rho_i\left(R_i^*(X_i)\right), \forall i \in \mathcal{N}.$$

Therefore, (6) is equivalent to

$$\inf_{\{I_i\}_{i=1}^n \in \mathcal{I}^n} \rho\left(\sum_{i=1}^n (I_i(X_i) - \lambda_i + \rho_i(X_i - I_i(X_i))) \right), \tag{7}$$

and a solution $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ to (7) leads to a PO contract $(\{I_i^*\}_{i=1}^n, \{\lambda_i - \rho_i(X_i - I_i^*(X_i))\}_{i=1}^n)$. Varying the constraint vector $\{\lambda_i\}_{i=1}^n \in \mathbb{R}^n$ traces the entire Pareto frontier.

3 | WELFARE GAINS FROM RISK SHARING

Throughout this section, let $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ be a given fixed solution for (3). For any $\{\pi_i\}_{i=1}^n \in \mathbb{R}^n$ that satisfies (1) and (2), define the welfare gain of the ith policy holder by

$$W_i := \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - \pi_i, \forall i \in \mathcal{N}.$$

By (1), $W_i \geq 0$, for all $i \in \mathcal{N}$.

We interpret W_i as a welfare gain for two reasons. First, it is the risk reduction for the ith policy holder from the status quo resulting from entering into the insurance contract:

$$W_i = \rho_i(X_i) - \rho_i\left(R_i^*(X_i) + \pi_i\right).$$

Second, it is also a risk-free amount that the ith policy holder saves from being indifferent between the status quo or entering into the contract:

$$\rho_i\left(R_i^*(X_i) + \pi_i + W_i\right) = \rho_i(X_i).$$

In fact, for each policy holder $i \in \mathcal{N}$, the premium payment and the welfare gain are in a one-to-one correspondence. In other words, for any $W_i \geq 0$, the premium payment of the ith policy holder to the centralized insurer can be equivalently defined as

$$\pi_i = \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - W_i. \tag{8}$$

Hence, we also interpret W_i as the premium discount that is provided on the indifference premium, which is $\rho_i(X_i) - \rho_i(R_i^*(X_i))$ for the ith policy holder.

To characterize premium payments that are nonnegative in a PO contract, consider the following cooperative game. A coalition in this risk-sharing market is any nonempty subset $\mathcal{S} \subseteq \mathcal{N}$, together with the single centralized insurer (also called central authority), abbreviated as $\mathcal{CA} := \{\text{CA}\}$. For any coalition $\mathcal{S} \cup \mathcal{CA}$, denote the coalition's maximum welfare gain from risk sharing as $v(\mathcal{S} \cup \mathcal{CA})$, which is defined below as the maximum total welfare gains of the policyholders in the coalition:

$$\sup_{(\{I_i\}_{i\in\mathcal{S}}, \{\pi_i\}_{i\in\mathcal{S}}) \in \mathcal{IR}_\mathcal{S}} \sum_{i\in\mathcal{S}} W_i = \sup_{(\{I_i\}_{i\in\mathcal{S}}, \{\pi_i\}_{i\in\mathcal{S}}) \in \mathcal{IR}_\mathcal{S}} \sum_{i\in\mathcal{S}} (\rho_i(X_i) - \rho_i(R_i(X_i)) - \pi_i), \tag{9}$$

where $\mathcal{IR}_\mathcal{S}$ is the set of all IR contracts, restricted to the set of policyholders in $\mathcal{S}$, together with the single centralized insurer. The set $\mathcal{IR}_\mathcal{S}$ particularly contains the individual rationality condition of the insurer, given by $\sum_{i\in\mathcal{S}} \pi_i \geq \rho\left(\sum_{i\in\mathcal{S}} I_i(X_i)\right)$ (see (2) for $\mathcal{S} = \mathcal{N}$). Since the objective in (9) is strictly decreasing in $\{\pi_i\}_{i\in\mathcal{S}}$, it follows that this rationality condition is binding, and so (9) can be written as

$$\begin{aligned}
v(\mathcal{S} \cup \mathcal{CA}) &= \sup_{\{I_i\}_{i\in\mathcal{S}} \in \mathcal{I}^{|\mathcal{S}|}} \left(\sum_{i\in\mathcal{S}} \rho_i(X_i) - \left(\sum_{i\in\mathcal{S}} \rho_i(R_i(X_i)) + \rho\left(\sum_{i\in\mathcal{S}} I_i(X_i) \right) \right) \right) \\
&= \sum_{i\in\mathcal{S}} \rho_i(X_i) - \inf_{\{I_i\}_{i\in\mathcal{S}} \in \mathcal{I}^{|\mathcal{S}|}} \left(\sum_{i\in\mathcal{S}} \rho_i(R_i(X_i)) + \rho\left(\sum_{i\in\mathcal{S}} I_i(X_i) \right) \right),
\end{aligned} \tag{10}$$

which is nonnegative since the no-risk-sharing allocation (the status quo) among $\mathcal{S} \cup \mathcal{CA}$ is admissible. The maximum welfare gain $v(\mathcal{S} \cup \mathcal{CA})$ is therefore the largest possible reduction in the total risk exposure of all agents in the coalition $\mathcal{S} \cup \mathcal{CA}$, before and after risk sharing. In particular, when $\mathcal{S} = \mathcal{N}$, by definition of $\{I_i^*\}_{i=1}^n$,

$$v(\mathcal{N} \cup \mathcal{CA}) = \sum_{i=1}^n \rho_i(X_i) - \sum_{i=1}^n \rho_i\left(R_i^*(X_i)\right) - \rho\left(\sum_{i=1}^n I_i^*(X_i) \right). \tag{11}$$

The couple $(\mathcal{N}, v)$ defines a cooperative game. A central problem for such a cooperative game is to find a solution $(b_1, b_2, ..., b_n, b_{CA}) \in \mathbb{R}_+^{n+1}$ such that $\sum_{i\in\mathcal{N}\cup\mathcal{CA}} b_i = v(\mathcal{N} \cup \mathcal{CA})$. A well-known set of such solutions in cooperative game theory is the core (e.g., Osborne & Rubinstein, 1994), which is given by

$$\text{core}(\mathcal{N}, v) \coloneqq \left\{ (b_1, b_2, ..., b_n, b_{CA}) \in \mathbb{R}_+^{n+1} : \sum_{i \in \mathcal{N} \cup \mathcal{CA}} b_i = v(\mathcal{N} \cup \mathcal{CA}), \right.$$

$$\left. \sum_{i \in \mathcal{S} \cup \mathcal{CA}} b_i \geq v(\mathcal{S} \cup \mathcal{CA}), \forall \varnothing \neq \mathcal{S} \subseteq \mathcal{N} \right\}. \tag{12}$$

Note that $\text{core}(\mathcal{N}, v) \neq \varnothing$ since $(0, 0, ..., 0, v(\mathcal{N} \cup \mathcal{CA})) \in \text{core}(\mathcal{N}, v)$. Moreover, each element in the core of this game is a nonnegative allocation of the maximum welfare gain $v(\mathcal{N} \cup \mathcal{CA})$ to the n policyholders and the centralized insurer.

Theorem 3.1. *Let* $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ *be a solution of* (3), *and define the associated game* $(\mathcal{N}, v)$ *as above. Let* $(b_1, b_2, ..., b_n, b_{CA}) \in \text{core}(\mathcal{N}, v)$. *If* $W_i = b_i$ *for all* $i \in \mathcal{N}$, *then* $(\{I_i^*\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{P}$, *where* π_i *is defined in* (8).

For any core element that allocates b_i to W_i for each $i \in \mathcal{N}$, if π_i is defined as in (8) then the resulting contract $(\{I_i^*\}_{i=1}^n, \{\pi_i\}_{i=1}^n)$ is PO. In fact, this result holds as long as an allocation of the maximum welfare gain $v(\mathcal{N} \cup \mathcal{CA})$ is nonnegative. Moreover, under an additional assumption of monotonicity of ρ, the following result shows that the contract constructed in Theorem 3.1 yields nonnegative premia.

Theorem 3.2. *Assume that the risk measure* ρ *is monotone. Let* $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ *be a solution of* (3), *and define the associated game* $(\mathcal{N}, v)$ *as above. Let* $(b_1, b_2, ..., b_n, b_{CA}) \in \text{core}(\mathcal{N}, v)$. *If* $W_i = b_i$ *for all* $i \in \mathcal{N}$, *then* $\pi_i \geq 0$ *for all* $i \in \mathcal{N}$, *where* π_i *is defined in* (8).

This nonnegativity property of premia is in line with practice and is a reasonable expectation. In the remainder of this section, we show that the selection of a core element in $\text{core}(\mathcal{N}, v)$ is one-to-one related to a coalitional stability criterion, which is defined below.

Definition 3.1. A contract $(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{IR}$ is coalitionally stable if there does not exist a nonempty subset $\mathcal{S} \subseteq \mathcal{N}$ and a contract $(\{\hat{I}_i\}_{i \in \mathcal{S}}, \{\hat{\pi}_i\}_{i \in \mathcal{S}}) \in \mathcal{I}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{S}|}$, such that

$$\rho_i(\hat{R}_i(X_i) + \hat{\pi}_i) \leq \rho_i(R_i(X_i) + \pi_i) \quad \forall i \in \mathcal{S}, \text{ and,}$$
$$\rho\left(\sum_{i \in \mathcal{S}} (\hat{I}_i(X_i) - \hat{\pi}_i) \right) \leq \rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i) \right),$$

with at least one strict inequality.

Theorem 3.3. *Consider a contract* $\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n \right) \in \mathcal{IR}$. *Then the following are equivalent.*

(i) $\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n \right) \in \mathcal{P}$ *and is such that*

$$\left(W_1, W_2, ..., W_n, v(\mathcal{N} \cup \mathcal{CA}) - \sum_{i=1}^n W_i \right) \in \text{core}(\mathcal{N}, v).$$

(ii) $(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n)$ is coalitionally stable.

Theorem 3.3 provides a direct interpretation of the core of $(\mathcal{N}, v)$. The centralized insurer has no incentive to work with only a strict subset of policyholders, while maintaining the same level of the risk measures for the policyholders. Also, no subset of policyholders has a joint incentive with the centralized insurer to exclude the policyholders that are not in the subset for the joint insurance transaction.

4 | AN ILLUSTRATIVE EXAMPLE

Armed with the market setting and the cooperative game of Sections 2 and 3, this section revisits the model of flood insurance coverage discussed in Section 1. We illustrate this proposal via an example, in which the centralized insurer (e.g., a representative insurer procured by FEMA) proposes flood risk coverage to two different tuples of states ($n = 3$): one set of policyholders is the tuple CAL,[6] NY, and TX, while the other set is the tuple AL, LA, and MS. We assume that the centralized insurer covers the flood losses of the states in each tuple for 1 month; each state in each tuple aggregates all of the flood losses from its eligible entities in a month, which then serves as its endowed risk to be insured. Therefore, (X_1, X_2, X_3) represents the monthly aggregate flood losses of the tuple CAL-NY-TX, or AL-LA-MS, and $I_i(X_i)$, $i = 1, 2, 3$, represent the indemnities provided by the insurer to the states in each tuple. Obviously, the states in the tuple AL-LA-MS are neighboring, while the states in the tuple CAL-NY-TX are geographically far apart.

To provide context for this example, we make use of one of the datasets in OpenFEMA, an open-source database on multiple aspects of emergency management in the United States, including NFIP. This regularly updated data set[7] contains all historical claim transactions between the policyholders in NFIP and FEMA, with the earliest record dating back to August 1970. When the dataset was accessed on April 2022, there were $2,570,089$ claim records in total. The following five variables are useful for our analysis.

- *dateOfLoss*: Date on which water first entered the insured building.
- *state*: The two-character alpha abbreviation of the state in which the insured property is located.
- *amountPaidOnBuildingClaim*: Dollar amount paid on the building claim.
- *amountPaidOnContentsClaim*: Dollar amount paid on the contents claim.
- *amountPaidOnIncreasedCostOfComplianceClaim*: Dollar amount paid on the increased cost of compliance (ICC) claim.

To model the future states of the world via historical outcomes of (X_1, X_2, X_3), we construct a new data set from the OpenFEMA data set. Each observation in the new data set is a history of

---

[6]This should not be confused with the abbreviation CA for the centralized insurer (or central authority).

[7]The data set was last accessed on September 1, 2022 from https://www.fema.gov/openfema-data-page/fima-nfip-redacted-claims-v1, which is the version 1 being deprecated and removed by FEMA in September 2023. Its version 2, which contains additional fields, as well as changes some field name and data type, is accessible from https://www.fema.gov/openfema-data-page/fima-nfip-redacted-claims-v2. Results of the illustrative example in this paper are based on the version 1.

TABLE 1 Summary statistics for the past losses in the states California, New York, and Texas.

	California	New York	Texas
Mean	1.0236×10^6	9.1203×10^6	2.8184×10^7
Median	1.1970×10^4	7.6706×10^4	1.9222×10^5
$\text{VaR}_{0.95}$	2.8586×10^6	4.5924×10^6	3.8441×10^7
$\text{VaR}_{0.99}$	3.5045×10^7	6.1796×10^7	2.9626×10^8
Standard deviation	5.9048×10^6	1.6957×10^8	3.8016×10^8
Mean (positive loss)	1.5656×10^6	1.0768×10^7	3.3148×10^7
Median (positive loss)	5.9073×10^4	1.3181×10^5	3.4433×10^5
$\text{VaR}_{0.95}$ (positive loss)	6.5199×10^6	5.4031×10^6	4.6923×10^7
$\text{VaR}_{0.99}$ (positive loss)	3.8516×10^7	9.2777×10^7	4.8090×10^8
Standard deviation (positive loss)	7.2474×10^6	1.8423×10^8	4.1214×10^8
Correlation coefficient	$\begin{bmatrix} 1 & -0.0082 & -0.0082 \\ -0.0082 & 1 & -0.0038 \\ -0.0082 & -0.0038 & 1 \end{bmatrix}$		

Note: Here, VaR_α refers to the α-Value-at-Risk (see, e.g., McNeil et al., 2015), and "positive loss" in between brackets means that we condition on the historical loss being positive.

monthly aggregate flood losses for each of the 51 states (including Washington DC) in each year and each month. That is, in this new data set, there are 53 variables, which are Year, Month, and Total Loss in each state. These are obtained by first listing the values for the Year and Month variables from August 1970 to April 2022. Then, by filtering out the historical flood claims for each year, month, and state in the OpenFEMA dataset using the dateOfLoss and state variables, the monthly aggregate flood loss is the sum of the filtered flood claims using all three amountPaid variables. Note that the aggregate flood loss is the sum of the three amountPaid variables for building, contents, and ICC claim types. All missing values in the amountPaid variables are assumed to be of 0 loss. Due to data availability, we assume that the flood loss claim amount coincides with the original flood loss amount.[8] Ultimately, there are 621 historical monthly aggregate flood loss vectors in the new data set, which acts as empirical future states of the world ($N = 621$). Tables 1 and 2 display the summary statistics of the historical monthly aggregate flood losses for the three states in each tuple.

From the summary statistics, we observe that the expected monthly aggregate losses in each of the tuples are roughly similar, although the states have different sizes. In the tuple AL-LA-MS, the states are more seasonally hit by hurricanes. Since the insurance policy $I_i(X_i)$, for $i = 1, 2, 3$, of each state depends on the joint distribution of (X_1, X_2, X_3), it naturally depends on the risk measures ρ_i, for $i = 1, 2, 3$, of the states and the risk measure ρ of the centralized insurer. We are interested in how a risk measure of one state affects the flood insurance policies of other states.

We also observe that the monthly aggregate losses for the tuple CAL-NY-TX are almost (linearly) independent, while those for the tuple AL-LA-MS are highly (linearly) dependent,

---

[8]That is, claim amounts are served as proxy for loss amounts, which were not available in the version 1 of the data set but exist in its version 2.

TABLE 2 Summary statistics for the past losses in the states Alabama, Louisiana, and Mississippi.

	Alabama	**Louisiana**	**Mississippi**
Mean	2.0044×10^6	3.4006×10^7	5.0488×10^6
Median	7.3614×10^3	1.3331×10^5	4.5026×10^4
VaR$_{0.95}$	2.4472×10^6	1.6898×10^7	4.6017×10^6
VaR$_{0.99}$	2.6303×10^7	4.3491×10^8	1.8670×10^7
Standard deviation	2.0287×10^7	5.4864×10^8	1.0179×10^8
Mean (positive loss)	3.0359×10^6	4.0072×10^7	6.6708×10^6
Median (positive loss)	4.4401×10^4	2.1769×10^5	1.0875×10^5
VaR$_{0.95}$ (positive loss)	5.2468×10^6	2.5612×10^7	6.0536×10^6
VaR$_{0.99}$ (positive loss)	6.0350×10^7	4.5813×10^8	2.9388×10^7
Standard deviation (positive loss)	2.4914×10^7	5.9544×10^8	1.1699×10^8
Correlation coefficient	$\begin{bmatrix} 1 & 0.5635 & 0.5782 \\ 0.5635 & 1 & 0.9785 \\ 0.5782 & 0.9785 & 1 \end{bmatrix}$		

Note: Here, VaR$_\alpha$ refers to the α-Value-at-Risk, and "positive loss" in between brackets means that we condition on the historical loss being positive.

with Louisiana and Mississippi almost perfectly (linearly) dependent. Intuitively, this is due to the geographical locations of the states in each tuple. California, New York, and Texas are located in the West, Northeast, and South regions, respectively, while Alabama, Louisiana, and Mississippi are all located in the Southeast region. When a hurricane causes serious coastal flooding, it is very unlikely that California, New York, and Texas are affected at the same time. However, it is highly likely that Alabama, Louisiana, and Mississippi will suffer from coastal flooding around the same time if the hurricane hits the Southeast region of the United States. Naturally, such differences in the dependence structure of (X_1, X_2, X_3) are expected to affect the offerings of flood insurance policies by the single centralized insurer.

We revisit this example in Section 6, once we are able to obtain a crisper characterization of Pareto optima in the next section for the case of coherent risk measures. This characterization will also allow us to discuss maximum welfare gain and coalitional stability in the context of this simple flood insurance example.

5 | PARETO-OPTIMAL CONTRACTS WITH COHERENT RISK MEASURES

Hereafter, we aim to characterize the collection of PO contracts, in the special case where the risk measures ρ and ρ_i, for $i \in \mathcal{N}$, are coherent. By Theorem 2.1, it is enough to solve for indemnity functions $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ that are optimal for (3), as there would exist $\{\pi_i^*\}_{i=1}^n \in \mathbb{R}^n$ such that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$.

5.1 | **Characterizing PO contracts**

Recall that a risk measure is said to be coherent if it satisfies translation invariance, monotonicity, positive homogeneity, and subadditivity. Being convex, such risk measures admit a robust representation as worst-case expectations. The following lemma recalls this scenario representation of coherent risk measures defined on the set of all random variables on the finite measurable space $(\Omega, \mathcal{F})$. The proof of this standard result can be found in Artzner et al. (1999).

Lemma 5.1. *Let $L^0(\Omega, \mathcal{F})$ be the set of all random variables on the finite measurable space $(\Omega, \mathcal{F})$. A risk measure $\varrho : L^0(\Omega, \mathcal{F}) \to \mathbb{R}$ is coherent if and only if there exists a closed and convex set $\mathcal{O}$ of probability measures $\mathbb{O}$ on $(\Omega, \mathcal{F})$ such that*

$$\varrho(Z) = \sup_{\mathbb{O} \in \mathcal{O}} \mathbb{E}^{\mathbb{O}}[Z], \forall Z \in L^0(\Omega, \mathcal{F}).$$

By applying this representation result to the risk measures ρ and ρ_i, for $i \in \mathcal{N}$, with respective sets of probability measures denoted by $\mathcal{Q}$ and $\mathcal{Q}_i$, for $i \in \mathcal{N}$, the following theorem characterizes the indemnity functions $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ that are optimal for (3) and hence lead to a PO contract (by Theorem 2.1). Denote also by $\mathcal{Q}_{1:n}$ the Cartesian product of the sets $\mathcal{Q}_i$, for $i \in \mathcal{N}$.

Theorem 5.2. *$\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ is optimal for (3) if and only if for each $i \in \mathcal{N}$ and $t \in (0, \max_{j \in \{1, \dots, N\}} x_{j,i})$ almost everywhere,*

$$\left(I_i^*\right)'(t) = \begin{cases} 0 & \text{if } \mathbb{Q}^*(X_i > t) > \mathbb{Q}_i^*(X_i > t), \\ h_i(t) & \text{if } \mathbb{Q}^*(X_i > t) = \mathbb{Q}_i^*(X_i > t), \\ 1 & \text{if } \mathbb{Q}^*(X_i > t) < \mathbb{Q}_i^*(X_i > t), \end{cases} \tag{13}$$

for some measurable $[0, 1]$-valued function h_i, and where $\{\mathbb{Q}_i^*\}_{i=1}^n \in \mathcal{Q}_{1:n}$ and $\mathbb{Q}^* \in \mathcal{Q}$ solve the following optimization problem:

$$\sup_{\{\mathbb{Q}_i\}_{i=1}^n \in \mathcal{Q}_{1:n}} \sup_{\mathbb{Q} \in \mathcal{Q}} \sum_{i=1}^n \int_0^\infty \min\{\mathbb{Q}(X_i > t), \mathbb{Q}_i(X_i > t)\} dt. \tag{14}$$

Hence, by Theorem 5.2, to characterize PO contracts, we need to find solutions $\{\mathbb{Q}_i^*\}_{i=1}^n \in \mathcal{Q}_{1:n}$ and $\mathbb{Q}^* \in \mathcal{Q}$ of (14). These are the "worst-case probability measures" of the $n + 1$ agents.

In (14), for $\mathbb{Q} \in \mathcal{Q}$ and each $\mathbb{Q}_i \in \mathcal{Q}_i$, with $i \in \mathcal{N}$, the function

$$\begin{aligned} \nu_i(\cdot; \mathbb{Q}_i, \mathbb{Q}) : \mathbb{R}^+ &\to [0, 1] \\ t &\mapsto \nu_i(t; \mathbb{Q}_i, \mathbb{Q}) := \min\{\mathbb{Q}(X_i > t), \mathbb{Q}_i(X_i > t)\} \end{aligned}$$

evaluates the upper-tail event $\{X_i > t\}$ of the individual loss X_i by the more optimistic of the two scenarios representing the centralized insurer's and the ith policy holder's beliefs. Since the function

$t \mapsto \nu_i(t; \mathbb{Q}_i, \mathbb{Q})$ is monotone, one can then generate a synthetic probability measure $\tilde{\mathbb{Q}}_i$ on the space $(\Omega, \mathcal{F})$ that depends on $\mathbb{Q}_i$ and $\mathbb{Q}$ (using standard measure extension tools), such that

$$\nu_i(t; \mathbb{Q}_i, \mathbb{Q}) = \tilde{\mathbb{Q}}_i(X_i > t), \forall \, t \geq 0,$$

which is the survival function of the individual loss X_i under the synthetic measure $\tilde{\mathbb{Q}}_i$. Therefore, the objective function in (14) is the sum of the integrals of these upper-tail probabilities of all the n policyholders, and can be interpreted as the sum of expected individual losses under the synthetic measures:

$$\sum_{i=1}^{n} \int_0^\infty \nu_i(t; \mathbb{Q}_i, \mathbb{Q}) dt = \sum_{i=1}^{n} \mathbb{E}^{\tilde{\mathbb{Q}}_i}[X_i].$$

It is important to note that, with the above interpretation of (14), the worst-case scenario of each policy holder depends only on her own individual loss (via the corresponding expected individual loss under the synthetic measure), while the worst-case scenario of the centralized insurer is determined by all the losses together (via all expected individual losses under the corresponding synthetic measures). This points to the importance of the dependence structure between individual losses in determining PO contracts.

5.2 | Worst-case scenarios

As mentioned above, the upper-tail probabilities evaluated by the n policyholders and the centralized insurer play a crucial role in identifying their worst-case scenarios, and consequently the PO contracts (by Theorems 2.1 and 5.2). Note, however, that, for each policy holder $i \in \mathcal{N}$, the realizations of their individual loss $x_{j,i} \geq 0$ in the corresponding future states ω_j, for $j \in \mathcal{N}'$, are not necessarily ordered. To this end, for $i \in \mathcal{N}$, define the permutation mapping $\Pi_i : \mathcal{N}' \to \mathcal{N}'$, such that $x_{\Pi_i^{-1}(j),i}$, for $j \in \mathcal{N}'$, is the ascendingly ordered version of the realizations of the individual loss X_i, and each shall be abbreviated as $x_{[j],i}$; that is,

$$x_{[0],i} = 0 \leq x_{[1],i} \leq x_{[2],i} \leq \cdots \leq x_{[N],i} < \infty = x_{[N+1],i}.$$

Therefore, (14) is equivalent to

$$\sup_{\{\mathbb{Q}_i\}_{i=1}^n \in \mathcal{Q}_{1:n}} \sup_{\mathbb{Q} \in \mathcal{Q}} \sum_{i=1}^{n} \sum_{j=1}^{N} \min\{\mathbb{Q}(X_i \geq x_{[j],i}), \mathbb{Q}_i(X_i \geq x_{[j],i})\}(x_{[j],i} - x_{[j-1],i}). \tag{15}$$

5.2.1 | Worst-case scenarios for the policyholders

Assume that $\rho_i, i \in \mathcal{N}$, are all distortion risk measures. Throughout this section, we fix the policy holder $i \in \mathcal{N}$. There exists a nondecreasing and concave function

$g_i : [0, 1] \to [0, 1]$ with $g_i(0) = 0$ and $g_i(1) = 1$, such that ρ_i is a convex distortion risk measure[9]:

$$\rho_i(Z) = \int Z d(g_i \circ \mathbb{P}) := \int_0^\infty (g_i \circ \mathbb{P})(Z > z)dz, \forall\, Z \in L^0(\Omega, \mathcal{F}).$$

Concavity of the distortion functions yield coherence of the distortion risk measures. Therefore, they admit a robust representation by Artzner et al. (1999). The following lemma recalls the identification of the set of probability measures in the scenario representation for convex distortion risk measures. The proof can be found in Föllmer and Schied (2016, theorem 4.94).

Lemma 5.3. $\rho_i : L^0(\Omega, \mathcal{F}) \to \mathbb{R}$ *is a convex distortion risk measure if and only if*

$$\rho_i(Z) = \max_{\mathbb{Q}_i \in \mathcal{Q}_{g_i}} \mathbb{E}^{\mathbb{Q}_i}[Z], \forall\, Z \in L^0(\Omega, \mathcal{F}),$$

where

$$\mathcal{Q}_{g_i} := \{\mathbb{Q}_i \in \mathcal{M}_1(\Omega, \mathcal{F}) : \mathbb{Q}_i(A) \le (g_i \circ \mathbb{P})(A) \text{ for all } A \in \mathcal{F}\}, \tag{16}$$

and $\mathcal{M}_1(\Omega, \mathcal{F})$ is the set of all probability measures on the finite measurable space $(\Omega, \mathcal{F})$.

The following theorem constructs the worst-case scenario of policy holder i in the set $\mathcal{Q}_{g_i}$.

Theorem 5.4. *Define the set function $\mathbb{Q}_i^* : \mathcal{F} \to [0, 1]$ by $\mathbb{Q}_i^*(\varnothing) = 0$ and, for $j \in \mathcal{N}'$,*

$$\mathbb{Q}_i^*(X_i = x_{[j],i}) = (g_i \circ \mathbb{P})(X_i \ge x_{[j],i}) - (g_i \circ \mathbb{P})(X_i \ge x_{[j+1],i}). \tag{17}$$

Then, $\mathbb{Q}_i^* \in \mathcal{Q}_i = \mathcal{Q}_{g_i}$, where $\mathcal{Q}_{g_i}$ is given in (16), and $\mathbb{Q}_i^*$ is optimal for problem (15).

By (17), for each $j \in \mathcal{N}$, $\mathbb{Q}_i^*(X_i \ge x_{[j],i}) = (g_i \circ \mathbb{P})(X_i \ge x_{[j],i})$, and thus (15) is equivalent to

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \sum_{i=1}^n \sum_{j=1}^N \min\{\mathbb{Q}(X_i \ge x_{[j],i}), (g_i \circ \mathbb{P})(X_i \ge x_{[j],i})\}(x_{[j],i} - x_{[j-1],i}). \tag{18}$$

5.2.2 | Worst-case scenario for CA

Suppose that the risk measure ρ of CA is also given by a convex distortion risk measure. Here, we are not able to use the same construction for individual policyholders as in (17), and it is difficult for the worst-case scenario of CA to be explicitly solved via (18) in general. This difficulty arises from the fact that the ascendingly ordered versions of the realizations of the individual losses might not be ranked in the same order among the n policyholders.

---

[9]Convex distortion risk measures use a concave distortion function.

However, in the special case where the loss random variables X_i, for $i \in \mathcal{N}$ are comonotonic, the ascendingly ordered versions of the realizations of the individual losses will be ranked in the same order among the n policyholders. In that case, the worst-case scenario of CA can be constructed similarly to Theorem 5.4. Nonetheless, in the case of comonotonic individual losses, the vector $(I_1(X_1), ..., I_n(X_n))$ is comonotonic for each $\{I_i\}_{i=1}^n \in \mathcal{I}^n$, by definition of the set $\mathcal{I}$. In this case, one can characterize PO contracts directly from (3) by a standard argument, without applying Theorem 5.2, since $\rho(\sum_{i=1}^n I_i(X_i)) = \sum_{i=1}^n \rho(I_i(X_i))$, which follows from additivity of ρ for comonotonic random variables (see Wang et al., 1997).

To overcome the aforementioned difficulty, we resort to constructing the worst-case scenario of CA state by state. To this end, assume that the set of probability measures $\mathcal{Q}$ in the scenario representation of the coherent risk measure ρ of CA is convexly spanned by a class of $m \in \mathbb{N}$ prior probability measures $\mathbb{Q}^{(k)}$, $k = 1, ..., m$, on the finite measurable space $(\Omega, \mathcal{F})$. That is, we assume that

$$\mathcal{Q} = \left\{ \mathbb{Q}^\theta := \sum_{k=1}^m \theta_k \mathbb{Q}^{(k)} : \theta = (\theta_1, \theta_2, ..., \theta_m)^T \in [0, 1]^m, \mathbf{1}^T \theta = 1 \right\},$$

which is closed and convex. Here, $[0, 1]^m$ denotes the set of m-dimensional vectors with components drawn from the interval $[0, 1]$. The probability measures $\mathbb{Q}^{(k)}$, $k = 1, ..., m$, are prior beliefs of CA. Then (18) is equivalent to

$$\sup_{\substack{\theta \in [0,1]^m: \\ \mathbf{1}^T \theta = 1}} \sum_{i=1}^n \sum_{j=1}^N \min\{\mathbb{Q}^\theta(X_i \geq x_{[j],i}), (g_i \circ \mathbb{P})(X_i \geq x_{[j],i})\}(x_{[j],i} - x_{[j-1],i}), \tag{19}$$

in which the objective function is concave in θ. The worst-case scenario of CA is $\mathbb{Q}^* = \mathbb{Q}^{\theta^*}$, where θ^* solves the maximization problem (19). In general, it is difficult to obtain a closed-form solution for (19), and numerical methods are used; in particular, this shall be solved by the cvx package of MATLAB in Section 6.

Proposition 5.5. *For $i \in \mathcal{N}$, let*

$$\mathcal{S}_i' := \{j \in \mathcal{N}' : (g_i \circ \mathbb{P})(X_i \geq x_{[j],i}) > \mathbb{Q}^{(k)}(X_i \geq x_{[j],i}) \text{ for some } k = 1, ..., m\},$$

and

$$\mathcal{S}_i'' := \left\{ j \in \mathcal{S}_i' : (g_i \circ \mathbb{P})(X_i \geq x_{[j],i}) > \mathbb{Q}^{(k)}(X_i \geq x_{[j],i}) \text{ for all } k = 1, ..., m \right\}.$$

Define $\mathcal{S}^{(1)}, \mathcal{S}^{(2)} \subseteq \mathcal{N}$ by

(i) *for $i \in \mathcal{S}^{(1)}, \mathcal{S}_i' = \mathcal{S}_i'' = \varnothing$, that is, for all $j = 1, ..., N$ and $k = 1, ..., m, (g_i \circ \mathbb{P})(X_i \geq x_{[j],i}) \leq \mathbb{Q}^{(k)}(X_i \geq x_{[j],i})$;*

(ii) *for $i \in \mathcal{S}^{(2)}, \mathcal{S}_i' = \mathcal{S}_i'' = \mathcal{N}$, that is, for all $j = 1, ..., N$ and $k = 1, ..., m, (g_i \circ \mathbb{P})(X_i \geq x_{[j],i}) > \mathbb{Q}^{(k)}(X_i \geq x_{[j],i})$.*

Then, the worst-case scenario of CA is $\mathbb{Q}^* = \mathbb{Q}^{\theta^*}$, where θ^* solves the following maximization problem:

$$\sup_{\substack{\theta \in [0,1]^m: \\ \mathbf{1}^T \theta = 1}} \sum_{i \in \mathcal{N} \setminus \mathcal{S}^{(1)}} \sum_{j \in \mathcal{S}'_i} \min\{\mathbb{Q}^\theta(X_i \geq x_{[j],i}), (g_i \circ \mathbb{P})(X_i \geq x_{[j],i})\}(x_{[j],i} - x_{[j-1],i}).$$

(20)

Moreover, $I_i^* \equiv 0$ for $i \in \mathcal{S}^{(1)}$, and $I_i^* \equiv \mathrm{Id}$ for $i \in \mathcal{S}^{(2)}$, where Id denotes the identity function.

While Proposition 5.5 does not solve for the worst-case scenario of CA explicitly, it provides important insights into its construction. First, CA does not have to factor in those losses of the policyholders who evaluate their upper-tail probabilities under their worst-case scenarios lower than under all CA's prior beliefs (the set $\mathcal{S}^{(1)}$). This is because, in this case, no indemnity shall be provided to these policyholders. Second, for each policy holder not belonging to $\mathcal{S}^{(1)}$, an upper-tail event of the policy holder's individual loss shall only be taken into account if at least one of CA's prior beliefs evaluates this upper-tail event lower than under that policy holder's worst-case scenario. In particular, if any of those policyholders belongs to the set $\mathcal{S}^{(2)}$, that is, assigns a larger likelihood to her upper-tail events than all of CA's prior beliefs, then that policy holder will be fully indemnified.

6 | ILLUSTRATIVE EXAMPLE REVISITED

Recall the setting of Section 4, in which there are 621 future states of the world. Assume that under the physical measure $\mathbb{P}$ all 621 realizations are equally likely; that is, $\mathbb{P}(\omega_j) = 1/621$ for $j = 1, \ldots, 621$. For each state $i = 1, 2, 3$, the distortion function is given by a Proportional Hazards (PH) transform (see Wang, 1995) distortion function $g_i(z) = z^{\alpha_i}$, for some $\alpha_i \in (0, 1]$, and hence the risk measures are all convex distortion risk measures. Thus, a larger value of α_i leads to a less concave distortion function, which means that state i is less averse to mean-preserving spreads, as shown by Yaari, 1987. In particular, when $\alpha_i = 1$, the state i would be risk neutral and the risk measure is simply expectation; when α_i is close to 0, the risk measure of the state i would be given by the maximum of the range of the risk. Therefore, the range $(0, 1]$ of distortion parameter α_i captures all possible PH distortion risk measure, with various degrees of concavity on the distortion function, between these two extremes cases. Suppose that the uncertainty set $\mathcal{Q}$ that generates the risk measure ρ of CA is the convex hull of $N = m = 621$ probability measures, which are defined as follows. For any $j, k = 1, \ldots, 621$,

$$\mathbb{Q}^k(\omega_j) = \begin{cases} 0.6 & \text{if } j = k, \\ 0.4/620 & \text{if } j \neq k. \end{cases}$$

6.1 | Flood insurance coverage for standalone states

Before we examine the PO flood insurance contracts written to all the three states in each tuple, as well as their maximum welfare gains, we first examine the shape of optimal indemnities and the maximum welfare gain if CA were to provide flood risk coverage only to the first state of

each tuple, that is, California or Alabama alone. These are solved via (10), with $\mathcal{S} = \{\text{CAL}\}$ or $\mathcal{S} = \{\text{AL}\}$. This shall particularly shed light on the benefits of risk aggregation for the proposed risk sharing with multiple policyholders; see Section 6.2.1 for details.

Figure 1 displays the PO flood insurance indemnification if CA only writes the policy to California for the tuple CAL-NY-TX. Clearly, CA would not provide much flood insurance coverage to California alone, unless the state is highly averse to mean-preserving spreads, that is, when the distortion parameter α_1 is very low. The larger the distortion parameter α_1, the less averse to mean-preserving spreads the state of California is, and hence the more willing the state is to retain the upper tail part of its flood risk exposure, and even eventually retain its entire exposure. Table 3 lists the maximum welfare gain from this risk-sharing arrangement. The welfare gain decreases as the state is less averse to mean-preserving spreads, since there would be less flood insurance coverage and the state would retain more of its flood risk exposure.

Figure 2 plots the PO flood insurance indemnity, and Table 4 lists the maximum welfare gain if CA only writes the policy to Alabama for the tuple AL-LA-MS. Same conclusions hold for this tuple.
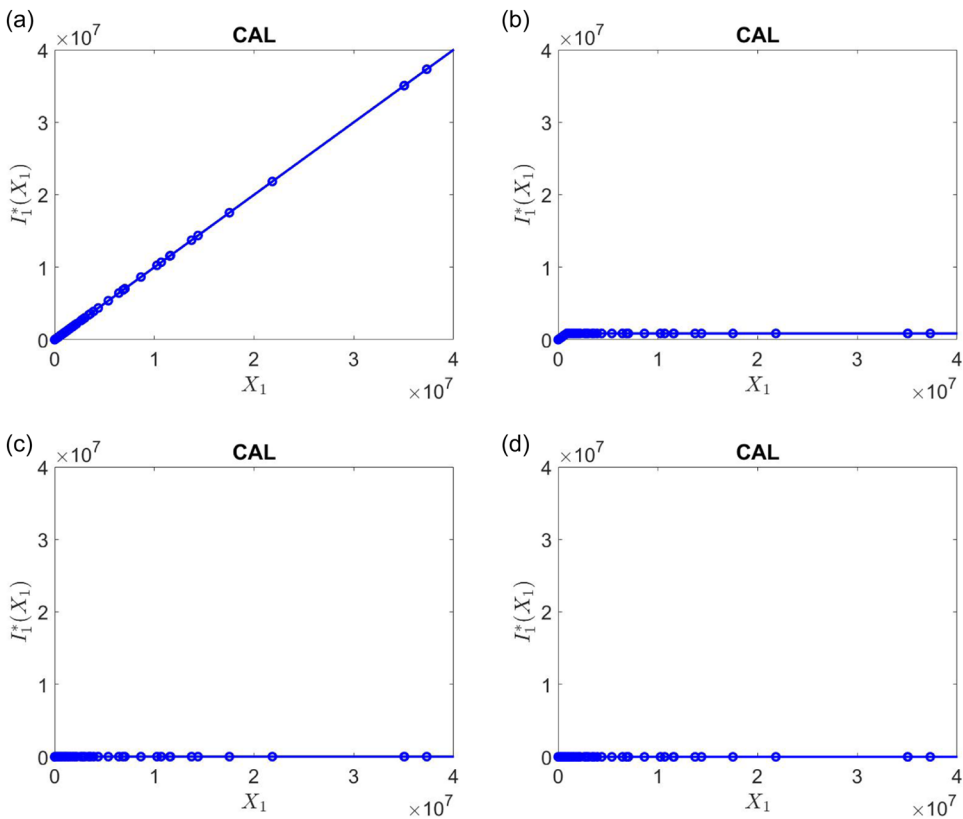

FIGURE 1 Pareto optimal insurance indemnity for California (CAL) when CA only writes a policy to California, with varying distortion parameter of California, which is solution in (10) with $\mathcal{S} = \{\text{CAL}\}$. The circles represent the historical data points. (a) $\alpha_1 = 0.1$, (b) $\alpha_1 = 0.2$, (c) $\alpha_1 = 0.3$, and (d) $\alpha_1 = 0.4, 0.5, ..., 1$.

TABLE 3 Maximum welfare gain for PO contract when CA only writes a policy to California (CAL).

$\mathcal{S}$	{CAL}
α_1	
0.1	2.2145×10^6
0.2	2.3718×10^4
0.3	301.53
0.4, 0.5, ..., 1	0

Note: The maximum welfare gain is defined by $v(\{\text{CAL}\} \cup \mathcal{CA})$ in (10).
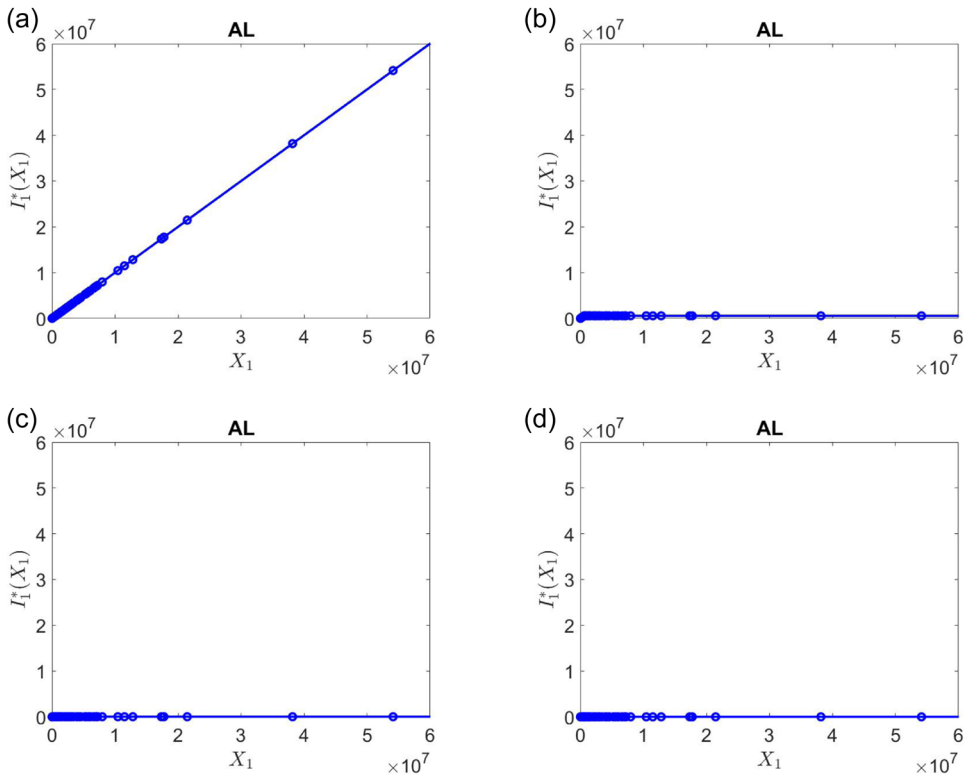

FIGURE 2 Pareto optimal insurance indemnity for Alabama (AL) when CA only writes a policy to Alabama, with varying distortion parameter of Alabama, which is solution in (10) with $\mathcal{S} = \{\text{AL}\}$. The circles represent the historical data points. (a) $\alpha_1 = 0.1$, (b) $\alpha_1 = 0.2$, (c) $\alpha_1 = 0.3$, and (d) $\alpha_1 = 0.4, 0.5, ..., 1$.

6.2 | Flood insurance coverage for the three states

This section showcases the benefit of pooling the flood risk to CA, and offering flood insurance via the PO contracts previously characterized. We also illustrate the effect of dependence between risk exposures on the level of coverage and on the maximum welfare

TABLE 4 Maximum welfare gain for PO contract when CA only writes a policy to Alabama (AL).

$\mathcal{S}$	{AL}
α_1	
0.1	2.2506×10^6
0.2	1.7748×10^4
0.3	204.77
0.4, 0.5, ..., 1	0

Note: The maximum welfare gain is defined by $v(\{AL\} \cup \mathcal{CA})$ in (10).

gains. Moreover, we investigate how a risk measure of one state affects the insurance coverage of other states.

Figures 3 and 4 display the PO flood insurance indemnities, solved via Theorem 5.2 with the worst-case scenarios of the policyholders and CA solved as in Sections 5.2.1 and 5.2.2, for the tuples CAL-NY-TX and AL-LA-MS, respectively, by varying the distortion parameter α_1 of the first state (California or Alabama). The distortion parameters α_2 and α_3 of the second state (New York or Louisiana) and the third state (Texas or Mississippi) are assumed to be 0.5 and 0.7, respectively. For better graphical illustration, the upper bounds of the y-axis are chosen as the $\mathrm{VaR}_{0.99}$ of the individual monthly aggregate positive flood losses.

6.2.1 | Benefits of risk aggregation

Comparing Figure 3 (coverage for California) to Figure 1 (see also Figure 5 which collates the two figures for California together), the benefit of offering flood risk policies using the PO contracts previously characterized is evident. For the same level of risk aversion for California in the tuple CAL-NY-TX, the centralized insurer provides no less, if not better (in terms of lower deductible and higher limit), coverage to California when it writes the flood insurance contract to the tuple than when it writes only to California. A similar conclusion can be drawn for the tuple AL-LA-MS, by comparing Figure 4 (coverage for Alabama) to Figure 2 (see also Figure 6 which collates the two figures for Alabama together), although the benefit of bundling coverage Alabama, Louisiana and Mississippi is less significant than that among California, New York and Texas, with the reason which shall be explained in the next section. These benefits can also be illustrated by the increase of maximum welfare gains for states from sharing risk with CA in various coalitions. Recall from Section 3 that the maximum welfare gain of a coalition $\mathcal{S} \subseteq \mathcal{N}$ are given by $v(\mathcal{S} \cup \mathcal{CA})$ in (10). Tables 5 and 6 display the welfare gains for all choices of $\mathcal{S}$, and for various values of α_1, respectively for the tuples CAL-NY-TX and AL-LA-MS. For example, in Table 5, the maximum welfare gain of writing flood coverage to only California is $v(\{CAL\} \cup \mathcal{CA}) \approx 301.53$, when $\alpha_1 = 0.3$, which is then substantially increased to $v(\{CAL, NY, TX\} \cup \mathcal{CA}) \approx 6.5734 \times 10^5$ when the flood coverage is written to the tuple CAL-NY-TX. While Table 6 holds similar conclusions, the benefits are less substantial, where the reason will be described next.
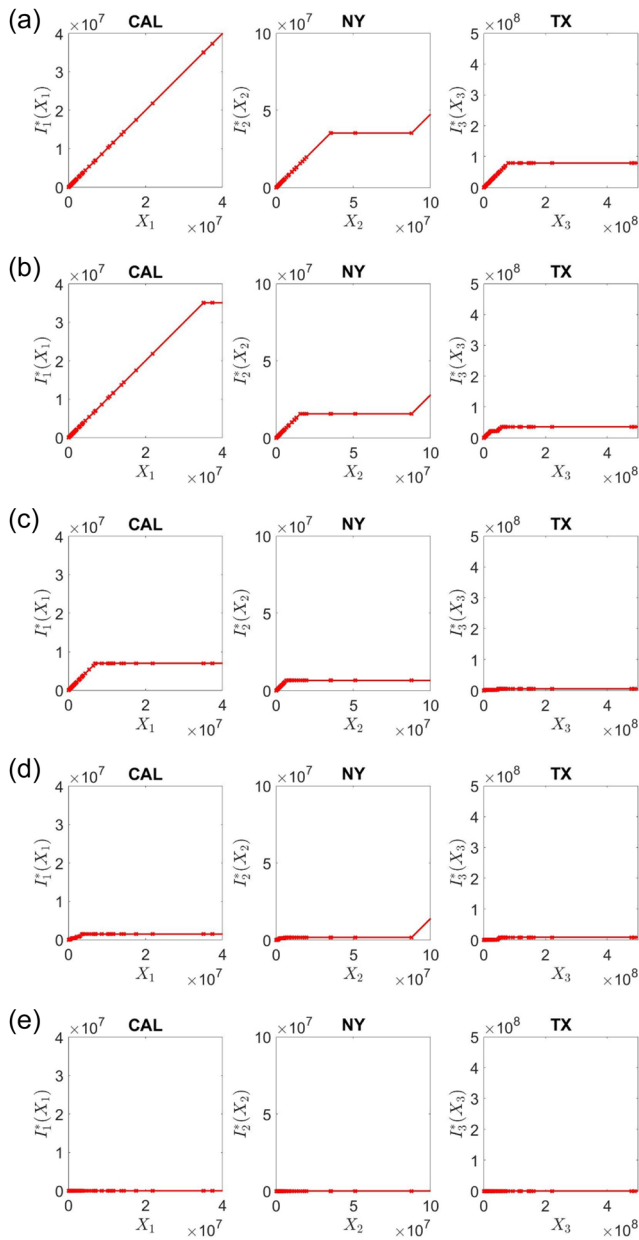
FIGURE 3 Pareto optimal insurance indemnities for California (CAL), New York (NY), and Texas (TX) when CA writes policies to the CAL-NY-TX tuple, with varying distortion parameter of California. The crosses represent the historical data points. (a) $\alpha_1 = 0.1$, (b) $\alpha_1 = 0.2$, (c) $\alpha_1 = 0.3$, (d) $\alpha_1 = 0.4$, and (e) $\alpha_1 = 0.5, 0.6, ..., 1$.

6.2.2 | Effect on benefits by geographical locations

As mentioned above, the benefit of offering flood risk policies to the tuple CAL-NY-TX over the standalone states is more significant than that to the tuple AL-LA-MS. This is due to different
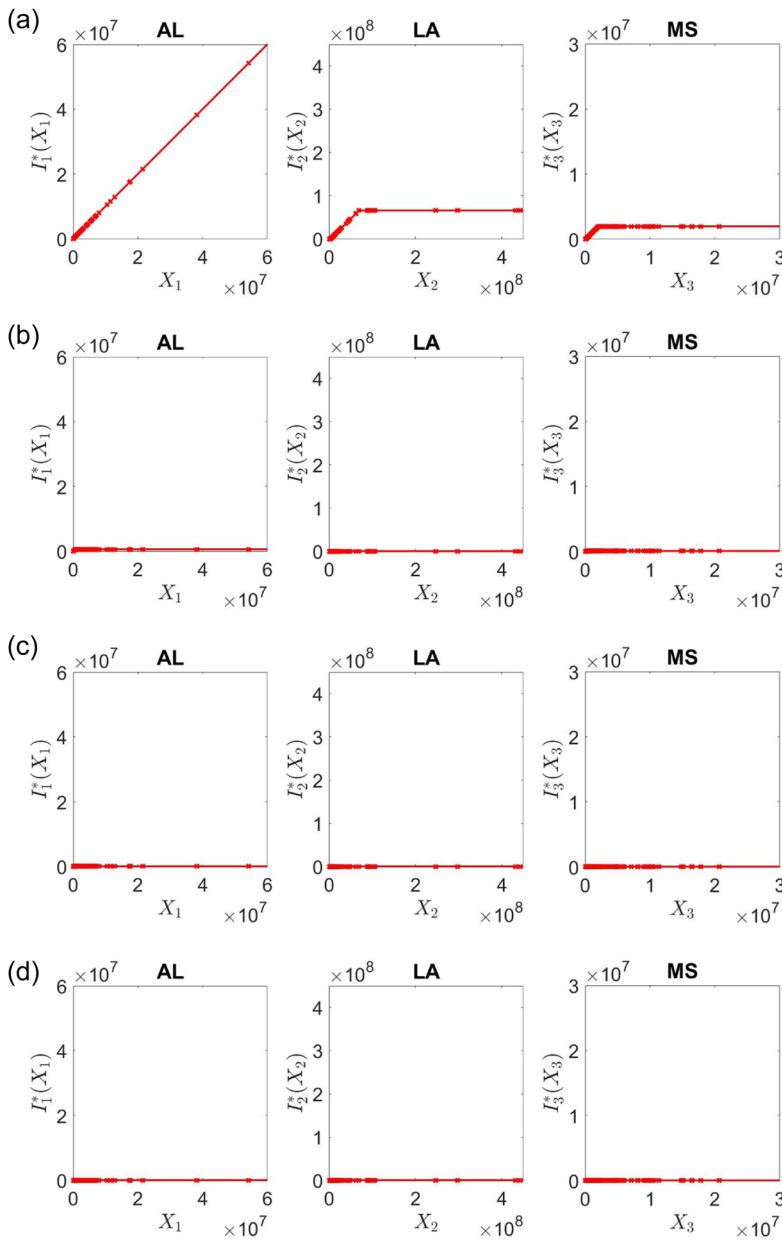
FIGURE 4 Pareto optimal insurance indemnities for Alabama (AL), Louisiana (LA), and Mississippi (MS) when CA writes policies to the AL-LA-MS tuple, with varying distortion parameter of Alabama. The crosses represent the historical data points. (a) $\alpha_1 = 0.1$, (b) $\alpha_1 = 0.2$, (c) $\alpha_1 = 0.3$, and (d) $\alpha_1 = 0.4, 0.5, ..., 1$.

dependence structures of (X_1, X_2, X_3) between the tuples. By comparing Figures 3 and 4 under the same distortion parameter α_1, we find that more insurance coverage is provided to the tuple CAL-NY-TX than to AL-LA-MS. This is intuitive since CA generally prefers losses that are less dependent to those that are positively correlated (recall the correlation coefficients in Tables 1
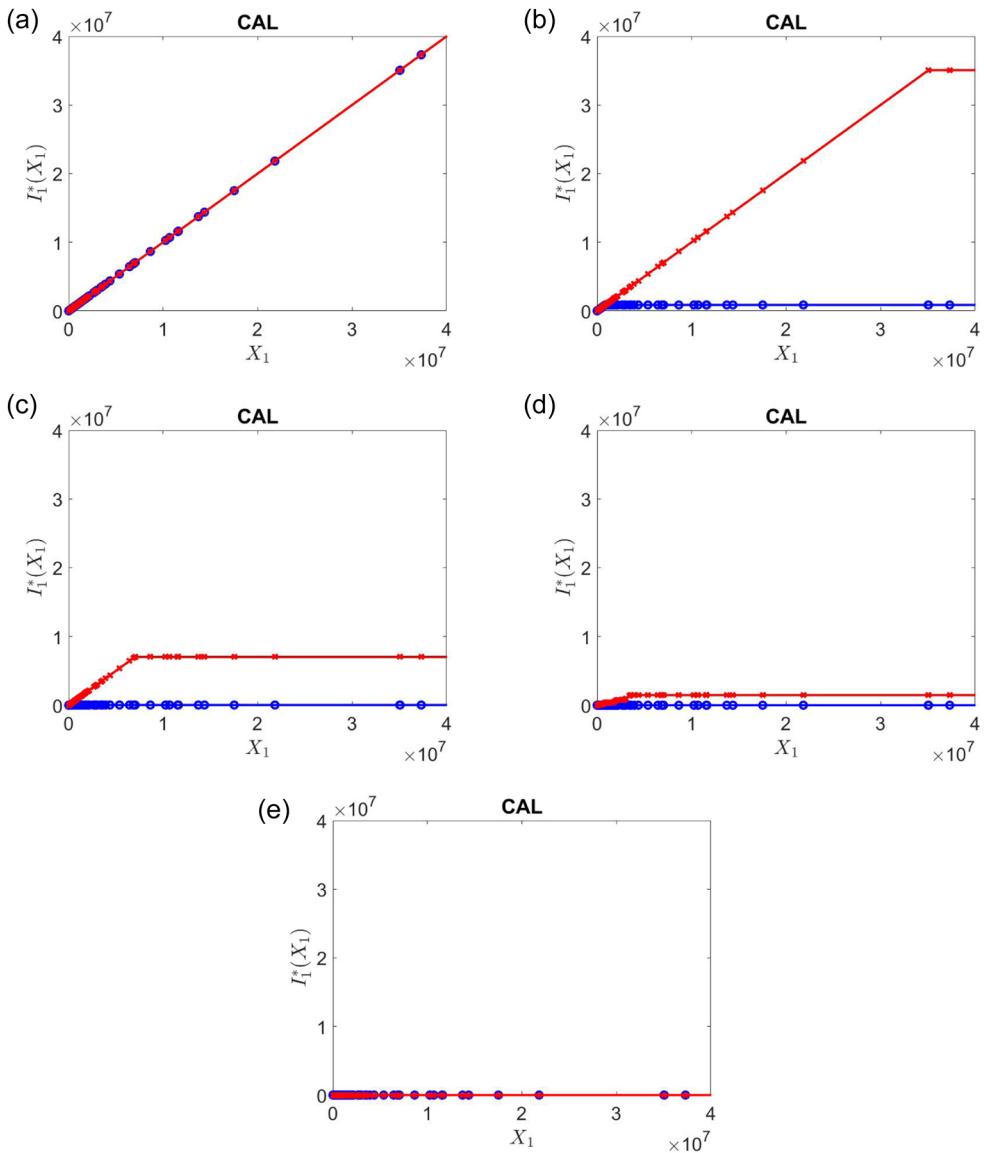
FIGURE 5 Pareto optimal insurance indemnity for California (CAL), with varying distortion parameter of California. The red (resp., blue) line with crosses (resp., circles) is solution in (10) with $\mathcal{S} = \{\text{CAL}, \text{NY}, \text{TX}\}$ (resp., $\mathcal{S} = \{\text{CAL}\}$). The circles represent the historical data points. (a) $\alpha_1 = 0.1$, (b) $\alpha_1 = 0.2$, (c) $\alpha_1 = 0.3$, (d) $\alpha_1 = 0.4$, and (e) $\alpha_1 = 0.5, 0.6, ..., 1$.

and 2, as well as the explanations in Section 4). Moreover, the indemnities for Louisiana and Mississippi are small even when $\alpha_1 = 0.1$, and this holds since the corresponding losses are almost perfectly dependent. By comparing Table 5 to Table 6, the maximum welfare gains from risk sharing are much more significant in the tuple CAL-NY-TX than in AL-LA-MS. This follows from the near independence of the loss variables in the tuple CAL-NY-TX. For this reason, expanding the set $\mathcal{S}$ is relatively more beneficial for this tuple.
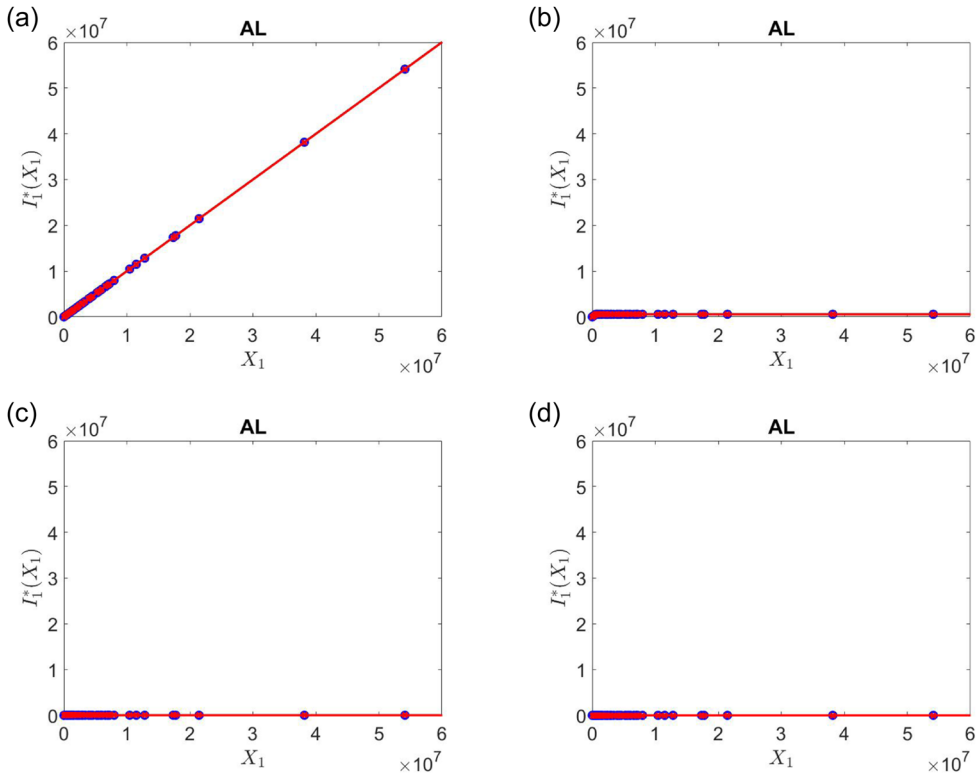
FIGURE 6 Pareto optimal insurance indemnity for Alabama (AL), with varying distortion parameter of Alabama. The red (resp., blue) line with crosses (resp., circles) is solution in (10) with $\mathcal{S} = \{\text{AL}, \text{LA}, \text{MS}\}$ (resp., $\mathcal{S} = \{\text{AL}\}$). The circles represent the historical data points. (a) $\alpha_1 = 0.1$, (b) $\alpha_1 = 0.2$, (c) $\alpha_1 = 0.3$, and (d) $\alpha_1 = 0.4, 0.5, ..., 1$.

TABLE 5 Maximum welfare gains for PO contracts in various subsets of the states California (CAL), New York (NY), and Texas (TX).

$\mathcal{S}$	{CAL}	{NY}	{TX}	{CAL,NY}	{CAL,TX}	{NY,TX}	{CAL,NY, TX}
α_1							
0.1	2.2145×10^6	0	0	1.0540×10^7	9.7386×10^6	0	1.8371×10^7
0.2	2.3718×10^4	0	0	1.1629×10^6	7.8016×10^5	0	3.6057×10^6
0.3	301.53	0	0	5.8073×10^4	1.5103×10^4	0	6.5734×10^5
0.4	0	0	0	989.98	0	0	3.4271×10^4
0.5, 0.6, ..., 1	0	0	0	0	0	0	0

Note: The maximum welfare gains are defined by $v(\mathcal{S} \cup \mathcal{CA})$ in (10), with $\mathcal{S} \subseteq \mathcal{N} = \{\text{CAL}, \text{NY}, \text{TX}\}$.

6.2.3 | Effect on indemnities by risk appetite of other states

From Figures 3 and 4, when α_1 increases (for California or Alabama), the level of insurance coverage for the second and third states (i.e., New York and Texas, or Louisiana and Mississippi)

TABLE 6 Maximum welfare gains for PO contracts in various subsets of the states Alabama (AL), Louisiana (LA), and Mississippi (MS).

$\mathcal{S}$	{AL}	{LA}	{MS}	{AL,LA}	{AL,MS}	{LA,MS}	{AL,LA,MS}
α_1							
0.1	2.2506×10^6	0	0	9.7993×10^6	2.6715×10^6	0	9.9208×10^6
0.2	1.7748×10^4	0	0	1.7748×10^4	1.8536×10^4	0	1.8536×10^4
0.3	204.77	0	0	204.77	204.77	0	204.77
0.4, 0.5, ..., 1	0	0	0	0	0	0	0

Note: The maximum welfare gains are defined by $v(\mathcal{S} \cup \mathcal{CA})$ in (10), with $\mathcal{S} \subseteq \mathcal{N} = \{\text{AL}, \text{LA}, \text{MS}\}$.

are generally reduced. This is the case because CA indemnifies the first state (i.e., California or Alabama) considerably less, and therefore there are less benefits from risk pooling. On the other hand, we observe that from Figure 3 for the tuple CAL-NY-TX, when α_1 increases from 0.3 to 0.4, the insurance coverage of California is reduced, while New York and Texas are covered in more tail losses. This may hold because CA indemnifies California considerably less, and therefore CA will have further capacity to accept tail losses from New York and Texas. These findings demonstrate that when the insurance contracts are written to multiple policyholders (or states) at the same time, the risk appetite of one policy holder could affect the insurance coverage of the others, unlike segregating coverages by several bilateral contracts. Moreover, high values of α_1 correspond to a less concave distortion function for the first state, and this leads to smaller welfare gains for the collective. A less concave distortion function leads to a smaller demand for insurance for the first state. This, in turn, leads to less diversification opportunities for CA, thereby reducing the insurance demand for the other states as well.

6.3 | Flood insurance premia for the three states

Once the PO insurance indemnities for the tuples CAL-NY-TX and AL-LA-MS are determined, the corresponding PO premia can then be chosen to satisfy (1) and (2). Recall that each state is endowed with the convex distortion risk measure with a concave PH distortion function, and thus satisfies the comonotonic additivity and nonnegative loading conditions. Therefore, by Proposition 2.2, PO premia can be chosen to be larger than the expected PO insurance indemnities.

In general, by Theorem 3.1, PO premia can be obtained by first allocating the welfare gains W_i, for $i \in \mathcal{N}$, through an element of the core (12), and then recovering the premia via (8).[10] Moreover, these PO premia are nonnegative by Theorem 3.2, since the risk measure of CA is

---

[10]In 2022, NFIP's pricing was reformed through the Risk Rating 2.0. Before the reform, a premium for a given flood insurance coverage in NFIP was solely based on flood zone, occupancy type, and elevation, without any consideration for the individual property's characteristics. With the reform, the premium now takes into account these missed characteristics, such as structure foundation type, flood frequency, and distance from water. Aligning with the Risk Rating 2.0, more informative features would provide for better predictive power regarding the loss distribution of X_i, which in turn leads to more actuarially sound pricing via the indifference premia. For an overview of NFIP's Risk Rating 2.0, we refer to https://crsreports.congress.gov/product/pdf/IN/IN11777 and https://sgp.fas.org/crs/homesec/R44593.pdf.

coherent, and hence monotone. For example, when $\alpha_1 = 0.2$, the following summarizes the cores (12) of the cooperative game for the tuples CAL-NY-TX and AL-LA-MS, respectively.

$$\text{core}(\{\text{CAL}, \text{NY}, \text{TX}\}, v) = \Big\{ (b_1, b_2, b_3, b_{CA}) \in \mathbb{R}_+^4 : b_2 \leq 2.8255 \times 10^6, \; b_3 \leq 2.4428 \times 10^6,$$
$$b_2 + b_3 \leq 3.5820 \times 10^6,$$
$$b_1 + b_2 + b_3 + b_{CA} = 3.6057 \times 10^6;$$

$$\text{core}(\{\text{AL}, \text{LA}, \text{MS}\}, v) = \Big\{ (b_1, b_2, b_3, b_{CA}) \in \mathbb{R}_+^4 : b_2 = 0, \; b_3 \leq 788,$$
$$b_1 + b_3 + b_{CA} = 1.8536 \times 10^4.$$

When we compare these two cores, it appears more beneficial for CA to insure the states in the tuple CAL-NY-TX; therein, an equal allocation of the maximum welfare gain leads to a core element, and is such that $b_{CA} = 9.014 \times 10^5$. However, CA would receive at most a welfare gain of $b_{CA} = 1.854 \times 10^4$ in $\text{core}(\{\text{AL}, \text{LA}, \text{MS}\}, v)$. It is interesting to note that the roles of CA and the first state (i.e., California or Alabama) are symmetric in both tuples in terms of their cores, which remain unchanged if we swap the labels of these two agents. With $\alpha_1 = 0.2$, the first state is the most averse to mean-preserving spreads among the policyholders. Without CA or the first state, no welfare gain is possible (see, also, Tables 5 and 6). Additionally, in core $(\{\text{AL}, \text{LA}, \text{MS}\}, v)$, the welfare gains of Louisiana and Mississippi are very small. This may be attributed to the high correlation between X_2 and X_3 (0.9785, as shown in Table 2), which reduces the negotiation power of both states.

7 | IMPLICATIONS FOR THE NATIONAL FLOOD INSURANCE PROGRAM

Recall that in NFIP, each individual eligible entity would either be indemnified or reimbursed from NFIF, through a DSA or a private insurance company in the WYO Program procured by FEMA. Therefore, through FEMA, the federal government of the United States acts as the single centralized insurer within NFIP, and it writes segregated bilateral contracts with eligible entities. The fact that these contracts are written individually and processed in real-time might lead to coverage schemes that do not necessarily consider the benefits of risk aggregation from the pool of policyholders. Moreover, this existing market structure is not compatible with the proposed multilateral risk sharing setting, in which coverage is determined periodically, at the beginning of each single period.

The theoretical results in this paper, and the insights gained from the illustrative example in Section 6 on the simpler flood insurance market model, could be used to improve the existing NFIP program. First, making use of the benefit of risk aggregation in NFIP would allow for a larger maximum total welfare gain at the aggregate level. To accomplish this, one of our suggestions is that the federal government write flood insurance contracts in NFIP using batch processing. Any eligible entity can join NFIP or renew their existing contract only at the beginning of each period. Such a period could be a month, a quarter, a half-year, a year, and so on. By doing so, the federal government could design flood insurance coverage schemes that take into consideration the benefits of risk aggregation from the existing pool of policyholders at the design stage. The policyholders in NFIP would then be indemnified at the end of the period. The resulting larger welfare gain, which is supported by both the theoretical results and the illustrative example in this paper, is then translated into a larger sum of premium

discounts, which are subtracted from the indifference premia in the flood insurance policies. This shall further improve NFIP's pricing, along with the pricing reform Risk Rating 2.0 in 2022. In addition, as illustrated in Figures 3 and 4, Pareto-optimal indemnities need not be only of the deductible and limit type, which is the case for the coverage provided in the existing NFIP to individual eligible entities.

8 | CONCLUSION

In this paper, we characterize Pareto-optimal contracts in centralized insurance markets where multiple policyholders interact with a centralized monopolistic insurer, without any assumption on the dependence structure among the risky endowments of the multiple policyholders. We show that, at a Pareto optimum, indemnification functions are those that minimize the total of all agents' risk exposure, and they depend on the agents' assessment of the likelihoods associated with their loss tail events. Premia in a Pareto-optimal contracts are then determined so that the individual rationality (market participation) constraints are satisfied. Through a naturally associated cooperative game, we provide another characterization of Pareto optima and show a tight link between the core of that game and the set of premia at a Pareto optimum.

Applying our results to an illustrative example, we shed light on the benefit of risk aggregation and risk diversification when sharing risks within the centralized insurance markets that we propose, compared with the segregated bilateral contracts currently used in the NFIP of the United States. The theoretical results in this paper and the lessons drawn from this example lead to important policy implications for NFIP.

DATA AVAILABILITY STATEMENT
Available upon request.

CONFLICTS OF INTEREST STATEMENT
The authors declare no conflicts of interest.

ORCID
Tim J. Boonen http://orcid.org/0000-0001-9186-416X
Mario Ghossoub http://orcid.org/0000-0002-6718-783X

REFERENCES
Abdikerimova, S., Boonen, T., & Feng, R. (2024). Multiperiod peer-to-peer risk sharing. *Journal of Risk and Insurance*. Forthcoming.

Artzner, P., Delbaen, F., Eber, J., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, *9*(3), 203–228.

Asimit, A., & Boonen, T. (2018). Insurance with multiple insurers: A game-theoretic approach. *European Journal of Operational Research*, *267*(2), 778–790.

Asimit, A., Boonen, T., Chi, Y., & Chong, W. (2021). Risk sharing with multiple indemnity environments. *European Journal of Operational Research*, *295*(2), 587–603.

Barbu, V., & Precupanu, T. (2012). *Convexity and optimization in banach spaces* (4th ed.). Springer.

Cai, J., & Wei, W. (2012). Optimal reinsurance with positively dependent risks. *Insurance: Mathematics and Economics*, *50*(1), 57–63.

Cheung, K., Sung, K., & Yam, S. (2014). Risk-minimizing reinsurance protection for multivariate risks. *Journal of Risk and Insurance*, *81*(1), 219–236.

Collier, B., & Ragin, M. (2020). The influence of sellers on contract choice: Evidence from flood insurance. *Journal of Risk and Insurance*, *87*(2), 523–557.

Collier, B., Schwartz, D., Kunreuther, H., & Michel-Kerjan, E. (2022). Insuring large stakes: A normative and descriptive analysis of households' flood insurance coverage. *Journal of Risk and Insurance*, *89*(2), 273–310.

Denuit, M., Dhaene, J., Ghossoub, M., & Robert, C. (2023). *Comonotonicity and Pareto optimality, with application to collaborative insurance*. Working Paper.

Denuit, M., & Dhaene, J. (2012). Convex order and comonotonic conditional mean risk sharing. *Insurance: Mathematics and Economics*, *51*(2), 265–270.

Denuit, M., Dhaene, J., & Robert, C. (2022). Risk-sharing rules and their properties, with applications to peer-to-peer insurance. *Journal of Risk and Insurance*, *89*(3), 615–667.

Denuit, M., & Robert, C. (2022). Collaborative insurance with stop-loss protection and team partitioning. *North American Actuarial Journal*, *26*(1), 143–160.

Denuit, M., & Vermandele, C. (1998). Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics*, *22*(3), 229–233.

Ermoliev, Y., Ermolieva, T., MacDonald, G., Norkin, V., & Amendola, A. (2000). A system approach to management of catastrophic risks. *European Journal of Operational Research*, *122*, 452–460.

Feng, R., Liu, C., & Taylor, S. (2022). Peer-to-peer risk sharing with an application to flood risk pooling. *Annals of Operations Research*, *321*(1–2), 813–842.

Feng, R., Liu, M., & Zhang, N. (2022). *A unified theory of decentralized insurance*. Working Paper.

Föllmer, H., & Schied, A. (2016). *Stochastic finance: An introduction in discrete time* (4th ed.). Walter de Gruyter.

Ghossoub, M., & Zhu, M. (2024). Stackelberg equilibria with multiple policyholders. *Insurance: Mathematics and Economics*, *116*(1), 89–201.

Guerra, M., & de Moura, A. (2021). Reinsurance of multiple risks with generic dependence structures. *Insurance Mathematics and Economics*, *101*(B), 547–571.

Hinck, S. (2024). Optimal insurance contract design with government disaster relief. *Journal of Risk and Insurance*. Forthcoming.

Kousky, C., & Cooke, R. (2012). Explaining the failure to insure catastrophic risks. *The Geneva Papers on Risk and Insurance—Issues and Practice*, *37*, 206–227.

Kousky, C., & Shabman, L. (2014). *Pricing flood insurance: How and why the NFIP differs from a private insurance company*. Resources for the Future Discussion Paper (pp. 14–37).

Li, H., & Su, J. (2024). Mitigating wildfire losses via insurance-linked securities: Modeling and risk management perspectives. *Journal of Risk and Insurance*. Forthcoming.

McNeil, A., Frey, R., & Embrechts, P. (2015). *Quantitative risk management: Concepts, techniques and tools—Revised edition*. Princeton University Press.

Osborne, M., & Rubinstein, A. (1994). *A course in game theory*. MIT Press.

Wang, S. (1995). Insurance pricing and increased limits ratemaking by proportional hazards transforms. *Insurance: Mathematics and Economics*, *17*(1), 43–54.

Wang, S., Young, V., & Panjer, H. (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics*, *21*(2), 173–183.

Yaari, M. (1987). The dual theory of choice under risk. *Econometrica*, *55*(1), 95–115.

APPENDIX A: PROOFS

Proof of Theorem 2.1. Note that $\mathcal{S}' \subseteq \mathcal{P}$ is obvious, and it can be easily proved by contradiction. To show the reverse inclusion, assume, by way of contradiction, that there exist $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$ such that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \notin \mathcal{S}'$. Then, there exist $(\{\tilde{I}_i\}_{i=1}^n, \{\tilde{\pi}_i\}_{i=1}^n) \in \mathcal{IR}$ such that

$$
\begin{aligned}
\sum_{i=1}^n \rho_i(\tilde{R}_i(X_i) + \tilde{\pi}_i) &+ \rho\left(\sum_{i=1}^n (\tilde{I}_i(X_i) - \tilde{\pi}_i)\right) \\
&< \sum_{i=1}^n \rho_i\left(R_i^*(X_i) + \pi_i^*\right) + \rho\left(\sum_{i=1}^n \left(I_i^*(X_i) - \pi_i^*\right)\right).
\end{aligned}
\tag{A1}
$$

Define, for $i \in \mathcal{N}$,

$$
\hat{\pi}_i := \tilde{\pi}_i + \left(\rho_i\left(R_i^*(X_i) + \pi_i^*\right) - \rho_i(\tilde{R}_i(X_i) + \tilde{\pi}_i)\right) = \rho_i\left(R_i^*(X_i)\right) + \pi_i^* - \rho_i(\tilde{R}_i(X_i)).
\tag{A2}
$$

Note that $\left(\{\tilde{I}_i\}_{i=1}^n, \{\hat{\pi}_i\}_{i=1}^n\right) \in \mathcal{IR}$. Indeed, for $i \in \mathcal{N}$, by (A2),

$$
\rho_i(\tilde{R}_i(X_i) + \hat{\pi}_i) = \rho_i\left(R_i^*(X_i) + \pi_i^*\right) \le \rho_i(X_i),
$$

and

$$
\begin{aligned}
\rho\left(\sum_{i=1}^n (\tilde{I}_i(X_i) - \hat{\pi}_i)\right) &= \rho\left(\sum_{i=1}^n (\tilde{I}_i(X_i) - \tilde{\pi}_i)\right) + \sum_{i=1}^n \rho_i(\tilde{R}_i(X_i) + \tilde{\pi}_i) \\
&\quad - \sum_{i=1}^n \rho_i\left(R_i^*(X_i) + \pi_i^*\right) \\
&< \rho\left(\sum_{i=1}^n \left(I_i^*(X_i) - \pi_i^*\right)\right) \le \rho(0) = 0,
\end{aligned}
$$

where the second last inequality is due to (A1), which contradicts the fact that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$. Thus, $\mathcal{P} \subseteq \mathcal{S}'$, and these show (i).

The equivalence in (ii) is simply by the translation-invariance property of ρ_i, for $i \in \mathcal{N}$, and ρ. Finally for (iii), as $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ solves (3) and the no-risk-sharing status quo is admissible,

$$
\sum_{i=1}^n \rho_i\left(R_i^*(X_i)\right) + \rho\left(\sum_{i=1}^n I_i^*(X_i)\right) \le \sum_{i=1}^n \rho_i(X_i).
$$

Then, with $\pi_i^* := \rho_i(X_i) - \rho_i(R_i^*(X_i))$, for $i \in \mathcal{N}$, $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ satisfies (1) and (2). $\qquad \square$

Proof of Proposition 2.2. First, note that the comonotonic additivity and nonnegative loading conditions ensure that the intervals are nonempty, since $\rho_i(X_i) - \rho_i(R_i^*(X_i)) = \rho_i(I_i^*(X_i)) \geq \mathbb{E}[I_i^*(X_i)]$, for $i \in \mathcal{N}$. Also, $\pi_i^* \geq \mathbb{E}[I_i^*(X_i)]$, for $i \in \mathcal{N}$, is obvious in both cases. By Theorem 2.1, $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$ if they satisfy (1) and (2).

When $\rho(\sum_{i=1}^n I_i^*(X_i)) \leq \mathbb{E}[\sum_{i=1}^n I_i^*(X_i)]$, (1) clearly hold for all $i \in \mathcal{N}$. As for (2), $\sum_{i=1}^n \pi_i^* \geq \mathbb{E}[\sum_{i=1}^n I_i^*(X_i)] \geq \rho(\sum_{i=1}^n I_i^*(X_i))$.

When $\mathbb{E}[\sum_{i=1}^n I_i^*(X_i)] \leq \rho(\sum_{i=1}^n I_i^*(X_i))$, (1) also clearly hold for all $i \in \mathcal{N}$. As for (2),

$$
\begin{aligned}
&\sum_{i=1}^n \pi_i^* \\
&\geq \sum_{i=1}^n \max\left\{\rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - \frac{1}{n}\left(\sum_{j=1}^n \left(\rho_j(X_j) - \rho_j\left(R_j^*(X_j)\right)\right) - \rho\left(\sum_{j=1}^n I_j^*(X_j)\right)\right),\right. \\
&\qquad\qquad \left.\mathbb{E}\left[I_i^*(X_i)\right]\right\} \\
&\geq \sum_{i=1}^n \left(\rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - \frac{1}{n}\left(\sum_{j=1}^n \left(\rho_j(X_j) - \rho_j\left(R_j^*(X_j)\right)\right) - \rho\left(\sum_{j=1}^n I_j^*(X_j)\right)\right)\right) \\
&= \sum_{i=1}^n \left(\rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right)\right) - \left(\sum_{j=1}^n \left(\rho_j(X_j) - \rho_j\left(R_j^*(X_j)\right)\right) - \rho\left(\sum_{j=1}^n I_j^*(X_j)\right)\right) \\
&= \rho\left(\sum_{j=1}^n I_j^*(X_j)\right). \qquad\qquad \square
\end{aligned}
$$

Proof of Theorem 2.3. For a given $\{\lambda_i\}_{i=1}^n \in \mathbb{R}^n$, consider the problem

$$
\inf_{(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{IR}} \left\{\rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right) : \rho_i(R_i(X_i) + \pi_i) \leq \lambda_i, \forall i \in \mathcal{N}\right\}. \tag{A3}
$$

Suppose first that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$, and let $\lambda_i^* := \rho_i(R_i^*(X_i) + \pi_i), \forall i \in \mathcal{N}$. We show that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is optimal for (A3), with parameters $\{\lambda_i^*\}_{i=1}^n$. Suppose that this is not the case. Then there exist some $(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{IR}$ such that

$$
\begin{aligned}
&\rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right) < \rho\left(\sum_{i=1}^n \left(I_i^*(X_i) - \pi_i^*\right)\right), \\
&\rho_i(R_i(X_i) + \pi_i) \leq \lambda_i^* = \rho_i\left(R_i^*(X_i) + \pi_i^*\right), \forall i \in \mathcal{N},
\end{aligned}
$$

hence contradicting the Pareto optimality of $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$.

Conversely, suppose that there exist $\{\lambda_i\}_{i=1}^n \in \mathbb{R}^n$ such that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is optimal for (A3), but that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is not PO. Then there exist some $(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{IR}$ such that

$$\rho_i(R_i(X_i) + \pi_i) \leq \rho_i\left(R_i^*(X_i) + \pi_i^*\right)(\leq \lambda_i), \forall i \in \mathcal{N},$$

$$\rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right) \leq \rho\left(\sum_{i=1}^n \left(I_i^*(X_i) - \pi_i^*\right)\right),$$

with at least one strict inequality. In particular, $\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\right)$ is feasible for (A3). If

$$\rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right) < \rho\left(\sum_{i=1}^n \left(I_i^*(X_i) - \pi_i^*\right)\right),$$

then this contradicts the optimality of $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ for (A3). Therefore,

$$\rho\left(\sum_{i=1}^n (I_i(X_i) - \pi_i)\right) = \rho\left(\sum_{i=1}^n \left(I_i^*(X_i) - \pi_i^*\right)\right),$$

and there is some $j \in \mathcal{N}$ such that

$$\rho_j(R_j(X_j) + \pi_j) < \rho_j\left(R_j^*(X_j) + \pi_j^*\right) \leq \lambda_j.$$

Let $\varepsilon_j := \rho_j\left(R_j^*(X_j) + \pi_j^*\right) - \rho_j(R_j(X_j) + \pi_j) > 0$, and $\bar{\pi}_j := \pi_j + \varepsilon_j > \pi_j$. Consider the contract

$$\bar{C} := (I_1, I_2, ..., I_n, \pi_1, ..., \pi_{j-1}, \bar{\pi}_j, \pi_{j+1}, ..., \pi_n).$$

Then $\bar{C} \in \mathcal{IR}$ is feasible for (A3) and

$$\rho\left(\sum_{i=1}^n I_i(X_i) - \sum_{i \neq j} \pi_i - \bar{\pi}_j\right) = \rho\left(\sum_{i=1}^n I_i(X_i) - \sum_{i=1}^n \pi_i - \varepsilon_j\right) = \rho\left(\sum_{i=1}^n I_i(X_i) - \sum_{i=1}^n \pi_i\right) - \varepsilon_j$$

$$= \rho\left(\sum_{i=1}^n I_i^*(X_i) - \sum_{i=1}^n \pi_i^*\right) - \varepsilon_j < \rho\left(\sum_{i=1}^n I_i^*(X_i) - \sum_{i=1}^n \pi_i^*\right),$$

contradicting the optimality of $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ for (A3). Hence, $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$.

We have therefore shown that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n) \in \mathcal{P}$ if and only if there exist some $\{\lambda_i\}_{i=1}^n \in \mathbb{R}^n$ such that $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is optimal for (A3). To conclude the proof of Theorem 2.3, it remains to show that for a given $\{\lambda_i\}_{i=1}^n \in \mathbb{R}^n$, problems (6) and (A3) have the same solutions. This, however, follows immediately from the fact that an optimum for (A3), all constraints have to bind. Indeed, suppose that for some $\{\lambda_i\}_{i=1}^n \in \mathbb{R}^n$, $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ is optimal for (A3), but that there is some $j \in \mathcal{N}$ such that $\rho_j(R_j^*(X_j) + \pi_j^*) < \lambda_j$. Let $\varepsilon_j := \lambda_j - \rho_j(R_j^*(X_j) + \pi_j^*) > 0$, and $\bar{\pi}_j^* := \pi_j^* + \varepsilon_j$. Consider the contract

$$\bar{C} := \left(I_1^*, I_2^*, ..., I_n^*, \pi_1^*, ..., \pi_{j-1}^*, \bar{\pi}_j^*, \pi_{j+1}^*, ..., \pi_n^*\right).$$

Then $\bar{C} \in \mathcal{IR}$ is feasible for (A3) and

$$
\begin{aligned}
\rho\left(\sum_{i=1}^{n} I_i^*(X_i) - \sum_{i \neq j} \pi_i^* - \bar{\pi}_j^*\right) &= \rho\left(\sum_{i=1}^{n} I_i^*(X_i) - \sum_{i=1}^{n} \pi_i^* - \varepsilon_j\right) \\
&= \rho\left(\sum_{i=1}^{n} I_i^*(X_i) - \sum_{i=1}^{n} \pi_i^*\right) - \varepsilon_j \\
&< \rho\left(\sum_{i=1}^{n} I_i^*(X_i) - \sum_{i=1}^{n} \pi_i^*\right),
\end{aligned}
$$

contradicting the optimality of $(\{I_i^*\}_{i=1}^n, \{\pi_i^*\}_{i=1}^n)$ for (A3). $\qquad \square$

Proof of Theorem 3.1. Fix $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ that solves (3). For any $(b_1, b_2, ..., b_n, b_{CA}) \in \text{core}(\mathcal{N}, v)$, if the welfare gain W_i of the ith policy holder is $b_i \geq 0$, then the welfare gain of the centralized insurer from risk sharing is given by

$$
\begin{aligned}
0 \leq b_{CA} &= \sum_{i \in \mathcal{N} \cup \mathcal{CA}} b_i - \sum_{i \in \mathcal{N}} b_i = v(\mathcal{N} \cup \mathcal{CA}) - \sum_{i=1}^{n} W_i \\
&= \sum_{i=1}^{n} \rho_i(X_i) - \sum_{i=1}^{n} \rho_i\left(R_i^*(X_i)\right) - \rho\left(\sum_{i=1}^{n} I_i^*(X_i)\right) \\
&\quad - \sum_{i=1}^{n} \left(\rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - \pi_i\right) \\
&= \sum_{i=1}^{n} \pi_i - \rho\left(\sum_{i=1}^{n} I_i^*(X_i)\right),
\end{aligned}
$$

and so $\rho(\sum_{i=1}^{n} I_i^*(X_i)) \leq \sum_{i=1}^{n} \pi_i$. That is, the vector $\{\pi_i\}_{i=1}^n$ defined by

$$
\pi_i := \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - b_i, \forall i \in \mathcal{N},
$$

satisfies (2). Moreover, for each $i \in \mathcal{N}$,

$$
0 \leq b_i = W_i = \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - \pi_i,
$$

and so

$$
\pi_i \leq \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right).
$$

That is, the vector $\{\pi_i\}_{i=1}^n$ also satisfies (1). Hence, $(\{I_i^*\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{IR}$. Thus, by Theorem 2.1, $(\{I_i^*\}_{i=1}^n, \{\pi_i\}_{i=1}^n) \in \mathcal{P}$. $\qquad \square$

Proof of Theorem 3.2. Fix $\{I_i^*\}_{i=1}^n \in \mathcal{I}^n$ that solves (3), and $(b_1, b_2, ..., b_n, b_{\mathcal{CA}}) \in \text{core}(\mathcal{N}, v)$. Define, for $i \in \mathcal{N}$,

$$\pi_i := \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - b_i.$$

Fix any $i \in \mathcal{N}$. Then,

$$\pi_i = \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - \sum_{j \in \mathcal{N} \cup \mathcal{CA}} b_j + \sum_{j \in (\mathcal{N}\backslash\{i\}) \cup \mathcal{CA}} b_j$$

$$\geq \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - v(\mathcal{N} \cup \mathcal{CA}) + v((\mathcal{N}\backslash\{i\}) \cup \mathcal{CA})$$

$$= \rho_i(X_i) - \rho_i\left(R_i^*(X_i)\right) - \left[\sum_{j=1}^n \rho_j(X_j) - \left(\sum_{j=1}^n \rho_j\left(R_j^*(X_j)\right) + \rho\left(\sum_{j=1}^n I_j^*(X_j)\right)\right)\right]$$

$$+ \left(\sum_{j \in \mathcal{N}\backslash\{i\}} \rho_j(X_j) - \inf_{\{I_j\}_{j \in \mathcal{N}\backslash\{i\}} \in \mathcal{I}^{n-1}}\left(\sum_{j \in \mathcal{N}\backslash\{i\}} \rho_j(R_j(X_j)) + \rho\left(\sum_{j \in \mathcal{N}\backslash\{i\}} I_j(X_j)\right)\right)\right)$$

$$\geq -\rho_i\left(R_i^*(X_i)\right) + \sum_{j=1}^n \rho_j\left(R_j^*(X_j)\right) + \rho\left(\sum_{j=1}^n I_j^*(X_j)\right)$$

$$- \left(\sum_{j \in \mathcal{N}\backslash\{i\}} \rho_j\left(R_j^*(X_j)\right) + \rho\left(\sum_{j \in \mathcal{N}\backslash\{i\}} I_j^*(X_j)\right)\right)$$

$$= \rho\left(\sum_{j=1}^n I_j^*(X_j)\right) - \rho\left(\sum_{j \in \mathcal{N}\backslash\{i\}} I_j^*(X_j)\right) \geq 0,$$

where the second inequality is due to the second condition in the core, the fourth inequality is because of

$$\inf_{\{I_j\}_{j \in \mathcal{N}\backslash\{i\}} \in \mathcal{I}^{n-1}}\left(\sum_{j \in \mathcal{N}\backslash\{i\}} \rho_j(R_j(X_j)) + \rho\left(\sum_{j \in \mathcal{N}\backslash\{i\}} I_j(X_j)\right)\right)$$

$$\leq \sum_{j \in \mathcal{N}\backslash\{i\}} \rho_j\left(R_j^*(X_j)\right) + \rho\left(\sum_{j \in \mathcal{N}\backslash\{i\}} I_j^*(X_j)\right),$$

and the last inequality is by the monotonicity property of ρ. $\qquad\square$

Proof of Theorem 3.3. We first prove that (ii) implies (i). Let $\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\right) \in \mathcal{IR}$ be coalitionally stable. By Definition 3.1 with $\mathcal{S} = \mathcal{N}$, it immediately follows that $\left(\{I_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n\right) \in \mathcal{P}$. Assume, by way of contradiction, that

$\left(W_1, W_2, ..., W_n, v(\mathcal{N} \cup \mathcal{CA}) - \sum_{i=1}^{n} W_i\right) \notin \text{core}(\mathcal{N}, v)$. Then there exists a nonempty $\mathcal{S} \subsetneq \mathcal{N}$ such that $\sum_{i \in \mathcal{S}} W_i + v(\mathcal{N} \cup \mathcal{CA}) - \sum_{i=1}^{n} W_i < v(\mathcal{S} \cup \mathcal{CA})$. This yields

$$
\begin{aligned}
&\sum_{i \in \mathcal{S}} (\rho_i(X_i) - \rho_i(R_i(X_i)) - \pi_i) - \rho\left(\sum_{i=1}^{n} (I_i(X_i) - \pi_i)\right) \\
&< \sum_{i \in \mathcal{S}} (\rho_i(X_i) - \rho_i(\tilde{R}_i(X_i))) - \rho\left(\sum_{i \in \mathcal{S}} \tilde{I}_i(X_i)\right),
\end{aligned}
\tag{A4}
$$

where $\{\tilde{I}_i\}_{i \in \mathcal{S}} \in \mathcal{I}^{|\mathcal{S}|}$ solves

$$
\inf_{\{I_i\}_{i \in \mathcal{S}} \in \mathcal{I}^{|\mathcal{S}|}} \left(\sum_{i \in \mathcal{S}} \rho_i(R_i(X_i)) + \rho\left(\sum_{i \in \mathcal{S}} I_i(X_i)\right)\right)
$$

in $v(\mathcal{S} \cup \mathcal{CA})$. Define, for $i \in \mathcal{S}, \tilde{\pi}_i := \rho_i(R_i(X_i) + \pi_i) - \rho_i(\tilde{R}_i(X_i))$, which is such that $\rho_i(\tilde{R}_i(X_i) + \tilde{\pi}_i) = \rho_i(R_i(X_i) + \pi_i)$. However, by (A4),

$$
\begin{aligned}
&\sum_{i \in \mathcal{S}} (\rho_i(X_i) - \rho_i(\tilde{R}_i(X_i) + \tilde{\pi}_i)) - \rho\left(\sum_{i \in \mathcal{S}} (\tilde{I}_i(X_i) - \tilde{\pi}_i)\right) \\
&> \sum_{i \in \mathcal{S}} (\rho_i(X_i) - \rho_i(R_i(X_i) + \pi_i)) - \rho\left(\sum_{i=1}^{n} (I_i(X_i) - \pi_i)\right),
\end{aligned}
$$

which implies that $\rho\left(\sum_{i \in \mathcal{S}} (\tilde{I}_i(X_i) - \tilde{\pi}_i)\right) < \rho\left(\sum_{i=1}^{n} (I_i(X_i) - \pi_i)\right)$. These then contradict the fact that $\left(\{I_i\}_{i=1}^{n}, \{\pi_i\}_{i=1}^{n}\right)$ is coalitionally stable.

We then prove that (i) implies (ii). Let $\left(\{I_i\}_{i=1}^{n}, \{\pi_i\}_{i=1}^{n}\right) \in \mathcal{P}$ and $\left(W_1, W_2, ..., W_n, v(\mathcal{N} \cup \mathcal{CA}) - \sum_{i=1}^{n} W_i\right) \in \text{core}(\mathcal{N}, v)$. Assume, by way of contradiction, that $\left(\{I_i\}_{i=1}^{n}, \{\pi_i\}_{i=1}^{n}\right)$ is not coalitionally stable. Then there exists a nonempty subset $\mathcal{S} \subseteq \mathcal{N}$ and $\left(\{\hat{I}_i\}_{i \in \mathcal{S}}, \{\hat{\pi}_i\}_{i \in \mathcal{S}}\right) \in \mathcal{I}^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{S}|}$, such that

$$
\begin{aligned}
&\rho_i(\hat{R}_i(X_i) + \hat{\pi}_i) \leq \rho_i(R_i(X_i) + \pi_i), \forall i \in \mathcal{S}, \\
&\rho\left(\sum_{i \in \mathcal{S}} (\hat{I}_i(X_i) - \hat{\pi}_i)\right) \leq \rho\left(\sum_{i=1}^{n} (I_i(X_i) - \pi_i)\right),
\end{aligned}
$$

with at least one strict inequality. If $\mathcal{S} = \mathcal{N}$, this contradicts the fact that $\left(\{I_i\}_{i=1}^{n}, \{\pi_i\}_{i=1}^{n}\right) \in \mathcal{P}$. If $\mathcal{S} \subsetneq \mathcal{N}$,

$$\sum_{i \in \mathcal{S}} W_i + v(\mathcal{N} \cup \mathcal{CA}) - \sum_{i=1}^{n} W_i$$

$$= \sum_{i \in \mathcal{S}} (\rho_i(X_i) - \rho_i(R_i(X_i)) - \pi_i) - \rho\left(\sum_{i=1}^{n} (I_i(X_i) - \pi_i)\right)$$

$$< \sum_{i \in \mathcal{S}} (\rho_i(X_i) - \rho_i(\hat{R}_i(X_i) + \hat{\pi}_i)) - \rho\left(\sum_{i \in \mathcal{S}} (\hat{I}_i(X_i) - \hat{\pi}_i)\right)$$

$$= \sum_{i \in \mathcal{S}} (\rho_i(X_i) - \rho_i(\hat{R}_i(X_i))) - \rho\left(\sum_{i \in \mathcal{S}} \hat{I}_i(X_i)\right)$$

$$\leq \sum_{i \in \mathcal{S}} \rho_i(X_i) - \inf_{\{I_i\}_{i \in \mathcal{S}} \in \mathcal{I}^{|\mathcal{S}|}} \left(\sum_{i \in \mathcal{S}} \rho_i(R_i(X_i)) + \rho\left(\sum_{i \in \mathcal{S}} I_i(X_i)\right)\right)$$

$$= v(\mathcal{S} \cup \mathcal{CA}),$$

which then contradicts the fact that $\left(W_1, W_2, ..., W_n, v(\mathcal{N} \cup \mathcal{CA}) - \sum_{i=1}^{n} W_i\right) \in \text{core}(\mathcal{N}, v)$.
□

Proof of Theorem 5.2. By Lemma 5.1, (3) is equivalent to

$$\inf_{\{I_i\}_{i=1}^{n} \in \mathcal{I}^n} \sup_{\{\mathbb{Q}_i\}_{i=1}^{n} \in \mathcal{Q}_{1:n}} \sup_{\mathbb{Q} \in \mathcal{Q}} \sum_{i=1}^{n} \mathbb{E}^{\mathbb{Q}_i}[R_i(X_i)] + \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n} I_i(X_i)\right]. \tag{A5}$$

Since a probability measure on the finite measurable space $(\Omega, \mathcal{F})$ assigns to each future state $\omega_j, j = 1, ..., N$, a value in $[0, 1]$, with the sum of these values being equal to 1, it can be seen as an element of the standard simplex in $\mathbb{R}^N$. The sets $\mathcal{Q}_i$, for $i \in \mathcal{N}$, and $\mathcal{Q}$ are thus all subsets of the standard simplex, which is bounded. Moreover, by Lemma 5.1, the sets $\mathcal{Q}_i$, for $i \in \mathcal{N}$, and $\mathcal{Q}$ are closed. Hence, by the Heine-Borel theorem for the finite-dimensional space $\mathbb{R}^N$, they are also compact. Since $\mathcal{I}^n$, $\mathcal{Q}_i$, for $i \in \mathcal{N}$, and $\mathcal{Q}$ are convex and compact, and the objective function in (A5) is continuous in $(\{I_i\}_{i=1}^{n}, \{\mathbb{Q}_i\}_{i=1}^{n}, \mathbb{Q}) \in \mathcal{I}^n \times \mathcal{Q}_{1:n} \times \mathcal{Q}$, by the Sion's Minimax Theorem (e.g., Barbu & Precupanu, 2012, theorem 2.132), (3) and (A5) are also equivalent to

$$\sup_{\{\mathbb{Q}_i\}_{i=1}^{n} \in \mathcal{Q}_{1:n}} \sup_{\mathbb{Q} \in \mathcal{Q}} \inf_{\{I_i\}_{i=1}^{n} \in \mathcal{I}^n} \sum_{i=1}^{n} \mathbb{E}^{\mathbb{Q}_i}[R_i(X_i)] + \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n} I_i(X_i)\right]. \tag{A6}$$

Moreover, by the compactness property of $\mathcal{Q}_i$, for $i \in \mathcal{N}$, and $\mathcal{Q}$, and the continuity property of the objective function in $\left(\{\mathbb{Q}_i\}_{i=1}^{n}, \mathbb{Q}\right) \in \mathcal{Q}_{1:n} \times \mathcal{Q}$, the suprema in (A6) are attained at some $(\{\mathbb{Q}_i^*\}_{i=1}^{n}, \mathbb{Q}^*) \in \mathcal{Q}_{1:n} \times \mathcal{Q}$. Additionally,

$$\inf_{\{I_i\}_{i=1}^n \in \mathcal{I}^n} \sum_{i=1}^n \mathbb{E}^{\mathbb{Q}_i^*}[R_i(X_i)] + \mathbb{E}^{\mathbb{Q}^*}\left[\sum_{i=1}^n I_i(X_i)\right]$$

$$= \sum_{i=1}^n \mathbb{E}^{\mathbb{Q}_i^*}[X_i] + \inf_{\{I_i\}_{i=1}^n \in \mathcal{I}^n} \sum_{i=1}^n \left(\mathbb{E}^{\mathbb{Q}^*}[I_i(X_i)] - \mathbb{E}^{\mathbb{Q}_i^*}[I_i(X_i)]\right)$$

$$= \sum_{i=1}^n \mathbb{E}^{\mathbb{Q}_i^*}[X_i] + \inf_{\{I_i\}_{i=1}^n \in \mathcal{I}^n} \sum_{i=1}^n \int_0^\infty \left(\mathbb{Q}^*(X_i > t) - \mathbb{Q}_i^*(X_i > t)\right)(I_i)'(t)dt,$$

which immediately implies (13). Finally, substituting (13) into (A6) gives

$$\sum_{i=1}^n \mathbb{E}^{\mathbb{Q}_i^*}\left[R_i^*(X_i)\right] + \mathbb{E}^{\mathbb{Q}^*}\left[\sum_{i=1}^n I_i^*(X_i)\right]$$

$$= \sum_{i=1}^n \mathbb{E}^{\mathbb{Q}_i^*}[X_i] + \sum_{i=1}^n \int_0^\infty \left(\mathbb{Q}^*(X_i > t) - \mathbb{Q}_i^*(X_i > t)\right)\left(I_i^*\right)'(t)dt$$

$$= \sum_{i=1}^n \int_0^\infty \mathbb{Q}_i^*(X_i > t)dt$$

$$+ \sum_{i=1}^n \int_{t \in (0, \max_{j=1,...,N} x_{j,i}) : \mathbb{Q}^*(X_i > t) < \mathbb{Q}_i^*(X_i > t)} \left(\mathbb{Q}^*(X_i > t) - \mathbb{Q}_i^*(X_i > t)\right)dt$$

$$= \sum_{i=1}^n \Bigg(\int_{t \in (0, \max_{j=1,...,N} x_{j,i}) : \mathbb{Q}^*(X_i > t) < \mathbb{Q}_i^*(X_i > t)} \mathbb{Q}^*(X_i > t)dt$$

$$+ \int_{t \in (0, \max_{j=1,...,N} x_{j,i}) : \mathbb{Q}^*(X_i > t) \geq \mathbb{Q}_i^*(X_i > t)} \mathbb{Q}_i^*(X_i > t)dt \Bigg)$$

$$= \sum_{i=1}^n \int_0^\infty \min\left\{\mathbb{Q}^*(X_i > t), \mathbb{Q}_i^*(X_i > t)\right\}dt,$$

which shows that $(\{\mathbb{Q}_i^*\}_{i=1}^n, \mathbb{Q}^*) \in \mathcal{Q}_{1:n} \times \mathcal{Q}$ solves (14). These show necessity. Sufficiency follows immediately by similar arguments. □

Proof of Theorem 5.4.   By the properties of the capacity $(g_i \circ \mathbb{P})$, it is obvious that $\mathbb{Q}_i^* \in \mathcal{M}_1(\Omega, \mathcal{F})$. Since the capacity $(g_i \circ \mathbb{P})$ is submodular, we have for $j = 1, ..., N$,

$$\mathbb{Q}_i^*(X_i = x_{[j],i}) = (g_i \circ \mathbb{P})(X_i \geq x_{[j],i}) - (g_i \circ \mathbb{P})(X_i \geq x_{[j+1],i}) \leq (g_i \circ \mathbb{P})(X_i = x_{[j],i}).$$

This can then show that, for any $A \in \mathcal{F}, \mathbb{Q}_i^*(A) \leq (g_i \circ \mathbb{P})(A)$. These imply that $\mathbb{Q}_i^* \in \mathcal{Q}_i = \mathcal{Q}_{g_i}$.

Fix an $I_i \in \mathcal{I}$. By (16), for any $\mathbb{Q}_i \in \mathcal{Q}_i = \mathcal{Q}_{g_i}$,

$$\mathbb{E}^{\mathbb{Q}_i}[R_i(X_i)] = \int_\Omega R_i(X_i)d\mathbb{Q}_i \leq \int_\Omega R_i(X_i)d(g_i \circ \mathbb{P}).$$

Moreover, by (17), for $j = 1, ..., N$, $\mathbb{Q}_i^*(X_i \geq x_{[j],i}) = (g_i \circ \mathbb{P})(X_i \geq x_{[j],i})$, and hence,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}_i^*}[R_i(X_i)] &= \int_0^\infty \mathbb{Q}_i^*(R_i(X_i) > z)dz = \int_0^\infty \mathbb{Q}_i^*\left(X_i > R_i^{-1+}(z)\right)dz \\
&= \int_0^\infty (g_i \circ \mathbb{P})\left(X_i > R_i^{-1+}(z)\right)dz = \int_0^\infty (g_i \circ \mathbb{P})(R_i(X_i) > z)dz \\
&= \int_\Omega R_i(X_i)d(g_i \circ \mathbb{P}),
\end{aligned}
$$

where $R_i^{-1+}(z) = \sup\{t \in [0, \max_{j=1, ..., N} x_{j,i}] : R_i(t) \leq z\}$, for $z \in \mathbb{R}_+$. Therefore, for any $\mathbb{Q}_i \in \mathcal{Q}_i = \mathcal{Q}_{g_i}$, $\mathbb{E}^{\mathbb{Q}_i}[R_i(X_i)] \leq \mathbb{E}^{\mathbb{Q}_i^*}[R_i(X_i)]$, which implies that $\mathbb{Q}_i^*$ solves (A5). By the equivalences, $\mathbb{Q}_i^*$ also solves (A6), (14), and (15). $\qquad\square$

Proof of Proposition 5.5.   Note that, for any $\theta \in [0, 1]^m$ with $\mathbf{1}^T\theta = 1$, for $i \in \mathcal{S}^{(1)}$ and $j \in \mathcal{N}'$, as well as for $i \in \mathcal{N}\backslash\mathcal{S}^{(1)}$ and $j \in \mathcal{N}'\backslash\mathcal{S}_i'$,

$$
\mathbb{Q}^\theta(X_i \geq x_{[j],i}) \geq (g_i \circ \mathbb{P})(X_i \geq x_{[j],i})\sum_{k=1}^m \theta_k = (g_i \circ \mathbb{P})(X_i \geq x_{[j],i}),
$$

and thus,

$$
\begin{aligned}
&\sum_{i=1}^n \sum_{j=1}^N \min\{\mathbb{Q}^\theta(X_i \geq x_{[j],i}), (g_i \circ \mathbb{P})(X_i \geq x_{[j],i})\}(x_{[j],i} - x_{[j-1],i}) \\
&= \left(\sum_{i\in\mathcal{S}^{(1)}} \sum_{j=1}^N + \sum_{i\in\mathcal{N}\backslash\mathcal{S}^{(1)}} \sum_{j\in\mathcal{S}_i'} + \sum_{i\in\mathcal{N}\backslash\mathcal{S}^{(1)}} \sum_{j\in\mathcal{N}'\backslash\mathcal{S}_i'}\right) \\
&\quad \min\{\mathbb{Q}^\theta(X_i \geq x_{[j],i}), (g_i \circ \mathbb{P})(X_i \geq x_{[j],i})\}(x_{[j],i} - x_{[j-1],i}) \\
&= \left(\sum_{i\in\mathcal{S}^{(1)}} \sum_{j=1}^N + \sum_{i\in\mathcal{N}\backslash\mathcal{S}^{(1)}} \sum_{j\in\mathcal{N}'\backslash\mathcal{S}_i'}\right)(g_i \circ \mathbb{P})(X_i \geq x_{[j],i})(x_{[j],i} - x_{[j-1],i}) \\
&\quad + \sum_{i\in\mathcal{N}\backslash\mathcal{S}^{(1)}} \sum_{j\in\mathcal{S}_i'} \min\{\mathbb{Q}^\theta(X_i \geq x_{[j],i}), (g_i \circ \mathbb{P})(X_i \geq x_{[j],i})\}(x_{[j],i} - x_{[j-1],i}).
\end{aligned}
$$

This implies that (19) is equivalent to (20).

Finally, regardless of θ^* being solved by (20), for $i \in \mathcal{S}^{(1)}$, $\mathbb{Q}^{\theta^*}(X_i \geq x_{[j],i}) \geq (g_i \circ \mathbb{P})(X_i \geq x_{[j],i})$ for all $j = 1, ..., N$; by Theorem 5.2, $I_i^*(\cdot) \equiv 0$. Similarly, for $i \in \mathcal{S}^{(2)}$, $\mathbb{Q}^{\theta^*}(X_i \geq x_{[j],i}) < (g_i \circ \mathbb{P})(X_i \geq x_{[j],i})$ for all $j = 1, ..., N$; by Theorem 5.2, $I_i^*(\cdot) \equiv \mathrm{Id}$. $\qquad\square$