Extension of inverses of Γ-equivariant holomorphic embeddings of bounded symmetric domains of rank ≥ 2 and applications to rigidity problems

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Abstract

Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$ being quasi-projective. By means of Gauss-Bonnet type integral formulas and ergodicity theory of semisimple Lie groups, the first author proved that any nonconstant Γ-equivariant holomorphic map $F : \Omega \to D$ into a bounded domain D is necessarily an embedding. Here we give a complete proof of the existence of an extension of $F^{-1} : F(\Omega) \to \Omega$ to a *bounded* holomorphic map $R : D \to \mathbb{C}^n$, called the Extension Theorem, with a proof that relies on Fatou's theorem on admissible boundary values of bounded holomorphic functions on $\mathbb{B}^m$, noting that the fibers of a Cayley projection of Ω onto one of its maximal boundary components are images of holomorphic isometric embeddings of some $\mathbb{B}^m$. As applications of a general form of the Extension Theorem we prove the Fibration Theorem for holomorphic maps $f : X = \Omega/\Gamma \to N$, in which N is a quasi-compact complex manifold whose universal covering space $\widetilde{N}$ admits enough bounded holomorphic functions, whenever f induces an isomorphism $f_* : \Gamma \xrightarrow{\cong} \Gamma' := \pi_1(N)$ on fundamental groups, by proving that $R(\widetilde{N}) \subset \Omega$ and that R descends to a retraction map $\rho : N \to X$. Furthermore, requiring $\widetilde{N}$ to be a bounded domain D (on a Stein manifold), under the same hypothesis on f_* but weakening quasi-compactness of N to the hypothesis $\mathrm{Volume}(N, \mu_N) < \infty$, μ_N being the Kobayashi volume form on N, we prove the Isomorphism Theorem ascertaining that $f : X \xrightarrow{\cong} N$ is a biholomorphism. The novelty in the proof lies in the use of the canonical complete Kähler-Einstein metric on the hull of holomorphy $\widehat{D} \supset D$ and a result of independent interest showing that $\widehat{D} - D$ is *a priori* of zero Lebesgue measure under the hypothesis $\mathrm{Volume}(N, \mu_N) < \infty$.

1 Introduction

Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. In [15], [17] and [34] Hermitian metric rigidity for the canonical Kähler-Einstein metric was established. In the locally irreducible case, it says that the latter is up to a normalizing constant the unique Hermitian metric on X of nonpositive curvature in the sense of Griffiths. This led to the rigidity result for nonconstant holomorphic mappings of X into Hermitian manifolds of nonpositive curvature in the sense of Griffiths, proving that up to a normalizing constant any such holomorphic mapping must be an isometric immersion totally geodesic with respect to the Hermitian connection.

With an aim to studying holomorphic mappings of X into complex manifolds which are of nonpositive curvature in a more generalized sense, for instance, quotients of *arbitrary* bounded domains in Stein manifolds by torsion-free discrete groups of automorphisms, a form of metric rigidity was established in [19] applicable to complex Finsler metrics, including especially induced Carathéodory metrics (defined using bounded holomorphic functions). By studying *extremal* bounded holomorphic functions in relation to certain complex Finsler metrics, rigidity theorems were established in [19] for nonconstant holomorphic mappings $f : X \to N$ into complex manifolds N whose universal covers admit sufficiently many bounded holomorphic functions. A new feature of the findings is that the liftings $F : \Omega \to \widetilde{N}$ to universal covers were shown to be holomorphic *embeddings*. (Here and in what follows by a holomorphic embedding we mean an injective holomorphic immersion, and it is not required that the image is a closed subset.) The latter result will be referred to as the Embedding

Theorem. In the survey article [20] of the first author, a strengthening of the Embedding Theorem was stated, as follows.

Theorem 1.1. (The Extension Theorem) *Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization, and $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be the decomposition of Ω into a Cartesian product of irreducible factors, $\Omega_i \Subset \mathbb{C}^{n_i}$ in which $n_i := \dim(\Omega_i)$. Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, and $X := \Omega/\Gamma$ be the finite volume quotient (with respect to the canonical metric), and $\pi : \Omega \to X$ be the universal covering map. Let N be a complex manifold and $\tau : \widetilde{N} \to N$ be its universal covering map. Suppose $f : X \to N$ is a holomorphic map. Write $F : \Omega \to \widetilde{N}$ for a lifting of f, i.e., $\tau \circ F \equiv f$. Assume furthermore that $(X, N; f)$ satisfies the following nondegeneracy condition:*

> $(\clubsuit)$: *For each $k\,(1 \leq k \leq m)$, there exists a bounded holomorphic function h_k on $\widetilde{N}$ and an irreducible factor subdomain $\Omega'_k \subset \Omega$ such that h_k is nonconstant on $F(\Omega'_k)$.*

Then, there exists a bounded holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$ such that $R \circ F = id_\Omega$.

Theorem 1.1 gives a solution to the Extension Problem in the sense that it gives an extension of the inverse map of the holomorphic embedding $F : X \to \widetilde{N}$. We note moreover that the proof of Theorem 1.1 is independent of the fact that F is an embedding, and it gives an alternative proof of the Embedding Theorem of [19]. In fact, the existence of $R : \widetilde{N} \to \mathbb{C}^n$ such that $R \circ F = id_\Omega$ implies *a fortiori* that F is injective and immersive, yielding

Corollary 1.2. (The Embedding Theorem) *F is a holomorphic embedding.*

A complete proof of Theorem 1.1 for the case of polydisks was given in [20], and a sketch of the proof for the general case was also given there. The key ingredients in [20] are Moore's ergodicity theorem and Korányi's notion and existence theorem on admissible limits for bounded holomorphic functions on bounded symmetric domains.

In this article, we give first of all a complete and a much simplified proof of Theorem 1.1 in which the use of admissible limits for bounded holomorphic functions on bounded symmetric domains is reduced by means of Cayley projections to the case where the domains are complex unit balls, which is more elementary and more familiar to complex analysts. For the proof of Theorem 1.1 we have to show that the identity map $id_\Omega = F^*R$ for some bounded holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$, $n := \dim(\Omega)$. This motivates us to consider bounded holomorphic functions $h = F^*h_1$, where $h_1 \in H^\infty(\widetilde{N})$, the vector space of bounded holomorphic functions on $\widetilde{N}$, noting that h can be chosen to be nonconstant by the nondegeneracy assumption $(\clubsuit)$. In case Ω is irreducible (and of rank ≥ 2), a Cayley projection is determined by a regular (cf. §4.4) pair (Φ, Ψ) of maximal boundary components (i.e., of rank $r-1$, or equivalently, lying on $\mathrm{Reg}(\partial\Omega)$). For almost every Cayley projection $\rho_{\Phi,\Psi} : \Omega \to \Phi$ we obtain by taking admissible boundary values of $h \in \mathscr{F} := F^*H^\infty(\widetilde{N})$ a bounded holomorphic function $s_{\Phi,\Psi}$ on Φ such that $\hat{h}_{\Phi,\Psi} := \rho^*_{\Phi,\Psi}s_{\Phi,\Psi} \in \mathscr{F}$. By the S^1-averaging argument of H. Cartan, we produce from $\rho^*_{\Phi,\Psi}s_{\Phi,\Psi}$ a nontrivial linear function on Ω. Writing $\Omega = G_0/K$, where $G_0 := \mathrm{Aut}_0(\Omega)$ and $K \subset G_0$ is the isotropy subgroup at $0 \in \Omega$, by a K-averaging argument, we show that $id_\Omega = F^*R$ for some bounded holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$, proving the Extension Theorem.

The Cayley projections $\rho_{\Phi,\Psi}$ are intimately related to nonstandard holomorphic isometric embeddings $\mathbb{B}^m \hookrightarrow \Omega$ constructed in Mok [24] by means of minimal rational curves. Fibers of the Cayley projections $\rho_{\Phi,\Psi} : \Omega \to \Phi$ are holomorphic isometric embeddings of unit balls and the existence of boundary values $s_{\Phi,\Psi}$ for almost every Cayley projection $\rho_{\Phi,\Psi} : \Omega \to \Phi$ in the above follows from a generalization of the classical Fatou's Theorem to the unit ball (special case of Korányi [6]).

In all the averaging arguments, various group actions are applied to spaces of (vector-valued) bounded holomorphic functions. Such group actions may produce functions which are not *a priori* inside the original spaces of functions under consideration. To resolve this problem, we observe that $\hat{h}_{\Phi,\Psi}$ is invariant under a 1-parameter group of transvections and apply a density result which follows from Moore's ergodicity theorem.

As applications of the Extension Theorem, we derive rigidity results on irreducible finite-volume quotients of bounded symmetric domains Ω of rank ≥ 2. This includes two principal results, viz., the Fibration Theorem and the Isomorphism Theorem, concerning the quotient $X = \Omega/\Gamma$ of Ω by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\Omega)$, as follows. Given a holomorphic mapping $f : X \to N$ into a quasi-compact complex manifold N such that $f_* : \pi_1(X) \xrightarrow{\cong} \pi_1(N)$ and such that $(X, N; f)$ satisfies the non-degeneracy condition (♣), we prove that f is a holomorphic embedding and that there exists a holomorphic fibration $\rho : N \to X$ such that $\rho \circ f \equiv id_X$, proving the Fibration Theorem (cf. Theorem 7.1). We note that in the special case where N is a compact Kähler manifold, the method for proving strong rigidity in Siu [33] using harmonic maps applies to give a proof of the Fibration Theorem, but our method applies in the quasi-compact case and without assuming N to be Kähler. For the Isomorphism Theorem (cf. Theorem 7.2), we impose the hypothesis of the existence of a holomorphic map $f : X \to N$ into an arbitrary quotient N of a bounded domain such that $f_* : \pi_1(X) \xrightarrow{\cong} \pi_1(N)$ and such that N is of finite intrinsic measure with respect to the Kobayashi volume form μ_N, and prove that $f : X \xrightarrow{\cong} N$ is a biholomorphism. The novelty in the proof lies in using the existence of canonical complete Kähler-Einstein metrics on bounded domains of holomorphy (on Stein manifolds), a result due to [4] and [28], which allows one to enlarge the complex manifold $N = D/\Gamma'$ to a complete Kähler manifold $\left(\widehat{N}, \omega_{_{KE}}\right)$ and a result of independent interest showing that $\widehat{D} - D$ is *a priori* of zero Lebesgue measure under the hypothesis $\mathrm{Volume}(N, \mu_N) < \infty$.

The organization of the article goes as follows. In §2 the boundary component theory for bounded symmetric domains of Wolf is briefly recalled. It serves to fix notations and recall several basic facts to be used in later chapters. In §3, the concept of Cayley projections on an irreducible bounded symmetric domain is introduced, and Cayley projections onto maximal boundary components are constructed via the use of varieties of minimal rational tangents. The Cayley projection depends on a one parameter group $H \subset \mathrm{Aut}_0(\Omega)$ of transvections and applies to any pair of maximal boundary components which are regular in the sense that their topological closures are disjoint. In §4 we consider special product subspaces $P \subset \Omega$, $P \cong \Delta \times \Omega'$, $\mathrm{rank}(\Omega') = r - 1$, in preparation for Cauchy integral formulas for almost all triples (Φ, Ψ, Z) consisting of a regular pair (Φ, Ψ) of maximal boundary components (Φ, Ψ) and a distinguished section Z of the unique special product subspace $P(\Phi, \Psi)$ containing Φ and Ψ in its closure. In §5, a well-known S^1-averaging argument of H. Cartan is recalled. We will also discuss a K-averaging argument. In §6, we give the proof of Theorem 1.1 (the Extension Theorem) relying on averaging over Cayley limits of pull-backs of bounded holomorphic functions by $F : \Omega \to \widetilde{N}$ In §7 and §8 we give applications of Theorem 1.1 to rigidity problems on irreducible finite-volume quotients of bounded symmetric domains of rank ≥ 2. In §7 we give statements of the applications, viz., Theorem 7.1 (the Fibration Theorem) and Theorem 7.2 (the Isomorphism Theorem), discuss comparison theorems on intrinsic metrics and intrinsic volume forms, and establish a lower bound for the Kobayashi volume form on a bounded domain. In §8 we give proofs of Theorem 7.1 and Theorem 7.2, especially showing how a lower estimate of the Kobayashi volume form makes it possible to pass to Kähler geometry by invoking the existence of canonical complete Kähler-Einstein metrics on bounded domains of holomorphy.

The current article grew out on the one hand from a self-contained solution of the Extension Problem made possible by the use of nonstandard holomorphic isometric embeddings of the complex unit ball via the use of minimal rational curves given by [24], and on the other hand on applications of the Extension Theorem to rigidity problems on $X = \Omega/\Gamma$. The solution of the Extension Problem constitutes a portion of the Ph.D. thesis of the second author written under the supervision of the first author, while applications of the solution of the Extension Problem are due to the second author.

The write-up of the article was delayed in part since a self-contained and complete proof relying on [24] was only made available by the thesis of the second author. In the current article, the latter proof is complemented by a self-contained geometric construction of the Cayley projection, a tool of fundamental importance for the proof of Theorem 1.1, jointly written by both authors using the geometric theory of varieties of minimal rational curves, and the full article has been thoroughly and jointly rewritten.

2 Boundary Structure of Bounded Symmetric Domains

We start with some preliminaries on bounded symmetric domains in part to fix terminology and notation. We are following essentially [35] and [1].

Let X_0 be the underlying space of an irreducible Hermitian symmetric space (X_0, h) of the semisimple and noncompact type, G_0 be the identity component of the group of holomorphic isometries of (X_0, h), equivalently the identity component of the group $\mathrm{Aut}(X_0)$ of biholomorphic automorphisms of X_0. Then, $X_0 = G_0/K$, where $K \subset G_0$ is the isotropy subgroup at a reference point $x_0 = eK \in X_0$. Let $\mathfrak{g}_0$ be the Lie algebra of G_0, and $\mathfrak{k}$ be the Lie algebra of K, so that we have the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{m}_0$. For a real vector space V we write $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ for its complexification. Write $\mathfrak{g} := \mathfrak{g}_0^{\mathbb{C}}$ and $\mathfrak{m} = \mathfrak{m}_0^{\mathbb{C}}$. Then $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}$ and $\mathfrak{g}_{\mathfrak{c}} = \mathfrak{k} \oplus i\mathfrak{m}_0$ is a compact real form of $\mathfrak{g}$. Denoting by $z \in \mathfrak{k}$ the central element that induces the complex structure $J = \mathrm{ad}(z)|_{\mathfrak{m}}$ on X_0 as well as on its compact dual X, we have the corresponding decomposition $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ into $(\pm i)$-eigenspaces of J. Write $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$, which is a parabolic subalgebra of $\mathfrak{g}$ consisting of nonnegative eigenspaces of $\mathrm{ad}(iz)$.

For Lie subalgebras of $\mathfrak{g}$, which are denoted by Gothic characters, the corresponding real analytic subgroups in G will be denoted by capital Roman letters. By Borel embedding theorem, we have a holomorphic embedding $X_0 = G_0/K \hookrightarrow X = G/P = G_c/K$ as an open orbit $G_0(x_c) \subset X$ given by $gK \mapsto g \cdot x_c$, where $x_c := eP \in G/P$. The topological boundary of X_0 in X will be denoted by ∂X_0. There is also a complex analytic diffeomorphism of $M^+ \times K^{\mathbb{C}} \times M^- \cong U$ onto a dense open subset $U \subset G$ given by $(m^+, k, m^-) \mapsto m^+km^-$, such that $U \supset G_0$. This induces the Harish-Chandra realization of the bounded symmetric domain $\Omega = \xi^{-1}X_0 \Subset \mathfrak{m}^+$, where the embedding map $\xi : \mathfrak{m}^+ \to X = G/P$ is given by $m \mapsto \exp(m)P$. We will study the topological boundary $\partial\Omega$ of Ω in $\mathfrak{m}^+ \cong \mathbb{C}^n$.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Then it is also a Cartan subalgebra in $\mathfrak{g}_0$ and $\mathfrak{g}_c$. The complexification $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t}^{\mathbb{C}})$ be a corresponding root system. If $\mu \in \Delta$, the corresponding root space is denoted by $\mathfrak{g}_\mu$. Then $\mathfrak{g} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\mu\in\Delta(\mathfrak{g},\mathfrak{t}^{\mathbb{C}})} \mathfrak{g}_\mu$. Recall that $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}$. Either $\mathfrak{g}_\mu \subset \mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{g}_\mu \subset \mathfrak{m}$. If $\mathfrak{g}_\mu \subset \mathfrak{m}$, then μ is called a noncompact root. Denote by Δ_M^+ the set of noncompact positive roots. We also have $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$. Fix a standard choice of ordering of roots as in [35, Section 3] so that $\mathfrak{m}^+ = \bigoplus\{g_\mu : \mu \in \Delta_M^+\}$.

For $\mu \in \Delta$, take $h_\mu \in i\mathfrak{t}$ so that $2\mu(h)/\langle\mu, \mu\rangle = \langle h, h_\mu\rangle$. Choose root vectors $e_\mu \in \mathfrak{g}_\mu$ that are normalized by $[e_\mu, e_{-\mu}] = h_\mu$. For each $\mu \in \Delta_M^+$, define $x_{\mu,0} = e_\mu + e_{-\mu}$ and $y_{\mu,0} = i(e_\mu - e_{-\mu})$. Then $\{x_{\mu,0}, y_{\mu,0} : \mu \in \Delta_M^+\}$ and $\{x_\mu = ix_{\mu,0}, y_\mu = iy_{\mu,0} : \mu \in \Delta_M^+\}$ are $\mathbb{R}$-basis of $\mathfrak{m}_0$ and $\mathfrak{m}_c = i\mathfrak{m}_0$ respectively. Note that the Lie algebra of the compact real form $\mathfrak{g}_c = \mathfrak{k} \oplus \mathfrak{m}_c$. Two roots $\theta, \psi \in \Delta$ are said to be strongly orthogonal if and only if $\theta \pm \psi \notin \Delta$. Let $\Psi \subset \Delta_M^+$ be a maximal strongly orthogonal set, i.e., a set of mutually strongly orthogonal noncompact positive roots of $\mathfrak{g}$ of maximal cardinality. For each $\gamma \in \Psi$, $c_\gamma := \exp(\frac{\pi}{4}y_\gamma) \in G_c$ defines a partial Cayley transform. If $\Gamma \subset \Psi$, then $c_\Gamma := \prod_{\gamma\in\Gamma} c_\gamma$. To each $\Gamma \subset \Psi$, there is a totally geodesic Hermitian symmetric subspace $X_{\Gamma,0} = G_{\Gamma,0}(x_0) \subset X_0$. Note that ∂X_0 decomposes into G_0 orbits of the form $G_0(c_{\Psi-\Gamma}x_0) = \bigcup_{k\in K} kc_{\Psi-\Gamma}X_{\Gamma,0}$, where $\Gamma \subsetneq \Psi$ and $kc_{\Psi-\Gamma}X_{\Gamma,0}$ is a boundary component of X_0. Each $kc_{\Psi-\Gamma}X_{\Gamma,0}$ is a Hermitian symmetric space of the noncompact type of rank $|\Gamma|$. Here $G_0(c_{\Psi-\Gamma}x_0) = G_0(c_{\Psi-\Sigma}x_0)$ if and only if $|\Gamma| = |\Sigma|$. Thus $\partial X_0 =$

$E_0 \cup E_1 \cup \cdots \cup E_{r-1}$ as a disjoint union of G_0 orbits E_i given by $\xi^{-1} k c_{\Psi-\Gamma} X_{\Gamma,0} = ad(k) \cdot \xi^{-1} c_{\Psi-\Gamma} X_{\Gamma,0}$. This also shows that the boundary components of Ω in $\mathfrak{m}^+$ are bounded symmetric domains of rank $|\Gamma|$. Identifying $\Omega = \xi^{-1} X_0 \Subset \mathfrak{m}^+$ with $X_0 \subset X$ we also write $\partial\Omega = E_0 \cup E_1 \cup \cdots \cup E_{r-1}$. The regular part $\mathrm{Reg}(\partial\Omega)$ of $\partial\Omega$ is exactly E_{r-1}, which is a union of boundary components of maximal dimension.

For $\Gamma \subsetneq \Psi$, the subset $N_{\Psi-\Gamma,0} = \{g \in G_0 : g c_{\Psi-\Gamma} X_{\Gamma,0} = c_{\Psi-\Gamma} X_{\Gamma,0}\} \subset G_0$ is the normalizer of the boundary component $c_{\Psi-\Gamma} X_{\Gamma,0}$ of X_0 in X. The Boundary Group Theorem and Boundary Flag Theorem [35, p.295, 299] say that $N_{\Psi-\Gamma,0}$ is conjugate to $N_{\Psi-\Sigma,0}$ in G_0 if and only if $|\Gamma| = |\Sigma|$; and since G is simple, $N_{\Psi-\Gamma,0}$ are maximal parabolic subgroups of G_0 and any maximal parabolic subgroups of G_0 is conjugate to $N_{\Psi-\Gamma,0}$ for some Γ. Moreover, there is a K-equivariant fibration $\pi : G_0(c_{\Psi-\Gamma} x_0) \to G_0/N_{\Psi-\Gamma,0} = K(c_{\Psi-\Gamma} x_0)$ and that $K(c_{\Psi-\Gamma} x_0)$ is the moduli space of all rank $|\Gamma|$ boundary components of X_0 in S.

In this article we will focus on boundary components of maximal dimension $\Phi \subset \mathrm{Reg}(\partial\Omega) = E_{r-1}$. Note that K acts transitively on the moduli space of Φ's.

3 Cayley Projections

In preparation for the proof of Theorem 1.1 solving the Extension Problem 1.1, we prove in this section a result constructing bounded holomorphic functions from their boundary values on maximal boundary components, i.e., from a boundary component lying on $\mathrm{Reg}(\partial\Omega)$. For a bounded holomorphic function continuous up to $\overline{\Omega}$, this is done by composing on the left by a holomorphic projection map first defined by Korányi-Wolf, which we call the Cayley projection, accompanying the inverse of the first partial Cayley transform. In [1, Chapter III, §3], Ash-Mumford-Rapoport-Tai gave a description of the Cayley projection (which they called the "natural projection") in group-theoretic terms. To make our article self-contained and to link up with the nonstandard holomorphic isometries in [24] of the complex unit ball constructed using varieties of minimal rational tangents, we will present a geometric construction of Cayley projections with a description of their fibers in terms of minimal disks. For our purpose we will apply Cayley projections to pull-backs of bounded holomorphic functions on the complex manifold $\widetilde{N}$ by the holomorphic mapping $F : \Omega \to \widetilde{N}$, for which it is not meaningful to introduce any continuity assumption up to $\overline{\Omega}$. We will introduce the notion of Cayley limits of such functions.

3.1 Geometric construction of Cayley projections from varieties of minimal rational tangents

To define Cayley projections and describe how they are constructed, we will need some basic theory on minimal rational curves in the setting of Hermitian symmetric spaces of the compact type, as can be found for example in [7].

Let Ω be an irreducible bounded symmetric domain of rank $r \geq 2$ and S be the compact dual of $X_0 \cong \Omega$ so that $\Omega \Subset \mathbb{C}^n \subset S$ incorporates both the Harish-Chandra realization $\Omega \Subset \mathbb{C}^n$ and the Borel embedding $\Omega \cong X_0 \subset S$. By the polydisk theorem (cf. [35, p. 280]), there exists a maximal polydisk $P \cong \Delta^r$, $r = \mathrm{rank}(\Omega)$, embedded into Ω as a totally geodesic complex submanifold, and moreover we have $\Omega = \bigcup_{k \in K} kP$.

The maximal polydisk $P \subset \Omega$ is obtained as follows. In the notation of §2, for $\mu \in \Delta_M^+$ write $\mathfrak{g}_0[\mu] := \mathrm{Span}_{\mathbb{R}}\{x_{\mu,0}, y_{\mu,0}, ih_\mu\}$, which is a real Lie algebra isomorphic to $\mathfrak{su}(1,1)$, and denote by $G_0[\mu] \subset G_0$ the associated real Lie subgroup $\cong \mathbb{P}\mathrm{SU}(1,1)$ in $G_0 = \mathrm{Aut}_0(\Omega)$. Let $\Psi \subset \Delta_M^+$ be a maximal strongly orthogonal set of positive noncompact roots, and define the real Lie algebra $\mathfrak{g}_0[\Psi] = \bigoplus \{\mathfrak{g}_0[\psi] : \psi \in \Psi\}$, and correspondingly the real Lie subgroup $G_0[\Psi] \subset G_0$. Then, the orbit of $0 = eK$ under $G_0[\Psi]$ gives a maximal polydisk $P \subset \Omega$. Moreover, writing $H_0 \subset G_0$ for the stabilizer subgroup of $P \subset \Omega$, then the restriction of H_0 to P induces the full automorphism group of $P \cong \Delta^r$. (In particular, all roots $\psi \in \Psi$ are of the same length. They are in fact all long roots.) By a maximal polydisk $Q \subset \Omega$ we mean a totally geodesic complex submanifold of maximal dimension biholomorphic to a polydisk, and it is the complexification of a maximal totally geodesic flat subspace of Ω. The Lie group G_0 acts transitively on the set of maximal polydisks on Ω.

For any $\psi \in \Psi$, $G_0[\psi]\cdot 0$ is a totally geodesic holomorphic disk on Ω and they are equivalent to each other under the action of G_0. In general a factor disk of a maximal polydisk $Q \subset \Omega$ will be referred to as a minimal disk, and G_0 acts transitively on the set of minimal disks on Ω. Given $P = G_0[\Psi]\cdot 0$ and $\psi \in \Psi$, we have $\mathfrak{g}_0[\psi] = \mathbb{R}x_{\psi,0} + \mathbb{R}y_{\psi_0} + i\mathbb{R}h_\psi \subset \mathbb{C}e_\psi + \mathbb{C}e_{-\psi} + \mathbb{C}h_\psi =: \mathfrak{g}[\psi] \cong \mathfrak{sl}(2,\mathbb{C})$. Writing $G[\psi] \subset G = \mathrm{Aut}_0(S)$ for the corresponding complex Lie subgroup, the orbit $G_0[\psi]\cdot 0$ in Ω gives a minimal disk $D(\psi)$, and the orbit $G[\psi]\cdot 0$ gives a minimal rational curve $\ell(\psi) \subset S$, and $D(\psi) \subset \ell(\psi)$ is the Borel embedding. Since G_0 acts transitively on the set of minimal disks, for any minimal disk $D \subset \Omega$ we always have $D = \ell \cap \Omega$ for a minimal rational curve ℓ on the compact dual manifold S of Ω.

Define

$$\mathcal{V}_x := \bigcup \big\{\ell : \ell \text{ is a minimal rational curve on } S \text{ passing through } x\big\} \subset S.$$

We call $\mathcal{V}_x$ the cone at x swept out by minimal rational curves and $x \in \mathcal{V}_x$ its vertex. Associated to minimal rational curves there is also the important notion of the variety of minimal rational tangents (VMRT) on S at a point $x \in S$, to be denoted by $\mathscr{C}_x(S)$, defined by $\mathscr{C}_x(S) := \big\{[T_x(\ell)] : \ell \subset S \text{ is a minimal rational curve passing through } x\big\}$. Writing $T_x(\ell) =: \mathbb{C}\eta$, $[\eta] \in \mathbb{P}T_x(S)$ is called a minimal rational tangent, and $\eta \in T_x(S)$ is called a minimal rational tangent vector.

$\mathcal{V}_x \subset S$ is a projective subvariety singular only at the isolated singularity $x \in \mathcal{V}_x$. $\mathcal{V}_x$ may be described in terms of the VMRT $\mathscr{C}_x(S)$, as follows. There is a holomorphic $\mathbb{P}^1$-bundle $\lambda : \mathscr{P} \to \mathscr{C}_x(S)$ with a tautological holomorphic section $\sigma : \mathscr{C}_x(S) \to \mathscr{P}$ and a natural evaluation map $\mu : \mathscr{P} \to S$ such that $\mu(\lambda^{-1}([\alpha]))$ is the minimal rational curve $\ell \subset S$ passing through x such that $T_x(\ell) = \mathbb{C}\alpha$. We have $\mathcal{V}_x = \mu(\mathscr{P})$, where $\sigma(\mathscr{C}_x(S))$ is collapsed by μ, giving $\mu(\sigma(\mathscr{C}_x(S)) = x$.

For any point $y \in \mathcal{V}_x$ distinct from x, write ℓ for the minimal rational curve passing through both x and y. We have $T_S|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$. Write $P_\ell := \mathcal{O}(2) \oplus \mathcal{O}(1)^p$. Although the Grothendieck summands are not uniquely determined, the filtration $T_\ell \subset P_\ell \subset T_S|_\ell$ is uniquely determined, where $T_\ell \cong \mathcal{O}(2)$. We call $P_\ell \subset T_S|_\ell$ the *positive* part of $T_S|_\ell$. From the deformation theory of rational curves we have $T_y(\mathcal{V}_x) = P_\ell$. Note that for a point $x' \in \ell$ distinct from x and y, $T_y(\mathcal{V}_{x'}) = P_\ell = T_y(\mathcal{V}_x)$. In other words, the cones $\mathcal{V}_x$ resp. $\mathcal{V}_{x'}$ at x resp. x' of minimal rational curves are tangent to each other at the point y.

We will need to make use of canonical Kähler-Einstein metrics. In what follows on an irreducible bounded symmetric domain $U \Subset \mathbb{C}^N$ we denote by g_U the canonical Kähler-Einstein metrics on U normalized so that the minimal disks are of constant Gaussian curvature -2. For $U = \mathbb{B}^n$ we will also write g_n for $g_{\mathbb{B}^n}$. Thus, $(\mathbb{B}^n, g_n)$ is of constant holomorphic sectional curvature -2.

Denote by R the curvature tensor of (Ω, g_Ω). For $\chi \in T_0(\Omega)$ we denote by H_χ the Hermitian bilinear form $H_\chi(\xi, \overline{\eta}) := R_{\chi\overline{\chi}\xi\overline{\eta}}$. Let now $\alpha \in T_0(D)$ be a unit vector, hence α is a minimal rational tangent vector of unit length. We have the decomposition of $T_0(\Omega)$ into an orthogonal direct sum of eigenspaces of H_α, viz., $T_0(\Omega) = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$ values -2 resp. -1 resp. 0. Then, there exists a (unique) totally geodesic complex submanifold $\Omega_0' \subset \Omega$ passing through 0 such that $T_0(\Omega_0') = \mathcal{N}_\alpha$. $\Omega_0' \subset \Omega$ is biholomorphically an irreducible bounded symmetric domain Ω' of rank $r-1$ embedded in a Euclidean space by means of the Harish-Chandra embedding. We have a holomorphic totally geodesic isometric embedding $\nu : \Delta \times \Omega' \to \Omega$ such that

$$\nu(\Delta \times \{0\}) = D \quad \text{and} \quad \nu(\{0\} \times \Omega') =: \Omega_0' \tag{1}$$

(cf. Mok-Tsai [26, Proposition 1.7]). A boundary component $\Phi \subset \mathrm{Reg}(\partial\Omega)$ may be taken to be of the form $\Phi = \nu(\{e^{i\theta}\} \times \Omega') =: \Sigma(\theta) \cong \Omega'$ for some $\theta \in \mathbb{R}$. In what follows we take the reference boundary component to be

$$\Sigma = \nu(\{1\} \times \Omega'). \tag{2}$$

Let $\alpha \in T_0(\Omega)$ be a minimal rational tangent of unit length. We choose Harish-Chandra coordinates $(z_1; z_2, \ldots, z_{p+1}; z_{p+2}, \ldots, z_n) = (z_1; z'; z'')$ with $\frac{\partial}{\partial z_i}, 1 \le i \le n$, corresponding to unit root vectors, such that $\alpha = \frac{\partial}{\partial z_1}$, and such that the ordering of coordinates $(z_1; z'; z'')$ corresponds to the orthogonal decomposition $\mathbb{C}^n = T_0(\Omega) = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$, $\mathcal{H}_\alpha \cong \mathbb{C}^p$, $\mathcal{N}_\alpha \cong \mathbb{C}^q, 1 + p + q = n$. We will call these privileged Harish-Chandra coordinates.

There is an injective Lie algebra homomorphism $\mathfrak{sl}(2;\mathbb{C}) \hookrightarrow \mathfrak{g}$ corresponding to an embedding of Dynkin diagrams which restricts to the real form $\mathfrak{su}(1,1) \hookrightarrow \mathfrak{g}_0$. Exponentiating we have a natural injective group homomorphism

$$\Theta : \mathbb{P}\mathrm{SU}(1;1) \cong \mathrm{Aut}(\Delta) \hookrightarrow \mathrm{Aut}_0(\Omega) =: G_0. \tag{3}$$

The minimal disk $\Delta_\alpha := \Delta \times \{(0;0)\} \subset \Omega$ is the image of a Θ-equivariant totally geodesic holomorphic embedding $\iota : \Delta \to \Omega$ of the Poincaré unit disk Δ into Ω. The embedding ι extends to $\iota : \mathbb{P}^1 \to S$ (recalling that $S \supset X_0 \cong \Omega$ is the compact dual of Ω) and we may require $\iota(0) = 0 \in \Omega$ and $\iota(1) = b_0 := (1; 0; 0) \in \Sigma$.

For $\Omega \Subset \mathbb{C}^N$ and a point $x \in \Omega$ we denote by σ_x^Ω the involution on Ω as a bounded symmetric domain. A boundary component $\Phi \subset \partial\Omega$ is an open subset of the smallest affine linear subspace W_Φ containing it. Identifying W_Φ with the complex linear subspace $W_\Phi - 0_\Phi$, where $0_\Phi \in \Phi$ is the center of Φ, i.e., the center of gravity of Φ with respect to the restriction of the Euclidean metric from $\mathbb{C}^N$, we may regard $\Phi \subset W_\Phi$ as a bounded symmetric domain on W_Φ. For a point $y \in \Phi$ we denote by σ_y^Φ the involution at $y \in \Phi$. We also define $\Phi' := \sigma_0^\Omega(\Phi)$ and call it the opposite boundary component of Φ with respect to (the center) $0 \in \Omega$. In terms of the chosen Harish-Chandra coordinates (depending on the choice of a minimal rational tangent $\alpha \in T_0(\Omega)$ of unit length), and given $b = (e^{i\theta}; 0; w) \in \Phi$, $w \in \Omega'$, we call $b' = (-e^{i\theta}; 0; w)$ the opposite boundary point of b. Equivalently, the opposite boundary point can be defined as $\sigma^\Omega_{0_{\Phi'}}(\sigma_0^\Omega(b))$. (Compare with the description of the natural projection in [1, Chapter III, §3].) Here $\sigma_0^\Omega(b) = (-e^{i\theta}; 0; -w)$, hence $\sigma^\Omega_{0_{\Phi'}}(\sigma_0^\Omega(b)) = \sigma^\Omega_{0_{\Phi'}}(-e^{i\theta}; 0; -w) = (-e^{i\theta}; 0; w) = b'$. We will write $b = b(w), w \in \Omega'$. We have the reference point $b_0 = (1; 0; 0)$ on $\Sigma \subset \mathrm{Reg}(\partial\Omega)$ whose opposite boundary point is $b_0' = (-1; 0; 0)$ on $\Sigma' := \nu(\{-1\} \times \Omega')$.

Let $a \in \mathrm{Reg}(\partial\Omega)$. Define $V_a := \mathcal{V}_a \cap \Omega$. Then $V_a \subset \Omega$ is a subvariety swept out by minimal disks, which we describe more precisely, as follows. Let $\mathcal{K}$ be the irreducible component of the Chow space $\mathrm{Chow}(S)$ consisting of minimal rational curves on S. We call $\mathcal{K} \subset \mathrm{Chow}(S)$ the minimal rational component on S. For $x \in S$, denote by $\mathcal{K}_x \subset \mathcal{K}$ the projective submanifold consisting of minimal rational curves passing through x. The tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(S)$, defined by $\tau_x([\ell]) = [T_x(\ell)]$, maps $\mathcal{K}_x$ biholomorphically onto the VMRT $\mathscr{C}_x(S) \subset \mathbb{P}T_x(S)$ as defined in §3.1. For $a \in \mathrm{Reg}(\partial\Omega)$ define $\mathscr{D}_a \subset \mathcal{K}_a$ by $\mathscr{D}_a := \{[\ell] \in \mathcal{K}_a : \ell \cap \Omega \neq \emptyset\}$, which is *a priori* a nonempty open subset in the complex topology. Then $V_a = \mathcal{V}_a \cap \Omega = \coprod \{\ell \cap \Omega : [\ell] \in \mathscr{D}_a\}$ is a disjoint union of minimal disks $D_\ell := \ell \cap \Omega \subset \Omega$ satisfying $a \in \overline{D_\ell}$.

Write $\mathscr{D} = \{[\ell] \in \mathcal{K} : \ell \cap \Omega \neq \emptyset\}$. We have $\mathscr{D}_a \subset \mathscr{D}$ for any $a \in \mathrm{Reg}(\partial\Omega)$. The space $\mathscr{D}$ parametrizes the set of minimal disks on Ω.

Remark 3.1. Strictly speaking we have the moduli space $\mathcal{U}_x$ of minimal rational curves marked at $x \in X$, and the universal family $\rho : \mathcal{U} \to \mathcal{K}$ as a holomorphic $\mathbb{P}^1$-bundle, and the evaluation map $\mu : \mathcal{U} \to X$ such that $\mathcal{U}_x := \mu^{-1}(x)$, and $\mathcal{K}_x := \rho(\mathcal{U}_x)$. The tangent map is defined by $\tau_x(u) = [T_x(\ell)] \in \mathbb{P}T_x(X)$, where $u \in \mathcal{U}$ is a minimal rational curve ℓ endowed with a marking at x. Here we are making an identification of $\mathcal{U}_x$ with $\mathcal{K}_x$, given that in the case of the Hermitian symmetric manifold X, at any point $x \in X$ we have a biholomorphism $\rho\big|_{\mathcal{U}_x} : \mathcal{U}_x \xrightarrow{\cong} \mathcal{K}_x \subset \mathcal{K}$ of $\mathcal{U}_x$ onto $\mathcal{K}_x$.

3.2 Holomorphic isometric embeddings of the unit ball

In what follows we adopt the privileged Harish-Chandra coordinates $(z_1, \cdots, z_n) = (z_1; z'; z'')$ introduced in §3.1, immediately after (2) above according to $T_0(\Omega) = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$, $\alpha = \frac{\partial}{\partial z_1}$, and write $z_i = x_i + \sqrt{-1}y_i$ as usual. Assume $a \in \mathrm{Reg}(\partial\Omega)$, $[\ell] \in \mathscr{D}_a$ and $c \in \partial D_\ell \subset \partial V_a$ be distinct from a. We have

Lemma 3.2. *∂V_a is smooth and strictly pseudoconvex at c.*

Proof. Since G_0 acts transitively on the space $\mathscr{D}$ of minimal disks on Ω, without loss of generality we may take $a = (1; 0; 0) = b_0$, ℓ to be the unique minimal rational curve ℓ_0 on X passing through $0 \in \Omega$ such that $T_0(\ell) = \mathbb{C}\frac{\partial}{\partial z_1}$, hence $D_\ell = \{(z_1; 0; 0) : |z_1| < 1\} = \Delta \times \{(0; 0)\} =: D_0$. Since the parabolic subgroup P_a of $\mathrm{Aut}(D_0) \hookrightarrow \mathrm{Aut}_0(\Omega) = G_0$ fixing a acts transitively on $\partial D_0 - \{a\}$, we may assume $c = (-1; 0; 0) = b_0'$.

For the cone $\mathcal{V}_x$ at x swept out by minimal rational curves, from the deformation theory of minimal rational curves we have $T_c(\mathcal{V}_a) = \mathbb{C}\frac{\partial}{\partial z_1} \oplus \mathrm{Span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_i} : 2 \le i \le p+1\right\}$. On the other hand, $c = b_0' = (-1; 0; 0)$ is a smooth point of $\partial\Omega$, and the line segment $\overline{0c}$ is the shortest distance from 0 to $\partial\Omega$ with respect to the Euclidean metric (as can be seen from the polydisk theorem), so that $T_c^{\mathbb{R}}(\partial\Omega)$ is the orthogonal complement of $\frac{\partial}{\partial x_1}$, i.e., $T_c^{\mathbb{R}}(\partial\Omega) = \mathbb{R}\frac{\partial}{\partial y_1} \oplus \mathrm{Re}\left(\mathrm{Span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_i} : 2 \le i \le n\right\}\right)$ isometrically. Now for any point $p \in \ell$ distinct from c, $T_c^{\mathbb{R}}(\mathcal{V}_p) \supset T_c^{\mathbb{R}}(\ell) \supset \mathbb{R}\frac{\partial}{\partial x_1}$, hence $\mathcal{V}_p$ intersects $\partial\Omega$ transversally at c. For $0 \in \ell \cap \Omega$, from the polydisk theorem it follows readily that $\mathcal{V}_0 \cap \partial\Omega = \mathbb{B}^n \cap \partial\Omega$, so that ∂V_0 is smooth and strictly pseudoconvex at c. Since $\mathcal{V}_a$ is tangent to $\mathcal{V}_0$ at c, ∂V_a is also tangent to ∂V_0 at c, hence ∂V_a is smooth and strictly pseudoconvex at c, as desired. □

In order to construct Cayley projections geometrically we will apply the following classical result of H. Cartan (cf. [29, p.78, Theorem 4])

Theorem 3.3. *Let $U \Subset \mathbb{C}^n$ be a bounded domain, and $\{\gamma_i : 1 \le i < \infty\}$ be a sequence of automorphisms $\gamma_i \in \mathrm{Aut}(U)$ such that γ_i converge as a sequence of holomorphic maps into $\mathbb{C}^n$, $h := \lim_{t\to\infty} \gamma_i(z)$ uniformly on compact subsets of U. Then, either $h \in \mathrm{Aut}(U)$, or $h : U \to \partial\Omega$.*

We are now ready to construct and describe Cayley projections geometrically. Recall that $\Theta : \mathrm{Aut}(\Delta) \cong \mathbb{P}\mathrm{SU}(1,1) \hookrightarrow \mathrm{Aut}_0(\Omega) =: G_0$ is the injective group homomorphism defined in (3). We write $\{\theta_t : -1 < t < 1\} \subset G_0$ for the 1-parameter subgroup of transvections.

Proposition 3.4. *Let $\psi_t \in \mathrm{Aut}(\Delta)$ be given by $\psi_t(z) := \frac{z+t}{1+tz}, t \in (-1,1)$, and $\theta_{t,\Sigma} := \Theta(\psi_t)$. Then, $\rho_\Sigma := \lim_{t\to 1} \theta_{t,\Sigma}$ exists, and $\rho_\Sigma : \Omega \to \Sigma$ is a holomorphic submersion onto the maximal boundary component $\Sigma = \{(1; 0)\} \times \Omega' \subset \partial\Omega$. If $b \in \Sigma$, then $\rho_\Sigma(x) = b$ for any $x \in V_{b'} = \mathcal{V}_{b'} \cap \Omega$, where b' is the opposite boundary point on Σ' of $b \in \Sigma$ with respect to 0, Σ' being the opposite boundary of Σ with respect to 0. Moreover, for each $x \in \Omega$, there exists a unique $b \in \Sigma$ such that $x \in V_{b'}$, so that $\rho_\Sigma^{-1}(b) = V_{b'}$ and the level sets $V_{b'} = \rho_\Sigma^{-1}(b), b \in \Sigma$, gives a decomposition of Ω into a disjoint union $\Omega = \coprod\{V_{b'} : b' \in \Sigma'\} = \coprod\{V_{b'} : b \in \Sigma\}$.*

Proof. For $-1 < t < 1$ write $\theta_t := \theta_{t,\Sigma}$. Since $\theta_t \in \mathrm{Aut}_0(\Omega)$, by Montel's theorem, there is a subsequence $\{\theta_{t_n}\}$, $t_n \to 1$, such that θ_{t_n} converges on compact subsets of Ω to a holomorphic map $\rho : \Omega \to \mathbb{C}^n$. Noting that $\lim_{t\to 1} \psi_t(z) = 1$ for any $z \in \Delta$. Choosing Harish-Chandra coordinates and the totally geodesic embedding of $\Delta \hookrightarrow \Omega$ as described above, we have $\rho(x_0) = b_0 \in \Sigma \subset \partial\Omega$. (Recall that we take $x_0 = 0$ and $b_0 = (1; 0; 0)$.) By a theorem of H. Cartan (Theorem 3.3 here), we know that $\rho : \Omega \to \partial\Omega$.

In what follows, by a holomorphic arc in $\mathbb{C}^n$ we will mean the holomorphic image of the unit disk under a nonconstant holomorphic map. Let $x \in \Omega$ and assume that $\rho(x) \ne \rho(x_0) = b_0$. There is a holomorphic map $f : \Delta \to \Omega$ such that $x_0 = f(z_0)$ and $x = f(z_1)$ for some $z_0, z_1 \in \Delta$, i.e., x_0 and x are connected by the holomorphic arc $f(\Delta)$. Since $\rho(f(z_1)) = \rho(x) \neq \rho(x_0) = b_0$, the holomorphic map $\rho \circ f : \Delta \to \mathbb{C}^N$ is nonconstant, and $\rho(x_0)$ and $\rho(x)$ are contained in the holomorphic arc $\rho(f(\Delta))$. Recall from [35] that a boundary component is exactly a holomorphic arc component, meaning that if two points are connected by a finite chain of holomorphic arcs, then they belong to the same holomorphic arc component. Hence, for any point $x \in \Omega$, we have $\rho(x) \in \Sigma$.

From the construction, $\theta_t \in \mathrm{Aut}_0(\Omega) = G_0, t \in (-1, 1)$, is a 1-parameter subgroup of transvections of Ω. From the natural embedding $G_0 \hookrightarrow G = \mathrm{Aut}_0(S)$, each θ_t extends to an automorphism of the compact dual S. Since any automorphism $\mu \in \mathrm{Aut}(S)$ preserves the set of minimal rational curves on S as a set, we have $\mu(\mathcal{V}_p) = \mathcal{V}_{\mu(p)}$ for any $p \in S$. In particular, we have $\mu(\mathcal{V}_p) = \mathcal{V}_p$ whenever $\mu(p) = p$. Now let $b \in \Sigma$ and let $b' \in \Sigma'$ be the opposite boundary point with respect to 0. Then, for $t \in (-1, 1)$ we have $\theta_t(b') = b'$, hence $\theta_t(\mathcal{V}_{b'}) = \mathcal{V}_{b'}$, and $\theta_t(V_{b'}) = \theta_t(\mathcal{V}_{b'} \cap \Omega) = \theta_t(\mathcal{V}_{b'}) \cap \theta(\Omega) = \mathcal{V}_{b'} \cap \Omega = V_{b'}$, so that $\{\theta_t : t \in (-1, 1)\}$, restricted to $V_{b'}$, acts as a 1-parameter group of automorphisms of $V_{b'}$.

Now for $b = (1; 0; w) \in \Sigma$ we have $\theta_t(z_1; 0; w) = \left(\frac{z_1+t}{1+tz_1}; 0; w\right)$, hence $\rho(0; 0; w) = \lim_{t\to 1} \theta_t(0; 0; w) = (1; 0; w) = b \in \Sigma$. By Lemma 3.2, $\partial V_{b'}$ is smooth and strictly pseudoconvex at b. Hence there exists a local plurisubharmonic peak function at b for $\partial V_{b'}$. In other words, there exists an open neighborhood U of b on $\mathcal{V}_{b'}$, and a plurisubharmonic function ψ on U such that $\psi(b) = 0$ and $\psi(x) < 0$ for any $x \in U \cap V_{b'}$. Pick now any point $y \in D_0 = \Delta \times \{(0; 0)\}$. Since $\rho(y) = b$, on some open neighborhood W of y, $\psi(\rho(z))$ is defined and plurisubharmonic, and it takes a maximal value at y, implying that $\psi(\rho(z)) = 0$ for any $z \in W$, so that $\rho(z) = b$ for $z \in W$ as ψ is a local peak function at b. Hence, $\rho(z) = b$ for any $z \in V_{b'}$ by the identity theorem for analytic functions. In other words, $V_{b'} \subset \rho^{-1}(b)$.

We claim that for any $x \in \Omega$ there exists some $b \in \Sigma$ such that $x \in V_{b'}$. Since G_0 acts transitively on Ω, there exists $\beta \in G_0$ such $\beta(x) = 0$. Consider the maximal boundary component $\beta(\Sigma)$. By Wolf [35], $K \subset G_0$ acts transitively on the set of all boundary components of the same rank, hence there also exists $\kappa \in K$ such that $\kappa(\beta(\Sigma)) = \Sigma$. Then $\gamma := \kappa\beta \in G_0$ is an automorphism of Ω such that $\gamma(x) = 0$ and $\gamma(\Sigma) = \Sigma$. Since the origin $0 \in \Omega$ and the boundary point $b_0' = (-1; 0; 0) \in \partial D \times \{(0; 0)\} \subset \mathrm{Reg}(\partial\Omega)$ belong to the same minimal rational curve, we have $x \in \mathcal{V}_{\gamma^{-1}(-1;0;0)}$, i.e., there exists a point $b \in \Sigma$ such that $x \in V_{b'} := \mathcal{V}_{b'} \cap \Omega$ for the boundary point b' opposite to b with respect to 0.

Note that $\rho\big|_{\Omega_0'} : \Omega_0' \xrightarrow{\cong} \Sigma$. Thus, $\rho : \Omega \to \Sigma$ is surjective, and ρ is a holomorphic submersion onto Σ. Up to this point $\rho = \lim_{n\to\infty} \theta_{t_n}$ depends on the choice of the sequence $t_n \to 1$. Note that any point $x \in \Omega$ belongs to V_b for a *unique* $b \in \Sigma$, we must have $\lim_{n\to\infty} \theta_{t_n}(x) = b$ for any choice of $t_n \to 1$ and the limit ρ does not depend on the choice of the sequence $\{t_n\}$. It follows in fact $\lim_{t\to 1} \theta_t = \rho$.

Finally, for distinct points $b_1, b_2 \in \Sigma$, we have $V_{b_1'} \cap V_{b_2'} = \emptyset$ since $\rho(V_{b_1'}) = \{b_1\} \neq \{b_2\} = \rho(V_{b_2'})$. It follows that $\rho^{-1}(b)$ is exactly equal to $V_{b'}$, and we have a decomposition of Ω into a disjoint union of level sets $V_{b'}, b' \in \Sigma'$, of Ω, b' being the opposite boundary point with respect to 0 of $b \in \Sigma$, as desired. The proof of Proposition 3.4 is complete. □

Remark 3.5. For the proof of $V_{b'} \subset \rho^{-1}(b)$, the function $x_1 = \mathrm{Re}(z_1)$ is in fact a *global* peak function, as follows from the proof of [25, Lemma 2.2.2]. Also, $V_{b'} \subset \rho^{-1}(b)$ follows from the nonexistence of holomorphic arcs on $\partial V_{b'}$ passing through b and from a slight generalization of Cartan's result used in which the bounded domain is replaced by a nonsingular bounded domain on a Stein space, which follows from the proof of the said result in [29].

When Ω is an irreducible bounded symmetric domain of rank ≥ 2, beyond Proposition 3.4 the subvarieties $V_a \subset \Omega$ play an important role in this article. By [22] we have

Theorem 3.6. *For any point $a \in \mathrm{Reg}(\partial\Omega)$, $(\mathcal{V}_a, g_\Omega|_{V_a})$ is biholomorphically isometric to the complex unit ball $(\mathbb{B}^{p+1}, g_{p+1})$.*

We examine now holomorphic isometries $\theta : V_a \xrightarrow{\cong} \mathbb{B}^{p+1}$ for an arbitrary given point $a \in \mathrm{Reg}(\partial\Omega)$. For $[\ell] \in \mathscr{D}$, i.e., $[\ell] \in \mathcal{K}$ and $\ell \cap \Omega \neq \emptyset$, $D_\ell := \ell \cap \Omega \subset \Omega$ is a totally geodesic holomorphic isometric image of the Poincaré disk. Moreover, from the monotonicity of holomorphic bisectional curvatures $(D_\ell, g_\Omega|_{D_\ell}) \hookrightarrow (V_a, g_\Omega|_{V_a})$ is also totally geodesic. Hence, writing $\Delta_\ell := \theta(D_\ell)$, we have $(\Delta_\ell, g_{p+1}|_{\Delta_\ell}) \hookrightarrow \big(\mathbb{B}^{p+1}, g_{p+1}\big)$. Writing $\theta_\ell := \theta|_{D_\ell}$, we have $\theta_\ell : D_\ell \xrightarrow{\cong} \Delta_\ell$, $\Delta_\ell \subset \mathbb{B}^{p+1}$ is also totally geodesic, i.e., Δ_ℓ is a minimal disk on $\mathbb{B}^{p+1}$, equivalently the nonempty intersection of a projective line on $\mathbb{P}^{p+1}$ with $\mathbb{B}^{p+1}$. Regarding the boundary behavior of the holomorphic isometry $\theta : (V_a, g_\Omega|_{V_a}) \xrightarrow{\cong} \big(\mathbb{B}^{p+1}, g_{p+1}\big)$ near the vertex a of $\mathcal{V}_a$ we have,

Proposition 3.7. *For every $[\ell] \in \mathscr{D}_a$, θ_ℓ extends to a biholomorphism $\theta_\ell^\sharp : \ell \xrightarrow{\cong} \Lambda$, where $\Lambda \subset \mathbb{P}^{p+1}$ is a projective line and $\Delta_\ell \subset \Lambda$ is the Borel embedding. There exists $u \in \partial\mathbb{B}^{p+1}$ such that $\theta_\ell^\sharp(a) = u$ for every $[\ell] \in \mathscr{D}_a$. Moreover, defining $\partial^\flat V_a = \coprod \{\partial D_\ell - \{a\} : [\ell] \in \mathscr{D}_a\}$, $\theta : V_a \xrightarrow{\cong} \mathbb{B}^{p+1}$ extends holomorphically to a biholomorphism $\theta^\sharp : W \xrightarrow{\cong} U$ where W is a neighborhood of $V_a \coprod \partial^\flat V_a$ in S and U is a neighborhood of $\mathbb{B}^{p+1} - \{u\}$ in $\mathbb{P}^{p+1}$. Furthermore, $\theta^\sharp\big|_{V_a \coprod \partial^\flat V_a}$ extends continuously to $\theta^\dagger : \overline{V_a} \to \mathbb{B}^{p+1}$.*

Proof. From the standard embedding $\mathbb{P}\mathrm{SU}(1,1) \hookrightarrow \mathbb{P}\mathrm{SL}(2;\mathbb{C})$ the isomorphism $\theta|_\ell : D_\ell \xrightarrow{\cong} \Delta_\ell$ extends to a biholomorphism $\theta_\ell^\sharp : \ell \xrightarrow{\cong} \Lambda$ from the compact dual ℓ of D_ℓ to the compact dual Λ of Δ_ℓ.

For $[\ell] \in \mathscr{D}_a$, write $u(\ell) := \theta_\ell^\sharp(a)$. We proceed to prove that $u(\ell)$ is independent of ℓ. Let $[\ell_1], [\ell_2] \in \mathscr{D}_a$ be distinct. For $i = 1, 2$ let $\gamma_i : [0, \infty) \to D_{\ell_i}$ be a geodesic ray such that $\lim_{t\to\infty} \gamma_i(t) = a$. Write $\delta_i(t) := \delta(\gamma_i(t))$ for the Euclidean distance from $\gamma_i(t)$ to $\partial\Omega$. Considering the inclusion map $\iota : \mathbb{B}^N(\gamma_i(t); \delta_i(t)) \hookrightarrow \Omega$. From the Schwarz lemma applied to the inclusion map ι and the canonical Kähler-Einstein metrics, we have

(†): There exists $C > 0$ such that $g_\Omega(\gamma_i(t)) \leq \frac{C}{\delta_i(t)^2} ds^2$,

where ds^2 stands for the standard Euclidean metric. Moreover, connecting ℓ_1 to ℓ_2 through a smooth family $\{\ell(s)\}_{0<s<3}$ of minimal rational curves belonging to $\mathscr{D}_a$, $\ell_i = \ell(i)$ for $i = 1, 2$, and hence obtaining a smooth 1-parameter family of geodesic rays $\gamma_s : [0, \infty) \to V_a$, $\mathrm{Image}(\gamma_s) \subset D_{\ell(s)}$, $\lim_{t\to\infty} \gamma_s(t) = a$, the estimate (†) remains valid for γ_s for some constant $C > 0$ independent of $s \in [1, 2]$. It follows by estimating the length of the curves $\mu_t : [1, 2] \to V_a$ defined by $\mu_t(s) = \gamma_s(t)$ that there exists $C' > 0$ such that $\mathrm{Length}(\mathrm{Image}(\mu_t)) < C'$ for $t \in [0, \infty)$.

Suppose there exist $[\ell_1], [\ell_2] \in \mathscr{D}_a$ such that $u(\ell_1) \neq u(\ell_2)$. Then, $\sigma_i := \theta_*(\gamma_i)$, $i = 1, 2$, are geodesic rays on Δ_1 resp. Δ_2 such that $\lim_{t\to\infty} d(\sigma_1(t), \sigma_2(t)) = \infty$ from the estimate $g_{p+1}(y) \geq \frac{c \cdot ds^2}{\delta(y; \partial\mathbb{B}^{p+1})}$ for some constant $c > 0$, where d stands for distances on $\big(\mathbb{B}^{p+1}; g_{p+1}\big)$, contradicting with the uniform bound $\mathrm{Length}(\theta_*\mu_t) = \mathrm{Length}(\mu_t) \leq C'$ for $t \in [0, \infty)$, proving by contradiction that $u(\ell) = u$ for some $u \in \partial\mathbb{B}^{p+1}$ and for all $[\ell] \in \mathscr{D}_a$.

For the existence of neighborhoods $W \supset V_a \coprod \partial^\flat V_a$ and $U \supset \mathbb{B}^{p+1} - \{u\}$ such that a holomorphic extension $\theta^\sharp : W \xrightarrow{\cong} U$ of θ exists, it suffices to take $W = \coprod \{\ell : [\ell] \in \mathscr{D}_a\}$ and $U = \mathbb{P}^{p+1} - H$, where $H \subset \mathbb{P}^{p+1}$ is the projective hyperplane passing through u such that H is tangent to $\partial\mathbb{B}^{p+1}$ at the point u, given the existence of $\theta_\ell^\sharp : \ell \xrightarrow{\cong} \Lambda$ as established above. It remains to prove that $\theta^\sharp\big|_{V_a \coprod \partial^\flat V_a}$ extends continuously to $\theta^\dagger : \overline{V_a} \to \mathbb{B}^{p+1}$.

Writing $\partial^\sharp V_a := \overline{V}_a - \big\{V_a \coprod \partial^\flat V_a\big\} = \partial V_a - \partial^\flat V_a$, we claim that the extension $\theta^\dagger$ defined by $\theta^\dagger\big|_{\partial^\sharp V_a} \equiv u$ is continuous. Letting $\{x_k\}_{1\leq k<\infty}$ be a sequence of points on $V_a \coprod \partial^\flat V_a$ such that $x_k \to c \in \partial^\sharp V_a$, we have to prove that $\theta^\sharp(x_k) \to u$. Given any point $p \in \partial D_\ell - \{a\} \subset \partial^\flat V_a$, there exists $x \in D_\ell$ such that $\|\theta^\sharp(x) - \theta^\sharp(p)\| \leq \frac{1}{k}$, hence we may assume that $x_k \in V_a$ for $k \geq 1$. Suppose on the contrary $\theta(x_k) \to u' \neq u$. Since θ is in particular proper, we have $u' \in \partial\mathbb{B}^{p+1}$, hence $u' = \theta^\sharp(b)$

for some $b \in \partial^\flat V_a$, and there exists a neighborhood $\mathcal{O}$ of b in $\mathcal{V}_a$ such that $\theta^\sharp|_{\mathcal{O}} : \mathcal{O} \xrightarrow{\cong} \mathcal{O}'$ for some neighborhood $\mathcal{O}'$ of u' in $\mathbb{C}^{p+1}$. We have $\theta^\sharp(\mathcal{O} \cap V_a) = \mathcal{O}' \cap \mathbb{B}^{p+1}$. We may assume that $\theta(x_k) \in \mathcal{O}' \cap \mathbb{B}^{p+1}$ for $k \geq 1$. Since θ is bijective it follows that $x_k \in \mathcal{O} \cap V_a$ and hence $x_k \to b$, contradicting with the assumption that $x_k \to c \notin \partial^\flat V_a$, proving the claim and completing the proof of the proposition. □

Remark 3.8. There is a unique maximal boundary component $\Pi \subset \mathrm{Reg}(\partial\Omega)$ which contains a given point $a \in \mathrm{Reg}(\partial\Omega)$. Writing S_Π for the Zariski closure of Π in S, S_Π is the compact dual of Π, and $\Pi \subset S_\Pi$ is the Borel embedding. Define $\mathcal{V}_a' \subset S_\Pi$ as the cone on S_Π swept out by minimal rational curves on S_Π containing the point a, and defining $V_a' := \mathcal{V}_a' \cap \Pi$, it can be proven that the topological boundary ∂V_a in $\mathbb{C}^N$ is the disjoint union of $\partial^\flat V_a$ and $\overline{V_a'}$.

Proposition 3.4 allows us to prove Theorem 1.1 (the Extension Theorem) by avoiding the technically more difficult harmonic analysis for higher rank symmetric spaces (cf. Korányi [11], [12]) and use instead Fatou-type results much more familiar to complex analysts for the rank-1 case of the complex unit ball, and this will be the starting point of the next subsection.

Let $\Phi \subset \mathrm{Reg}(\partial\Omega)$ be a maximal boundary component. Then, there exists $k \in K$ such that $\Phi = k\Sigma$. If we define $\theta_{t,\Phi} := k\theta_{t,\Sigma}k^{-1}$, then we obtain $\rho_\Phi := \lim_{t\to 1}\theta_{t,\Phi} = k\rho_\Sigma k^{-1}$, which is a holomorphic submersion $\rho_\Phi : \Omega \to \Phi$. With the boundary component $\Phi' = \sigma_0^\Omega(\Phi)$ opposite to Φ with respect to 0 as defined, Proposition 3.4 remains valid with Σ being replaced by Φ, ρ_Σ being replaced by ρ_Φ and $b' \in \Phi'$ meaning the opposite point of $b \in \Phi$ with respect to 0 as defined by $b' = \sigma_{0_\Phi}^\Omega(\sigma_0^\Omega(b))$.

Definition 3.9. The holomorphic submersion $\rho = \rho_\Sigma : \Omega \to \Sigma$ in Proposition 3.4 onto Σ, or more generally $\rho = \rho_\Phi = k\rho_\Sigma k^{-1}$, $\rho : \Omega \to \Phi$ as in the last paragraph, is called a standard Cayley projection.

Remark 3.10. In [1] the standard Cayley projection was called the canonical projection, and it was described in group theoretic terms. The properties of the standard Cayley projection as described in Proposition 3.4 can be captured from [1, Chap. III, §3].

We proceed to describe more general Cayley projections $\rho_{\Phi,c} : \Omega \to \Phi$ depending on the choice of a maximal boundary component $\Phi \subset \mathrm{Reg}(\partial\Omega)$ and the choice of an appropriate point $c \in \mathrm{Reg}(\partial\Omega)$. The standard Cayley projection is the special case where, taking some point $b \in \Phi$, c is taking to be its opposite point $b' \in \Phi'$ on the opposite boundary component Φ' with respect to 0 as defined above. Recall from (1) that the construction of the standard Cayley projection starts with a holomorphic totally geodesic embedding

$$\nu : \Delta \times \Omega' \hookrightarrow \Omega \tag{4}$$

where the image passes through the origin. Denote by

$$\upsilon : \mathrm{Aut}(\Delta) \times \mathrm{Aut}_0(\Omega') \to G_0 \tag{5}$$

the group monomorphism accompanying the embedding $\nu : \Delta \times \Omega' \hookrightarrow \Omega$. We have

Lemma 3.11. *Let now $a, c \in \mathrm{Reg}(\partial\Omega)$ be distinct points. Let Φ resp. Ψ be the unique maximal boundary component containing a resp. c. Assume that there exists $[\ell] \in \mathscr{D}$ such that both a and c lie on the boundary circle of the minimal disk $D_\ell = \ell \cap \Omega$. Then, there exists $g \in G_0$ such that $g\Sigma = \Phi$, $g\Sigma' = \Psi, g(a) = b_0 = (1; 0; 0), g(c) = b_0' = (-1; 0; 0)$.*

Proof. Since G_0 acts transitively on the set of all minimal disks on Ω, there exists $\mu \in G_0$ such that $\mu(D_\ell) = \nu(\Delta \times \{0\}) =: D_0$. In terms of privileged Harish-Chandra coordinates we have

$$\partial D_0 = \left\{(e^{i\theta}; 0; w) : \theta \in \mathbb{R}, w \in \Omega'\right\}. \tag{6}$$

For $\theta \in \mathbb{R}$ write $\Sigma(\theta) = \nu\left(\{e^{i\theta}\} \times \Omega'\right)$. Since $\Sigma(\theta)$ is the unique maximal boundary component passing through $(e^{i\theta}; 0; w) \in \partial D_0$, we have $\mu(\Phi) = \Sigma(\theta_1)$ and $\mu(\Psi) = \Sigma(\theta_2)$ such that $e^{i\theta_1} \neq e^{i\theta_2}$. Since $\mathrm{Aut}(\Delta)$ acts doubly transitively on ∂D_0 there exists $\lambda \in \upsilon\left(\mathrm{Aut}(\Delta) \times \{id_{\Omega'}\}\right)$ such that $\lambda(\Sigma(\theta_1)) = \Sigma$ and $\lambda(\Sigma(\theta_2)) = \Sigma'$. Thus, writing $\gamma = \lambda \circ \mu$ we have $\gamma(\Phi) = \Sigma$ and $\gamma(\Psi) = \Sigma'$. Since $\gamma(D_\ell) = D_0$, we have $\gamma(a) \in \partial D_0 \cap \Sigma$ so that $\gamma(a) = b_0 = (1; 0; 0)$. Similarly, we have $\gamma(c) \in \partial D_0 \cap \Sigma'$ so that $\gamma(a) = b_0 = (-1; 0; 0)$. Putting $g = \gamma^{-1}$, we have established the lemma. $\square$

In the notation of Lemma 3.11, define now $\xi_t := g\theta_{t,\Sigma}g^{-1}$. Then, $\{\xi_t\}_{-1<t<1}$ is a 1-parameter group of transvections on Ω such that ξ_t converges as $t \to 1$ to a holomorphic submersion $\sigma : \Omega \to \Phi$. We may write $\xi_t =: \theta_{t,\Phi,c}$ and $\sigma =: \rho_{\Phi,c}$, which will be referred to as a Cayley projection. Clearly, ξ_t and σ are determined by Φ and the unique boundary component Ψ containing c. We may therefore also write the Cayley projection as $\rho_{\Phi,\Psi}$, and $\theta_{t,\Phi,c}$ as $\theta_{t,\Phi,\Psi}$, noting that $\rho_\Phi = \rho_{\Phi,\Phi'}$.

If $u \in G_0$ satisfies $u\Sigma = \Sigma$, $u\Sigma' = \Sigma'$, then $u = \upsilon(\eta, \varphi)$ for some $\eta \in H_0$ and $\varphi \in \mathrm{Aut}_0(\Omega')$, where $H_0 = \{\psi_t : -1 < t < 1\}$. We have $\upsilon(H_0 \times \{id_{\Omega'}\}) = H$, $\upsilon(\{id_\Delta\} \times \mathrm{Aut}_0(\Omega')) =: G_0'$ and H commutes with G_0'. Thus for $u = \upsilon(\eta, \varphi)$, $u\theta_{t,\Sigma,\Sigma'}u^{-1} = \xi\theta_{t,\Sigma,\Sigma'}\xi^{-1}$, where $\xi = \upsilon(\eta, id_{\Omega'}) \in H$ and $\theta_{t,\Sigma,\Sigma'} \in H$ so that they commute with each other, and $u\theta_{t,\Sigma,\Sigma'}u^{-1} = \theta_{t,\Sigma,\Sigma'}$. Now if the pair (Φ, Ψ) is given, and g_1, g_2 are such that $g_i\Sigma = \Phi$, $g_i\Sigma' = \Psi$ for $i = 1, 2$, we must have $g_2 = g_1 u$ for some u preserving both Σ and Σ' as in the above. Since u preserves $\theta_{t,\Sigma,\Sigma'}$ for $t \in (-1, 1)$, we deduce that $\theta_{t,\Phi,\Psi} = g\theta_{t,\Sigma,\Sigma'}g^{-1}$ is well-defined independent of the choice of g satisfying $g\Sigma = \Sigma$, $g\Sigma' = \Sigma'$, hence also the Cayley projection $\rho_{\Phi,\Psi} : \Omega \to \Phi$ is well-defined.

Noting that by convention $\theta_\Sigma = \theta_{\Sigma,\Sigma'}$ and $\rho_\Sigma = \rho_{\Sigma,\Sigma'}$, we have proven

Proposition 3.12. *The analogue of Proposition 3.4 remains valid when the pair (b, b') is replaced by (a, c), $\theta_{t,\Sigma}$ is replaced by $\theta_{t,\Phi,\Psi}$, and $\rho_\Sigma : \Omega \to \Sigma$ is replaced by $\rho_{\Phi,\Psi} : \Omega \to \Phi$.*

We reformulate the requirement on the pair of points (a, c) on $\mathrm{Reg}(\partial\Omega)$ as a property (%) on the pair (Φ, Ψ) of maximal boundary components, as follows.

(%): There exists some point $(a, c) \in \Phi \times \Psi$ such that $a \in \partial^\flat V_c \cap \Phi$.

When the pair (Φ, Ψ) of maximal boundary components of Ω satisfies the condition (%), we will simply say that (Φ, Ψ) is a pair of maximal boundary components for which $\rho_{\Phi,\Psi}$ can be defined. It will be shown in §4.4 that the condition (%) is equivalent to the condition $\overline{\Phi} \cap \overline{\Psi} = \emptyset$.

3.3 Admissible limits in relation to Cayley projections

3.3.1 Admissible limits for $\mathbb{B}^n$

We will need the notion of admissible limits in $\mathbb{B}^n$ in the sense of Korányi [10, 11] (cf. [30, section 5.4]), which will be recalled as follows.

Definition 3.13. For $\alpha > 1$ and $\xi \in \partial\mathbb{B}^n$, let $D_\alpha(\xi) := \{z \in \mathbb{B}^n : \frac{|1-\langle z,\xi\rangle|}{1-|z|^2} < \frac{\alpha}{2}\}$, where $\langle\cdot,\cdot\rangle$ is the standard Hermitian inner product. Let f be a complex-valued function on $\mathbb{B}^n$. We say that f has an admissible limit (or boundary value) $\lambda \in \mathbb{C} \cup \{\infty\}$ at ξ, or that f converges admissibly to λ at ξ if for any $\alpha > 1$ and any sequence $\{p_n\} \subset D_\alpha(\xi)$ such that $p_n \to \xi$ as $n \to \infty$, we have $\lim_{p_n \to \xi} f(p_n) = \lambda$.

Combining [6] and [10], we have the following generalization of the classical Fatou's theorem:

Theorem 3.14. *Let f be a bounded holomorphic function on $\mathbb{B}^n$. Then f converges admissibly to some bounded function $g \in L^\infty(\partial\mathbb{B}^n)$ almost everywhere on $\partial\mathbb{B}^n$*

Suppose $\{\eta_t\} \subset \mathrm{Aut}(\mathbb{B}^n)$ is a 1-parameter subgroup of transvections. It follows from the theorem of Cartan (Theorem 3.3 here) and the strong pseudoconvexity of $\mathbb{B}^n$ that a subsequence of $\{\eta_t\}$ converges uniformly on compact subsets to a constant function $\eta : \mathbb{B}^n \to \{\xi\} \subset \partial\mathbb{B}^n$. Let $p \in \mathbb{B}^n$. We claim that the trajectory $\{\eta_t(p)\}$ lies inside an admissible domain. First note that $D_\alpha(\xi)$ is invariant under unitary transformations, so we only need to consider the boundary point $\xi = e_1 := (1, 0, \ldots, 0) \in \partial\mathbb{B}^n$ and $\eta_t(z) := (\frac{z_1+t}{1+tz_1}, \frac{\sqrt{1-t^2}}{1+tz_1}z_2, \ldots, \frac{\sqrt{1-t^2}}{1+tz_1}z_n)$, $(0 \le t < 1)$, for $z = (z_1, \ldots, z_n) \in \mathbb{B}^n$. Write $w = (w_1, \ldots, w_n) = \eta_t(z)$. Note that

$$\frac{|1 - \langle w, e_1\rangle|}{1 - |w|^2} = \frac{|1 - w_1|}{1 - |w|^2} = \frac{|1 + tz_1|}{1 + t}\frac{|1 - z_1|}{1 - |z|^2} := \beta(t, z_1)\frac{|1 - z_1|}{1 - |z|^2},$$

where β is positive continuous on $[0, 1] \times \Delta$. For each fixed $p = (p_1, \ldots, p_n) \in \mathbb{B}^n$, there exists $M_{p_1} > 0$ such that $\beta(t, p_1) < M_{p_1} < \infty$. So

$$\{\eta_t(p) \mid 0 \le t < 1\} \subset D_{\beta_p}(e_1), \quad \text{where } \beta_p = 2M_{p_1}\frac{|1 - p_1|}{1 - |p|^2}.$$

If f is a bounded holomorphic functions, then $\lim_{t\to1} \eta_t^* f(z)$ exists provided that the admissible limit of f exists at e_1. Now for each $\xi \in \partial\mathbb{B}^n$, one obtains a corresponding 1-parameter subgroups $\{\eta_{t,\xi}\} \subset \mathrm{Aut}(\mathbb{B}^n)$, for which a convergent subsequence converges to the constant map $\eta_\xi : \mathbb{B}^n \to \{\xi\}$. Theorem 3.14 says that $\lim_{t\to1} \eta_{t,\xi}^* f(z)$ exists for almost all $\xi \in \partial\mathbb{B}^n$.

3.3.2 Cayley limits

Let Ω be a bounded symmetric domain of rank $r \ge 2$, and (Φ, Ψ) be a pair of maximal boundary components, by Proposition 3.4 and the paragraph preceding Definition 3.9 we obtain the Cayley projection $\lim_{t\to1} \theta_{t,\Phi,\Psi} = \rho_{\Phi,\Psi} : \Omega \to \Phi$, where $\{\theta_{t,\Phi,\Psi}\} \subset \mathrm{Aut}(\Omega)$ is a 1-parameter subgroup of transvections. In analogy to the notion of admissible limits on $\mathbb{B}^n$, we have

Definition 3.15. Let h be a function defined on a bounded symmetric domain Ω of rank $r \ge 2$. Let $\{\theta_{t,\Phi,\Psi}\}_{-1<t<1} \subset \mathrm{Aut}(\Omega)$ be the 1-parameter subgroup of transvections as defined in §3.1 so that $\lim_{t\to1} \theta_{t,\Phi,\Psi} = \rho_{\Phi,\Psi} : \Omega \to \Phi$ is a Cayley projection. Then, for each point $x \in \Omega$, we write $\hat{h}_{\Phi,\Psi}(x) := \lim_{t\to1} \theta_{t,\Phi,\Psi}^* h(x) := \lim_{t\to1} h(\theta_{t,\Phi}(x))$ if the limit exists, and call it the Cayley limit of h at x with respect to $\rho_{\Phi,\Psi}$. We say that the Cayley limit $\hat{h}_{\Phi,\Psi}$ exists on Ω if $\hat{h}_{\Phi,\Psi} = \lim_{t\to1} \theta_{t,\Phi,\Psi}^* h$ in the sense of uniform convergence on compact subsets of Ω.

Recall from §2 that the moduli space of boundary components $\Phi \subset \mathrm{Reg}(\partial\Omega) = E_{r-1}$ is of the form G_0/N for some maximal parabolic subgroup $N \subset G_0$.

Let (Φ, Ψ) be a pair of maximal boundary components for which $\rho_{\Phi,\Psi}$ is defined, in which case $(\Phi, \Psi) = (g\Sigma, g\Sigma')$ for some $g \in G_0$. As in the description preceding Proposition 3.12, we write $\xi_t = g\theta_t g^{-1} = \theta_{t,\Phi,\Psi}$ for $t \in (-1, 1)$ and for the 1-parameter subgroup $\{\theta_t\}_{-1<t<1} = \Theta \subset G_0$ of transvections. We have

Lemma 3.16. *Fix $h \in H^\infty(\Omega)$. Suppose (Φ, Ψ) is a pair of maximal boundary components of Ω for which $\rho_{\Phi,\Psi}$ is defined. Writing $\partial^\flat V_c \cap \Phi =: \{a(c)\}$ assume that the admissible boundary values $h^\sharp_{a(c),c}$ and $h^\sharp_{c,a(c)}$ both exist for almost all $c \in \Psi$. Then, for* **every** *point $c \in \Psi$ the admissible boundary values $h^\sharp_{a(c),c}$ and $h^\sharp_{c,a(c)}$ both exist. Moreover, $\hat{h}_{\Phi,\Psi} := \lim_{t\to1} h \circ \xi_t$ exists in the sense of uniform convergence on compact subsets of Ω, given by $\hat{h}_{\Phi,\Psi} = \rho^*_{\Phi,\Psi} s_{\Phi,\Psi}$, where $s_{\Phi,\Psi} \in H^\infty(\Phi)$ is given by $s_{\Phi,\Psi}(a) := h^\sharp_{a,c}$, for $\partial^\flat V_c \cap \Phi = \{a\}$.*

Proof. Since the Cayley projection $\rho_{\Phi,\Psi}$ can be defined, there exists $g \in G_0$ such that $(\Phi, \Psi) = (g\Sigma, g\Sigma')$. Re-coordinatizing Ω by re-labelling $g \cdot z$ as z, without loss of generality we may assume $(\Phi, \Psi) = (\Sigma, \Sigma')$.

By assumption, for almost all $b'(w) = (b'; 0; w) \in \Sigma'$, the admissible boundary value $h^\sharp_{b(w),b'(w)}$ exists. Since any subset of full Lebesgue measure of Ω' is necessarily dense in Ω' we can find a dense sequence of points $\{w_k\}_{0\le k<\infty}$ on Ω' such that $h^\sharp_{b(w_k),b'(w_k)}$ exists for $k \in \mathbb{N}$. Fix $w \in \Omega'$ and let $\{z_\ell\}$ be an arbitrary sequence of points on $V_{b'(w)}$ converging admissibly to $b(w)$. Write $z_\ell = (z^1_\ell; z'_\ell; w)$ in privileged Harish-Chandra coordinates.

Let now $\{w_{\tau(k)}\}$ be a subsequence of $\{w_k\}$ such that $w_{\tau(k)}$ converges to w. By the standard 3ϵ argument in the proof of Montel's theorem involving Cauchy's estimates for first derivatives, comparing $h(z_\ell) = h(z^1_\ell; z'_\ell; w)$ with $h(z^1_\ell; z'_\ell; w_{\tau(k)})$ we conclude that $h(z_\ell)$ is also a Cauchy sequence and hence the admissible boundary value $h^\sharp_{b(w),b'(w)}$ also exists. It follows that for any admissible sequence of points $(z^1_\ell; z'_\ell; w)$ on $V_{b'(w)}$, and defining $h_\ell(w) := h(z^1_\ell, z'_\ell, w)$ we must have $h_\ell(w)$ converging to $s_{\Sigma,\Sigma'}(b(w))$ for $s_{\Sigma,\Sigma'}(b(w)) := h^\sharp_{b(w),b'(w)}$.

Hence, for every point $z_0 \in \Omega$, and for any sequence $t_n \in (-1, 1)$ such that $t_n \to 1$ we have $\lim\limits_{t\to1} h(\theta_{t_i}(z_0)) = h^\sharp_{b(w_0),b'(w_0)}$ where $b(w_0) = (1; 0; w_0) = \rho_{\Sigma,\Sigma'}(z_0)$. For a polydisk $\Delta^n(z_0; \epsilon) \Subset \Omega$, with polyradii $(\epsilon, \cdots, \epsilon)$, $\epsilon > 0$, expressing the holomorphic function $h(\theta_{t_n}(z))$ in z for $z \in \Delta^n(z_0; \epsilon)$ in terms of the Cauchy integral of its restriction to the distinguished boundary $\delta\Delta^n(z_0; \epsilon)$, by dominated convergence and by covering Ω by such polydisks with variable $\epsilon > 0$ we conclude that $h(\theta_{t_n}(z))$ converges uniformly on compact subsets to a bounded holomorphic function of the form $\hat{h}_{\Sigma,\Sigma'}(z) = s_{\Sigma,\Sigma'}(\rho_{\Sigma,\Sigma'}(z))$, where $s_{\Sigma,\Sigma'}(b(w)) := h^\sharp_{b(w),b'(w)}$, implying at the same time that $s_{\Sigma,\Sigma'} : \Sigma \to \mathbb{C}$ is a bounded holomorphic function since $s_{\Sigma,\Sigma'}(b(w)) = \hat{h}_{\Sigma,\Sigma'}(0; 0; w)$, where $b(w) = (1; 0; w) \in \Sigma$, $b'(w) = (-1; 0; w) \in \Sigma'$. Since the arguments work for any choice of $t_i \to 1$, we conclude that $\lim\limits_{t\to1} \theta^*_t h = \rho^*_{\Sigma,\Sigma;} s_{\Sigma,\Sigma'}$ uniformly on compact subsets of Ω.

Finally, when $t \to -1$ the above arguments show *verbatim* that $h(\xi_t(z))$ converges admissibly to $\rho^*_{\Sigma',\Sigma} s_{\Sigma',\Sigma}(b'(w))$ where $s_{\Sigma',\Sigma}(b'(w)) := h^\sharp_{b'(w),b(w)}$. The proof of the lemma is complete. □

On a metric space (M, d) we say that two curves $\gamma_i : [0,\infty) \to M$ are asymptotically coincident if and only if the following holds true: As $s \to \infty$, there exists s' depending on s such that $\lim\limits_{s\to\infty} d(\gamma_1(s), \gamma_2(s')) = 0$, and the same holds with γ_1 and γ_2 switched.

Given a holomorphic function h on Ω, a pair (Φ, Ψ) of maximal boundary components for which the Cayley projection $\rho_{\Phi,\Psi}$ is defined, and given $a \in \partial^\flat V_c, c \in \Sigma'$, in principle the admissible boundary value $h^\sharp_{a,c}$ of $h|_{V_c}$ at a depends on Ψ. From estimates on intrinsic metrics we have nonetheless the following result.

Proposition 3.17. *Let* $[\Phi] \in G_0/N$*, and* $[\Psi_1], [\Psi_2] \in G_0/N - \{[\Phi]\}$ *for which Cayley projections* $\rho_i := \rho_{\Phi,\Psi_i}$*,* $i = 1, 2$*, are defined. Suppose* $a \in \Phi$ *and* $c_i \in \Psi_i$ *for* $i = 1, 2$*. Assume that, for* $i = 1, 2$*,* $\lim\limits_{k\to\infty} \xi^*_t h$ *exists for all sequences* $\{x_k\}_{0\le k<\infty}$ *on* V_{c_i} *converging admissibly to* a*, and denote the common limit (for* i *fixed) by* $h^\sharp_{a,c_i}(b)$*. Then,* $h^\sharp_{a,c_1} = h^\sharp_{a,c_2}$*. As a consequence, in the conclusion of Lemma 3.16 we have* $\lim\limits_{t\to1} \theta^*_t h = \rho^*_{\Phi,\Psi} s_\Phi$ *for* $s_\Phi \in H^\infty(\Phi)$ *independent of the choice of* Ψ*.*

Proof. For $i = 1, 2$ we have in particular $\lim\limits_{t\to1} h(\theta_{t,\Phi,\Psi_i}(0)) = h^\sharp_{a,c_i}$. There is a unique minimal rational curve ℓ_i belonging to $\mathscr{D}_a$ such that $c_i \in \ell_i$. Write $D_i := D_{\ell_i} = \ell_i \cap \Omega$. Writing $\gamma_i(s) := \theta_{t(s),\Phi,\Psi_i}(0)$, where the geodesic distance between 0 and $t(s)$ on the unit disk Δ endowed with the Poincaré metric is equal to s. Then, $\gamma_i : [0,\infty) \to \Omega$ is a geodesic ray parametrized by arc length such that $\lim\limits_{s\to\infty} \gamma_i(s) = a$. Denote by d_Ω the distance function on (Ω, g_Ω). We claim the validity of the following statement $(\flat)$.

($\flat$): The two geodesic rays γ_1 and γ_2 are asymptotically coincident.

By Theorem 3.6 and Proposition 3.7, we have a biholomorphism $\theta : V_a \xrightarrow{\cong} \mathbb{B}^{p+1}$ which extends to a continuous mapping $\theta^\dagger : \overline{V_a} \to \overline{\mathbb{B}^{p+1}}$. Moreover, $\theta^\dagger|_{\overline{D_i}}$ extends to a biholomorphism from ℓ_i to the minimal rational curve $\Lambda_i \subset \mathbb{P}^{p+1}$ containing Δ_i such that, assuming $\Psi_1 \neq \Psi_2$, we have $\overline{\Delta_1} \cap \overline{\Delta_2} = \{u\}$ for some point $u \in \partial\mathbb{B}^{p+1}$. Denote by d_{V_a} the distance function on V_a with respect to $g_\Omega|_{V_a}$, and by δ the distance function on $\left(\mathbb{B}^{p+1}, g_{p+1}\right)$. Since $\theta : V_a \xrightarrow{\cong} \mathbb{B}^{p+1}$, for $x, y \in V_a$ we have $d_\Omega(x,y) \le d_{V_a}(x,y) = \delta(\theta(x), \theta(y))$. Thus, to prove the claim ($\flat$) it suffices to show the validity of ($\sharp$), as follows.

($\sharp$): For any point $u \in \partial\mathbb{B}^{p+1}$, any 2 geodesic rays $\mu_i : [0,\infty) \to \mathbb{B}^{p+1}$, $i = 1, 2$, converging to u must be asymptotically coincident.

Note that μ_i must lie on a unique minimal disk Δ_i.

Any two geodesic rays ν and ν' on the upper half plane $\mathcal{H}$ converging to some point $p \in \partial\mathcal{H}$ must be tangent to each other at p. Given that on $\mathcal{H}$ the Poincaré metric is given by $ds^2_{\mathcal{H}} := \frac{1}{2\mathrm{Im}(\tau)^2}\mathrm{Re}(d\tau \otimes d\overline{\tau})$, it follows readily that ν and ν' are asymptotically coincident. Thus to prove the claim ($\sharp$), it suffices to prove its validity for *specific* choices of geodesic rays $\mu_i : [0,\infty) \to \Delta_i$.

The real unit ball $\mathbb{B}^{p+1}_{\mathbb{R}} = \mathbb{B}^{p+1} \cap \mathbb{R}^{p+1}$ is the fixed point set of the isometry with respect to $\left(\mathbb{B}^{p+1}, g_{p+1}\right)$ defined by $z \mapsto \overline{z}$, and it is thus a totally geodesic submanifold. We may assume $u = e_1 := (1, 0, \cdots, 0) \in \partial\mathbb{B}^{p+1}_{\mathbb{R}}$, and that, writing Π_i for the complex affine line containing Δ_i we have $\Pi_1 = \mathbb{C}e_1$, $T_u(\Pi_2) = \mathbb{C}(1, \beta_2, \cdots, \beta_{p+1})$. Applying a unitary transformation in the variables $(z_2, \cdots, z_{p+1})$ we may assume that $(\beta_2, \cdots, \beta_{p+1}) \in \mathbb{R}^p$.

Now for $i = 1, 2$ the intersection $L_i := \Delta_i \cap \mathbb{R}^{p+1}$ of two totally geodesic submanifolds is a totally geodesic curve on $\left(\mathbb{B}^{p+1}, g_{p+1}\right)$. As a set L_i is an open line segment on $\mathbb{R}^{p+1}$. Choose now the geodesic rays $\mu_i : [0,\infty) \to \mathbb{B}^{p+1}$ such that $\mu_i([0,\infty)) \subset \ell_i \subset \mathbb{B}^{p+1}_{\mathbb{R}}$ and such that $\lim_{s\to\infty} \mu_i(s) = e_1$. On $\mathbb{B}^{p+1}_{\mathbb{R}}$ we have $g_{p+1}\big|_{\mathbb{B}^{p+1}_{\mathbb{R}}} = \frac{2}{1-r^2}d\sigma_p^2 + \frac{2}{(1-r^2)^2}dr^2$ in terms of spherical coordinates, where $d\sigma_p^2$ is the spherical metric on the unit sphere $S^p = \partial\mathbb{B}^{p+1}_{\mathbb{R}}$. For ϵ sufficient small $\ell_i \cap \partial\mathbb{B}^{p+1}_{\mathbb{R}}(0; 1-\epsilon)$ is a unique point $x_i(\epsilon)$. There is a smooth curve $\lambda(\epsilon)$ joining $x_1(\epsilon)$ to $x_2(\epsilon)$ on the sphere $\partial\mathbb{B}^{p+1}_{\mathbb{R}}(0; 1-\epsilon)$ such that $\lambda(\epsilon)$ is of Euclidean length $\le C\epsilon$ for some fixed $C > 0$. On the other hand, since the restriction of g_{p+1} to the sphere $\partial\mathbb{B}^{p+1}_{\mathbb{R}}(0; 1-\epsilon)$ is given by $\frac{2}{1-r^2}d\sigma_p^2$ for some constant $C' > 0$, we must have Length $(\lambda(\epsilon); g_{p+1}) \le C'\frac{\epsilon}{\sqrt{\epsilon}} = C'\sqrt{\epsilon} \to 0$ as $\epsilon \to 0$, from which it follows that for $0 \le s < \infty$ there exists s' such that $d(\mu_1(s), \mu_2(s')) \to 0$ as $s \to \infty$, and the same is true when μ_1 and μ_2 are interchanged. Hence, μ_1 and μ_2 are asymptotically coincident on $\mathbb{B}^{p+1}$, proving the claim ($\sharp$).

We proceed now to deduce that the admissible boundary values $h^\sharp_{a,c_1} = h^\sharp_{a,c_2}$. We have $h(\gamma_i(t)) \to h^\sharp_{a,c_i}(b)$ for $i = 1, 2$ by assumption. As $s \to \infty$ choose now s' depending on s so that $d_\Omega(\gamma_1(s), \gamma_2(s')) = \epsilon(s) \to 0$ as $s \to \infty$. Writing $\Gamma_s : [0, \epsilon(s)] \to \Omega$ for the geodesic curve parametrized by arc length joining $\gamma_1(s)$ to $\gamma_2(s')$, for some constant $C > 0$ we have

$$
\begin{aligned}
h(\gamma_2(s')) - h(\gamma_1(s)) = \int_0^{\epsilon(s)} dh, \text{ hence} \\
|h(\gamma_2(s')) - h(\gamma_1(s))| \le \epsilon(s)\sup\left\{|dh(\Gamma_s'(t))| : t \in [0, \epsilon_s]\right\} \le C_1\epsilon(s) \to 0
\end{aligned} \tag{7}
$$

as $s \to \infty$, where we have $|dh(\Gamma_s'(t)| \le C_1$ for some $C_1 > 0$ as a consequence of the Schwarz lemma applied to the map $h : (\Omega, g_\Omega) \to \left(\Delta(R); g_{\Delta(R)}\right)$ assuming $h(\Omega) \subset \Delta(R)$, and $g_{\Delta(R)}$ is the Poincaré metric on $\Delta(R)$ of constant Gaussian curvature -2. From the choice of s' clearly $s' \to \infty$ as $s \to \infty$. Hence from (7), we have $h^\sharp_{a,c_1} = h^\sharp_{a,c_2}$. Taking $a \in \Phi$ such that $a \in \partial^\flat V_{c_1} \cap \partial^\flat V_{c_2}$. From $s_{\Phi,\Psi_i}(a) = h^\sharp_{a,c_i}$ for $a \in \partial^\flat V_{c_i}, i = 1, 2$, and $h^\sharp_{a,c_1} = h^\sharp_{a,c_2}$ whenever both admissible boundary values at a exists, it follows readily that $s_{\Phi,\Psi_1} = s_{\Phi,\Psi_2}$ as holomorphic functions on Φ, and we may write $s_\Phi := s_{\Phi,\Psi_1} = s_{\Phi,\Psi_1}$ as given in the proposition, as desired. The proof of Proposition 3.17 is complete. $\square$

Remark 3.18. Although Proposition 3.17 is not needed for the proof of Theorem 1.1, we deem it interesting to include the statement and its proof for its intrinsic value. It also serves to justify the notation s_Φ in place of $s_{\Phi,\Psi}$.

4 Parametrizing S^1-families of Cayley Projections

4.1 The universal space

In order to parametrize all Cayley projections we consider first of all the set of all varieties V_c swept out by minimal disks with vertices at $c \in \mathrm{Reg}(\partial\Omega)$. For a set A we write $\mathfrak{D}(A) := \{(a,a) : a \in A\}$ for the diagonal of $A \times A$. Let

$$\mathcal{S} := \{(a,c) \mid c \in \partial^\flat V_a - \{a\}\} \subset \mathrm{Reg}(\partial\Omega) \times \mathrm{Reg}(\partial\Omega) - \mathfrak{D}(\mathrm{Reg}(\partial\Omega)).$$

Recall from §2 that the G_0-orbit consisting of all rank $r-1$ boundary components is exactly $E_{r-1} = \mathrm{Reg}(\partial\Omega)$. Any boundary component of rank $r-1$ is biholomorphic to a fixed irreducible bounded symmetric domain Ω' of rank $r-1$. Such boundary components will also be referred to as maximal boundary components of Ω. The moduli space of maximal boundary components $\Phi \subset \mathrm{Reg}(\partial\Omega)$ is a compact G_0-homogeneous manifold, hence there is a maximal parabolic subgroup $N \subset G_0$ such that we have the fibration

$$\pi : \mathrm{Reg}(\partial\Omega) \to G_0/N, \quad \pi^{-1}(gN) \cong \Omega'. \tag{8}$$

Note that the fiber dimension of π is equal to $\dim(\Omega') = 2q$, so that $\dim(G_0/N) = \dim(\mathrm{Reg}(\partial\Omega)) - \dim(\Omega') = (2n-1) - 2q = (2(p+q+1) - 2q = 2p+1$. Here and in what follows "dim" means $\dim_\mathbb{R}$.

Using the projection of the second factor in $\mathcal{S}$, we get the following fibration:

$$\pi_2 : \mathcal{S} \to \mathrm{Reg}(\partial\Omega), \quad \pi_2^{-1}(c) = \partial^\flat V_c \cong \partial\mathbb{B}^{p+1} - \{u\}, u \in \partial\mathbb{B}^{p+1}, \tag{9}$$

where the latter isomorphism is a CR diffeomorphism between strictly pseudoconvex CR manifolds, cf. Proposition 3.7. Note that $a \in \partial^\flat V_c$ if and only if a and c lie on the boundary of a minimal disk $D_\ell = \Omega \cap \ell$ belonging to $\mathscr{D}$, hence it follows that $a \in \partial^\flat V_c$ if and only if $c \in \partial^\flat V_a$.

We have $\dim(\mathcal{S}) = \dim(\mathrm{Reg}(\partial\Omega)) + \dim(\partial\mathbb{B}^{p+1}) = (2n-1) + (2p+1) = 2n+2p$. On the other hand we have a projection $\sigma : \mathcal{S} \to G_0/N$ defined by $\sigma(a,c) = [\Phi] \in G_0/N$ where $\Phi \subset \mathrm{Reg}(\partial\Omega)$ is the unique maximal boundary component containing a. We have $\sigma^{-1}([\Phi]) = \{(a,c) : a \in \Phi \cap \partial^\flat V_c\}$, hence the fiber dimension of σ is $2q+2p+1$, since for every $c \in \partial^\flat V_a$ we have $a \in \partial^\flat V_c$. The base dimension of σ is $\dim(G_0/N) = \dim(\mathcal{S}) - (2q+2p+1) = (2n+2p) - (2q+2p+1) = 2(n-q)+1 = 2p+1$, which coincides with the computation of $\dim(G_0/N)$ obtained using $\pi : \mathrm{Reg} \to G_0/N$.

Let h be a bounded holomorphic function on Ω. For a point $(a,c) \in \mathcal{S}$. Let Φ resp. Ψ be the unique maximal boundary component containing a resp. c. From Proposition 3.7, we have $\theta^\sharp : V_c \coprod \partial^\flat V_c \xrightarrow{\cong} \overline{\mathbb{B}^{p+1}} - \{u\}$ for some point $u \in \partial\mathbb{B}^{p+1}$. We consider the restriction of h to V_c, and we write $h^\sharp_{a,c} = \lambda \in \mathbb{C}$ to mean that $(\theta^{-1})^* h$ has an admissible boundary value equal to λ at the point $\theta^\sharp(b) \in \partial\mathbb{B}^{p+1} - \{u\}$. Given $(a,c) \in \mathcal{S}$, for any point $x \in V_c$, $\rho_{\Phi,\Psi}(x)$ converges to a admissibly, hence $\lim_{t\to 1} \theta_{t,\Phi,\Psi}(x) = h^\sharp_{a,c}$.

Define now $E \subset \mathcal{S}$ to be the subset consisting of points (a,c) such that $h^\sharp_{a,c}$ does not exist. By Fatou's theorem for the complex unit ball (Theorem 3.14 here), we know that $E \cap \partial^\flat V_c$ is a null set, i.e., it is of zero measure with respect to any smooth volume form on $\partial^\flat V_c$. Provided that E is proven to be a measurable set, we can apply Fubini's theorem to $\pi_2 : \mathcal{S} \to \mathrm{Reg}(\partial\Omega)$ to conclude that $E \subset \mathcal{S}$ is a null set. Applying the latter statement to the fibration $\sigma : \mathcal{S} \to G_0/N$, we conclude that $E \cap \sigma^{-1}([\Phi])$ is a null set on Φ for almost every $[\Phi] \in G_0/N$.

We will next give a general discussion on null sets in relation to a double fibration on a smooth manifold, and then give a proof of the measurability of E in order to obtain admissible boundary values on almost every maximal boundary component on $\mathrm{Reg}(\partial\Omega)$.

4.2 Null sets on the total space of a double fibration

Definition 4.1. Let M be a smooth manifold and $E \subset M$. We say that E is a measurable subset of M if and only if it is measurable with respect to some smooth (hence any) smooth volume form $d\mu$ on M. $E \subset M$ is said to be a null set on M if and only if $E \subset M$ is measurable and $\mathrm{Volume}(E, dV_h) = 0$ for some (hence any) Riemannian metric h on M. A statement is said to hold true almost everywhere on M if and only if it holds true for all points $x \in M$ lying outside some null set on M.

We have the following lemma concerning null sets on the total space of a double fibration which follows readily from Fubini's theorem.

Lemma 4.2. *Let $\mathscr{P}$ be a smooth manifold which is the total space of a double locally trivial smooth fibration $\pi_i : \mathscr{P} \to B_i$, $i = 1, 2$. For $b_i \in B_i$ write $F^i_{b_i} := \pi_i^{-1}(b_i)$. Suppose $E \subset \mathscr{P}$ is a measurable subset. Assume that for almost every base point $b_1 \in B_1$, $E \cap F^1_{b_1}$ is a null set. Then, for almost every base point $b_2 \in B_2$, $E \cap F^2_{b_2} \subset F^2_{b_2}$ is a null set.*

4.3 Measurability of the set of boundary points having admissible limits

Recall the double fibration $\pi_2 : \mathcal{S} \to \mathrm{Reg}(\partial\Omega)$ and $\sigma : \mathcal{S} \to G_0/N$, for which we have defined in §4.1 a subset $E \subset \mathcal{S} \subset \mathrm{Reg}(\partial\Omega) \times \mathrm{Reg}(\partial\Omega) - \mathfrak{D}(\mathrm{Reg}(\partial\Omega))$ in terms of nonexistence of admissible boundary values of h.

Fix an arbitrary point $u \in \partial\mathbb{B}^{p+1}$. For $\pi_2 : \mathcal{S} \to \mathrm{Reg}(\partial\Omega)$, the fiber $\pi_2^{-1}(c) = \partial^\flat V_c \cong \partial\mathbb{B}^{p+1} - \{u\}$. When $c \in \mathrm{Reg}(\partial\Omega)$ is fixed we have by Proposition 3.7 a biholomorphism $\theta^\sharp : W \xrightarrow{\cong} U$, where $W \supset V_c \coprod \partial^\flat V_c$, $U \supset \overline{\mathbb{B}^{p+1}} - \{u\}$. Write $\theta^\flat := \theta^\sharp|_{V_c \coprod \partial^\flat V_c} : V_c \coprod \partial^\flat V_c \to \overline{\mathbb{B}^{p+1}} - \{u\}$, which extends the biholomorphism $\theta : V_c \xrightarrow{\cong} \mathbb{B}^{p+1}$. θ (and hence $\theta^\flat$) is uniquely determined up to composition on the left by an element $\xi \in G'_0 = \mathrm{Aut}(\mathbb{B}^{p+1}) \hookrightarrow \mathrm{Aut}(\mathbb{P}^{p+1})$, where $\xi(u) = u$. Let $Q \subset G'_0$ be the isotropy subgroup at u, so that $G'_0/Q \cong \partial\mathbb{B}^{p+1}$. By Proposition 3.7, $\theta^\flat$ admits a continuous extension $\theta^\dagger : \overline{V} \xrightarrow{\cong} \overline{\mathbb{B}^{p+1}}$. Vary c over $\mathrm{Reg}(\partial\Omega)$ and write H_c for the set of biholomorphisms $\eta_c : V_c \xrightarrow{\cong} \mathbb{B}^{p+1}$ such that $\eta^\dagger(c) = u$ (where we use the same convention for extensions of η_c as we did for $\theta =: \theta_c$), then for $c_1, c_2 \in \mathrm{Reg}(\partial\Omega)$ we have $H_{c_2} = g^*_{21} H_{c_1}$ for some $g_{21} \in G'_0$. Defining $\mathcal{H} := \{(c, \eta) : \eta \in H_c\}$, it follows readily that the canonical projection onto the first factor $\tau : \mathcal{H} \to \mathrm{Reg}(\partial\Omega)$ defines naturally on $\mathcal{H}$ the structure of a real analytic locally trivial fiber bundle with fibers isomorphic to the space of biholomorphisms from V_{b_0} to $\mathbb{B}^{p+1}$ whose continuous extension to $\overline{V_{b_0}}$ maps b_0 to u.

Defining $\mathfrak{B} := \{(x, c) \in \Omega \times S : c \in \mathrm{Reg}(\partial\Omega), x \in V_c\}$, we write $\lambda : \mathfrak{B} \to \mathrm{Reg}(\partial\Omega)$ for the projection to the second Cartesian factor of $\Omega \times S$. If for each $c \in \mathrm{Reg}(\partial\Omega)$ we choose on some neighborhood $\mathcal{O}_c$ of c in $\mathrm{Reg}(\partial\Omega)$ a real analytic section of $\tau : \mathcal{H} \to \mathrm{Reg}(\partial\Omega)$ over $\mathcal{O}_c$, then we obtain a trivialization over $\mathcal{O}_c$ of $\lambda : \mathfrak{B} \to \mathrm{Reg}(\partial\Omega)$ as $\mathbb{B}^{p+1} \times \mathcal{O}_c$, hence $\lambda : \mathfrak{B} \to \mathrm{Reg}(\partial\Omega)$ is a real analytic locally trivial fiber bundle with fibers isomorphic to $\mathbb{B}^{p+1}$. Considering the typical fiber of λ as $\mathbb{B}^{p+1} \subset \mathbb{P}^{p+1}$, we have an associated real analytic fiber bundle $\mu : \mathfrak{P} \to \mathrm{Reg}(\partial\Omega)$ with typical fiber $\cong \mathbb{P}^{p+1}$, $\mathfrak{P} \supset \mathfrak{B}$, and $\delta : \partial\mathfrak{B} \to \mathrm{Reg}(\partial\Omega)$ with typical fiber $\cong \partial\mathbb{B}^{p+1}$, $\partial\mathfrak{B}$ being the boundary of $\mathfrak{B}$ in $\mathfrak{P}$. Since the subgroup $Q \subset G'_0 = \mathrm{Aut}(\mathbb{B}^{p+1})$ fixes u, the fiber bundle $\delta : \partial\mathfrak{B} \to \mathrm{Reg}(\partial\Omega)$ admits a canonical section $\sigma : \mathrm{Reg}(\partial\Omega) \to \partial\mathfrak{B}$ defined by setting $\sigma(c)$ to be the point in $\partial(\lambda^{-1}(c))$ corresponding to the point u fixed by Q, and $\pi_2 : \mathcal{S} \to \mathrm{Reg}(\partial\Omega)$ can be realized as an open subset of $\delta : \partial\mathfrak{B} \to \mathrm{Reg}(\partial\Omega)$ by removing the the image of σ.

Note that on $[0,1]\times[0,1]$ there exists a non-measurable subset Σ such that $\Sigma\cap\{t\}\times[0,1]$ is of zero Lebesgue measure (cf. Sierpiński [32]). In order to apply Lemma 4.2 to our situation of $E\subset\mathcal{S}$, for lack of a proper reference we proceed to prove the *a priori* measurability of E and hence that it is a null set. Consider the complex unit ball $\mathbb{B}^m$, $m\geq 1$. We give a proof of the measurability of the set where admissible limits exist, not just for a single bounded holomorphic function, but for a family of such functions, as follows.

For notational convenience, in the following lemma, given a bounded measurable function h on $\mathbb{B}^m$ and $\xi\in\partial\mathbb{B}^m$, the admissible limit $\lim\{h(x): x\to\xi \text{ admissibly}\}$ will be denoted by $h^\sharp(\xi)$ in place of $h^\sharp_\xi$.

Lemma 4.3. *Let $U\subset\mathbb{B}^s$ be a connected open set and $h: U\times\mathbb{B}^m\to\mathbb{C}$ be a bounded continuous function such that $h|_{\{u\}\times\mathbb{B}^m}:\mathbb{B}^m\to\mathbb{C}$ is a holomorphic function. For $(u,x)\in U\times\mathbb{B}^m$ write $h_u(x):=h(u,x)$. Define $A:=\Big\{(u,\xi)\in U\times\partial\mathbb{B}^m: h^\sharp_u(\xi)\text{ exists}\Big\}$. Then, $A\subset U\times\partial\mathbb{B}^m$ is a measurable subset. As a consequence, $E:=(U\times\partial\mathbb{B}^m)-A$ is a null set on $U\times\partial\mathbb{B}^m$.*

Proof. We will prove the measurability of $A\subset U\times\mathbb{B}^m$ assuming only that $h: U\times\mathbb{B}^m\to\mathbb{C}$ is a bounded continuous function. Fix $\xi_0\in\partial\mathbb{B}^m$. For a continuous function f on $\mathbb{B}^m$, $\sigma\in U(m+1)$ and $\xi\in\partial\mathbb{B}^m$, f converges admissibly at $\sigma(\xi_0)$ if and only if $f\circ\sigma$ converges admissibly at ξ_0. For $\xi\in\partial\mathbb{B}^m$ choose a locally closed $(2m-1)$-dimensional smooth submanifold $\Sigma\subset U(m+1)$ diffeomorphic to a Euclidean domain such that the mapping $U(m+1)\to\partial\mathbb{B}^m$ defined by $\sigma\mapsto\sigma(\xi_0)$ maps Σ diffeomorphically onto a neighborhood of ξ on $\partial\mathbb{B}^m$. Thus, replacing h by $h':(U\times\Sigma)\times\mathbb{B}^m\to\mathbb{C}$ defined by $h'((u,\sigma),x)=h(u,\sigma(x))$, to prove the measurability of $A\in U\times\partial\mathbb{B}^m$ it is sufficient to prove that the set A' of all points where $h'_{u'}:\mathbb{B}^m\to\mathbb{C}$ converges admissibly at ξ_0, $u'=(u,\sigma)$, is a measurable subset of $U'=U\times\Sigma$. We will now change notations so that U,h,u will mean U',h',u' respectively and the problem is reduced to proving that the set $A'\subset U$ where $h_u:\mathbb{B}^m\to\mathbb{C}$ converges admissibly at ξ_0 is a measurable subset of U.

Recall that $h_u:\mathbb{B}^m\to\mathbb{C}$ converges admissibly at ξ_0 to c if and only if for any $\alpha>1$ and for any sequence of points $\{y_\ell\}_{0\leq\ell<\infty}$ on $D_\alpha(\xi_0)$ converging to ξ_0 we have $\lim\limits_{\ell\to\infty}h(u,y_\ell)=c$. Take $\xi_0=(1,0,\cdots,0)$. Let $\{x_k\}_{0\leq k<\infty}$ be a dense sequence of points on $\mathbb{B}^m$. Fix $\alpha>1$. For each positive integer n define $D^n_\alpha(\xi_0)=\big\{z=(z_1,\cdots,z_m)\in D_\alpha(\xi_0): |z_1-1|<\frac{1}{n}\big\}$. Define now $s^\beta_n: U\to\mathbb{R}$ by $s^\beta_n(u)=\sup\{|h(u,x_k)-h(u,x_\ell)|: x_k,x_\ell\in D^n_\alpha(\xi_0)\}$. When n is fixed, $s^\alpha_n(u)$ is the limit of a bounded non-decreasing sequence of nonnegative continuous functions, hence the monotone pointwise limit $s^\alpha_n(u)$ must be bounded and measurable. As $n\to\infty$, $s^\alpha_n(u)$ is non-increasing, and the monotone pointwise limit $s^\alpha(u)=\lim\limits_{n\to\infty}s^\beta_n(x)\geq 0$ exists as a bounded measurable function on U. Now $s^\alpha(u)=0$ if and only if $h(u,x_k)$ is a Cauchy sequence, i.e., if and only if $\lim\limits_{k\to\infty}h(u,x_k)=: h^\sharp(u)$ exists. Since h_u is continuous, by the density of $\{x_k\}_{0\leq k<\infty}$ in Ω it follows that for any sequence $\{y_\ell\}_{0\leq\ell<\infty}$ lying on $D_\alpha(\xi_0)$, we have $\lim\limits_{\ell\to\infty}h(u,y_\ell)=h^\sharp(u)$. Define $A'_\alpha\subset U$ to be the zero set of $s^\alpha: U\to\mathbb{R}$. It follows that $A'=\bigcap\Big\{A'_{\beta_k}\Big\}$ for any increasing sequence of real numbers β_k diverging to ∞, hence $A'\subset U$ is a measurable subset.

In the setting of the lemma we have proven that $A\subset U\times\partial\mathbb{B}^m$ is measurable for any bounded continuous function h on $U\times\mathbb{B}^m$, hence also its complement E in $U\times\mathbb{B}^m$ is measurable. When h_u is holomorphic on $\mathbb{B}^m$ for each $u\in U$, by Theorem 3.14 the set $E\cap(\{u\}\times\mathbb{B}^m)$ is of zero Lebesgue measure. Applying Fubini's theorem to the characteristic function χ_E we deduce that $E\subset U\times\mathbb{B}^m$ is of zero Lebesgue measure, proving the lemma. $\square$

Recall that $h\in H^\infty(\Omega)$ is given, and $E\subset\mathcal{S}\subset\mathrm{Reg}(\partial\Omega)\times\mathrm{Reg}(\partial\Omega)-\mathfrak{D}(\mathrm{Reg}(\partial\Omega)$ is the subset of all $(a,c)\in\mathcal{S}$ such that $h^\sharp_{a,c}$ does not exist. We will write $\mathcal{S}_{\mathrm{adm}}(h):=\{(a,c)\in\mathcal{S}\text{ such that }h^\sharp_{a,c}\text{ exists}\}=\mathcal{S}-E$.

We define $\mathfrak{s} : \mathcal{S} \to \mathcal{S}$ by $\mathfrak{s}(a, c) := (c, a)$, and for a subset $A \subset \mathcal{S}$ we write $A' = A \cap \mathfrak{s}(A)$. Observe that $A' \subset \mathcal{S}$ is of full measure whenever $A \subset \mathcal{S}$ is of full measure. In particular, $(\mathcal{S}_{\mathrm{adm}}(h))' \subset \mathcal{S}$ is of full measure. For notational convenience we will write $\mathcal{S}'_{\mathrm{adm}}(h)$ for $(\mathcal{S}_{\mathrm{adm}}(h))'$.

Define $G_{0,\mathrm{adm}}(h) \subset G_0$ to consist of all elements $g \in G_0$ such that $(g \cdot b_0, g \cdot b_0') \in \mathcal{S}_{\mathrm{adm}}(h)$. Writing $\tau : G \times \mathcal{S} \to \mathcal{S}$ for the action of G_0 on $\mathcal{S}$, and $\tau_{a,c}(g) := (g \cdot a, g \cdot c)$, then $G_{0,\mathrm{adm}}(h) = \tau^{-1}_{b_0,b_0'}(\mathcal{S}_{\mathrm{adm}}(h))$ for the reference pair of points $(b_0, b_0') \in \mathcal{S}$. Since G_0 act transitively on $\mathcal{S}$, $\tau_{b_0,b_0'} : G_0 \to \mathcal{S}$ is a locally trivial smooth fibration. Since $\mathcal{S}_{\mathrm{adm}}(h) \subset \mathcal{S}$ is of full measure, by Fubini's theorem $G_{0,\mathrm{adm}}(h) \subset G_0$ is also of full measure. We also write $G'_{0,\mathrm{adm}}(h) := \tau^{-1}_{b_0,b_0'}(\mathcal{S}'_{\mathrm{adm}}(h))$, and again $G'_{0,\mathrm{adm}}(h) \subset G_0$ is of full measure for the same reason.

4.4 Regular pairs of maximal boundary components

We say that the pair $(\Phi, \Psi) \in G_0/N \times G_0/N$ is a regular pair (of maximal boundary components) if and only if the condition (%) after Proposition 3.12 is satisfied, equivalently, that for some $c_0 \in \Psi$, $\partial^\flat V_{c_0} \cap \Phi \neq \emptyset$. By Lemma 3.11, this happens if and only if there exists $g \in G_0$ such that $g\Phi = \Sigma$ and $g\Psi = \Sigma'$, implying that (Φ, Ψ) is a regular pair if and only if (Ψ, Φ) is a regular pair. Moreover, (Φ, Ψ) is a regular pair if and only if for any point $c \in \Psi$, $\partial^\flat V_c \cap \Phi = \{a(c)\}$ for a unique point $a(c) \in \Phi$. Denote by

$$\mathscr{C} \subset G_0/N \times G_0/N$$

the set of regular pairs (Φ, Ψ) of maximal boundary components, and write $\mathscr{E} := (G_0/N \times G_0/N) - \mathscr{C}$. In what follows we will refer to a maximal boundary component Φ as $[\Phi]$ when it is necessary to think of the maximal boundary component as a point (an element) in the moduli space G_0/N of such objects, and when we adopt set-theoretical notations such as in the statement "$[\Phi] \in G_0/N$". The same notational convention has been and will be applied to objects in moduli spaces that we encountered throughout the article. We have

Lemma 4.4. *G_0 acts transitively on $\mathscr{C}$ by $\alpha(g, ([\Phi], [\Psi])) = ([g\Phi], [g\Psi])$. Moreover, a pair (Φ, Ψ) of maximal boundary components on $\partial\Omega$ is a regular pair if and only if $\overline{\Phi} \cap \overline{\Psi} = \emptyset$. As a consequence, $\mathscr{E} \subsetneq G_0/N \times G_0/N$ is a proper (real) algebraic subset. In particular $\mathscr{C}$ is an open (and dense) subset in $G_0/N \times G_0/N$ of full measure.*

Proof. (Φ, Ψ) is a regular pair of maximal boundary components on $\partial\Omega$ if and only if there exists $g \in G_0$ such that $(g\Phi, g\Psi) = (\Sigma, \Sigma')$, hence by definition G_0 acts transitively on $\mathscr{C}$. Moreover, for a pair (Φ, Ψ) we have $\overline{\Phi} \cap \overline{\Psi} = g^{-1}(\overline{\Sigma} \cap \overline{\Sigma'}) = \emptyset$.

Conversely, suppose $\overline{\Phi} \cap \overline{\Psi} = \emptyset$. Let $c \in \Psi$ and denote by $\{x_k\}_{0 \le k < \infty}$ a sequence of points on Ω such that $x_k \to c$ as $k \to \infty$. By Proposition 3.4 there is a unique point $a_k \in \Phi$ such that x_k and a_k lie on some minimal rational curve ℓ_k on the Hermitian symmetric space S dual to Ω, $\Omega \subset S$ being the Borel embedding. Passing to a subsequence if necessary we may assume that $x_k \to c$, $a_k \to a \in \overline{\Phi}$ and ℓ_k converges in $\mathcal{K}$ to some minimal rational curve ℓ passing through c and a.

Suppose $\ell \cap \Omega \neq \emptyset$. Pick $x \in \ell \cap \Omega$. Noting that for any minimal rational curve Λ passing through 0, $\Lambda \cap \partial\Omega \subset \mathrm{Reg}(\partial\Omega)$, as can be seen from $\Lambda \cap \Omega \subset \mathbb{B}^n(0; 1)$, and noting also that G_0 acts transitively on Ω, we conclude that $\ell \cap \Omega \neq \emptyset$ implies

$$a \in \overline{\Phi} \cap \mathrm{Reg}(\partial\Omega) = \Phi,$$

so that (Φ, Ψ) belongs to $\mathscr{C}$. If however $\ell \cap \Omega = \emptyset$, then the germ of ℓ at c must lie on $\mathrm{Reg}(\partial\Omega)$. Since the germ of any holomorphic curve through c lying on $\mathrm{Reg}(\partial\Omega)$ must lie on the maximal boundary component passing through c, hence $\ell \cap \overline{\Omega}$ must be a closed disk lying on $\overline{\Phi}$, and it follows that $a \in \overline{\Phi} \cap \overline{\Psi}$, contradicting with the hypothesis $\overline{\Phi} \cap \overline{\Psi} = \emptyset$, and we conclude that $\overline{\Phi} \cap \overline{\Psi} = \emptyset$ implies (Φ, Ψ) is a regular pair of maximal boundary components.

For the last statement, with respect to the Borel embedding $\Omega \subset S$ into the compact dual S of Ω, write $S_\Phi \supset \Phi$ resp. $S_\Psi \supset \Phi$ for the Zariski closure of Φ resp. Ψ in S. Then, $\Phi \subset S_\Phi$ and $\Psi \subset S_\Psi$ are Borel embeddings. If (Φ, Ψ) is a regular pair, then $S_\Phi \cap S_\Psi = \emptyset$ as can be seen in the case $(\Phi, \Psi) = (\Sigma, \Sigma')$, given the embedding $\mathbb{P}^1 \times S_\Sigma \subset S$, $\Delta \times \Omega \subset \mathbb{P}^1 \times S_\Sigma$ being the Borel embedding. Denote by $\mathcal{Q}$ the irreducible component of the Chow space $\mathrm{Chow}(S)$ to which the cycle $[S_\Sigma]$ belongs. Then we have the embedding $\beta : G_0/N \hookrightarrow \mathcal{Q}$ as a closed (real) algebraic submanifold defined by $\beta([\Phi]) := [S_\Phi] \in \mathcal{Q}$. Let $\mathscr{B} \subset \mathcal{Q} \times \mathcal{Q}$ be the subset consisting of pairs of cycles (W, Z) such that $W \cap Z \neq \emptyset$. Then, $([W], [Z]) \in \mathscr{B}$ if and only if there exists a point in $(W \times Z) \cap \mathfrak{D}(\mathcal{Q})$, hence by the proper mapping theorem and the fact that $\mathscr{C}$ is nonempty, $\mathscr{B} \subset \mathcal{Q} \times \mathcal{Q}$ is a proper algebraic subset, and $\mathscr{E} = \beta^{-1}(\beta(G_0/N \times G_0/N) \cap \mathscr{B})$ is a proper algebraic subset of $G_0/N \times G_0/N$, hence $\mathscr{E} \subset G_0/N \times G_0/N$ is a closed null set, and $\mathscr{C} \subset G_0/N \times G_0/N$ is an open subset of full measure, hence also a dense subset, as desired. The proof of Lemma 4.4 is complete. $\square$

4.5 Special product subspaces equipped with distinguished sections

Definition 4.5. Write $P_0 = \nu(\Delta \times \Omega') \subset \Omega$. A submanifold $P \subset \Omega$ is called a special product subspace in Ω if and only if $P = gP_0$ for some $g \in G_0$.

Since G_0 acts transitively on $\mathscr{C}$, given any pair (Φ, Ψ) belonging to $\mathscr{C}$ there exists $g \in G_0$ such that $g\Sigma = \Phi$ and $g\Sigma' = \Psi$.

Write $\partial^\dagger P_0 := \nu(\partial\Delta \times \Omega') \subset \partial P_0$, and for $P = gP_0$ write $\partial^\dagger P := g(\partial^\dagger P_0)$. We also write $P = P(\Phi, \Psi)$ for the special product subspace of Ω corresponding to $([\Phi], [\Psi]) \in \mathscr{C}$. Considering the case where $(\Phi, \Psi) = (\Sigma, \Sigma')$, we have

Lemma 4.6. *Let $u \in G_0$ be such that $u\Sigma = \Sigma$ and $u\Sigma' = \Sigma'$. Then $uP_0 = P_0$.*

Proof. For any point $b(w) = (1; 0; w) \in \Sigma$, its opposite point with respect to 0 is $b'(w) = (-1; 0; w)$, and we have $\partial^\flat V_{b(w)} \cap \Sigma' = b'(w)$, $\partial^\flat V_{b'(w)} \cap \Sigma = b(w)$. For $w \in \Omega'$ denote by $\Lambda(w) = (-1, 1) \times \{0\} \times \{w\}$ the unique geodesic curve lying on a minimal disk having boundary points $b(w)$ and $b'(w)$. For $w \in \Omega'$, $g(D(w))$ must be a minimal disk having $u(b(w)) =: b(\widetilde{w}_1)$ and $u(b'(w)) = b'(\widetilde{w}_2)$ in its closure. Thus, $b'(\widetilde{w}_2) \in \partial^\flat V_{b(\widetilde{w}_1)}$ and we must have $\widetilde{w}_2 = \widetilde{w}_1 =: \widetilde{w}$, hence $u(\Lambda(w)) = \Lambda(\widetilde{w})$ and by complexification $u(D(w)) = u(D(\widetilde{w}))$ so that $u(P_0) = P_0$, as desired. $\square$

Suppose $g_i\Sigma = \Phi$ and $g_i\Sigma' = \Psi$ for $i = 1, 2$. Then $g_2 = g_1 u$ for some $u \in G_0$ such that u restricts to an automorphism on P_0, and such that $u(1; 0; w) = (1; 0; \widetilde{w})$ and $u(-1; 0; w) = (-1; 0; \widetilde{w})$, where $\widetilde{w} = \varphi(w)$ for some $\varphi \in \mathrm{Aut}(\Omega')$. Any automorphism σ of P_0 satisfying $\sigma(1, w) = (1, \varphi(w))$ and $\sigma(-1, w) = (-1, \varphi(w))$ must be of the form $\sigma(z_1; w) = (\sigma_1(z_1), \sigma_2(w))$, and it follows that $u(z_1; 0; w) = (\eta_t(z_1), 0; \varphi(w))$ for the transvection $\eta_t(z_1) = \frac{z_1 + t}{1 + tz_1}$ for some $t \in (-1, 1)$.

For the reference pair of regular maximal components (Σ, Σ'), and the associated special product subspace $P_0 = \nu(\Delta \times \Omega')$, noting that $\Sigma, \Sigma' \subset \partial\Delta \times \Omega \subset \partial^\dagger P_0$, we define an accompanying subset $\mathcal{S}_{P_0;\Omega'_0} := \{(b_\theta(w), b'_\theta(w)) : \theta \in \mathbb{R}, w \in \Omega'\} \subset \mathcal{S}$. The natural projection $\varepsilon : \mathcal{S}_{P_0;\Omega'_0} \to \mathfrak{D}(\Omega')$ realizes $\mathcal{S}_{P_0;\Omega'_0}$ as a trivial S^1-bundle, and we may regard $S_{P_0;\Omega'_0}$ as a Cartesian product $S_0 \times \mathfrak{D}(\Omega')$ where $S_0 \subset \partial\Delta \times \partial\Delta$ is defined as $S_0 = \{(e^{i\theta}, -e^{i\theta}) : \theta \in \mathbb{R}\}$.

In general, given any pair (P, Z) consisting of a special product subspace $P \subset \Omega$ and a distinguished section $Z \subset P$, there exists $g \in G_0$ such that $gP_0 = P$ and $g\Omega'_0 = Z$. Now, if the latter equalities are satisfied for both $g = g_1$ and $g = g_2$, then $g_2 = g_1 u$ for some group element $u \in G_0$ such that $uP_0 = uP_0$ and $u\Omega'_0 = \Omega'_0$. Clearly, any such element $u \in G_0$ must satisfy $u(z_1; 0; w) = (e^{i\theta} z_1; 0; \varphi(w))$ for some $\theta \in \mathbb{R}$ and some $\varphi \in \mathrm{Aut}(\Omega') =: G'_0$. Since $\mathcal{S}_{P_0,\Omega'_0}$ is invariant under any such group element $u \in S^1 \times G'_0$, the subset $\mathcal{S}_{P,Z} := g\mathcal{S}_{P_0,\Omega'_0} = \{(g \cdot x, g \cdot y) : (x, y) \in \mathcal{S}_{P_0,\Omega'_0}\}$ is independent of the particular choice of $g \in G_0$ satisfying $gP_0 = P$ and $g\Omega'_0 = Z$. This gives a uniquely defined subset $\mathcal{S}_{P,Z} \subset \mathcal{S}$. The latter subset will now be used to prove the following lemma.

Lemma 4.7. *Fix $h \in H^\infty(\Omega)$. Then, for almost all $g \in G_0$ the following holds true. For almost all $\theta \in \mathbb{R}$, both $(g\Sigma(\theta), g\Sigma'(\theta))$ and $(g\Sigma'(\theta), g\Sigma(\theta))$ are regular pairs of h-admissible maximal boundary components.*

Proof. Let $\mathcal{M}$ be the moduli space of all special product subspaces $P \subset \Omega$. Fix the reference special product subspace $P_0 = P(\Sigma, \Sigma')$, and denote by $R \subset G_0$ the stabilizer subgroup of P_0. Then, $\mathcal{M} = G_0/R$ as a homogeneous space. Let $\widetilde{\mathcal{M}}$ be the moduli space of pairs (P, Z), where $Z \subset P$ is a distinguished section, and denote by $\widetilde{R} \subset R \subset G_0$ the subgroup of all P_0-stabilizing automorphisms fixing $\Omega'_0 \subset P_0$ (as a set). We have $\widetilde{R} = S^1 \times G'_0$.

Writing $\mathcal{S}_{P_0,\Omega'_0} = \{((e^{i\theta}; 0; w), (-e^{i\theta}; 0; w) : \theta \in \mathbb{R}, w \in \Omega'\}$. When $(P, Z) = (gP_0, g\Omega'_0)$, we define $\mathcal{S}_{P,Z} := \{(g \cdot b, g \cdot b') : (b, b') \in \mathcal{S}_{P_0,\Omega'_0}$. If $g' \in G_0$ also satisfies $(P, Z) = (g'P_0, g'\Omega'_0)$, then $g' = gu$ for some $u \in G_0$ preserving P_0 such that $u\Omega'_0 = \Omega'_0$, implying that $u(z_1; 0; w) = (e^{i\theta}z_1; 0; \varphi(w))$ for some $\theta \in \mathbb{R}$ and some $\varphi \in G'_0$, and it follows readily that u preserves $\mathcal{S}_{P_0,\Omega'_0}$. It follows that $\mathcal{S}_{P,Z}$ is well defined independent of the choice of $g \in G_0$ satisfying $(P, Z) = (gP_0, g\Omega'_0)$

To each (P, Z) belonging to $\widetilde{\mathcal{M}}$ we associate now the subset $\mathcal{S}_{P,Z} \subset \mathcal{S}$ as defined. Since $g \in G_0$ fixes both P_0 and Ω'_0 if and only if it fixes $\mathcal{S}_{P_0,\Omega'_0}$, $\widetilde{\mathcal{M}}$ also serves as the moduli space of $\mathcal{S}_{P,Z} \subset \mathcal{S}$.

Consider the universal family $\chi : \mathscr{P} \to \widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{M}}$, $\mathscr{P} \subset \mathcal{S} \times \widetilde{\mathcal{M}}$, as a moduli space in the latter sense, and denote by $\mu : \mathscr{P} \to \mathcal{S}$ the associated evaluation map. We apply Lemma 4.2 to the total space $\mathscr{P}$ of a double fibration. Define $\mathscr{P}'_{\mathrm{adm}}(h) := \mu^{-1}(\mathcal{S}'_{\mathrm{adm}}(h))$. By Fubini's theorem, $\mathscr{P}'_{\mathrm{adm}}(h)$ is of full measure in $\mathscr{P}$. By Fubini's theorem again for almost all $\mathcal{S}_{P,Z}$ belonging to $\widetilde{\mathcal{M}}$, $\mathcal{S}_{P,Z} \cap \mathscr{P}'_{\mathrm{adm}}(h)$ is of full measure in $\mathcal{S}_{P,Z}$. Write $\widetilde{\mathcal{M}}'_{\mathrm{adm}}(h \subset \widetilde{\mathcal{M}}$ for the subset consisting of all $[\mathcal{S}_{P,Z}] \in \widetilde{\mathcal{M}}$ such that $\chi^{-1}([S_{P,Z}]) \cap \mathscr{P}'_{\mathrm{adm}}(h)$ is of full measure. Then, $\widetilde{\mathcal{M}}'_{\mathrm{adm}}(h) \subset \widetilde{\mathcal{M}}$ is of full measure. For each member $[S_{P,Z}] \in \widetilde{\mathcal{M}}$, $\mathcal{S}_{P,Z} \cap \mathcal{S}'_{\mathrm{adm}}(h)$ is of full measure in $\mathcal{S}_{P,Z}$ if and only if $[S_{P,Z}] \in \widetilde{\mathcal{M}}'_{\mathrm{adm}}(h)$.

Finally $\widetilde{R} = S^1 \times G'_0$ acts transitively on $\mathcal{S}_{P_0,\Omega'_0}$, G_0 acts transitively on $\mathscr{P}$. Consider the reference point $(b_0, b'_0) \in \mathcal{S}_{P_0,\Omega'_0}$. Suppose $u \in G_0$ fixes each of P_0, $\Omega'_0 \subset P_0$ and $(b, b') \in \mathcal{S}_{P_0,\Omega'_0}$, then $u(z_1; 0; w) = (e^{i\theta_0}z_1; 0; u'(w)) \in \widetilde{R} = S^1 \times G'_0$ for some $\theta_0 \in \mathbb{R}$ and $u' \in G'_0$, and it satisfies $u(b_0) = b_0$ and $u(b'_0) = b'_0$, forcing $e^{i\theta_0} = 1$. It follows that $\mathscr{P} = G_0/K'$, where K' is the isotropy subgroup of $G'_0 \cong \mathrm{Aut}(\Omega')$ at $0 \in \Omega'_0$.

Denote by $\omega : G_0 \to G_0/K'$ the canonical projection, and write $G^\sharp_{0,\mathrm{adm}}(h) := \omega^{-1}(\mathscr{P}'_{\mathrm{adm}}(h))$. Then, by Fubini's Theorem $G^\sharp_{0,\mathrm{adm}}(h) \subset G_0$ is of full measure, as desired. The proof of Lemma 4.7 is complete. □

Remark 4.8. We could define $\mathscr{P}_{\mathrm{adm}}(h) = \mu^{-1}(\mathcal{S}_{\mathrm{adm}}(h))$ and define $\mathcal{M}_{\mathrm{adm}}(h)$ by replacing $\mathscr{P}'_{\mathrm{adm}}(h)$ in the definition $\mathcal{M}'_{\mathrm{adm}}(h)$ by $\mathscr{P}_{\mathrm{adm}}(h)$, but only the symmetrized versions $\mathscr{P}'_{\mathrm{adm}}(h)$ and $\mathcal{M}'_{\mathrm{adm}}(h)$ are needed in the article.

Recall that given $([\Phi], [\Psi]) \in \mathscr{C}$ we have the special product subspace $P(\Phi, \Psi)$ consisting of minimal disks $D(w)$ parametrized by $w \in \Omega'$ and the subset $\Lambda(\Phi, \Psi) \subset P(\Phi, \Psi)$ consisting of geodesic curves $\Lambda(w)$ on $D(w)$ with the two limit points lying on Φ resp. Ψ. Consider triples (Φ, Ψ, Z) where (Φ, Ψ) is a regular pair of maximal boundary components and Z is a distinguished section of $P(\Phi, \Psi)$ lying on $\Lambda(\Phi, \Psi)$. The moduli space $\widetilde{\mathscr{C}}$ of triples (Φ, Ψ, Z) is related to $\widetilde{\mathcal{M}}$, as follows. For each triple (Φ, Ψ, Z) belonging to $\widetilde{\mathscr{C}}$ we can associate the pair $(P(\Phi, \Psi), Z)$ belonging to $\widetilde{\mathcal{M}}$, written $([P(\Phi, \Psi)], [Z]) = \alpha([\Phi], [\Psi], [Z])$. Writing $(\Phi, \Psi, Z) = (g\Sigma, g\Sigma', g\Omega'_0)$, we have an S^1-family of triples $(g\Sigma(\theta), g\Sigma'(\theta), g\Omega'_0)$, $\theta \in \mathbb{R}$, and we have $\alpha([\Phi'], [\Psi'], [Z']) = ([P(\Phi, \Psi)], [Z])$ if and only if $(\Phi', \Psi', Z') = (g\Sigma(\theta), g\Sigma'(\theta), g\Omega'_0)$ for some $\theta \in \mathbb{R}$. Thus, we have a natural map $\alpha : \widetilde{\mathscr{C}} \to \widetilde{\mathcal{M}}$ realizing $\widetilde{\mathscr{C}}$ as a circle bundle over $\widetilde{\mathcal{M}}$.

G_0 acts transitively on $\widetilde{\mathscr{C}}$ since the 1-parameter group of transvections $\psi_t(z) = \frac{z+t}{1+tz}$, $t \in (-1, 1)$ on Δ, embedded as a subgroup $H \subset G_0$, acts transitively on $\nu((-1, 1) \times \{0\})$ preserving P_0. Moreover,

$(g\Sigma, g\Sigma', g\Omega'_0) = (\Sigma, \Sigma', \Omega')$ if and only $g \in G'_0$, hence $\widetilde{\mathscr{C}} = G_0/G'_0$ as a homogeneous space.

To each triple (Φ, Ψ, Z) belonging to $\widetilde{\mathscr{C}}$ we define $\mathcal{S}_{\Phi,\Psi,Z} := \mathcal{S}_{\beta([\Phi],[\Psi],[Z])} = \mathcal{S}_{P(\Phi,\Psi),Z}$. From Lemma 4.7 we have readily

Lemma 4.9. *Fix $h \in H^\infty(\Omega)$. Then, for almost every $g \in G_0$, $\mathcal{S}_{g\Sigma,g\Sigma',g\Omega'} \cap \mathcal{S}'_{\mathrm{adm}}(h) \subset \mathcal{S}_{g\Sigma,g\Sigma',g\Omega'}$ is of full measure.*

We adopt the following terminology concerning h-admissibility.

Definition 4.10. Fix $h \in \mathscr{F}$. A regular pair (Φ, Ψ) of maximal boundary components is said to be h-admissible, written $([\Phi], [\Psi]) \in \mathscr{C}_{\mathrm{adm}}(h)$ if and only if for almost all $c \in \Psi$, and writing $\partial^\flat V_c \cap \Phi =: \{a(c)\}$, the admissible boundary value $h^\sharp_{a(c),c}$ exists. For a triple (Φ, Ψ, Z) consisting of $([\Phi], [\Psi]) \in \mathscr{C}$ and a distinguished section Z of the special product subspace $P(\Phi, \Psi)$ such that $Z \subset \Lambda(\Phi, \Psi)$, we say that $([\Phi], [\Psi], [Z]) \in \widetilde{\mathscr{C}}$ is h-admissible, written $([\Phi], [\Psi], [Z]) \in \widetilde{\mathscr{C}}_{\mathrm{adm}}(h)$ if and only if $S_{\Phi,\Psi,Z} \cap S'_{\mathrm{adm}}(h) \subset S_{\Phi,\Psi,Z}$ is of full measure.

We denote by $\mathfrak{S} : \mathscr{C} \to \mathscr{C}$ by $\mathfrak{S}([\Phi], [\Psi]) := ([\Psi], [\Phi])$, and, for a subset $\mathcal{A} \subset \mathscr{C}$ we also write $\mathcal{A}' = \mathcal{A} \cap \mathfrak{S}(\mathcal{A})$. Writing $\mathscr{C}'_{\mathrm{adm}}(h)$ for $(\mathscr{C}_{\mathrm{adm}}(h))'$, we have $([\Phi], [\Psi]) \in \mathscr{C}'_{\mathrm{adm}}(h)$ if and only if both (Φ, Ψ) and (Ψ, Φ) are h-admissible, i.e., if and only if both $h^\sharp_{a(c),c}$ and $h^\sharp_{c,a(c)}$ exist for almost all $c \in \Psi$.

Concerning h-admissible triples, we have

Lemma 4.11. *Fix $h \in H^\infty(\Omega)$. Then, for almost every $g \in G_0$, the triple $(g\Sigma, g\Sigma', g\Omega'_0)$ is h-admissible.*

Proof. Given $g \in G_0$, $S_{gP_0,gZ} \cap \mathcal{S}'_{\mathrm{adm}}(h) \subset S_{gP_0,gZ}$ is of full measure if and only if $g \in G^\sharp_{0,\mathrm{adm}}$. But now $\mathcal{S}_{g\Sigma,g\Sigma',g\Omega'_0} = \mathcal{S}_{gP_0,g\Omega'_0}$, hence the triple $(g\Sigma, g\Sigma', g\Omega'_0)$ is h-admissible if and only if $g \in G^\sharp_{0,\mathrm{adm}}(h)$, and the proof of the lemma is completed by recalling from the proof of Lemma 4.7 that $G^\sharp_{0,\mathrm{adm}} \subset G_0$ is of full measure. □

Note that the statements $([\Phi], [\Psi], [Z]) \in \widetilde{\mathscr{C}}_{\mathrm{adm}}(h)$ and $([\Phi], [\Psi]) \in \mathscr{C}_{\mathrm{adm}}(h)$ are not related to each other in any obvious way.

5 Averaging Arguments for Holomorphic Maps

In this section, we discuss averaging arguments on vector-valued bounded holomorphic maps, which will be applied in the proof of Theorem 1.1 to the linearization of properly chosen Cayley projections of bounded holomorphic maps.

5.1 A theorem of H. Cartan

The following statement is a classical result on holomorphic maps defined on bounded circular domains due to H. Cartan (cf. [3]).

Theorem 5.1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded complete circular domain. Suppose $F : \Omega_1 \to \mathbb{C}^m$ is a holomorphic map. Then*

$$F(z) = e^{-i\theta} F(e^{i\theta} z), \quad \forall e^{i\theta} \in S^1, \forall z \in \Omega_1,$$

if and only if F is a linear transformation $\mathbb{C}^n \to \mathbb{C}^m$.

Proof. Write $F = (F^1, \cdots, F^m)$. For $1 \le k \le m$, expanding both sides of $F^k(z) = e^{-i\theta} F^k(e^{i\theta} z)$ as Taylor series at $0 \in \Omega$, grouping all monomials of the same total degree together, and comparing term by term after the regrouping, we obtain from the identity theorem for holomorphic functions that F is a linear transformation, as desired. □

5.2 Linearization of holomorphic maps on complete circular domains

We are going to construct S^1-equivariant maps from given holomorphic maps by taking averages with respect to a natural S^1-action and obtain linear transformations as a consequence of Theorem 5.1, as follows.

Lemma 5.2. *Suppose $S : \Omega \to \mathbb{C}^m$ is a holomorphic map defined on a complete circular domain Ω. Define*

$$\widetilde{S}(z) := \int_{-\pi}^{\pi} e^{i\theta} S(e^{-i\theta} z) \frac{d\theta}{2\pi}.$$

Then, $\widetilde{S}$ is S^1-equivariant in the sense that $\widetilde{S}(e^{i\theta} z) = e^{i\theta}\widetilde{S}(z)$ for all $\theta \in \mathbb{R}$. Hence, $\widetilde{S}$ is a linear transformation.

5.3 K-equivariant holomorphic maps on $\Omega = G/K$

Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain in its Harish-Chandra realization. We have $\Omega = G_0/K$ as a homogeneous space, where $G_0 = \mathrm{Aut}_0(\Omega)$, $K \subset G_0$ being the isotropy subgroup at $0 \in \Omega$. K is a closed subgroup of the unitary group $\mathrm{U}(n)$, and as such it acts on $\mathbb{C}^n$ as a group of unitary transformations. The set of all maximal boundary components of Ω, i.e., all boundary components of rank $r-1$ form an orbit $\mathrm{Reg}(\partial\Omega) = E_{r-1} = G_0(c_\Gamma x_0)$ for some partial Cayley transform c_Γ, cf. §2. Geometrically the orbit $\mathrm{Reg}(\partial\Omega)$ is a disjoint union of boundary components which are biholomorphically isomorphic bounded symmetric domains. Let $\Phi \subset \partial\Omega$ be an arbitrarily chosen maximal boundary component and write $\mathrm{Reg}(\partial\Omega) = \coprod_{k\in K} k\Phi$. (Recall that each boundary component is of the form $\xi^{-1}kc_{\Psi-\Gamma}X_{\Gamma,0} = ad(k)\xi^{-1}c_{\Psi-\Gamma}X_{\Gamma,0}$). Recall that G_0 acts transitively on the set of all maximal boundary components, which is in one-to-one correspondence with G_0/N for some parabolic subgroup $N \subset G_0$. For $k \in K$, let $\Lambda : \Omega \to \mathbb{C}^n$ be a bounded holomorphic map. Let $(k\Lambda) : \Omega \to k\Phi$ be defined by $(k\Lambda)(z) := k(\Lambda(z))$ for each $z \in \Omega$, recalling that K acts on $\mathbb{C}^n$ as a group of linear transformations. We have

Lemma 5.3. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain in its Harish-Chandra realization. Let $\Lambda : \Omega \to \mathbb{C}^n$ be a bounded holomorphic map. Take $d\mu(k)$ to be the Haar measure on K of unit volume and define*

$$\widetilde{\Lambda}(z) := \int_K k\Lambda(k^{-1}z) d\mu(k).$$

Then $\widetilde{\Lambda} : \Omega \to \mathbb{C}^n$ is either the zero map or a nonzero constant multiple of the identity map id_Ω.

Proof. For any element $k_0 \in K$ we have

$$\begin{aligned}\widetilde{\Lambda}(k_0 z) := \int_K k\Lambda(k^{-1}(k_0 z)) d\mu(k) &= \int_K k_0(k_0^{-1}k)\Lambda((k_0^{-1}k)^{-1}(z)) d\mu(k) \\ &= k_0 \int_K k\Lambda(k^{-1}(z)) d\mu(k) = k_0\big(\widetilde{\Lambda}(z)\big)\end{aligned} \tag{10}$$

so that $\widetilde{\Lambda} : \Omega \to \mathbb{C}^n$ is a K-equivariant holomorphic map between domains in $\mathbb{C}^n$. Because the center of K is S^1, $K = (S^1 \times K_s)/Z$, where K_s is the semisimple part of K, $K_s = \exp(\mathfrak{k}_s)$, $\mathfrak{k}_s := [\mathfrak{k}, \mathfrak{k}]$, and $Z \subset S^1 \times K_s$ is a finite group. By Fubini's Theorem $\widetilde{\Lambda}$ can be performed by first integrating over S^1 and then over K_s. By Lemma 5.2, Λ is the restriction to Ω of a linear transformation on $\mathbb{C}^n$. Moreover, since the isotropy subgroup K acts irreducibly on $T_0(\Omega) \cong \mathbb{C}^n$, by the Schur lemma, the map $\widetilde{\Lambda}$ can either be 0 or a nonzero constant multiple of the identity map. □

For our purpose, we need to make sure that such a K-averaging process produces a non-zero map. Another technical issue in the proof of Theorem 1.1 involving the averaging argument is that such an averaging process may *a priori* produce functions outside the algebra under consideration. These problems can be addressed by some density arguments, cf. [21, p.26 Lemma 3], which will be taken up in the next section in the key arguments come from Moore's ergodicity for lattices on semisimple groups and on density arguments resulting therefrom.

6 The Solution to the Extension Problem

Let X be a smooth manifold, and G be a Lie group. Suppose μ is a smooth volume form on X and G acts on X by a smooth group action $\alpha : G \times X \to X$. We say that G acts ergodically on (X, μ) if and only if for any G-invariant measurable subset $A \subset X$ we have either $\mu(A) = 0$ or $\mu(X - A) = 0$. Note that we do not require that μ is invariant under the action of G, but clearly for any $g \in G$, $\mu(A) = 0$ if and only if $\mu(gA) = 0$ since G necessarily acts on X as a group of diffeomorphisms on X given by $g \cdot x = \alpha(g, x)$.

6.1 Moore's ergodicity theorem and its applications

For the proof of Theorem 1.1, we need Moore's ergodicity theorem in the following form (cf. [37, Theorem 2.2.6],).

Theorem 6.1. *Let $G = \prod G_i$ be a connected semisimple real Lie group, where each G_i is a connected non-compact simple Lie group with finite center. Let $\Gamma \subset G$ be an irreducible lattice. Let $H \subset G$ be a noncompact closed subgroup. Then, H acts ergodically on $\Gamma \backslash G$.*

We need the following lemma (cf. Zimmer [37, Proposition 2.1.7]]) regarding H-orbits in $\Gamma \backslash G$.

Lemma 6.2. *In the notation of Theorem 6.1, there exists a subset $E \subset \Gamma \backslash G$ of zero measure such that the H-orbit $xH \subset \Gamma \backslash G$ is dense in $\Gamma \backslash G$ whenever $x \in (\Gamma \backslash G) - E$.*

Returning to our situation of a bounded symmetric domain $\Omega \Subset \mathbb{C}^n$, we denote by $\varpi : \mathrm{Aut}_0(\Delta^r) \to \mathrm{Aut}_0(\Omega) = G_0$ the canonical group homomorphism. We have

Lemma 6.3. *Let Ω be a bounded symmetric domain of rank $r \geq 2$ and $\Gamma \subset \mathrm{Aut}_0(\Omega) = G_0$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let $P \subset \Omega$ be a maximal polydisk of Ω, which induces a canonical embedding $\mathrm{Aut}(\Delta^r) \hookrightarrow \mathrm{Aut}_0(\Omega)$. Write $\eta_t(z) = \frac{z+t}{1+tz}$ so that, writing $\theta_t = \varpi(id_{\Delta^{r-1}}, \eta_t)$, $H := \{\theta_t : -1 < t < 1\} \hookrightarrow \mathrm{Aut}_0(\Omega) = G_0$ is a noncompact 1-parameter closed subgroup of transvections. Suppose $g \in G_0$ and the coset $\Gamma g H$ is dense in $\Gamma \backslash G_0$. Then, there exists a discrete sequence $\{\gamma_k\} \subset \Gamma$ such that $\gamma_k = \left(g\theta_{t_k} g^{-1}\right) \delta_k$ for some $\delta_k \in \mathrm{Aut}_0(\Omega)$ and $t_k \in (-1, 1)$ satisfying $\delta_k \to id_{G_0}$ and either $t_k \to 1$ or $t_k \to -1$.*

Proof. We assume first of all that $\Omega = G_0/K$ is an irreducible bounded symmetric domain.

By the hypothesis $\Gamma g H$ is dense in $\Gamma \backslash G_0$. For two subsets $A, B \subset G_0$ we write $A \cdot B := \{ab \in G_0 : a \in A, b \in B\}$. Then $\Gamma g H$ is dense in $\Gamma \backslash G_0$ if and only if $\Gamma \cdot g \cdot H$ is dense in G_0. We will not make a clumsy notational distinction between $\Gamma g H \subset \Gamma \backslash G_0$ as a subset of right cosets and $\Gamma \cdot g \cdot H$ as a subset of G_0, writing both as $\Gamma g H$. (In the text of the proof, the latter will actually mean a subset of G_0.)

Let $\widetilde{g} \in G_0$. Then, there exists $\mu_\ell \in \Gamma$ and $\epsilon_\ell \in G_0$ satisfying $\epsilon_\ell \to id_{G_0}$ as $\ell \to \infty$ such that

$$\mu_\ell g \theta_{s_\ell} = \epsilon_\ell \widetilde{g} \tag{11}$$

Taking inverses we have

$$\theta_{-s_\ell} g^{-1} \nu_\ell = \widetilde{g}^{-1} \epsilon'_\ell; \quad \text{hence} \quad \widetilde{g} \theta_{-s_\ell} g^{-1} \nu_\ell = \epsilon'_\ell \,, \tag{12}$$

where $\nu_\ell := (\mu_\ell)^{-1}$ and $\epsilon'_\ell := (\epsilon_\ell)^{-1}$, and note that $\theta_t^{-1} = \theta_{-t}$ for $t \in (-1,1)$. Let $k > 0$ be an integer. Choose now $\widetilde{g}_k$ such that $\widetilde{g}_k \notin \Gamma g H$. For each positive integer k we have in analogy to (12)

$$\mu_{\ell,k} g \theta_{s_{\ell,k}} = \epsilon_{\ell,k} \widetilde{g}_k \,, \tag{13}$$

where we write $\mu_{\ell,k}$ to indicate its dependence on k, etc. Fixing k we have $\epsilon_{\ell,k} \to id_{G_0}$ as $\ell \to \infty$. Now we have

$$(g\theta_{-s_{\ell,k}} g^{-1})\nu_{\ell,k} = (g\widetilde{g}_k^{-1})(\widetilde{g}_k \theta_{-s_{\ell,k}} g^{-1})\nu_{k,\ell} = (g\widetilde{g}_k^{-1})\epsilon'_{k,\ell} \,, \tag{14}$$

Choose a sequence $\widetilde{g}_k$ on G_0 such that $\widetilde{g}_k \notin \Gamma g H$ and such that $\widetilde{g}_k \to g$ as $k \to \infty$. Let $\alpha : (-1,1) \to (-\infty, \infty)$ be a strictly increasing diffeomorphism such that, writing $\theta_t = \sigma_{\alpha(t)}$, $\{\sigma_{t'} : -\infty < t' < \infty\}$ gives a re-parametrization of H so that $\sigma : \mathbb{R} \to H$ is a group homomorphism, i.e., $\sigma_0 = id_\Omega$, $\sigma_{a+b} = \sigma_a \sigma_b$ for $a, b \in \mathbb{R}$. Define $\mathbf{I}_k := \{\sigma_{t'} : t' \in [-k,k]\}$For each $\gamma \in \Gamma$, $\gamma g \cdot \mathbf{I}_k$ is a closed subset of G_0. Since $\Gamma \subset G_0$ is discrete, $\Gamma g \cdot \mathbf{I}_k \subset G_0$ is a closed subset. Hence, with respect to an auxiliary Riemannian metric on G_0, $\widetilde{g}_k$ is at a positive distance from $\Gamma g \cdot \mathbf{I}_k$. Hence there exists a positive integer $\ell(k)$ such that $\epsilon_{\ell(k),k}$ is close enough to id_{G_0} so that $\epsilon_{\ell(k),k}\widetilde{g}_k \notin \Gamma g \cdot \mathbf{I}_k$. It follows therefore from (9) that $|\alpha(s_{\ell(k),k})| > k$. For each positive integer k define now $t_k := s_{\ell,k} \in (-1,1)$ so that $|\alpha(t_k)| > k$, $\gamma_k = \nu_{\ell(k),k}$, $\delta_k := (g\widetilde{g}_k^{-1})\epsilon'_{\ell(k),k} \to id_{G_0}$ as $k \to \infty$. We have

$$g\theta_{-t_k} g^{-1}\gamma_k = \delta_k \quad \text{hence} \quad \gamma_k = (g\theta_{t_k} g^{-1})\delta_k \,, \tag{15}$$

where $|\alpha(t_k)| > k$, $\gamma_k \in \Gamma$ and $\delta_k \to id_{G_0}$ as $k \to \infty$.

For $-1 < t < 1$ we sometimes write $\xi_t = g\theta_t g^{-1}$, so that $\Xi = \{\xi_t : t \in (-1,1)\}$ is a 1-parameter group of transvections fixing $\Phi = g\Sigma$ and $\Psi = g\Sigma'$ and flowing from Ψ to Φ

Assume now that Ω is reducible, and $\Omega = \Omega_1 \times \cdots \times \Omega_m$, $m \geq 2$, is the decomposition of Ω into a product of irreducible bounded symmetric domains, and $\Gamma \subset \Omega$ is an irreducible lattice. For $1 \leq i \leq m$ write $G_i := \mathrm{Aut}_0(\Omega_i)$, so that $G_0 = G_1 \times \cdots \times G_m$. Suppose now $g = (g_1, \cdots, g_m) \in G_0$ is such that $\Gamma g H$ is dense in G_0, where H is a 1-parameter group of transvections arising from a minimal disk of one of the irreducible factors $\Omega_i, 1 \leq i \leq m$, then we perform exactly the same argument as in the locally irreducible case by freezing the variables belonging to the other irreducible factors. The proof of the lemma is complete. □

Remark 6.4. At the end of the proof, note that although θ_t only acts on Ω_1, so that one may write $\xi_t = (\xi_t^1, id_{G_2} \cdots, id_{G_m})$, the group elements $\gamma_k \in \Gamma$ obtained of the form $\gamma_k = (\gamma_k^1, \gamma_k^2, \cdots \gamma_k^m)$ where, for $2 \leq i \leq m$, γ_k^i are in general non-identity elements close to and converging to id_{G_i}. Since the proof of the locally reducible but global irreducible case of Theorem 1.1 can be obtained by arguing factor by factor according to the decomposition of Ω as in the above, we will from now on restrict our arguments to the case where Ω is irreducible and of rank ≥ 2 except at the end of the proof of Theorem 1.1.

Lemma 6.5. *In the notation of Lemma 6.3 and its proof, assuming that $\Gamma g H$ is dense in G_0, there exists a discrete sequence of group elements $\gamma_k \in \Gamma$ such that the sequence of holomorphic maps $\gamma_k(z)$ on Ω either converges to the Cayley projection $\rho_{g\Sigma, g\Sigma'}$, or to the Cayley projection $\rho_{g\Sigma', g\Sigma}$*

Proof. As $t \to 1$, $(g\theta_t g^{-1})(z) = g\theta_t(g^{-1}(z)) \to g\rho_{\Sigma,\Sigma'}(g^{-1}(z))$. The latter is a Cayley projection with image being on $g\Sigma$ and with vertices in $g\Sigma'$, i.e, $\rho_{g\Sigma, g\Sigma'}$. On the other hand, as $t \to -1$, $(g\theta_t g^{-1})(z)$ converges to $g\rho_{\Sigma',\Sigma}(g^{-1}(z)) = \rho_{g\Sigma', g\Sigma}$ by exactly the same argument. □

6.2 The algebra $\mathscr{F}$ of pull-backs of bounded holomorphic functions by $F : \Omega \to \widetilde{N}$

We define now $\mathscr{F}$ to be the algebra of bounded holomorphic functions on Ω obtained by pulling back bounded holomorphic functions on $\widetilde{N}$ by $F : \Omega \to \widetilde{N}$, i.e., $\mathscr{F} = F^* H^\infty(\widetilde{N})$. The algebra $\mathscr{F}$ will be crucial for the proof of Theorem 1.1 (the Extension Theorem). In fact, the latter is equivalent to showing that the identity map id_Ω belongs to $\mathscr{F}^n$. To start with we collect some general properties of $\mathscr{F}$ in the following lemma.

Lemma 6.6. *The algebra $\mathscr{F} = F^*(H^\infty(\widetilde{N})$ possesses the following properties:*

(a) *For any $h \in \mathscr{F}$ and for any group element $\gamma \in \Gamma$, we have $\gamma^* h \in \mathscr{F}$.*

(b) *Suppose $h_i = F^* u_i \in \mathscr{F}$ for $0 \le i < \infty$ such that $\{u_i\}$ is a uniformly bounded sequence of holomorphic functions and such that h_i converges uniformly on compact subsets to $h \in H^\infty(\Omega)$. Then, $h \in \mathscr{F}$.*

(c) *Let Q be a manifold equipped with a smooth volume form $d\mu$ such that* $\mathrm{Volume}(Q, d\mu) < \infty$. *Suppose $H : \Omega \times Q \to \mathbb{C}$ is a bounded measurable function on $\Omega \times Q$ such that for almost every parameter $t \in Q$, the function $h_t : \Omega \to \mathbb{C}$ is a bounded holomorphic function belonging to $\mathscr{F}$. Then, the function $f : \Omega \to \mathbb{C}$ defined by $f(z) = \int_Q H(z,t)d\mu(t)$ also belongs to $\mathscr{F}$.*

Proof. By definition $F : \Omega \to \widetilde{N}$ is a lifting of the holomorphic map $f : X \to N$, where $X = \Omega/\Gamma$. The holomorphic map $f : X \to N$ induces a group homomorphism $f_* : \Gamma = \pi_1(X) \to \pi_1(N)$. Thus, identifying $\pi_1(N) = \Gamma'$ as a group of biholomorphic covering transformations of $\widetilde{N}$ and writing $\sigma_\gamma := f_*(\gamma)$, for $z \in \Omega$ and $\gamma \in \Gamma$ we have $F(\gamma z) = \sigma_\gamma(F(z))$.

For the proof of (a) suppose now $h = F^* u$ for a holomorphic function $u \in H^\infty(\widetilde{N})$. Then, for $\gamma \in \Gamma$ we have $\gamma^* h(z) = h(\gamma(z)) = u(F(\gamma(z)) = u(\sigma_\gamma(F(z)) = F^*(u \circ \sigma_\gamma)$, hence $\gamma^* h = F^*(\sigma_\gamma^* u) \in \mathscr{F}$ since $\sigma_\gamma^* u \in H^\infty(\widetilde{N})$.

As to (b), given $h_i = F^* u_i$ such that there exists a uniform bounded $M > 0$ such that $|u_i(w)| \le M$ for all $i \in \mathbb{N}$, by Montel's theorem there is a subsequence $u_{i(k)}$ which converges uniformly on compact subsets to a bounded holomorphic function $u \in H^\infty(\widetilde{N})$. It follows that

$$h = \lim_{k\to\infty} h_{i(k)} = \lim_{k\to\infty} F^* u_{i(k)} = F^*\Big(\lim_{k\to\infty} u_{i(k)}\Big) = F^* u \,, \tag{16}$$

hence $h \in \mathscr{F}$.

Clearly $\mathscr{F}$ is a complex vector space (in fact a $\mathbb{C}$-algebra). An integral of a measurable family of uniformly bounded holomorphic functions in $\mathscr{F}$ is the limit of a uniformly bounded sequence of weighted finite sums of bounded holomorphic functions belonging to $\mathscr{F}$, and as such the integral $f(z) = \int_Q H(z,t)d\mu(t)$ belongs to $\mathscr{F}$ by (b), completing the proof of the lemma. □

Theorem 6.7. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of rank $r \ge 2$ in its Harish-Chandra realization, and $\partial\Omega$ be the topological boundary of Ω in $\mathbb{C}^n$. Let h be a bounded holomorphic function on Ω. Then, for almost all $g \in G_0$ the following holds true. For almost all* maximal *boundary components $\Phi \subset \partial\Omega$, h admits the Cayley limit $\hat{h}_\Phi(z) = \lim_{t\to 1} h(\theta_{t,\Phi,\Psi}(z))$ with respect to the Cayley projection $\rho_{g\Sigma, g\Sigma'}$. Moreover, $\hat{h}_\Phi = \rho^*_{\Phi,\Psi} s_\Phi$ for some bounded holomorphic function $s_\Phi \in H^\infty(\Phi)$ independent of Ψ.*

For the notations in what follows, cf. Definition 4.10 and the paragraph immediately after it. As an application of Lemma 6.6, we have

Proposition 6.8. *Fix $h \in \mathscr{F}$. Suppose $([\Phi],[\Psi]) \in \mathscr{C}'_{\mathrm{adm}}(h)$. Then, either $\rho^*_{\Phi,\Psi} s_\Phi$ or $\rho^*_{\Psi,\Phi} s_\Psi$ belongs to $\mathscr{F}$.*

Proof. Since G_0 acts transitively on $\mathscr{C}$ there exists $g \in G_0$ such that $g\Sigma = \Phi$ and $g\Sigma' = \Psi$. By Lemma 6.3, there exists $t_k \to 1$ or $t_k \to -1$ such that, writing $\xi_t = g\theta_t g^{-1}$, there exists $\gamma_k \in \Gamma$ such that $\gamma_k = \xi_{t_k}\delta_k$ for $k \in \mathbb{N}$ such that $\delta_k \to id_\Omega$. We assume $t_k \to 1$ and proceed to prove that $\rho^*_{\Phi,\Psi} s_\Phi \in \mathscr{F}$. The case where $t_k \to -1$ will lead to $\rho^*_{\Psi,\Phi} s_\Psi \in \mathscr{F}$ by exactly the same argument. We have

$$\begin{aligned}
\left|\xi^*_{t_k} h(z) - \gamma^*_k h(z)\right| &= \left|(h \circ \xi_{t_k})(z) - (h \circ \gamma_k)(z))\right| \\
&= \left|(h \circ \xi_{t_k})(z) - (h \circ \xi_{t_k})(\delta_k(z))\right| \\
&= \left|\int_z^{\delta_k(z)} (h \circ \xi_{t_k})'(\xi) d\xi\right| \\
&\leq C \left\|h \circ \xi_{t_k}\right\|_{H^\infty(\Omega)} \left\|\delta_k(z) - z\right\| \quad \text{(Cauchy's estimate)} \\
&\leq C \left\|h\right\|_{H^\infty(\Omega)} \left\|\delta_k(z) - z\right\| \to 0,
\end{aligned}$$

for some constant $C > 0$, since $\delta_k \to id$ as $k \to \infty$. $\square$

6.3 Proof of the Extension Theorem: existence of a nonconstant element of $\mathscr{F}$ invariant under a conjugate of H

Recall the subgroup $H \subset G_0$ defined explicitly in terms of Harish-Chandra coordinates. This is the reference 1-parameter group of transvections preserving the reference special product subspace $P_0 = \nu(\Delta \times \Omega')$ and flowing from $b'_0 = (-1; 0; w)$ to $b_0 = (1; 0; w)$ for $w \in \Omega'$. Our approach is to change the Harish-Chandra coordinates so that z in the new coordinates corresponds to $g_1 \cdot z$ in the original Harish-Chandra coordinates. For $g \in G_0$ write $h_g := g^*h$. An automorphism $g_1 \in G_0$ will be chosen so that among other things the pair (P_0, Ω'_0) is h'-admissible for $h' = h_{g_1}$. Thus the reference objects $b_\theta(w)$, $b'_\theta(w)$, $\Sigma(\theta)$, $\Sigma'(\theta)$, P_0, Ω'_0 are written explicitly in a choice of Harish-Chandra coordinates determined by $g_1 \in G_0$ satisfying certain conditions to be made precise.

By Lemma 6.2, there is a subset $G_{0,\mathrm{den}} \subset G_0$ of full measure such that $\Gamma g_0 H$ is dense in G_0 if and only if $g_0 \in G_{0,\mathrm{den}}$. Consider the multiplication map $\mu : G_0 \times S^1 \to G_0$ given by $\mu(g_1, e^{i\theta}) := g_1\beta_\theta$, where $\beta_\theta := \upsilon(e^{i\theta} id_\Delta, id_{\Omega'})$ for the group homomorphism υ as in (5). The map $\mu : G_0 \times S^1 \to G_0$ realizes $G_0 \times S^1$ as the total space of a smooth locally trivial fibration with fibers diffeomorphic to S^1. Thus, the subset $\mu^{-1}(G_{0,\mathrm{den}}) \subset G_0 \times S^1$ is of full measure. Equipping $G_0 \times S^1$ also with the canonical projection $\pi_1 : G_0 \times S^1 \to G_0$ onto the first Cartesian factor we have the structure of a double locally trivial smooth fibration on $G_0 \times S^1$, and it follows by Lemma 4.2 that for almost all $g_1 \in G_0$ we have

(†): $g_1\beta_\theta \in G_{0,\mathrm{den}}$ for almost all $\theta \in \mathbb{R}$.

We now make the requirement on g_1 at the end of the first paragraph precise. Denote by $G^\dagger_{0,\mathrm{den}}$ the set of all $g_1 \in G_0$ satisfying (†). We require

(♯): $g_1 \in G_{0,\mathrm{adm}}(h) \cap G^\dagger_{0,\mathrm{den}}$

to hold true. For $\theta_1 \in \mathbb{R}$ write $H_1 := \beta_{\theta_1} H \beta_{-\theta_1}$. The condition (♯) allows us to take admissible boundary values of h_{g_1} for the regular pairs $(\Sigma(\theta_1), \Sigma'(\theta_1))$ and $(\Sigma'(\theta_1), \Sigma(\theta_1))$ of maximal boundary components for almost all $\theta_1 \in \mathbb{R}$, so that at least one of the two functions $\hat{h}_{g_1,\Sigma(\theta_1),\Sigma'(\theta_1)}$ or $\hat{h}_{g_1,\Sigma'(\theta_1),\Sigma(\theta_1)}$ belongs to $\mathscr{F}' := g_1^*\mathscr{F}$, by Proposition 6.8. At the same time, we will show in Proposition 6.10 that θ_1 can be chosen so that furthermore both functions are nonconstant.

We adopt a change of Harish-Chandra coordinates determined by $g_1 \in G_0$ as explained. In so doing, $h \in H^\infty(\Omega)$ is replaced by $h' := h_{g_1}$, $F : \Omega \to \widetilde{N}$ is replaced by $F' := F \circ g_1 : \Omega \to \widetilde{N}$, and hence $\mathscr{F}$ is replaced by $(F \circ g_1)^* H^\infty(\widetilde{N}) = g_1^*(F^* H^\infty(\widetilde{N})) = g_1^*\mathscr{F} = \mathscr{F}'$. In the statement of results, whenever there is no danger of confusion we will refer to h' as h, treating the latter as a generic symbol for a bounded holomorphic function on Ω.

Proposition 6.9. *Suppose $g_1 \in G^{\dagger}_{0,\mathrm{den}}$ and suppose $\theta_1 \in \mathbb{R}$ is chosen to ensure that $g_1\beta_{\theta_1} \in G_{0,\mathrm{den}}$, so that, in terms of the new Harish-Chandra coordinates with respect to which $z \in \Omega$ corresponds to $g_1 \cdot z$ in the original Harish-Chandra coordinates, $\Gamma'\beta_{\theta_1}H$ is dense in G_0 for $\Gamma' = g_1^{-1}\Gamma g_1$. Writing $H_1 := \beta_{\theta_1}H\beta_{-\theta_1}$, let $h \in \mathscr{F}' = g_1^*\mathscr{F}$ be H_1-invariant in the sense that, $h(\xi(z)) = h(z)$ for any element $\xi \in H_1$. Then, for every element $g \in G_0$, g^*h is a $g^{-1}H_1g$-invariant bounded holomorphic function belonging to $\mathscr{F}'$.*

Proof. Let $g_0 \in G_0$. Applying the density of $\Gamma'\beta_{\theta_1}H$ in G_0 to the group element $g_0^{-1} \in G_0$, there exist $\mu_k \in \Gamma'$ and $\eta_k \in H$ for each natural number k such that $\lim\limits_{k\to\infty} \mu_k\beta_{\theta_1}\eta_k = g_0^{-1}$. In other words, $g_0^{-1} = \epsilon_k\mu_k\beta_{\theta_1}\eta_k$ such that $\epsilon_k \to id_\Omega$. Taking inverses we have $g_0 = \xi_k\beta_{-\theta_1}\gamma_k\delta_k$ where $\delta_k = \epsilon_k^{-1}$, $\xi_k = \eta_k^{-1}$ and $\gamma_k = \mu_k^{-1}$. We have

$$(\beta_{\theta_1}g_0)^*h(z) = h(\beta_{\theta_1}\xi_k\beta_{-\theta_1}(\gamma_k(\delta_k(z)))) = h(\gamma_k(\delta_k(z))\,. \tag{17}$$

since $\beta_{\theta_1}\xi_k\beta_{-\theta_1} \in \beta_{\theta_1}H\beta_{-\theta_1} = H_1$ and $h \in \mathscr{F}'$ is assumed to be H_1-invariant. Since $h \in \mathscr{F}'$, by Lemma 6.6, for each natural number k, we have $h \circ \gamma_k = \gamma_k^*h \in \mathscr{F}'$. Write $g := \beta_{\theta_1}g_0$. By the same argument as in the proof of Proposition 6.8, for any compact subset $Q \subset \Omega$ we have

$$\begin{gathered}\lim_{k\to\infty} \sup\left\{\left|(h\circ\gamma_k)(z) - (h\circ\gamma_k)(\delta_k(z))\right| : z \in Q\right\} = 0\,, \quad \text{hence}\\ g^*h = \lim_{k\to\infty} \gamma_k^*h \ \text{ uniformly on compact subsets}\end{gathered} \tag{18}$$

By Lemma 6.6, we conclude that $g^*h \in \mathscr{F}'$. Finally, for $\eta \in H_1$ we have

$$g^*h(g^{-1}\eta g(z)) = h(g(g^{-1}\eta g)(z)) = h(\eta(g(z)) = h(g(z)) = g^*h(z)\,, \tag{19}$$

where the second last equality results from the H_1-invariance of h. It follows that g^*h is $g^{-1}H_1g$-invariant, proving the lemma. □

Starting with a nonconstant $h \in \mathscr{F}$, we will now prove the existence of a nonconstant holomorphic function $\hat{h} \in \mathscr{F}$ which is invariant under some conjugate of $H \subset G_0$. Here and in what follows by a conjugate of $H \subset G_0$ we mean a subgroup $\widetilde{H} = \alpha^{-1}H\alpha \subset G_0$ for some element $\alpha \in G_0$.

For the notations in what follows, recall that for $-1 < t < 1$, we write $\psi_t(z_1) = \frac{z_1+t}{1+tz_1}$ for $z_1 \in \Delta$, and $\chi_t := \upsilon(\psi_t, id_{\Omega'})$ using (5), and for $\theta_1 \in \mathbb{R}$, $H_1 = \beta_{\theta_1}H\beta_{-\theta_1}$, we write $\xi_t = \beta_{\theta_1}\chi_t\beta_{-\theta_1}$. We note that χ_t was denoted by $\theta_t := \Theta(\psi_t)$ using (3) in §3.2, but here we change the notation in order to avoid confusion with the notation θ representing an angle, e.g., in $e^{i\theta}$.

Proposition 6.10. *Let $g_1 \in G_0$ be the automorphism chosen as in Proposition 6.9. Write $\Gamma' = g_1^{-1}\Gamma g_1$, $\mathscr{F}' := g_1^*\mathscr{F}$. Let $h \in \mathscr{F}'$ be nonconstant. Assume that $\Gamma'\beta_\theta H$ is dense in G_0 for almost all $\theta \in \mathbb{R}$ and that $([\Sigma(\theta)], [\Sigma'(\theta)]) \in \mathscr{C}'_{0,\mathrm{adm}}(h)$ for almost all $\theta \in \mathbb{R}$. Then, there exists $\theta_1 \in \mathbb{R}$ and a nonconstant bounded holomorphic function $\hat{h} \in \mathscr{F}'$ on Ω which is invariant under $H_1 = \beta_{\theta_1}H\beta_{-\theta_1}$.*

Proof. To prove Proposition 6.10, it suffices to produce a single nonconstant holomorphic function $\hat{h} \in \mathscr{F}'$ which is invariant under $H_1 = \beta_{\theta_1}H\beta_{-\theta_1}$ for some $\theta_1 \in \mathbb{R}$. For $P_0 = \nu(\Delta \times \Omega')$, recall that $b_\theta(w) = (e^{i\theta}; 0; w) \in \Sigma(\theta)$ and $b'_\theta(w) = (-e^{i\theta}; 0; w) \in \Sigma'(\theta)$. We claim that the following statement holds true.

(††): There exists a subset $A \subset [0, 2\pi)$ of positive Lebesgue measure such that for any $\theta_1 \in A$, both $h^\sharp_{b_\theta(w),b'_\theta(w)}$ and $h^\sharp_{b'_\theta(w),b_\theta(w)}$ exist as bounded holomorphic functions on Ω' and both functions are nonconstant on Ω'.

By assumption $\Gamma'\beta_\theta H \subset G_0$ is dense for almost all $\theta \in \mathbb{R}$, Assuming that the claim (††) is valid, and choose $\theta_1 \in A$. Note that $\Gamma' H_1 = \Gamma'\beta_{\theta_1} H \beta_{-\theta_1} = (\Gamma'\beta_{\theta_1} H)\beta_{-\theta_1} \subset G_0$ is dense in G_0. Hence, as in the proof of Proposition 6.9, for $k \in \mathbb{N}$ there exists $\xi_{t_k} \in H_1 = \beta_{\theta_1} H \beta_{-\theta_1}$, and $\gamma_k \in \Gamma'$ such that $\gamma_k \xi_{-t_k} \to id_\Omega$, i.e., $\xi_{t_k} = \gamma_k \delta_k$ such that $\delta_k \to id_\Omega$ and such that either $t_k \to 1$ or $t_k \to -1$ as $k \to \infty$. In the former case we have $\lim_{t\to 1} h(\xi_t(z)) = \rho^*_{\Sigma(\theta),\Sigma'(\theta)} h^\sharp_{b_\theta(w),b'_\theta(w)}$, and in the latter case we have $\lim_{t\to -1} h(\xi_t(z)) = \rho^*_{\Sigma'(\theta),\Sigma(\theta)} h^\sharp_{b'_\theta(w),b_\theta(w)}$, and in either case we have obtained a nonconstant bounded holomorphic function, to be denoted by $\hat{h}$, such that $\hat{h} \in \mathscr{F}'$, by Proposition 6.8. Let $\eta = \xi_s \in H_1$, $s \in (-1,1)$. In the former case we have $\hat{h}(\eta z) = \lim_{t\to 1} h(\xi_t(\eta z)) = \lim_{t\to 1} h(\xi_{t'} z)$, where $t' \to 1$ as $t \to 1$, and it follows that $\hat{h}(\eta z) = \hat{h}(z)$ for any $z \in \Omega$ and for any $\eta \in H_1$. The same argument works verbatim to show that $\hat{h}(\eta z)$ in the latter case where $t_k \to -1$. In any event, we have obtained an H_1-invariant bounded holomorphic function $\hat{h}$ such that $\hat{h}$ is the limit uniformly on compact subsets of Ω of $\gamma_k^* h$, hence $\hat{h} \in \mathscr{F}'$, by Lemma 6.6.

Fix $\theta_1 \in \mathbb{R}$. By assumption, for almost every point $w \in \Omega'$, in particular for a dense set of points $w \in \Omega'$, $h(\xi_t(z))$ converges admissibly to $h^\sharp_{b_{\theta_1}(w),b'_{\theta_1}(w)}$ as $t \to 1$ while $h(\xi_t(z))$ converges admissibly to $h^\sharp_{b'_{\theta_1}(w),b_{\theta_1}(w)}$ as $t \to -1$. It follows from the proof of Montel's theorem, cf. Lemma 3.16 that $h(\xi_t(z))$ converges admissibly to $h^\sharp_{b_{\theta_1}(w),b'_{\theta_1}(w)}$ for all $w \in \Omega'$ as $t \to 1$, and to $h^\sharp_{b'_{\theta_1}(w),b_{\theta_1}(w)}$ as $t \to -1$, and in fact the convergence is uniform on compact subsets of Ω'.

For $1 \le j \le q$ write ∂_j for $\frac{\partial}{\partial w_j}$. From Cauchy estimates on first derivatives, on the special product subspace $P_0 = \nu(\Delta \times \Omega') \subset \Omega$, $\partial_j h(\psi_t(z_1); 0; w)$ converges admissibly on P_0 to $\partial_j h^\sharp_{b_{\theta_1}(w),b'_{\theta_1}(w)}$ when $t \to 1$, and it converges to $\partial_j h^\sharp_{b'_{\theta_1}(w),b_{\theta_1}(w)}$ when $t \to -1$, uniformly on compact subsets as functions in $w \in \Omega'$.

By the hypothesis, for almost every $\theta \in \mathbb{R}$, $\Gamma'\beta_\theta H$ is dense in G_0, where $\beta_\theta := \upsilon(e^{i\theta} id_\Delta, id_{\Omega'}) \in G_0$. Choose $\theta = \theta_1$ such that $\Gamma'\beta_{\theta_1} H$ is dense in G_0. Hence $\Gamma' H_1$ is dense in G_0 and there exists a sequence $\xi_{t_k} \in H_1$ and elements $\gamma_k \in \Gamma'$ such that $\xi_{t_k}\gamma_k \to \beta_{\theta_1}$ (which one sees by taking inverses on both sides). As in the proof of Proposition 6.8, after passing to a subsequence we may assume without loss of generality that either $t_k \to 1$ or $t_k \to -1$.

For $1 \le j \le q$, define on P_0 the holomorphic function u_j by

$$u_j(z_1; 0; w) = \partial_j h(z_1; 0; w) \cdot \partial_j h(-z_1; 0; w) \tag{20}$$

For almost all $\theta_1 \in \mathbb{R}$, and for any sequence x_k of points on Δ such that x_k converges admissibly to $e^{i\theta_1}$, we know that $h(x_k; 0; w)$ converges as holomorphic functions in w uniformly on compact subsets of Ω' to $h^\sharp(e^{i\theta_1}; 0; w) := h^\sharp_{b_{\theta_1}(w),b'_{\theta_1}(w)}$, and at the same time $h(-x_k; 0; w)$ converges in the same sense to $h^\sharp(-e^{i\theta_1}; 0; w) := h^\sharp_{b'_{\theta_1}(w),b_{\theta_1}(w)}$. It follows from Cauchy estimates for first derivatives that $\partial_j h(x_k; 0; w)$ converges as (not necessarily bounded) holomorphic functions in w uniformly on compact subsets of Ω' to $\partial_j h^\sharp(e^{i\theta_1}; 0; w)$, while $\partial_j h(-x_k; 0; w)$ converges in the same sense to $\partial_j h^\sharp(-e^{i\theta_1}; 0; w)$.

It follows that the holomorphic function $u_j(z_1; 0; w)$ on the special product subspace $P_0 \subset \Omega$ as defined in (20), bounded as a holomorphic function on $\nu(\Delta \times V)$ for any open subset $V \Subset \Omega'$, admits admissible boundary values in the z_1 variable at $(e^{i\theta_1}; 0; w)$ for almost all $\theta_1 \in \mathbb{R}$. By the Cauchy integral formula for admissible boundary limits of a bounded holomorphic function on Δ, we have

$$u_j(z_1; 0; w) = \int_{\partial\Delta} \frac{u_j^\sharp(\zeta; 0, w)}{\zeta - z_1} d\zeta\,, \tag{21}$$

where for $\zeta = e^{i\theta} \in \partial\Delta$ we define $u_j^\sharp(\zeta; 0, w) := \partial_j h^\sharp(e^{i\theta}; 0, w)\partial_j h^\sharp(-e^{i\theta}; 0, w)$.

Suppose (††) fails. Then, for almost all $\theta \in \mathbb{R}$, either $h^\sharp_{b_\theta(w),b'_\theta(w)}$ or $h^\sharp_{b_\theta(w),b'_\theta(w)}$ is constant as a function in $w \in \Omega'$. Then, $\partial_j h^\sharp_{b_\theta(w),b'_\theta(w)} \equiv 0$ for $1 \le j \le q$, or $\partial_j h^\sharp_{b_\theta(w),b'_\theta(w)} \equiv 0$ for $1 \le j \le q$, as functions in $w \in \Omega'$, so that for almost all $\theta \in \mathbb{R}$, $u^\sharp_j(e^{i\theta};0;w) = 0$ as a function in $w \in \Omega'$. Then (21) implies that $u_j(z_1;0;w) = 0$ for $(z_1;w) \in \Delta \times \Omega'$. As a consequence, for $1 \le j \le q$ we have $\partial_j h(z_1;0,w) = 0$ for all $(z_1,w) \in \Delta \times \Omega'$, by (21). In other words, $h(z_1;0,w) = v(z)$ for some bounded holomorphic function v on Δ. Writing $\Omega'(z_0) := \nu(\{z_0\} \times \Omega')$, h is constant on each $\Omega'(z_0)$ for $z_0 \in \Delta$. In particular, h is constant on $\Omega'_0 := \Omega'(0)$.

For $g \in G^\sharp_{0,\mathrm{adm}}(h)$, $(g\Sigma, g\Sigma', g\Omega'_0)$ is an h-admissible triple. Denoting g^*h by h_g, we have either

(a) there exists some $g_1 \in G^\sharp_{0,\mathrm{adm}}(h)$, and some $\theta_1 \in \mathbb{R}$ such that $g_2 := g_1\beta_{\theta_1} \in G'_{0,\mathrm{adm}}(h)$, and such that both $\hat{h}_{g_1,\Sigma(\theta_1),\Sigma'(\theta_1)}$ and $\hat{h}_{g_1,\Sigma'(\theta_1),\Sigma(\theta_1)}$ (which are known to exist by the choice of $g_2 := g_1\beta_{\theta_1}$) are nonconstant; or

(b) for all $g_1 \in G^\sharp_{0,\mathrm{adm}}(h)$, and for all $\theta_1 \in \mathbb{R}$ such that $g_2 := g_1\beta_{\theta_1} \in G'_{0,\mathrm{adm}}(h)$, either $\hat{h}_{g_1,\Sigma(\theta_1),\Sigma'(\theta_1)}$ or $\hat{h}_{g_1,\Sigma'(\theta_1),\Sigma(\theta_1)}$ are constant on Ω'_0.

When Alternative (a) occurs, putting either $\hat{h} := \hat{h}_{g_1,\Sigma(\theta_1),\Sigma'(\theta_1)}$ or $\hat{h} := \hat{h}_{g_1,\Sigma'(\theta_1),\Sigma(\theta_1)}$, we know that $\hat{h}$ is nonconstant and $g_1^{-1}H_1g_1$-invariant under the 1-parameter group of transvections $\{\xi_t : -1 < t < 1\}$ preserving $P_0 = \nu(\Delta \times \Omega')$ flowing from $b'_\theta(w) = (-e^{i\theta};0;w)$ to $b_\theta(w) = (e^{i\theta};0;w)$ for $w \in \Omega'$ as $t \to 1$ and from $b_\theta(w)$ to $b'_\theta(w)$ as $t \to -1$, for $w \in \Omega'$. Thus, $\hat{h}$ is invariant under $\beta_{\theta_1}H\beta_{-\theta} = H_1$, as desired.

When Alternative (b) occurs, by the argument using the parametrized Cauchy integral formula (21) on $P_0 = \nu(\Delta \times \Omega')$ we conclude that h_{g_1} is constant on Ω'_0, i.e., h is constant on $g_1\Omega'_0$. Since $G^\sharp_{0,\mathrm{adm}}(h) \subset G_0$ is of full measure, hence in particular dense in G_0, we conclude in this case that h is constant on $g\Omega'_0$ for any $g \in G_0$.

As can be easily seen from the polydisk theorem, given any two distinct points $x, y \in \Omega$, there exists a finite sequence $x_0, \cdots, x_{s+1}$ of points on Ω, $x_0 = x$, $x_{s+1} = y$, $1 \le s \le r = \mathrm{rank}(\Omega)$, $x_i \ne x_{i+1}$ for $0 \le i \le s-1$, and a chain of s minimal disks $D_1, \cdots, D_s$ such that $x_{j-1}, x_j \in D_j$ for $1 \le j \le s$. Now each minimal disk D on Ω can be embedded into some characteristic subspace $g\Omega'_0$, and taking $x_0 = 0$ it follows from the above that, assuming that Alternative (b) could occur, we would have $h(y) = h(0)$ for any $y \in \Omega$ so that h would be a constant function, contradicting with the hypothesis of the proposition and ruling out Alternative (b). Thus, only Alternative (a) occurs, and the proof of the proposition is complete. □

Remark 6.11. Note that for $\hat{h} = h_{g_1,\Sigma(\theta),\Sigma'(\theta)} = \rho^*_{\Sigma(\theta),\Sigma'(\theta)}s_{\Sigma(\theta)}$, the regular pair of maximal boundary components $(\Sigma(\theta), \Sigma'(\theta))$ must necessarily belong to $\mathscr{C}_{\mathrm{adm}}(h_{g_1})$, and an analogous statement holds when $\Sigma(\theta)$ and $\Sigma'(\theta)$ are interchanged.

6.4 End of the proof of Theorem 1.1

We are now ready to complete the proof of Theorem 1.1.

Proof. Consider $\hat{h} \in \mathscr{F}'$ invariant under $H_1 = \beta_{\theta_1}H\beta_{-\theta_1}$ as guaranteed by Proposition 6.10. We will now replace the notation $\hat{h}$ by h, again treating the latter symbol as a generic symbol for a bounded holomorphic function on Ω, this time endowed with the special property that h is H_1-invariant.

We consider the *final* change of coordinates on Ω such that $z \in \Omega$ corresponds to $g_2{\cdot}z$ in the *initial* Harish-Chandra coordinates for $g_2 = g_1\beta_{\theta_1}$. Then, Γ in the initial Harish-Chandra coordinates (which was replaced by $\Gamma' = g_1^{-1}\Gamma g_1$ in the *intermediate* Harish-Chandra coordinates determined by g_1) is now replaced by $\Gamma'' := g_2^{-1}\Gamma g_2 = \beta_{-\theta_1}\Gamma'\beta_{\theta_1}$. Write $\mathscr{F}'' = g_2^*\mathscr{F} = \beta_{\theta_1}^*\mathscr{F}'$, and define $\widetilde{h}(z) := \beta_{\theta_1}^* h$. For all $\chi \in H$ and $z \in \Omega$ we have

$$\widetilde{h}(\chi z) = \beta_{\theta_1}^* h(\chi z) = h(\beta_{\theta_1}\chi(z)) = h(\beta_{\theta_1}\chi\beta_{-\theta_1}(\beta_{\theta_1}(z)) = h(\beta_{\theta_1}(z)) = \widetilde{h}(z)\,,$$

where for the second last equality we have used the invariance of h (originally denoted as $\hat{h}$ in Proposition 6.10) under H_1. Hence, in terms of the final Harish-Chandra coordinates we have an H-invariant bounded holomorphic function $\widetilde{h}$ belonging to $\mathscr{F}''$. By Remark 6.11, (Σ, Σ') is a regular pair of maximal boundary components belonging to $\mathscr{C}'_{\mathrm{adm}}(\widetilde{h})$. Moreover by our choice of θ_1, it follows that the admissible boundary values $\widetilde{h}^\sharp_{b(w),b'(w)}$ are nonconstant.

Consider now the subgroup $G_0' = \upsilon(\{id_\Delta\} \times \mathrm{Aut}_0(\Omega')) \subset G_0$. Then, G_0' fixes each of Σ and Σ' as a set, and both $(g^*\widetilde{h})^\sharp_{b(w),b'(w)} = \widetilde{h}^\sharp_{b(g(w)),b'(g(w))}$ and $(g^*\widetilde{h})^\sharp_{b'(w),b(w)} = \widetilde{h}^\sharp_{b(g(w)),b'(g(w))}$ exist. (Here we use the notation g both as an element of $G_0' \subset G_0$ and as an element of $\mathrm{Aut}_0(\Omega')$ in the obvious way.) By Proposition 6.9, $g^*\widetilde{h} \in \mathscr{F}$. Since $\widetilde{h}(0;0;w) = \widetilde{h}^\sharp(1;0;w)$ is nonconstant, there exists $g \in G_0'$ such that $d(g^*\widetilde{h})(0) \neq 0$, hence without loss of generality we may assume that $d\widetilde{h}(0) \neq 0$.

In what follows, for the proof of Theorem 1.1 in the case where Ω is irreducible, we will assume without loss of generality that ΓH is dense in G_0, (P_0, Ω_0') is h-admissible, (Σ, Σ') belongs to $\mathscr{C}'_{\mathrm{adm}}(h)$, $\hat{h}(z) = \lim\limits_{t\to 1} h(\chi_t(z)) = s_\Sigma(\rho_{\Sigma,\Sigma'}(z))$ for some nonconstant $s_\Sigma \in H^\infty(\Sigma)$. (Replacing β_θ by $\beta_{\theta+\pi}$ if necessary, we assume here that $\hat{h}$ is obtained by taking $t \to 1$.) In other words, we use the symbol Γ to mean Γ'', $\mathscr{F}$ for $\mathscr{F}''$, etc., in the last paragraphs.)

Define now $\Lambda_0 : \Omega \to \mathbb{C}^n$ by $\Lambda_0 = [\widetilde{h}, 0, \cdots, 0]^t$ as a column n-vector of bounded holomorphic functions, hence $\Lambda_0 \in \mathscr{F}^n$. Since $\widetilde{h}$ and hence $\Lambda_0 : \Omega \to \mathbb{C}^n$ is H-invariant, by Proposition 6.9 we have $g^*\Lambda \in \mathscr{F}^n$ for every $g \in G_0$. We claim that the following statement ($\flat$) holds true.

($\flat$): There exists a linear transformation $A \in \mathrm{End}(\mathbb{C}^n)$ such that, writing $\Lambda = A\Lambda_0 \in \mathscr{F}^n$, we have $\mathrm{Tr}(d\Lambda(0)) \neq 0$.

Since $d\widetilde{h}(0) \neq 0$ there exists i, $1 \leq i \leq n$, such that $\frac{\partial h}{\partial z_i}(0) \neq 0$. Let A be the elementary n-by-n matrix such that for an n-by-n matrix X the map $X \to AX$ has the effect of switching the first row and the i-th row. Then $(A(d\Lambda_0(0))_{ii} \neq 0$ while by definition all k-th rows of $A(d\Lambda_0(0))$ are 0 except for $k = i$. Hence, for $\Lambda = A\Lambda_0$ we have

$$\mathrm{Tr}(d\Lambda(0)) = \mathrm{Tr}(d(A\Lambda_0)(0) = Ad\Lambda_0(0)_{ii} \neq 0\,,$$

proving the claim. Now for $z \in \Omega$ define

$$\widetilde{\Lambda}(z) := \int_K k\Lambda(k^{-1}z)d\mu(k)$$

Since obviously Λ is H-invariant, by Proposition 6.9 we have $g^*\Lambda \in \mathscr{F}^n$ for every $g \in G_0$. It follows that the vector-valued function $\Lambda(k^{-1}z)$ belongs to $\mathscr{F}^n$. On the other hand, $\mathscr{F}$ is closed under left multiplication by a constant n-by-n matrix, since that gives simply an element of $(H^\infty(\Omega))^n$ whose entries are linear combinations of functions belonging to $\mathscr{F}$. It follows by Lemma 6.6 that $\widetilde{\Lambda} \in \mathscr{F}^n$.

Now $0 \in \Omega$ is fixed by K. For every $k \in K$, writing $\Lambda_k(z) = k\Lambda(k^{-1}z)$ we have $d\Lambda_k(0) = kd\Lambda(0)k^{-1}$ so that $\mathrm{Tr}(d\Lambda_k(0)) = \mathrm{Tr}(d\Lambda(0)) \neq 0$, hence $\mathrm{Tr}(d\widetilde{\Lambda}(0)) \neq 0$. As explained in §4.2, $\widetilde{\Lambda}$ is necessarily a linear function. Since K acts irreducibly as a group of unitary transformations on $\mathbb{C}^n$, it follows from $\mathrm{Tr}(d\widetilde{\Lambda}(0)) \neq 0$ and the Schur lemma that $\widetilde{\Lambda} = \frac{1}{c}id_\Omega$ for some constant $c \neq 0$. Hence, $\frac{1}{c}id_\Omega = F^*\mu = F^*[\mu_1, \ldots, \mu_n]^t$, where $\mu_i \in H^\infty(\widetilde{N})$ for $1 \leq i \leq n$. The proof of the Extension Theorem (Theorem 1.1) is completed by taking $R = c\mu_\Phi : \widetilde{N} \to \mathbb{C}^n$.

Assume now Ω is reducible (and the lattice $\Gamma \subset \mathrm{Aut}_0(\Omega)$ is irreducible), and $\Omega = \Omega_1 \times \cdots \times \Omega_m$, $m \geq 2$, is the decomposition of Ω into a Cartesian product of irreducible bounded symmetric domains $\Omega_j \Subset \mathbb{C}^{n_j} \subset S_j$ being the standard inclusions incorporating the Harish-Chandra realization $\Omega_j \Subset \mathbb{C}^{n_j}$ and the Borel embedding $\Omega_j \Subset S_j$ of Ω_j into its dual irreducible Hermitian symmetric space S_j of the compact type. Denote by $\pi_j : \Omega \to \Omega_j$ the canonical projection of Ω onto the j-th Cartesian factor.

For each j, $1 \leq j \leq m$, by the hypothesis (♣) in Theorem 1.1 there exists a bounded holomorphic function φ_j on $\widetilde{N}$ such that $h_j := F^*\varphi_j$ is nonconstant. Write $z = (z^1, \cdots, z^m)$, where $z^j = (z_1^j, (z^j)', w^j)$ are the privileged Harish-Chandra coordinates on Ω_j according to the eigenspace decomposition at $0 \in \Omega_j$ of the Hermitian bilinear form H_α with respect to some unit minimal rational tangent vector α of S_j, $\Omega_j \Subset S_j$ being the Borel embedding.

Let $\nu_j : \Delta \times \Omega_j' \to \Omega_j$ be the holomorphic embedding giving the reference special product domain $P_0 = \nu_j(\Delta \times \Omega_j')$ on Ω_j, where by convention Ω_j' is a single point in the case where Ω_j is of rank 1, and by $\upsilon_j : \mathrm{Aut}_0(\Delta) \to \mathrm{Aut}_0(\Omega_j)$ the natural group monomorphism. Recall that for $-1 < t < 1$ we write $\psi_t \in \mathrm{Aut}(\Delta)$ for the transvection $\psi_t(z) = \frac{z+t}{1+tz}$. Let H^j be the one-parameter group of transvection defined by

$$H^j := \Big\{ \big(id_{\Omega_1}, \cdots, id_{\Omega_{j-1}}, \chi_t^j, id_{\Omega_{j+1}}, \cdots, id_{\Omega_m}\big) : -1 < t < 1 \Big\}, \text{ where}$$

$$\chi_t^j(z_1^j, 0, w^j) = \big(\psi_t(z_1^j), 0, w^j\big) = \left(\frac{z_1^j + t}{1 + t z_1^j}, 0, w^j \right).$$

Exactly the same argument as in the case where Ω is irreducible produces a holomorphic map $R_j : \widetilde{N} \to \Omega_j$ such that $R_j \circ F = \pi_j$. Then, defining $R := (R_1, \cdots, R_m) : \widetilde{N} \to \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_m}$, we have $R \circ F(z) = (\pi_1(z), \cdots, \pi_m(z)) = z$, as desired. The proof of Theorem 1.1 is complete. □

7 Applications to Rigidity Theorems: Preliminaries, Statements of Results, and First Arguments

The first link between bounded holomorphic functions and rigidity problems was given by the Embedding Theorem in [19]. In Theorem 1.1 we solved the Extension Problem, which is the problem of *inverting* the holomorphic embedding $F : \Omega \to \widetilde{N}$ as a bounded holomorphic map, i.e., finding a holomorphic extension $R : \widetilde{N} \to \mathbb{C}^n$ of the inverse $i : F(\Omega) \to \Omega \Subset \mathbb{C}^n$ as a *bounded* holomorphic map. As a first application of Theorem 1.1, we are going to prove for $X = \Omega/\Gamma$ (in the notation of Theorem 1.1) a factorization result called the Fibration Theorem for holomorphic mappings $f : X \to N$ inducing isomorphisms on fundamental groups which says that there exists a holomorphic fibration $\rho : N \to X$ such that $\rho \circ f \equiv id_X$. After lifting f to $F : \Omega \to \widetilde{N}$, quasi-compactness of N (cf. §7.1) is used to show that certain bounded plurisubharmonic functions constructed on $\widetilde{N}$ have to be constant, which allows us to show that R descends from $\widetilde{N}$ to N.

A primary objective in our research relating bounded holomorphic functions to rigidity problems is to study holomorphic mappings from X into target manifolds N which are uniformized by an *arbitrary* bounded domain D on a Stein manifold, $N := D/\Gamma'$. We study further the situation where $f : X \to N = D/\Gamma'$ induces an isomorphism on fundamental groups and look for necessary and sufficient conditions which guarantee that $f : X \to N$ is a biholomorphism. We are going to show $f : X \xrightarrow{\cong} N$ in Theorem 7.2 (the Isomorphism Theorem) under the assumption that N is of finite intrinsic measure with respect to the Kobayashi volume form. When N is a complete Kähler manifold of finite volume, we have at our disposal the tool of integration by parts. We resort to such techniques, by passing first of all to the hull of holomorphy $\widehat{D}$ of D and making use of the canonical Kähler-Einstein metric constructed by [4] and shown to be complete in [28]. We exploit the hypothesis that $N = D/\Gamma'$ is of finite intrinsic measure with respect to the Kobayashi volume form to prove that N can be enlarged to a complete Kähler-Einstein manifold of *finite volume*, which is enough to show that the bounded plurisubharmonic functions constructed are constant. The hypothesis on N in Theorem 7.2 appears to be the most natural geometric condition, as the notion of the Kobayashi pseudo-)volume form μ_M, unlike the canonical Kähler-Einstein metric, is elementary and defined for any complex manifold M, and the finiteness of the intrinsic measure with respect to μ_N is a necessary condition for the target manifold N of $f: X \to N$ to be quasi-compact. The passage from a quotient of finite intrinsic measure of a bounded domain to a complete Kähler-Einstein manifold of finite volume involves an elementary *a-priori* lower estimate of independent interest on the Kobayashi volume form on arbitrary bounded domains in terms of the Euclidean distance to the boundary.

For further discussion we fix some notational conventions consistent with that introduced in §3.1 and some terminology.

Let $g_{\mathbb{B}^n}$ be the Kähler-Einstein metric on the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ normalized to have constant Ricci curvature $-(n+1)$, and denote its volume form by $dV_{\mathbb{B}^n}$. On a complex manifold M let $\mathcal{K}_M$ be the space of all holomorphic maps $f: \mathbb{B}^n \to M$. For a holomorphic n-vector η at a point $x \in M$ its norm with respect to the Kobayashi (pseudo-)volume form μ_M is given by $\|\eta\|_{\mu_M} = \inf\{\|\xi\|_{dV_{\mathbb{B}^n}} : f_*\xi = \eta \text{ for some } f \in \mathcal{K}_M\}$. In the case where $M = U$ is a bounded domain $\pi: U \to Z$ spread over a Stein manifold Z, μ_U is a volume form, i.e., $\mu_U > 0$ everywhere, and the same is true for any quotient manifold of U by a torsion-free discrete group of automorphisms.

A complex manifold N is said to be quasi-compact if and only if there exists a compact complex manifold $N^\sharp$ such that $N \subset N^\sharp$ and $N^\sharp - N \subset N^\sharp$ is a complex analytic subvariety. Equivalently, N is quasi-compact if and only if it is an open subset of $N^\sharp$ with respect to the Zariski topology on $N^\sharp$ where the closed subsets are exactly the complex analytic subvarieties of $N^\sharp$. (We will simply say that N is Zariski open in $N^\sharp$.)

We are going to apply the solution to the Extension Problem given in Theorem 1.1 to holomorphic mappings $f: X \to N$ which induce isomorphisms on fundamental groups. We assume that N is quasi-compact. In this case we prove that there is a holomorphic retraction of N onto $f(X)$. More precisely, we have

Theorem 7.1. (The Fibration Theorem) *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let N be a quasi-compact complex manifold and denote by $\widetilde{N}$ its universal covering space, $N = \widetilde{N}/\Gamma'$. Let $f: X \to N$ be a holomorphic mapping into N inducing an isomorphism $f_*: \Gamma \xrightarrow{\cong} \Gamma'$ on fundamental groups. Suppose $(X, N; f)$ satisfies the nondegeneracy condition (♣) (as in the statement of Theorem 1.1). Then, $f: X \to N$ is a holomorphic embedding, and there exists a holomorphic fibration $\rho: N \to X$ such that $\rho \circ f = id_X$.*

In the Fibration Theorem, since $f_*: \Gamma \xrightarrow{\cong} \Gamma'$, there is a smooth map $g_0: N \to X$ such that $(g_0)_* = (f_*)^{-1}$ on fundamental groups. In the case where N is compact and equipped with a Kähler metric, and where X is also assumed to be compact, there is a harmonic map $g: N \to X$ homotopic to g_0, and by the method of strong rigidity starting with [33], g gives the holomorphic fibration $\rho: N \to X$. The method of harmonic maps can no longer be applied when we drop the Kähler condition on N, even when N is assumed to be compact. Thus, even in the compact case the strength of the Fibration Theorem lies on the use of bounded holomorphic functions on the universal covering space $\widetilde{N}$ of N in place of the Kähler condition on N. When N is quasi-compact but not compact then the method of harmonic maps does not always work even when N is equipped with a complete Kähler metric.

One of our primary objectives in relating bounded holomorphic functions to rigidity problems is to develop a theory applicable to holomorphic mappings from irreducible finite-volume quotients of bounded symmetric domains of rank ≥ 2 by torsion-free lattices to complex manifolds N uniformized by arbitrary bounded domains. In this case the nondegeneracy condition (♣) in Theorem 1.1 (the

Extension Theorem) is always satisfied for any nonconstant holomorphic mapping $f : X \to N$. We will apply Theorem 1.1 to such mappings assuming now as in the Fibration Theorem that the induced map on fundamental groups is an isomorphism. We look for some natural geometric condition on N which allows us to establish an analogue of the Fibration Theorem, in which case one expects the fibers on Ω to reduce to single points. We establish the following main result in §8 yielding a biholomorphism under the assumption that the target manifold $N = D/\Gamma'$ is of finite intrinsic measure with respect to the Kobayashi volume form μ_N, a condition necessary for N to admit a realization as a Zariski open subset of some compact complex manifold.

Theorem 7.2. (The Isomorphism Theorem) *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let D be a bounded domain on a Stein manifold, Γ' be a torsion-free discrete group of automorphisms on D, $N := D/\Gamma'$. Suppose N is of finite intrinsic measure with respect to the Kobayashi volume form μ_N, and $f : X \to N$ is a holomorphic map which induces an isomorphism $f_* : \Gamma \xrightarrow{\cong} \Gamma'$. Then, $f : X \xrightarrow{\cong} N$ is a biholomorphic map.*

Remark 7.3. We note that in the statement of the Isomorphism Theorem we do not need to assume that D is simply connected. We need a slight variation in the formulation of the Theorem 1.1 (the Extension Theorem). In the proof of the latter result it is not essential to use the universal covering space $\widetilde{N}$. We may use any regular covering $\tau : \widetilde{N} \to N$ and still have the same conclusion. Note that when this variation of Theorem 1.1 is applied to the holomorphic mapping $f : X \to N = D/\Gamma'$ in Theorem 7.2, f still lifts to $F : \Omega \to D$. The slight variation of Theorem 1.1 with the weakened assumption that $\tau : \widetilde{N} \to N$ is a regular covering and the same conclusion will be taken as known and used in the rest of the article.

7.1 Complete Kähler-Einstein metrics and estimates on the Kobayashi volume form

For the Isomorphism Theorem we are interested in the case where the target manifold N is uniformized by a bounded domain D on a Stein manifold. In our study of such manifolds we will need to resort to the use of canonical complete Kähler metrics. When D is assumed furthermore to be a domain of holomorphy, we have the canonical Kähler-Einstein metric. The existence of the metric was established by [4], and its completeness by [28]. More precisely, we have

Theorem 7.4. (Existence Theorem on Kähler-Einstein metrics) *Let Z be a Stein manifold of dimension n and $U \Subset Z$ be a bounded domain of holomorphy on Z. Then, there exists on U a unique complete Kähler-Einstein metric $g_{_{KE}}$ of Ricci curvature $-(n+1)$. The metric is furthermore invariant under $\mathrm{Aut}(U)$.*

Remark 7.5. The existence theorem was only stated for bounded domains on $\mathbb{C}^n$ but the same proof goes through when $\mathbb{C}^n$ is replaced by a Stein manifold Z, noting that Z can holomorphically embedded as a complex submanifold of some Euclidean space $\mathbb{C}^N$. We note that the invariance of $g_{_{KE}}$ under $\mathrm{Aut}(U)$ follows from the Ahlfors-Schwarz lemma for volume forms (cf. Yau [36]). Furthermore, the existence of $g_{_{KE}}$ on a bounded domain $U \Subset Z$ implies that U is a domain of holomorphy. It was in fact proven in [28] that any bounded domain on $\mathbb{C}^n$ admitting a complete Kähler-Einstein metric of negative Ricci curvature satisfies the Kontinuitätssatz of Oka's, and must therefore be a domain of holomorphy, and the same proof is valid when $\mathbb{C}^n$ is replaced by a Stein manifold Z.

In the formulation of the Isomorphism Theorem we assume that the target manifold $N = D/\Gamma'$ is of finite intrinsic measure with respect to the Kobayashi volume form μ_N. For the proof of the Isomorphism Theorem we need to work with the complete Kähler-Einstein metric. This is done by first passing to the hull of holomorphy $\widehat{D}$ of D. For the passage from D to $\widehat{D}$ and for estimates in the proof it is necessary to compare various canonical metrics and volume forms, as given in the following Comparison Lemma which results from the Ahlfors-Schwarz lemma for Kähler metrics and for volume forms (cf. [16] and the references given there). In what follows, for a complex manifold M let $\mathcal{C}_M$ denote the set of all holomorphic maps $h : M \to \Delta$. In the current article by the Carathódory metric κ_M on a complex manifold M we always mean the infinitesimal Carathéodory metric on M, i.e., the complex Finsler metric κ_M on M defined by $\|\xi\|_{\kappa_M} = \sup\{\|dh(\xi)\|_{g_\Delta} : h \in \mathcal{C}_M\}$ for $\xi \in T_M^{1,0}$, where the Poincaré metric g_Δ on the unit disk is normalized to be of constant Gaussian curvature -2 as in the above.

Lemma 7.6. (The Comparison Lemma) *Let U be a bounded domain of holomorphy on some n-dimensional Stein manifold, $g_{_{KE}}$ be the canonical complete Kähler-Einstein metric of constant Ricci curvature $-(n+1)$ on U, and denote by dV_{KE} its volume form. Then, for the infinitesimal Carathéodory metric κ_U and the Kobayashi volume form μ_U on U, we have*

$$g_{_{KE}} \geq \frac{2\kappa_U}{n+1}\ , \quad dV_{KE} \leq \mu_U.$$

We will need the following estimate for the Kobayashi volume form μ_U on a bounded domain $U \Subset \mathbb{C}^n$ in terms of distances to the boundary.

Proposition 7.7. *Let $U \Subset \mathbb{C}^n$ be a bounded domain, and denote by μ_U the Kobayashi volume form on U. For $z \in U$ denote by $\delta(z)$ the Euclidean distance of z from the boundary ∂U. Write dV for the Euclidean volume form on $\mathbb{C}^n$. Then, there exists a positive constant c depending only on n and the diameter of U such that*

$$\mu_U(z) \geq \frac{c}{\delta(z)}\, dV.$$

Proof. We will first deal with the case where $n = 1$. In this case, the Kobayashi volume form is the same as the infinitesimal Kobayashi metric, which agrees with the Poincaré metric, and we have the stronger lower estimate where $\frac{c}{\delta(z)}$ is replaced by $\frac{c}{\delta^2(z)(\log\delta)^2}$ (cf. [28]). The latter estimate relies on the Uniformization Theorem and does not carry over to the case of general n. We will instead prove the weaker estimate as stated in Proposition 7.7 for $n = 1$ using the maximum principle and Rouché's Theorem and then give the necessary modification to deduce the estimate for general n.

Let $z \in U$ and $f : \Delta \to U$ be a holomorphic function such that $f(0) = z$. Denote by w the Euclidean coordinate on Δ. We will show that for some absolute constant C to be determined, we have $|f'(0)| \leq C\sqrt{\delta(z)}$, which gives the estimate $\|\frac{\partial}{\partial w}\|^2_{\mu_U} \geq \frac{c}{\delta(z)}$ for $c = \frac{1}{C^2}$, regarding μ_U as the infinitesimal Kobayashi metric. Let $b \in \partial U$ be such that $|z - b| = \delta(z)$. To get an upper estimate for $|f'(0)|$ we are going to show that if $|f'(0)|$ were too large, then b would lie in the image f, leading to a contradiction. To this end consider the function $h(w) := f(w) - b$, $h : \Delta \to \Delta(2R)$ assuming $U \Subset \Delta(R), R < \infty$. The affine linear part of h at 0 is given by $L(w) = h'(0)w + h(0) = f'(0)w + (z-b)$, noting the trivial estimate $|f'(0)| \leq R$ by the maximum principle. Write $h(w) = L(w) + E(w)$. We claim that there is a constant $a > 0$ for which the following holds if $|f'(0)| \geq C\sqrt{\delta(z)}$ for any constant $C > \frac{3}{a}$.

$$(\&) : \quad \begin{cases} (a) \quad |L(w)| > 2\delta(z), & \text{whenever} \quad |w| = a\sqrt{\delta(z)}; \\ (b) \quad |E(w)| < \delta(z), & \text{whenever} \quad |w| = a\sqrt{\delta(z)} \end{cases}$$

From (a) and (b) it follows that $|h(w)| > \delta(z)$ whenever $|w| = a\sqrt{\delta(z)}$. To prove (b) of the claim observe that the "error" term $E(w)$ satisfies $E(0) = E'(0) = 0$ and $|E(w)| \leq |h(w)| + |L(w)| \leq$

$|h(w)| + |f'(0)||w| + |z-b| \leq 5R$ for $|w| < 1$, by the maximum principle applied to $\frac{E(w)}{w^2}$, so that (b) is valid whenever $(5R)a^2 < 1$. As to (a) choose now the constant C such that $C > \frac{3}{a}$. Then, for $|w| = a\sqrt{\delta(z)}$,

$$\big|L(w)\big| \geq \big(C\sqrt{\delta(z)}\,\big) \cdot |w| - \delta(z) > \frac{3}{a}\sqrt{\delta(z)}\big(a\sqrt{\delta(z)}\big) - \delta(z) > 2\delta(z) \tag{22}$$

so that (a) holds for $C > \frac{3}{a}$, completing the proof of the claim (&). For Proposition 7.7 in the case of $n = 1$ we can conclude by applying Rouché's Theorem to reach a contradiction whenever $|f'(0)| > C\sqrt{\delta(z)}$. In view of the generalization to several complex variables, we give the argument here. Assume $|f'(0)| > C\sqrt{\delta(z)}$. Consider $h_t(w) = L(w) + tE(w)$ for t real $0 \leq t \leq 1$. From (22) it follows that for $0 \leq t \leq 1$ we have $|h_t(w)| > (2-t)\delta(z) > 0$ whenever $|w| = a\sqrt{\delta(z)}$. For $t = 0$ the affine-linear function L admits a zero at $w = w_0 := \frac{b-z}{f'(0)}$, $|w_0| \leq \frac{\delta(z)}{C\sqrt{\delta(z)}} = \frac{\sqrt{\delta(z)}}{C} < \frac{a\sqrt{\delta(z)}}{3}$. In particular, $w_0 \in \Delta\big(a\sqrt{\delta(z)}\big)$ for the zero w_0 of $L(w) = h_0(w)$. For $0 \leq t \leq 1$ the number of zeros of h_t on the disk $\Delta\big(a\sqrt{\delta(z)}\big)$ is counted, by the Argument Principle, by the boundary integral

$$\frac{1}{2\pi}\int_{\partial\Delta\left(a\sqrt{\delta(z)}\right)} \sqrt{-1}\;\overline{\partial}\log|h_t|^2 = \frac{1}{2\pi}\int_{\Delta\left(a\sqrt{\delta(z)}\right)} \sqrt{-1}\partial\overline{\partial}\log|h_t|^2 \tag{23}$$

The boundary integral is well-defined, takes integral values, and varies continuously with t, so that it is independent of t, implying that there exists a zero of h_t on the disk $\Delta\big(\sqrt{\delta(z)}\big)$; $0 \leq t \leq 1$. In particular, for $t = 1$, $h_1(w) = h(w) = f(w) - b$, and $f(w) = b$ has a solution on $\Delta\big(\sqrt{\delta(z)}\big)$, contradicting with the assumption that $b \in \partial U$. This gives Proposition 7.7 for the special case where $n = 1$.

We now generalize the argument to several complex variables. Let $f : \mathbb{B}^n \to U$ be such that $f(0) = z$. Let again $b \in \partial U$ be a point such that $\delta(z) = b$. Consider the linear map $df(0)$. Assume $U \Subset \mathbb{B}^n(R)$, $R < \infty$. Considering $h(w) := f(w) - z$, $h(\mathbb{B}^n) \Subset \mathbb{B}^n(2R)$, by the Cauchy integral formula for the first derivative $\|df(0)(\eta)\| \leq R\|\eta\|$ for any $\eta \in T_0(\mathbb{B}^n) \cong \mathbb{C}^n$, where $\|\cdot\|$ denotes the Euclidean norm. To prove Proposition 7.7 for an arbitrary dimension n it suffices to get an estimate $|\det(df(0))| \leq C\sqrt{\delta(z)}$ for some constant $C > 0$ depending on U. In analogy to (a) and (b) in the case of $n = 1$, for the purpose of arguing by contradiction (in order to establish the estimate $|\det(df(0))| \leq C\sqrt{\delta(z)}\,$) we claim that for $U \Subset \mathbb{B}^n(R) \subset \mathbb{C}^n$ there exist constants $a, C > 0$ depending only on n and R for which the following holds assuming $|\det(df(0))| \geq C\sqrt{\delta(z)}$.

$$(\&)_n : \quad \begin{cases} (a)_n \quad \|L(w)\| > 2\delta(z) \quad \text{whenever} \quad \|w\| = a\sqrt{\delta(z)}; \\ (b)_n \quad \|E(w)\| < \delta(z) \quad \text{whenever} \quad \|w\| = a\sqrt{\delta(z)}. \end{cases}$$

Noting that $\|df(0)(w)\| \leq R\|w\|$ the argument for $(b)_n$ is the same as in the case of $n = 1$, and it suffices to choose a such that $(5R)a^2 < 1$. As for $(a)_n$, considering $|w| \in \partial\mathbb{B}^n(a\sqrt{\delta(z)})$ we have

$$\big|L(w)\big| \geq \|df(0)(w)\| - \|z - b\| = \|df(0)(w)\| - \delta(z). \tag{24}$$

To relate $\|df(0)(w)\|$ to $|\det(df(0)|$ suppose $df(0)(w) = \xi$ with $\|\xi\| = \alpha\|w\|$. Denoting by $w^\perp$ resp. $\xi^\perp$ the orthogonal complements of the nonzero vectors w and ξ in $\mathbb{C}^n$, we consider the linear map $\Lambda : w^\perp \to \xi^\perp$ given by $\Lambda = \pi \circ df(0)|_{w^\perp}$ where $\pi : \mathbb{C}^n \to \xi^\perp$ is the orthogonal projection. With respect to orthonormal bases of the $(n-1)$-dimensional complex vector spaces $w^\perp$ resp. $\eta^\perp$ we have $|\det(\Lambda)| \leq R^{n-1}$ by the Schwarz lemma while $|\det(df(0)| = \alpha(\det(\Lambda))$, giving $\alpha \geq \frac{|\det(df(0)|}{R^{n-1}}$. Choosing now any positive constant C such that $\frac{C}{R^{n-1}} > \frac{3}{a}$ for $\|w\| = a\sqrt{\delta(z)}$ and $|\det(df(0))| > C\sqrt{\delta(z)}$ we have

$$\alpha \geq \frac{|\det(df(0))|}{R^{n-1}} > \frac{C\sqrt{\delta(z)}}{R^{n-1}} > \frac{3}{a}\sqrt{\delta(z)}\,. \tag{25}$$

Thus, for $\|w\| = a\sqrt{\delta(z)}$ and assuming $|\det(df(0)| > C\sqrt{\delta(z)}$ we have by (24) and (25)

$$\|L(w)\| \geq \frac{3}{a}\sqrt{\delta(z)}\big(a\sqrt{\delta(z)}\big) - \delta(z) > 2\delta(z)\ , \tag{26}$$

yielding $(a)_n$ and completing the proof of the claim $(\&)_n$ for any n. From $(a)_n$ and $(b)_n$ it follows that $\|h(w)\| > \delta(z)$ whenever $\|w\| = a\sqrt{\delta(z)}$. In terms of the Euclidean coordinates $w = (w_1, ..., w_n)$ of the domain manifold define as for $n = 1$ the holomorphic map $h(w) = f(w) - b$. Decomposing $h(w) = L(w) + E(w)$ as in the case of $n = 1$, $L(w) = df(0)(w) + (z - b)$, and using exactly the same argument there we have a real 1-parameter family of holomorphic maps $h_t(w) = L(w)+tE(w)$. Hence, for $0 \leq t \leq 1$ we have $\|h_t(w)\| \geq \|L(w)\| - t\|E(w)\| \geq (2-t)\delta(z) > \delta(z)$ for $w \in \partial\mathbb{B}^n(a\sqrt{\delta(z)})$, hence $h_t(w) \neq 0$ for any $w \in \partial\mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$. For the analogue of Rouché's Theorem we note that the affine linear function $L(w) = df(0)(w)+(z-b)$ admits a unique zero at $w = w_0 = (df(0))^{-1}(b-z)$ on $\mathbb{C}^n$. By the hypothesis (for argument for contradiction) we have $\|df(0)(\eta)\| > \frac{3}{a}\sqrt{\delta(z)}\|\eta\|$ for $\eta \in T_0(\mathbb{B}^n) \cong \mathbb{C}^n$, hence $\|df(0)^{-1}(\xi)\| < \frac{a\|\xi\|}{3\sqrt{\delta(z)}}$ for $\xi \in T_z(U) \cong \mathbb{C}^n$, so that in particular $w_0 \in \mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$ and $h_0(w) = L(w)$ has a unique solution on $\mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$. Suppose for some t, $0 \leq t \leq 1, h_t(w) = 0$ is not solvable on $\mathbb{B}^n(a\sqrt{\delta(z)})$. Writing $h_t(w) = (h_{t,1}(w), \cdots, h_{t,n}(w))$, the coefficients $h_{t,k}(w)$ cannot be simultaneously zero, so that $[h_t] : \mathbb{B}^n \to \mathbb{P}^{n-1}$ is well-defined, and $\big(\sqrt{-1}\partial\,\overline{\partial}\log|h_t|^2\big)^n \equiv 0$, since the (1,1)-form inside the parenthesis is nothing other than the pull-back of the Kähler form of the Fubini-Study metric on $\mathbb{P}^{n-1}$, which is everywhere degenerate. If that happened, by Stokes' theorem we would have

$$I(t) := \frac{1}{(2\pi)^n}\int_{\partial\mathbb{B}^n\left(a\sqrt{\delta(z)}\right)} \sqrt{-1}\overline{\partial}\log|h_t|^2 \wedge \big(\sqrt{-1}\partial\overline{\partial}\log|h_t|^2\big)^{n-1} = 0\ . \tag{27}$$

The boundary integral is well-defined for $0 \leq t \leq 1$, with $I(0) = 1$. Obviously $I(t)$ varies continuously with t, but it is less clear that $I(t)$ is an integer for each t. To reach a contradiction to the assumption $b \in \partial U$ (as in the use of Rouché's Theorem for $n = 1$), we proceed as follows. The mapping $h_t := L+tE : \mathbb{B}^n \to \mathbb{C}^n$ is defined for any real t, and, for ϵ sufficiently small, in the interval $-\epsilon \leq t \leq 1+\epsilon$, h_t is not equal to 0 on $\partial\mathbb{B}^n\big(a\sqrt{\delta(t)}\big)$. Hence, the boundary integral $I(t)$ remains well-defined. The integral $I(t)$ then varies as a real-analytic function in t. For t sufficiently small, h_t is a biholomorphism of $\mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$ onto its image. The current $\big(\sqrt{-1}\partial\overline{\partial}\log|h_t|^2\big)^n$ over $\mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$ is given by $(2\pi)^n\delta_{x(t)}$, where $x(t)$ is the unique zero of h_t, and δ_x denotes the delta measure at x. Hence $I(t) = 1$ for t sufficiently small. It follows that $I(t) = 1$ for $0 \leq t \leq 1$ by real-analyticity, and we have a contradiction to (27) at $t = 1$. The proof of Proposition 7.7 is complete. □

For a bounded domain $U \Subset \mathbb{C}^n$ the Kobayashi volume form μ_U can be localized using Cauchy estimates, as follows. Suppose b lies on the boundary ∂U, and let B be a Euclidean coordinate ball centered at b, $B := \mathbb{B}^n(b;\rho)$. Suppose $0 < \epsilon < 1$, and write $B' := \mathbb{B}^n(b;(1-\epsilon)\rho)$. Then, there exists $r > 0$ depending only on the diameter of U and on ϵ such that for any holomorphic map $f : \mathbb{B}^n \to U$ satisfying $f(0) \in B' \cap U$ we have $f(\mathbb{B}^n(r)) \subset B \cap U$. This leads to an upper bound on the Kobayashi volume form $\mu_{B\cap U}\big|_{B'\cap U}$ in terms of $\mu_U\big|_{B'\cap U}$. For our application to the proof of the Isomorphism Theorem (Theorem 7.2) we formulate the localization estimate obtained by the same argument in a more general form as follows, noting the monotonicity property of the Kobayashi volume form.

Lemma 7.8. (Localization Lemma for the Kobayashi volume form) *Let $\pi : U \to Z$ be a bounded Riemann domain spread over a Stein manifold Z, $U_0 \subset U$ be a connected open subset, and $b \in \partial U_0$, the boundary of U_0 in U. Let $B \subset U$ be a neighborhood of b on U such that $\pi|_B : B \to Z$ is an open embedding of B onto a coordinate neighborhood $\pi(B)$ of $\pi(x)$ in Z, and $B' \Subset B$ be a neighborhood of b relatively compact in B. Then, there exists a positive constant C such that*

$$\mu_U(z) \leq \mu_{U\cap B}(z) \leq C\mu_U(z)$$

for any point $z \in B' \cap U$.

Next, for a bounded Riemann domain $\pi : U \to Z$ spread over a Stein manifold Z we will make use of the above Localization Lemma and Proposition 7.7 to study open subsets $U_0 \subset U$ of locally finite volume with respect to μ_U at any boundary point $b \in U - U_0$.

Proposition 7.9. *Let $\pi : U \to Z$ be a bounded Riemann domain spread over a Stein manifold Z, and $U_0 \subset U$ be an open subset. Assume $b \in U - U_0$ and let $B \subset U$ be a neighborhood of b in U such that $\pi|_B : B \to Z$ is an open embedding of B onto a coordinate neighborhood $\pi(B)$ of $\pi(x)$ in Z. Identify B as a Euclidean domain via the implicitly chosen local holomorphic coordinates on $\pi(B)$ and endow B hence with the Lebesgue measure λ. Suppose* Volume$(B' \cap U_0, \mu_{B \cap U_0}) < \infty$ *for any open subset $B' \Subset B$ relatively compact in B. Then, the closed subset $B - U_0 \subset B$ is of zero Lebesgue measure.*

Proof. We identify B with $\pi(B)$, $\pi|_B : B \xrightarrow{\cong} \pi(B)$, and then $\pi(B)$ via the holomorphic coordinate $\Phi : \pi(B) \xrightarrow{\cong} W \subset \mathbb{C}^n$, $\Phi \circ \pi|_B : B \xrightarrow{\cong} W$, so that B is identified with W. Write $W_0 := \Phi(\pi(B \cap U_0)) \subset W$. Using a locally finite collection of relatively compact open subsets covering W and applying the Localization Lemma for the Kobayashi volume form (Lemma 7.8), the proof of Proposition 7.9 can be reduced to that of the following special situation.

Identify $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way, and hence $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Let I denote the closed unit interval $[0, 1]$ and $I_0 := (0, 1)$ be the open unit interval, $\overline{I_0} = I$. Let α be a positive real number, and write $J := [-\alpha, 1+\alpha]$, $J_0 := (-\alpha, 1+\alpha)$, $\overline{J_0} = J$. Assume that $J^{2n} \subset W$. Write $E' = J^{2n} - \Phi(\pi(U_0))$. $E := I^{2n} - \Phi(\pi(U_0))$. Assume that $E \subset I^{2n-1} \times [\epsilon, 1]$ for some $\epsilon, 0 < \epsilon < 1$, so that $I^{2n-1} \times [0, \epsilon) \subset I^{2n} - E$. On $J^{2n} - E'$ denote by δ the Euclidean distance to E', i.e., $\delta(x) = \sup\{r : \mathbb{B}^n(x; r) \cap E' = \emptyset\}$ for $x \notin E'$. By Proposition 7.7, for some constant C we have

$$\int_{I^{2n}-E} \frac{dV}{\delta} \leq C \int_{I_0^{2n}-E} \mu_{J_0^{2n}-E'} < \infty \tag{28}$$

where dV denotes the Euclidean volume form on $\mathbb{R}^{2n}$. Then, we need to prove that $\lambda(E) = 0$ for the Lebesgue measure λ on $\mathbb{R}^{2n}$. Let $S \subset I^{2n-1}$ be the closed subset consisting of all parameters $s \in I^{2n-1}$ such that $\big(\{s\} \times I\big) \cap E \neq \emptyset$. Denote by t the Euclidean variable for the last direct factor of I^{2n}. For each parameter $s \in S$ we claim that

$$\int_{(\{s\}\times I)-E} \frac{dt}{\delta} = \infty\ . \tag{29}$$

To see this note that for each $s \in S$, $\delta(s; t) \leq |t - t_0|$ for any t_0 such that $(s; t_0) \in E$. For $s \in S$, taking $t_0 \in [\epsilon, 1]$ to be the smallest number such that $(s; t_0) \in E$, we have

$$\int_{(\{s\}\times I)-E} \frac{dt}{\delta} \geq \int_0^{t_0} \frac{dt}{t_0 - t} = \infty\ , \tag{30}$$

justifying the claim (2). By Fubini's theorem, the closed subset $S \subset I^{2n-1}$ is of zero Lebesgue measure, hence $E \subset S \times I$ is of zero Lebesgue measure, as desired. □

Given any unramified covering map $\chi : \widetilde{M} \to M$, by the fact that $\mathbb{B}^n$ is simply-connected, $\chi^* \mu_M$ on M agrees with $\mu_{\widetilde{M}}$ since any map $f \in \mathcal{K}_M$ lifts to $\widetilde{f} \in \mathcal{K}_{\widetilde{M}}$. Using Proposition 7.7 we deduce the following result crucial to the proof of the Isomorphism Theorem (Theorem 7.2). It relates the covering domain D to its hull of holomorphy $\widehat{D}$, and allows us to enlarge N to a complex manifold admitting a complete Kähler-Einstein metric of finite volume.

Proposition 7.10. *Let $D \subset Z$ be an a bounded domain on a Stein manifold Z, $\Gamma' \subset \mathrm{Aut}(D)$ be a torsion-free discrete group of automorphisms of D such that $N = D/\Gamma'$ is of finite measure with respect to μ_D. Let $\pi : \widehat{D} \to Z$ be the hull of holomorphy of D. Then, Γ' extends to a torsion-free discrete group of automorphisms Γ' of $\widehat{D}$ such that, writing $\widehat{N} := \widehat{D}/\widehat{\Gamma}'$, $\widehat{N}$ is of finite volume with respect to $\mu_{\widehat{N}}$.*

Proof. From Cartan's theorem on limits of automorphisms of bounded domains (cf. Narasimhan [29]), which generalizes readily to bounded domains on Stein manifolds, one deduces that $\Gamma' \subset \mathrm{Aut}(D)$ extends canonically to a discrete subgroup $\widehat{\Gamma}' \subset \mathrm{Aut}(\widehat{D})$, $\widehat{\Gamma}' \cong \Gamma'$ of automorphisms of the hull of holomorphy $\widehat{D} \supset D$ (cf. Mok-Wong [27], Lemma in (1.1) therein). Since torsion-freeness of Γ' is an algebraic property of the abstract group Γ', which is isomorphic to $\widehat{\Gamma}'$, the latter group is also torsion-free. Thus, $\widehat{\Gamma}'$ acts properly discontinuously on $\widehat{D}$ without fixed points and $\widehat{N} := \widehat{D}/\widehat{\Gamma}'$ is a complex manifold.

Since $\mu_{\widehat{N}} \le \mu_N$ on N, $\mathrm{Volume}(N, \mu_{\widehat{N}}) \le \mathrm{Volume}(N, \mu_N) < \infty$. On the other hand, $\mathrm{Volume}(\widehat{N} - N, \mu_{\widehat{N}})$ is obtained by integrating $\mu_{\widehat{N}}$ over $\widehat{N} - N$. Covering $\widehat{N}$ by a countable number of open subsets $B_\alpha \subset \widehat{N}$, $\alpha \in A$, such that each B_α is biholomorphic to a Euclidean domain W_α, and denoting by λ_α the Lebesgue measure on B_α thus obtained from $W_\alpha \subset \mathbb{C}^n$, we have $\mu_{\widehat{N}}\big|_{B_\alpha} = \theta_\alpha \cdot \lambda_\alpha$ for some locally bounded measurable function θ_α on B_α such that $B_\alpha \cap (\widehat{N} - N)$ is of zero Lebesgue measure with respect to λ_α, by Proposition 7.9. We conclude that $\mathrm{Volume}(\widehat{N}, \mu_{\widehat{N}}) = \mathrm{Volume}(N, \mu_{\widehat{N}}) < \infty$, as desired. □

By Proposition 7.9 it follows readily that $\widehat{D} - D$ is locally of zero Lebesgue measure at any boundary point b of D in $\widehat{D}$. In other words, for any such point b there is a coordinate neighborhood U of b on $\widehat{D}$ such that $U - D$ is a zero Lebesgue measure. In particular, $\widehat{D} - D$ does not contain any nonempty open subset, and it follows that the Riemann domain $\widehat{D}$ spread over the Stein manifold Z is schlicht, hence we may from now on identify $\widehat{D}$ as a bounded domain on Z.

Since $\widehat{D} \Subset Z$ is a bounded domain of holomorphy, by the Existence Theorem on canonical Kähler-Einstein metrics (Theorem 7.4) on bounded domains of holomorphy on Stein manifolds there is a unique complete Kähler-Einstein metric g_{KE} and the accompanying Kähler form ω_{KE} on $\widehat{D} \Subset Z$ which is invariant under $\mathrm{Aut}(\widehat{D})$. By Γ'-invariance, g_{KE} and ω_{KE} descend to $\widehat{N} = \widehat{D}/\Gamma'$, and we use the same notations to denote the Kähler metric resp. Kähler form on the quotient manifold $\widehat{N} \supset N$.

Corollary 7.11. *Let $\widehat{N} \supset N$ be the complex manifold as in Proposition 7.10. Then $\widehat{N}$ admits a unique complete Kähler-Einstein metric g_{KE} of finite volume and of constant Ricci curvature $-(n+1)$, $n = \dim N$.*

Proof. It remains to prove that $(\widehat{N}, g_{KE})$ is of finite volume. This follows from $\mathrm{Volume}(\widehat{N}, \omega_{KE}) \le \mathrm{Volume}(\widehat{N}, \mu_{\widehat{N}}) < \infty$, where the first inequality comes from the Comparison Lemma (Lemma 7.6) and the second inequality is in Proposition 7.10. □

8 Proofs of the Fibration Theorem and the Isomorphism Theorem

8.1 Proof of the Fibration Theorem

We deduce first of all the Fibration Theorem (Theorem 7.1) from the Extension Theorem (Theorem 1.1).

Proof. (The Fibration Theorem) By the hypothesis Ω is of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ is an irreducible torsion-free lattice. By Margulis [13], Γ is an arithmetic lattice, and the minimal compactification $\overline{X}_{\min} \supset X$ of Satake [31] and Baily-Borel [2] is projective, so that $X \subset \overline{X}_{\min}$ inherits naturally the structure of a quasi-projective manifold. Also by the hypothesis $N = \widetilde{N}/\Gamma'$ is quasi-compact, i.e., N is a dense Zariski open subset of some compact complex manifold $N^\sharp$.

It is further assumed that $f : X \to N$ induces an isomorphism $f_* : \pi_1(X) = \Gamma \xrightarrow{\cong} \Gamma' = \pi_1(N)$ on fundamental groups and that $(X, N; f)$ satisfies the condition (♣) concerning pull-backs of bounded holomorphic functions on $\widetilde{N}$ by the lifting $F : \Omega \to \widetilde{N}$ of $f : X \to N$ to universal covers. Let $R : \widetilde{N} \to \Omega$ be the holomorphic mapping given by Theorem 1.1 such that $R \circ F = id_\Omega$. We are going to prove that R descends to $\rho : N \to X$, in other words, that we have the following commutative diagram.

$$\begin{tikzcd} \Omega \arrow[r, "F"] \arrow[d] & \widetilde{N} \arrow[l, bend right, "R"'] \arrow[d] \\ X \arrow[r, "f"] & N \arrow[l, bend left, dashed, "\rho"] \end{tikzcd}$$

Assume first of all that Ω is irreducible (and of rank ≥ 2). We have first to prove that $R(\widetilde{N}) \subset \Omega$. Let $\alpha \in T_0(\Omega)$ be a minimal rational tangent vector of unit length at the origin 0, $\Delta_\alpha \subset \Omega$ be the minimal disk passing through 0 such that $\alpha \in T_0(\Delta_\alpha)$. Let $L_\alpha : \mathbb{C}^n \to \mathbb{C}\alpha$ be the Euclidean orthogonal projection, which projects Ω onto Δ_α. We identify $\mathbb{C}\alpha$ isometrically with $\mathbb{C}$ and hence Δ_α with Δ. We claim that $R(\widetilde{N}) \subset \Omega$. Denote by $\tau : \widetilde{N} \to N$ the universal covering map. For any bounded holomorphic function ψ on $\widetilde{N}$, consider the function $\psi_\theta : N \to \mathbb{R}$ defined by $\psi_\theta(q) = \sup\{|\theta(p)| : \tau(p) = q\}$. Defining $\widetilde{\psi} : \widetilde{N} \to \mathbb{R}$ by $\widetilde{\psi}(p) = \psi(\tau(p))$, we have

$$\widetilde{\psi}(p) = \sup\left\{|\theta(\gamma(p))| : \gamma \in \Gamma'\right\} = \sup\left\{|(\theta \circ \gamma)(p)| : \gamma \in \Gamma'\right\}. \tag{31}$$

Hence, $\widetilde{\psi}$ is the supremum of the absolute values $|\theta_\gamma|$, $\gamma \in \Gamma'$, on a family of holomorphic functions $\theta_\gamma : \widetilde{N} \to \Delta$, $\theta_\gamma := \theta \circ \gamma$. From Cauchy estimates on first derivatives $\widetilde{\psi}_\theta$ is uniformly Lipschitz and hence continuous. Thus, $\psi : N \to \mathbb{R}$ is a bounded continuous plurisubharmonic function. Since $N \subset N^\sharp$ is quasi-compact, ψ_θ extends by the Riemann extension theorem for bounded plurisubharmonic functions to $N^\sharp$, and is hence a constant function by the maximum principle. Applying the same argument now to the function $\theta_\alpha := L_\alpha \circ R$ we conclude that ψ_{θ_α} must be identically equal to 1, since $\psi_{\theta_\alpha}(p) = 1$ for any $p \in f(X)$. Taking all possible minimal rational tangent vectors α of unit length at 0 one concludes readily that $R(\widetilde{N}) \subset \overline{\Omega}$, as can be seen for instance from the polydisk theorem. It remains to show that $R(\widetilde{N}) \cap \partial\Omega = \emptyset$. Identifying Ω as an open subset of $T_0(\Omega)$ by the Harish-Chandra embedding, we have $\overline{\Omega} = \{\eta \in T_0(\Omega) : \|\eta\|_{\kappa_\Omega} \leq 1\}$ for the Carathéodory metric κ_Ω on Ω. If $R(\widetilde{N}) \cap \partial\Omega \neq \emptyset$, then the plurisubharmonic function $\theta(w) := \|R(w)\|_{\kappa_\Omega}$ on $\widetilde{N}$ attains its maximum value 1, and must therefore be identically equal to 1, so that $R(\widetilde{N}) \subset \partial\Omega$. But this contradicts with the fact that $R(p) \in \Omega$ for any point $p \in F(\Omega)$, and proves $R(\widetilde{N}) \subset \Omega$ via argument by contradiction, as desired.

Using $f_* : \Gamma \xrightarrow{\cong} \Gamma'$ we identify Γ with Γ'. For every $\gamma \in \Gamma$ and any $p \in F(\Omega), p = F(x)$, we have $R(\gamma(p)) = \gamma(R(p)) = \gamma(x)$ by definition. Consider now the vector-valued holomorphic map $T_\gamma : \widetilde{N} \to \mathbb{C}^n$ given by $T_\gamma(p) = R(\gamma(p)) - \gamma(R(p))$. Then T_γ vanishes identically on $F(\Omega)$. Considering the plurisubharmonic function $\|T_\gamma\|$ on $\widetilde{N}$ and descending to N by taking suprema over fibers of $\tau : \widetilde{N} \to N$ we conclude using Riemann extension and the maximum principle as in the above that T_γ vanishes identically on $\widetilde{N}$, i.e., we have the identity $R \circ \gamma \equiv \gamma \circ R$ on all of $\widetilde{N}$. It follows that the holomorphic mapping $R : \widetilde{N} \to \Omega$ descends to $\rho : N \to X$. Since $R \circ F \equiv id_\Omega$ we conclude that $\rho \circ f \equiv id_X$, proving Theorem 7.1 in the case where Ω is irreducible. For the general case where Ω may be reducible it suffices to consider pull-backs of bounded holomorphic functions which are nonconstant on irreducible factor subdomains of Ω and the proof follows *verbatim*. □

8.2 Proof of the Isomorphism Theorem

For the proof of the Theorem 7.2 (Isomorphism Theorem) we proceed now to justify the same line of argument by first proving the constancy of analogous functions ψ_θ. This will be demonstrated by integrating by parts on complete Kähler manifolds, for which purpose we will pass to the hull of holomorphy $\widehat{D}$ of D and make use of complete Kähler-Einstein metrics as explained in §2. A further argument, again related to the vanishing of certain bounded plurisubharmonic functions, will be needed to show that the holomorphic fibration obtained is trivial. For the proof of the Isomorphism Theorem along brown these lines of thought we will need

Lemma 8.1. *Let (Z,ω) be an s-dimensional complete Kähler manifold of finite volume, and u be a nonnegative uniformly Lipschitz bounded plurisubharmonic function on Z. Then, u is a constant function.*

Proof. Fix a base point $z_0 \in Z$. For $R > 0$ denote by B_R the geodesic ball on (Z,ω) of radius R centered at z_0. There exists a smooth nonnegative function $\rho_{_R}$ on Z, $0 \le \rho_{_R} \le 1$, such that $\rho_{_R} \equiv 1$ on B_R, $\rho_{_R} \equiv 0$ outside B_{R+1}, so that $\mathrm{Supp}(d\rho_{_R}) \subset B_{R+1} - \overline{B_R}$, and such that $\|d\rho_{_R}\| \le 2$. By Stokes' theorem, we have

$$0 = \int_Z \sqrt{-1}d(\rho_{_R} u) \wedge \overline{\partial} u \wedge \omega^{s-1} + \int_Z \rho_{_R}\sqrt{-1}u\partial\overline{\partial}u \wedge \omega^{s-1} \ . \tag{32}$$

Here $\sqrt{-1}\partial\overline{\partial}u \ge 0$ in the sense of currents, hence it has coefficients which are complex measures when expressed in terms of local holomorphic coordinates, and $\sqrt{-1}u\partial\overline{\partial}u$ is well-defined since u is a bounded function. We have

$$\begin{aligned} \int_{B_R} \sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \omega^{s-1} &\le \int_Z \rho_{_R}\sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \omega^{s-1} \\ = \ -\int_Z \sqrt{-1}u\partial\rho_{_R} \wedge \overline{\partial}u \wedge \omega^{s-1} &- \int_Z \rho_{_R}\sqrt{-1}u\partial\overline{\partial}u \wedge \omega^{s-1} \ . \end{aligned} \tag{33}$$

Here and in what follows, $\|\cdot\|$ will denote norms on Z arising from $\omega_{_{KE}}$. By assumption $\|du\|$ is uniformly bounded. Furthermore, $\|d\rho_{_R}\| \le 2$, and its support is contained in $Z - B_R$, so that the second last term of (33), up to a fixed constant, is bounded by $\mathrm{Volume}(Z - B_R, \omega)$, which decreases to 0 as $R \to \infty$ since $\mathrm{Volume}(Z,\omega) < \infty$ by assumption. On the other hand the last integral is nonnegative since $u \ge 0$ and u is plurisubharmonic. Fix any $R_0 > 0$. It follows from (1) that for some constants $C_1, C_2 > 0$, we have

$$\begin{aligned} \int_{B_R} \|\partial u\|^2 \le C_1 \int_Z \left\| u\partial\rho_{_R} \wedge \overline{\partial}u \wedge \omega^{s-1} \right\| &\le C_2 \mathrm{Volume}\Big(\mathrm{Supp}(d\rho_{_R}), \omega_{_{KE}}\Big) \\ \le C_2\mathrm{Volume}\Big(B_{R+1} - \overline{B_R}, \omega_{_{KE}}\Big) &< \mathrm{Volume}\Big(Z - \overline{B_R}, \omega_{_{KE}}\Big) \to 0 \end{aligned} \tag{34}$$

as $R \to \infty$, since $\mathrm{Volume}(Z, \omega_{_{KE}}) < \infty$ by the hypothesis. As a consequence, we have $\partial u \equiv 0$, hence $du \equiv 0$ since u is real-valued. In other words, the real-valued function u is a constant function, as desired. □

We are now ready to prove the main application of Theorem 1.1 (the Extension Theorem), as follows.

Proof. (The Isomorphism Theorem) By Proposition 7.10, we can enlarge D to a bounded domain of holomorphy $\widehat{D}$ on a Stein manifold and extend Γ' to a torsion-free discrete group of automorphisms $\widehat{\Gamma}'$, such that for $\widehat{N} = \widehat{D}/\widehat{\Gamma}'$ we have $\mathrm{Volume}\big(\widehat{N}, \mu_{\widehat{N}}\big) < \infty$ for the Kobayashi volume form $\mu_{\widehat{N}}$.

By Corollary 7.11, $\widehat{N}$ carries a (unique) complete Kähler-Einstein metric g_{KE} of constant Ricci curvature $-(n+1)$, $n = \dim(N)$, with Kähler form ω_{KE} such that $\mathrm{Volume}(\widehat{N}, \omega_{KE}) < \infty$.

Consider now the holomorphic map $f : X \to N \subset \widehat{N}$ as having image in $\widehat{N}$. Applying Theorem 1.1 (the Extension Theorem), we can extend the inverse map $i : F(\Omega) \xrightarrow{\cong} \Omega$ to $\widehat{R} : \widehat{D} \to \mathbb{C}^n$ as a bounded holomorphic map. We claim, in analogy to the proof of Theorem 7.1 (the Fibration Theorem), that $\widehat{R}(\widehat{D}) \subset \Omega$. Recall that there for any bounded holomorphic function θ on $\widetilde{N}$, $\tau : \widetilde{N} \to N$ being the universal covering map, we defined $\psi_\theta(q) = \sup\{|\theta(p)| : \tau(p) = q\}$. To establish $\widehat{R}(\widetilde{N}) \subset \Omega$ in the proof of the Fibration Theorem, a key point was to show that the bounded plurisubharmonic function ψ_θ on $\widetilde{N}$ is a constant.

In the current situation of Theorem 7.2 (the Isomorphism Theorem), we have $N = D/\Gamma' \subset \widehat{D}/\widehat{\Gamma}' = \widehat{N}$. Since the Carathéodory metric $\kappa_{\widehat{D}}$ is invariant under $\mathrm{Aut}(\widehat{D})$, it descends to a complex Finsler metric on $\widehat{N}$, to be denoted by $\nu_{\widehat{N}}$. Write $\widehat{\tau} : \widehat{D} \to \widehat{N}$ for the canonical projection, which is a regular covering map. In analogy to the proof of the Fibration Theorem here, for the Isomorphism Theorem we have to find an inverse of the holomorphic map $f : X \to N \subset \widehat{N}$, i.e., to find an inverse holomorphic map $\widehat{\rho} : \widehat{N} \to X$ making the following diagram commutative.

$$
\begin{tikzcd}
\Omega \arrow[r, "F"] \arrow[d] & D \arrow[r, hook] \arrow[d, "\tau"] & \widehat{D} \arrow[d, "\widehat{\tau}"] \arrow[ll, bend right, "\widehat{R}"'] \\
X \arrow[r, "f"] & N \arrow[r, hook] & \widehat{N} \arrow[ll, dashed, bend left, "\widehat{\rho}"]
\end{tikzcd}
$$

For a bounded holomorphic function θ on $\widehat{D}$ we define now $\widehat{\psi}_\theta(q) := \sup\{|\theta(p)| : \widehat{\tau}(p) = q\}$. As in (31), $\widehat{\psi}_\theta$ is uniformly Lipschitz, hence differentiable almost everywhere, and, where defined, $\|d\widehat{\psi}_\theta\|_{\kappa^*_{\widehat{D}}}$ is uniformly bounded. Here and in what follows for a complex Finsler metric σ on a complex manifold M, a point $y \in M$ and a $(1,0)$-covector β at y, we define $\|\beta\|_{\sigma^*} := \Big\{\sup|\beta(\eta)| : \eta \in T^{1,0}_y(M), \|\eta\|_\sigma \le 1\Big\}$, which defines σ^* as a complex Finsler metric on the holomorphic cotangent bundle $T^*_M \cong T^{1,0}(M)^*$. In case σ is the norm function of a Hermitian metric h on T_M, then σ^* is the norm function of the dual Hermitian metric h^* of h on T^*_M. By the Comparison Lemma (Lemma 7.6), g_{KE} dominates a constant multiple of $\kappa_{\widehat{D}}$, so that $\|d\widehat{\psi}_\theta\|_{g^*_{KE}}$ is uniformly bounded on $\widehat{N}$. By Lemma 8.1, it follows that ψ_θ is a constant, so that $\widehat{R}(\widehat{D}) \subset \Omega$. The same argument applied to the bounded vector-valued holomorphic functions $T_\gamma = \widehat{R} \circ \gamma - \gamma \circ \widehat{R}$ yields $T_\gamma \equiv 0$ and hence the equivariance of $\widehat{R}$ under Γ. As a consequence, the analogue of the Fibration Theorem remains valid, i.e., there exists a holomorphic map $\widehat{\rho} : \widehat{N} \to X$ such that $\widehat{\rho} \circ f \equiv id_X$.

To complete the proof of the Isomorphism Theorem, it remains to show that $f : X \to N$ is an open embedding. Knowing this, we will have $f \circ \widehat{\rho} \equiv id_{\widehat{N}}$ by the identity theorem for holomorphic functions, so that f maps X biholomorphically onto $\widehat{N}$. But, by the hypothesis $f(X) \subset N$, so that $\widehat{N} = N$ and we will have established that $f : X \to N$ is a biholomorphism.

We proceed to prove

($\sharp$): $f : X \to N \subset \widehat{N}$ is an open embedding.

Suppose otherwise. Then, $n = \dim(N) > \dim(X) := m$ and the fibers $\widehat{\rho}^{-1}(x)$ of $\widehat{\rho} : \widehat{N} \to X$ are positive-dimensional. Let $x_0 \in X$ be a regular value of $\widehat{\rho} : \widehat{N} \to X$, and $L \subset \widehat{\rho}^{-1}(x_0)$ be an irreducible component, $\dim(L) = n - m > 0$. We claim first of all that

($\flat$): L lifts in a univalent way to $\widehat{D}$, i.e., $\widehat{\tau}|_{\widehat{L}} : \widehat{L} \xrightarrow{\cong} L$ is a biholomorphism.

From ($\flat$) we will deduce ($\sharp$).

To establish ($\flat$) let $\widetilde{x}_0 \in \Omega$ be such that $\pi(\widetilde{x}_0) = x_0$ and $\widehat{L} \subset \widehat{D}$ be a connected component of $\widehat{R}^{-1}(\widetilde{x}_0)$, so that $\widehat{\tau}(\widehat{L}) = L$ for the covering map $\widehat{\tau} : \widehat{D} \to \widehat{N}$. Suppose $\gamma \in \Gamma$ acts as a covering transformation on $\widehat{D}$ such that $\gamma(\widehat{L}) = \widehat{L}$. (Recall that Γ is identified with Γ' via $f : \Gamma \xhookrightarrow{\cong} \Gamma'$.) The statement ($\flat$) will follow if we show that γ must necessarily be the identity element $id_{\widehat{D}}$. Since $\gamma(\widehat{L}) = \widehat{L}$, for $p \in \widehat{L}$ we have $\gamma(p) \in \widehat{L}$, so that $\widehat{R}(\gamma(p)) = \widehat{R}(p) = x_0$. On the other hand, by the Γ-equivariance of $\widehat{R}$ we have $\widehat{R}(\gamma(p)) = \gamma(\widehat{R}(p))$, and it follows that $\gamma(\widehat{R}(p)) = \widehat{R}(p)$, hence $x_0 = \widehat{R}(p)$ is fixed by $\gamma \in \Gamma'$. Denoting by $\iota : D \to \widehat{D}$ the natural inclusion map. Since $\Gamma \cong \Gamma' \subset \mathrm{Aut}(D) \xrightarrow{\iota_*} \mathrm{Aut}(\widehat{D})$ is torsion-free, $\gamma \in \Gamma' \cong \Gamma$ has no fixed point on $\widehat{D}$ unless $\gamma = id_{\widehat{D}}$, and we have proven that $\gamma(\widehat{L}) = \widehat{L}$ implies $\gamma = id_{\widehat{D}}$, i.e., we have established the statement ($\flat$) that $\widehat{\tau}|_{\widehat{L}}$ maps $\widehat{L}$ bijectively onto L.

We are going deduce ($\sharp$) from ($\flat$). More precisely, from the assumption that $n = \dim(N) > \dim(X) = m$, and from the existence of a *single* irreducible component $\widehat{L} \subset \Omega$ of $\widehat{\rho}^{-1}(x_0)$ for some regular value $x_0 \in X$ of $\widehat{\rho} : \widehat{N} \to X$ such that $\widehat{\tau}|_{\widehat{L}} : \widehat{L} \xrightarrow{\cong} L\ (\subset \widehat{N})$ is a biholomorphism and such that $\mathrm{Volume}(\widehat{L}, \omega_{_{KE}}) < \infty$, we are going to establish ($\sharp$) via argument by contradiction.

$$
\begin{tikzcd}
\Omega \arrow[r, "F"] \arrow[d, "\pi"] & D \arrow[r, hook, "\iota"] \arrow[d, "\tau"] & \widehat{D} \arrow[d, "\widehat{\tau}"] \arrow[ll, bend right, "\widehat{R}"'] & \supset & \widehat{L} \arrow[d, "\widehat{\tau}|_{\widehat{L}}", "\cong"'] \\
X \arrow[r, "f"] & N \arrow[r, hook] & \widehat{N} \arrow[ll, bend left, "\widehat{\rho}"] & \supset & L
\end{tikzcd}
$$

Let θ be a bounded holomorphic function on the bounded domain of holomorphy $\widehat{D}$ on the Stein manifold Z such that $\theta|_{\widehat{L}}$ is not identically a constant. Then, $u := |\theta|^2$ gives a nonnegative smooth plurisubharmonic function on $\widehat{L} \cong L$ such that du is uniformly bounded with respect to $\omega_{_{KE}}$. If we know $\mathrm{Volume}\,(L, \omega_{_{KE}}) < \infty$, then Lemma 8.1 applies to yield a contradiction. We only know $\mathrm{Volume}(\widehat{N}, \omega_{_{KE}}) < \infty$, and proceed now to prove that there exists some choice of regular value $x_0 \in X$ of $\widehat{\rho} : \widehat{N} \to X$ such that $(L, \omega_{_{KE}})$ is indeed of finite volume, and this will be sufficient for our argument. The proof will proceed via Fubini's theorem, and our proof will in fact show that for almost all regular values x of $\widehat{\rho}$, $q := f(x)$, $\mathrm{Volume}\,(L_q, \omega_{_{KE}}) < \infty$ for the irreducible component $L_q \subset \widehat{N}$ of $\widehat{\rho}^{-1}(x)$ passing through q.

Let again $x_0 \in X$ be a regular value of $\widehat{\rho} : \widehat{N} \to X$, $q_0 := f(x_0)$. Let V be a simply connected neighborhood of x_0 in X. For $x \in V$ denote by $L_q \subset \widehat{\rho}^{-1}(x) \subset \widehat{N}$ the irreducible component of $\widehat{\rho}^{-1}(x)$ containing $q := f(x)$. Since $\widehat{\rho} \circ f \equiv id_X$, $\widehat{\rho}\big|_{f(V)} : f(V) \xrightarrow{\cong} V$ is nothing other than the inverse map of $f\big|_V : V \xrightarrow{\cong} f(V)$, so that $\widehat{\rho}$ is a submersion at any point $q \in f(V)$, hence the irreducible complex analytic subvariety $L_q \subset \widehat{N}$ is of complex dimension $n - m$ for any point $q \in f(V)$. By Sard's theorem, for almost all $x \in V$, x is a regular value of $\widehat{\rho} : \widehat{N} \to X$, and $L_q \subset \widehat{N}$ is an $(n-m)$-dimensional complex submanifold. Let $W \subset \widehat{N}$ be the connected component of the open subset $\widehat{\rho}^{-1}(V) \subset \widehat{N}$ which contains $f(V)$.

Let now θ be a bounded holomorphic function on $\widehat{D}$. Recall that $\kappa_{\widehat{D}}$ is the Carathéodory metric on $\widehat{D}$. By the Comparison Lemma (Lemma 7.6), $g_{_{KE}} \geq \mathrm{Const.} \times \kappa_{\widehat{D}}$. Since $\partial u = \overline{\theta}\partial\theta$ and θ is bounded, we have

$$
\begin{aligned}
\|\partial u(y)\|_{g^*_{KE}} &\leq \mathrm{Const.} \times \|\partial\theta(y)\|_{g^*_{KE}} \\
&= \mathrm{Const.} \times \sup\left\{|\partial\theta(\eta)| : \eta \in T_y(\widehat{D}), \|\eta\|_{g_{_{KE}}} \leq 1\right\} \qquad (35)\\
&\leq \mathrm{Const.}' \times \sup\left\{|\partial\theta(\eta)| : \eta \in T_y(\widehat{D}), \|\eta\|_{\kappa_{\widehat{D}}} \leq 1\right\} < \infty\ ,
\end{aligned}
$$

where the last line follows from the inequality $g_{KE} \geq \frac{2\kappa_U}{n+1}$ for a bounded domain $U \subset Z$ on a Stein manifold Z in the Comparison Lemma (Lemma 7.6). Denote by $\mathfrak{R}_{\widehat{\rho}} \subset f(V)$ the subset of all $q = f(x)$, where $x \in V$ is a regular value of $\widehat{\rho}$. By Sard's Theorem $\mathscr{S}_1 := f(V) - \mathfrak{R}_{\widehat{\rho}}$ is of zero Lebesgue measure. Consider the holomorphic map $\widehat{R} : \widehat{D} \to \Omega$. Then, by the infinitesimal distance decreasing property on Carathéodory metrics for holomorphic maps, we have $\kappa_{\widehat{D}} \geq R^*(\kappa_\Omega)$. Hence,

$$g_{KE} \geq \text{Const.} \times \widehat{R}^* \kappa_\Omega \ . \tag{36}$$

Descending now to $\widehat{N}$, we denote by ω_{KE} the Kähler form of g_{KE} both for $\widehat{D}$ and its quotient manifold $\widehat{N} = \widehat{D}/\widehat{\Gamma}'$, and consider the fibration $\widehat{\rho}|_W : W \to V$. In what follows we impose the condition that the open set V is relatively compact in X and denote by $d\lambda$ the restriction of a smooth volume form on X to V. We have by (36) the inequality

$$\omega_{KE}^n \geq (\text{Const.} \times \widehat{\rho}^* d\lambda) \wedge \omega_{KE}^{n-m} \ . \tag{37}$$

By Fubini's theorem we conclude from the estimates that

$$\begin{aligned} &\int_{q \in \mathfrak{R}_{\widehat{\rho}}} \text{Volume}\,(L_q, \omega_{KE})\, d\lambda(q) \\ &\leq \text{Const.} \times \text{Volume}\,(W, \omega_{KE}) \leq \text{Const.} \times \text{Volume}\big(\widehat{N}, \omega_{KE}\big) < \infty \ . \end{aligned} \tag{38}$$

Hence, denoting by $\mathscr{S}_2 \subset f(V)$ the set of all $q = f(x) \in f(V)$ such that x is a regular value of $\widehat{\rho}$ and $\text{Volume}\big(L_q, \omega_{KE}\big) = \infty$, $\mathscr{S} := \mathscr{S}_1 \cup \mathscr{S}_2 \subset f(V)$ must be of zero Lebesgue measure. There exists therefore some $q^\sharp \in f(V) - \mathscr{S}$. Fix such a choice of $q^\sharp$. Write $L := L_{q^\sharp}$ and $\widehat{L} := \widehat{L}_{\widetilde{q}}$, where $\widehat{\tau}(\widetilde{q}) = q^\sharp$. Choose now a holomorphic function θ on $\widehat{D}$ such that $\theta|_{\widehat{L}}$ is nonconstant. Since $\text{Volume}\big(L, \omega_{KE}\big) < \infty$, applying Lemma 7.6 to the complete Kähler manifold $\big(L, \omega_{KE}\big|_L\big)$ of finite volume (so that Lemma 8.1 applies) and to the bounded smooth plurisubharmonic function $u = |\theta|^2$ on L we obtain a contradiction to Lemma 8.1 (which says that $u = |\theta|^2$ and hence θ is constant), proving via argument by contradiction that $f : X \to N$ is an open embedding, with which we have completed the proof of the Isomorphism Theorem (Theorem 7.2). □

8.3 A variant of the Isomorphism Theorem

We have the following variant of Theorem 7.2 when the fundamental groups of X and N are only assumed to be isomorphic as abstract groups.

Theorem 8.2 (Variation of the Isomorphism Theorem)**.** *Suppose in the statement of Theorem 7.2 in place of assuming that $f_* : \Gamma \xrightarrow{\cong} \Gamma'$ we assume instead that $\Gamma \cong \Gamma'$ as abstract groups and that $f : X \to N$ is nonconstant. Then, $f : X \to N$ is a biholomorphism.*

Proof. Let $G_0 = \text{Aut}_0(\Omega)$ be the identity component of the automorphism group of Ω. By the hypothesis, there exists a group isomorphism $\alpha : \Gamma \xrightarrow{\cong} \Gamma'$. Replacing Γ (and hence Γ') by a subgroup of finite index we may assume without loss of generality that $\Gamma \subset G_0$. Denoting by $\iota : \Gamma \hookrightarrow G_0$ the inclusion map regarded as a group monomorphism, $\iota \circ \alpha^{-1} : \Gamma' \to G_0$ identifies Γ' also as a subgroup of G_0. Since G_0 is semisimple, connected and of real rank ≥ 2, and $\Gamma \subset G_0$ is an irreducible lattice, by the Margulis superrigidity theorem [14], either $f_*(\Gamma)$ is finite, or else $f_* : \Gamma \to \Gamma'$ extends to a group automorphism $\theta : G_0 \to G_0$. In the former case, denoting by $\Gamma_0 := \text{Ker}(f_*) \subset \Gamma$, which is a (normal) subgroup of finite index, and defining $f_0 : X_0 =: \Omega/\Gamma_0 \xrightarrow{\xi} \Omega/\Gamma = X \xrightarrow{f} N$, where the first map $\xi : X_0 \to X$ is the canonical unramified covering map, we would have a lifting $\widetilde{f_0} : X_0 \to D$ of $f_0 : X \to N$ to the covering bounded domain $D \Subset Z$ of N, which would force f_0 to be constant by the Riemann extension theorem for holomorphic functions and by the maximum principle. In other words, the nonconstancy of f forces $f_* : \Gamma \to \Gamma'$ to extend to a group automorphism $\theta : G_0 \to G_0$. In particular, f_* is injective. With respect to a fixed Haar measure on the semisimple Lie group G_0, which is invariant under the automorphism θ, Volume(G_0/Γ) must agree with Volume$(G_0/f_*(\Gamma))$. Since $f_*(\Gamma) \subset \Gamma'$, it follows that $f_*(\Gamma) = \Gamma'$, so that $f_* : \Gamma \xrightarrow{\cong} \Gamma'$, and we are back to the original formulation of the the Isomorphism Theorem. The proof of Theorem 8.2 is complete. $\square$

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