Quantum networks boosted by entanglement with a control system

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Networks of quantum devices with coherent control over their configuration offer promising advantages in quantum information processing, including quantum communication, computation, and sensing. So far, the investigation of these advantages assumed that the control system was initially uncorrelated with the data processed by the network. Here, we explore the power of quantum correlations between data and control, showing two communication tasks that can be accomplished with information-erasing channels if and only if the sender shares prior entanglement with a third party (the "controller") controlling the network configuration. The first task is to transmit classical messages without leaking information to the controller. The second task is to establish bipartite entanglement with a receiver, or, more generally, to establish multipartite entanglement with a number of spatially separated receivers.

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I. INTRODUCTION

A remarkable feature of quantum particles is the ability to undergo multiple evolutions simultaneously, in a coherent quantum superposition [1–7]. In a seminal work [8], Gisin, Linden, Massar, and Popescu showed that the interference of multiple quantum evolutions could be used to filter out noise in quantum communication, with potential benefits for quantum key distribution and other quantum communication tasks. Recently, the interference of multiple quantum evolutions has been studied in terms of communication capacities both theoretically [4,5,9,10], and experimentally [11,12], in quantum computing [13] and also in quantum sensing [14,15].

The superposition of quantum evolutions is generated by introducing a control system, which determines the evolution undergone by a target system. Quantum networks equipped with control systems provide a new paradigm for quantum information processing, and at the same time are an interesting toy model for investigating new causal structures that could potentially arise in a quantum theory of gravity [16–18]. A concrete example of such a new causal structure is the quantum SWITCH [19], a higher-order operation that connects two variable channels in an order determined by the state of a quantum system, giving rise to a feature called causal nonseparability [20,21]. Over the past decade, the quantum SWITCH stimulated several experimental investigations [22–26] (see also Ref. [27] for a review) and was found to offer information processing advantages in many tasks, including classification of quantum channels [28,29], communication complexity [30], quantum communication [9,31–37], quantum metrology [38,39], and quantum thermodynamics [40–42].

Previous studies on the quantum SWITCH and other coherently controlled quantum networks explored the benefits of quantum superpositions of states of the control corresponding to definite configurations. In all these studies, the control was assumed to be initially uncorrelated with the target. It is possible, however, to consider a more general situation in which the control and the target share prior correlations. In this situation, the data processed by the network becomes correlated with its evolution, potentially giving rise to new phenomena that could not be observed in the traditional setting.

In this paper, we explore the power of quantum correlations between control and target, showing that they enable two communication tasks that are impossible with an uncorrelated control, or even with a classically correlated one. The tasks involve the assistance of a third party (the "controller") who has access to the control system and shares initial quantum correlations with the sender. The role of the controller is to assist the receiver by providing classical information gathered from the control system. For example, the controller could be a quantum communication company responsible for the connection between the sender and receiver. More generally, the controller could be any party who has access to the outcomes of measurements performed on the control system.

Our tasks involve communication through noisy channels that completely erase information when used in a definite configuration. The first task is the communication of a classical message without leaking information to the controller. We show that this task can be perfectly achieved with information-erasing channels if and only if the sender and the controller initially share a maximally entangled state. The second task is to establish bipartite entanglement between a sender and a receiver, or, more generally, to establish multipartite entanglement network between the sender and a number of spatially separated receivers. In this case, we show that perfect entanglement can be established via information-erasing channels if and only if the target and the control are initially in a maximally entangled state. Note that the term "quantum network" as used in the present paper has a broader meaning than just a quantum communication network in the conventional sense. Our protocols can be used to transfer information and establish entanglement in every set of interconnected quantum devices including quantum computers and quantum sensors.

II. QUANTUM COMMUNICATION WITH ENTANGLED CONTROL

We start by reviewing the mathematical description of coherent control over the configurations of quantum devices, focusing in particular on coherent control over the choice of quantum devices and over their order. For simplicity, we discuss the case of $N = 2$ qubit channels, leaving the details of the general cases to the Appendix.

The action of a quantum device is mathematically described by a quantum channel; that is, a completely positive, trace-preserving linear map, acting on the density matrices of a given quantum system [43]. Quantum channels can be conveniently expressed in the Kraus representation $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$, where the Kraus operators $\{E_i\}$ satisfy the normalization condition $\sum_i E_i^\dagger E_i = I$, I denoting the identity matrix on the system's Hilbert space. Control over the order of two devices is described by the quantum SWITCH [19], an operation that combines two channels $\mathcal{E}$ and $\mathcal{F}$ acting on a target system, generating a new channel $\mathcal{S}(\mathcal{E}, \mathcal{F})$ acting jointly on the target and a control system. In the simplest version of the quantum SWITCH, the channel $\mathcal{S}(\mathcal{E}, \mathcal{F})$ executes the two channels $\mathcal{E}$ and $\mathcal{F}$ either in the order $\mathcal{E} \circ \mathcal{F}$ or in the order $\mathcal{F} \circ \mathcal{E}$, depending on whether the control qubit is initialized in the state $|0\rangle$ or $|1\rangle$, respectively. Explicitly, the control-order channel $\mathcal{S}(\mathcal{E}, \mathcal{F})$ is specified by the relation

$$\mathcal{S}(\mathcal{E}, \mathcal{F})(\rho) = \sum_{i,j} S_{ij} \rho S_{ij}^\dagger, \tag{1}$$

with

$$S_{ij} = F_i E_j \otimes |0\rangle\langle 0| + E_j F_i \otimes |1\rangle\langle 1|, \tag{2}$$

with $\{E_j\}$ and $\{F_i\}$ being the Kraus operators corresponding to the channels $\mathcal{E}$ and $\mathcal{F}$, respectively. Note that the control-order channel $\mathcal{S}(\mathcal{E}, \mathcal{F})$ depends only on the input channels $\mathcal{E}$ and $\mathcal{F}$ and not on the specific Kraus decompositions used in Eq. (A3).

Control over the choice of a noisy channel can be described in a similar way. A quantum channel $\mathcal{T}$ that executes either channel $\mathcal{E}$ or channel $\mathcal{F}$ depending the state of a control system has the form [3–5]

$$\mathcal{T}(\rho) = \sum_{ij} T_{ij} \rho T_{ij}^\dagger, \tag{3}$$

with

$$T_{ij} = E_i \beta_j \otimes |0\rangle\langle 0| + F_j \alpha_i \otimes |1\rangle\langle 1|, \tag{4}$$

α_i and β_j being complex amplitudes satisfying the normalization conditions $\sum_i |\alpha_i|^2 = \sum_j |\beta_j|^2 = 1$.

An important difference between control over the choice of two devices and control over their order is that, while the control-order channel $\mathcal{S}(\mathcal{E}, \mathcal{F})$ depends only on the channels $\mathcal{E}$ and $\mathcal{F}$, the control-choice channel $\mathcal{T}$ depends also on the amplitudes α_i and β_j [2–7]. The physical reason for this dependence is that controlling the channel choice means choosing which channel is *not* used or, equivalently, which channel is fed a trivial input, such as, e.g., the vacuum state [5]. Modeling the trivial input as a state $|\mathrm{triv}\rangle$ orthogonal to all states of the target system, the choice-controlled channel $\mathcal{T}$ can be regarded as a function of two *extended channels* $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ with Kraus operators $\widetilde{E_i} = E_i + \alpha_i |\mathrm{triv}\rangle\langle \mathrm{triv}|$ and $\widetilde{F_j} = F_j + \beta_j |\mathrm{triv}\rangle\langle \mathrm{triv}|$, respectively [5,7]. For this reason, in the following we will use the notation $\mathcal{T}(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$.

In contrast with previous studies that assume uncorrelated target and control, we consider a scenario where the target and control are correlated due to preshared entanglement between the sender and a third party, named the controller, which controls the configuration of the network. In our communication scenario, the sender (Alice) has access to a local system, entangled with a control system in the hands of the controller (Charlie). Alice encodes information by performing local operations on her side of the entangled state, producing as output a target system, which will be sent to the receiver (Bob), and an auxiliary system, kept in her laboratory. Then, the target system travels to Bob via a noisy communication channel, while the control system is held by Charlie, whose assistance will be limited to one round of classical communication to Bob. A schematic of this communication scenario is illustrated in Fig. 1.

The initial entanglement between target and control can then be regarded as an offline resource independent of the message to be sent. Note that, assuming this resource is generally different from assuming entanglement directly with the receiver. Indeed, in a communication network, different nodes may be connected by channels of different quality. In our paper, we consider the scenario where the communication channel between the sender and the receiver is severely affected by noise, whereas the channel between the sender and the control is ideally noiseless and can be used to establish entanglement. This situation could arise, for example, when the controller is a server located in proximity of the sender, whereas the receiver is a far-away client. In the following, we assume that the roles of the sender, receiver, and controller are fixed. In a general network, of course, the role of the nodes can vary over time, and a party that acts as a receiver at a given moment may become a sender at a later times. In this more general scenario, one could imagine that the party that acts as sender at a given moment is aided by a server located in its proximity and plays the role of the controller in our analysis. In other words, one would have a network of local controllers aiding the communication between senders and receivers. Note that communication between two distant controllers would likely suffer from the same limitations as communication between senders and receivers. For this reason, the local controllers could not be used to create noiseless side channels that directly connect the senders to the receivers.

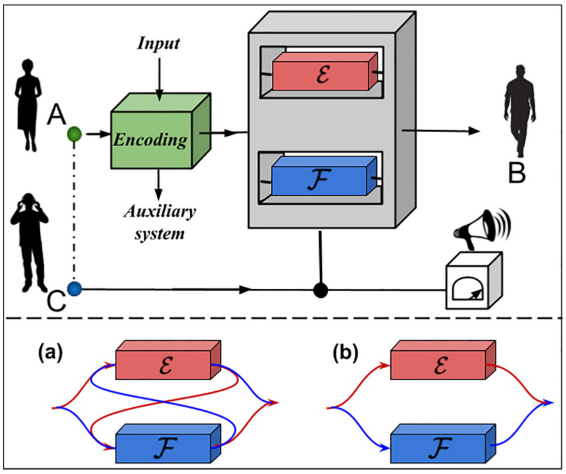

FIG. 1. Quantum communication with the assistance of correlations with a control system. Sender A communicates to receiver B through two noisy channels with the assistance of a third party C, who controls the configuration of the two channels. We focus on the case where the configuration is either (a) the order of the noisy channels, or (b) the choice of which channel is used. The controller and the sender initially share an entangled state (dotted line on the top left). Then, the sender encodes some input data by performing local operations on her part of the entangled state. The output of these operations is a signal that is sent through the network, and possibly some auxiliary systems that the sender will keep in her laboratory. After transmission, the controller assists the receiver by providing him classical information extracted from the control system.

To highlight the power of quantum correlations, we consider the extreme case where the channels $\mathcal{E}$ and $\mathcal{F}$ completely erase information, producing a fixed pure state for every possible initial state of the target system. These channels play a fundamental role in quantum thermodynamics, where they serve as the basis for extending Landauer's principle to the quantum domain [44] and for evaluating the work cost of quantum processors [45]. We refer to these channels as *information-erasing channels*. Taken in isolation, information-erasing channels have no ability to transmit any type of information, be it classical or quantum. In the following we focus on the case where $\mathcal{E}$ and $\mathcal{F}$ are *orthogonal* information-erasing channels that output orthogonal pure states, hereafter denoted as $|0\rangle$ and $|1\rangle$, respectively. In the case of control over the choice we consider the extended channels $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ with Kraus operators $\widetilde{E}_0 = |0\rangle\langle 0| + |\mathrm{triv}\rangle\langle\mathrm{triv}|$, $\widetilde{E}_1 = |0\rangle\langle 1|$, $\widetilde{F}_0 = |1\rangle\langle 0|$, and $\widetilde{F}_1 = |1\rangle\langle 1| + |\mathrm{triv}\rangle\langle\mathrm{triv}|$, respectively. The benefit of this setting is that the control-order and control-choice channels coincide, namely,

$$\mathcal{S}(\mathcal{E},\mathcal{F}) = \mathcal{T}(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}) =: \mathcal{K}, \tag{5}$$

as one can readily verify from the definitions. This observation allows us to treat the order and the choice in a unified way. It is worth stressing, however, that the identification in Eq. (5) holds only for specific extensions $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$, and that these extensions are *not* information-erasing channels on the larger space spanned by the three states $|0\rangle$, $|1\rangle$, and $|\mathrm{triv}\rangle$.

III. PRIVATE CLASSICAL COMMUNICATION

A sender Alice wants to communicate a bit of classical information to a distant receiver Bob. She wants the communication to be secure, in the sense that no other party except Bob can access the message. Unfortunately, Alice and Bob do not share a secret key, and therefore protocols like the one-time pad are not viable. Also, they do not have access to a sufficiently clean quantum communication channel, which could be used to establish a secret key via quantum key distribution [46,47]. Still, Alice has the assistance of a third party, Charlie, who controls the configuration of two communication channels, as in Fig. 1. Charlie can share entangled states with Alice and can assist the communication by sending classical information to Bob. However, Charlie should not be able to extract any information about Alice's message, otherwise the privacy requirement would be compromised.

We now show that the desired task can be achieved perfectly using coherently controlled information-erasing channels. The crucial observation is that the channel $\mathcal{K}$ in Eq. (5) has a decoherence-free subspace [48–51] spanned by the states $|0\rangle\otimes|0\rangle$ and $|1\rangle\otimes|1\rangle$ (see Appendix A for a detailed analysis). This subspace contains Bell states $|\Phi^{\pm}\rangle = (|0\rangle\otimes|0\rangle \pm |1\rangle\otimes|1\rangle)/\sqrt{2}$, which can be generated from $|\Phi^{+}\rangle$ by performing local unitary operations. Hence, Alice can encode a bit $x\in\{+,-\}$ in one of the states $|\Phi^{\pm}\rangle$ and send it through the channel $\mathcal{K}$ without encountering any noise. On the other hand, Charlie has no access to the value of the bit, because the states $|\Phi^{\pm}\rangle$ cannot be distinguished using only measurements on the control system. In the end, Charlie measures the control on the Fourier basis $\{|+\rangle, |-\rangle\}$, with $|\pm\rangle := (|0\rangle \pm |1\rangle)/\sqrt{2}$, and communicates the outcome to Bob, who also measures on the Fourier basis. If Charlie's outcome is $+$, then Bob's outcome is Alice's original bit. If Charlie's outcome is $-$, then Bob only needs to flip the value of his bit, thus obtaining the value of Alice's bit.

In Appendix A, we show that maximal entanglement between target and control is strictly necessary: for information-erasing channels $\mathcal{E}$ and $\mathcal{F}$, Alice can perfectly communicate a bit in a way that is oblivious to Charlie *only if* Alice and Charlie initially share a maximally entangled two-qubit state. In addition, we provide an extension of the above results from qubits to general d-dimensional systems:

Theorem 1. A classical $d$it can be communicated, with no leakage to the controller, through d orthogonal information-erasing channels in d coherently controlled configurations if and only if the control and target are initially in a d-dimensional maximally entangled state.

Theorem 1 highlights the advantage of quantum correlations between the target and control systems. Moreover, it also highlights a fundamental difference between protocols using control over the channels configurations and protocols using the noisy channels $\mathcal{E}$ and $\mathcal{F}$ in a fixed configuration, while allowing control over operations performed before and after each noisy channel [52], as illustrated in Fig. 2. These protocols allow Alice to send classical information to Bob through the control in a way that is completely independent of the noisy channels $\mathcal{E}$ and $\mathcal{F}$ [53]. However, this kind of protocols generally leak information to Charlie, violating the privacy requirement of our communication task. When $\mathcal{E}$ and $\mathcal{F}$ are

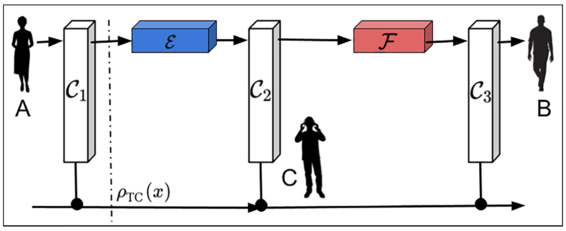

FIG. 2. General protocol with two noisy channels $\mathcal{E}$ and $\mathcal{F}$ in a fixed configuration and controlled operations before and after $\mathcal{E}$ and $\mathcal{F}$. Protocols of this type can perfectly transmit classical messages from a sender to a receiver, but necessarily leak information to the controller.

information-erasing channels, the leakage of information to Charlie is strictly necessary:

Theorem 2. Protocols with fixed configurations of the channels $\mathcal{E}$ and $\mathcal{F}$ cannot achieve private communication.

Proof. Let x be the bit value encoded by Alice, and let $\rho_{\mathrm{TC}}(x)$ be the joint state of the target and control after the first controlled operation in Fig. 2 of the main text. With the action of the information-erasing channel $\mathcal{E}$, the target system is erased and reset it to the fixed state $|0\rangle$, leaving Charlie's system in the marginal state $\rho_{\mathrm{C}}(x) := \mathrm{Tr}_{\mathrm{T}}[\rho_{\mathrm{TC}}(x)]$. Now, the state of all systems at later times of the protocol depends only on the states $\rho_{\mathrm{C}}(x)$. For Bob to retrieve Alice's message, the states $\rho_{\mathrm{C}}(0)$ and $\rho_{\mathrm{C}}(1)$ must be perfectly distinguishable. But if they are perfectly distinguishable, they can be copied by Charlie, who can read Alice's message without being discovered. ■

We now demonstrate how to extend our protocol to situations where the initial channels are partially information-erasing.

Controlled configuration of partial information-erasure channels

Let us now consider the examples of two *partial information-erasure channels*, defined as follows:

$$\begin{aligned} \mathcal{E} &: \rho \mapsto p\rho + (1-p)|0\rangle\langle 0|, \\ \mathcal{F} &: \rho \mapsto q\rho + (1-q)|1\rangle\langle 1|. \end{aligned} \tag{6}$$

Here we use the term "partial information-erasing channels" to distinguish these channels from the "erasure channels" used in earlier works on quantum Shannon theory [54]. The latter are of the form $\mathcal{N} : \rho \mapsto \mathcal{N}(\rho) = p\rho + (1-p)|\text{erasure}\rangle\langle\text{erasure}|$, where the erasure state $|\text{erasure}\rangle$ is orthogonal to all the states of the input system. In other words, these channels map a d-dimensional input system into a $(d+1)$-dimensional output system. The reason why we focus on partial information-erasing channels rather than erasure channels is that the sequential composition of two erasure channels is not well-defined, due to the dimensionality mismatch between their inputs and outputs. To consider two erasure channels in an indefinite causal order, one would have to introduce a third channel in between them, the choice of which, however, is highly nonunique and would considerably complicate our analysis.

For the partial information-erasing channels in Eq. (6), the Kraus operators are given by

$$\begin{aligned} \mathcal{E} &: E_0 = \sqrt{p}\mathbb{I}, \quad E_1 = \sqrt{1-p}|0\rangle\langle 0|, \\ &\quad E_2 = \sqrt{1-p}|0\rangle\langle 1|, \\ \mathcal{F} &: F_0 = \sqrt{q}\mathbb{I}, \quad F_1 = \sqrt{1-q}|1\rangle\langle 0|, \\ &\quad F_2 = \sqrt{1-q}|1\rangle\langle 1|. \end{aligned} \tag{7}$$

Now, the Kraus operators for the controlled operations can be obtained from Eq. (5) of the main text. When a maximally entangled state $|\Phi^+\rangle$ (and similarly the state $|\Phi^-\rangle$) is subjected to this controlled channel configuration, the output state is given as

$$\begin{aligned} |\Phi^\pm\rangle \to\ & (pq + (1-p)(1-q))|\Phi^\pm\rangle\langle\Phi^\pm| \\ & + \tfrac{1}{2}p(1-q)|1\rangle\langle 1| \otimes \mathbb{I}_2 + \tfrac{1}{2}q(1-p)|0\rangle\langle 0| \otimes \mathbb{I}_2, \end{aligned}$$

where $\mathbb{I}_n$ denotes the $n \times n$ identity matrix. To understand this better, let us consider $p = q = r$. In that case,

$$|\Phi^\pm\rangle \to \rho_\pm = [r^2 + (1-r)^2]|\Phi^+\rangle\langle\Phi^\pm| + \tfrac{1}{2}r(1-r)\mathbb{I}_4,$$

which is a Werner state with probability $r^2 + (1-r)^2$. When Alice encodes the classical information by σ_z, we end up distinguishing between the states ρ_+ and ρ_-. The minimum error probability comes out to be

$$p_{\mathrm{err}} = \frac{1 - r^2 - (1-r)^2}{2}. \tag{8}$$

This minimum error probability can be achieved if Charlie performs a measurement in the $\{|+\rangle, |-\rangle\}$ basis and communicates the measurement result to Bob who also measures in the same basis. The number of private bits that can be sent with these partial information-erasure channels is given by the mutual information between Alice's encoded data and the message inferred by Bob. It reads as

$$C = 2 - H(\mathbf{X}) \leqslant 1, \tag{9}$$

where $\mathbf{X} = \{\frac{1+y}{4}, \frac{1+y}{4}, \frac{1-y}{4}, \frac{1-y}{4}\}$, with $y = r^2 + (1-r)^2$. $H(\mathbf{X})$ denotes the Shannon entropy corresponding to the probability distribution $\mathbf{X}$.

It is interesting to observe that, while these channels do not completely erase the input information, their coherently controlled configuration is unable to accomplish the task perfectly. In contrast, the worst possible version of these channels, which erase the input information completely (i.e., $p = q = 0$), can be used to perfectly communicate the private bit. This is because the complete information-erasing channels as in Eq. (5) possess a perfect decoherence-free subspace spanned by the states $|0\rangle \otimes |0\rangle$ and $|1\rangle \otimes |1\rangle$, as already pointed out in Sec. III. This feature is absent for partial information-erasure.

IV. ESTABLISHING ENTANGLEMENT

Our second task is to establish entanglement between the sender and a receiver, or, more generally, a number of spatially separated receivers. We take each of the sender-receiver links to consist of complete information erasing channels as considered, and we show that, nevertheless, the entanglement between sender and controller can be used to achieve a high-quality distribution of entanglement. An alternative way to achieve long-distance entanglement would be to use quantum repeaters [55,56], which, in the language of our paper, would amount to entanglement shared between the sender and a sequence of intermediate nodes.

A. One sender and one receiver

Let us consider first the case of a single receiver, Bob. Initially, Alice and Charlie share a maximally entangled state. Then, Alice converts it into the Greenberger-Horne-Zeilinger (GHZ) state $(|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle)/\sqrt{2}$ by applying a CNOT gate on the target qubit and on an additional reference qubit, present in her laboratory and initially in the state $|0\rangle$. Alice keeps the reference qubit with her and sends the target qubit through the controlled channel $\mathcal{K}$. The presence of the decoherence-free subspace $\mathsf{Span}\{|0\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$ guarantees that the channel preserves the GHZ state. At this point, Charlie measures the control qubit on the Fourier basis $\{|+\rangle, |-\rangle\}$ and announces the result to Bob, who does nothing if the result is $+$, and performs a Pauli Z correction if the outcome is $-$. The net result of the protocol is that Alice and Bob share the maximally entangled state $|\Phi^+\rangle$, which can later be used for quantum communication.

The above protocol can be generalized to dimension d, using d orthogonal information-erasing channels and quantum control over d orders. Also in this case, we show that maximal entanglement between target and control is strictly necessary:

Theorem 3. Coherent control on the configuration of d orthogonal information-erasing channels enables perfect establishment of a maximally entangled two-qudit state if and only if the sender and controller initially share a d-dimensional maximally entangled state.

The proof is too lengthy to present here. See Appendix B for the details of the proof.

B. One sender and multiple receivers

We now extend the protocol to the case of N spatially separated receivers, each of which is connected to the sender through coherently controlled information-erasing channels, as in Fig. 3.

The generalization to $N > 1$ receivers has two important features. First, we show that the dimension of the control system can be kept constant, independently of N. In other words, the amount of control required by the protocol is asymptotically negligible in the large-N limit. The second feature is that our protocol transmits perfect $(N + 1)$-partite GHZ states, which can be used as a primitive in many applications, including communication complexity [57], multiparty cryptography [58], secret sharing and entanglement verification [59], and quantum sensor networks [60–62]. In the context of quantum communication, GHZ states can be used to achieve a task known as random receiver quantum communication (RRQC) [35], where the goal is to transfer quantum information to one of many receivers, whose identity is disclosed only after the transmission phase. Strikingly, entanglement with the control allows us to achieve RRQC with information-erasing channels, whereas in the lack of

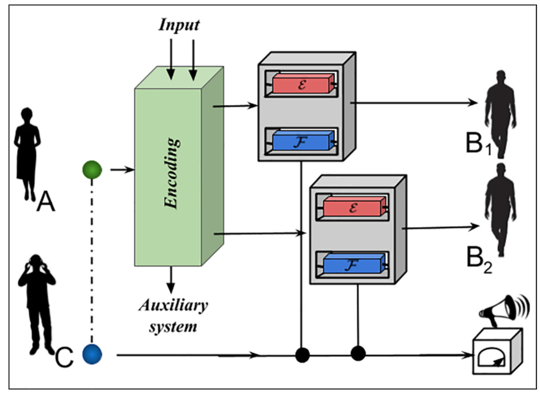

FIG. 3. Distribution of entanglement to $N = 2$ spatially separated parties through coherently controlled information-erasing channels. The task can be perfectly achieved with the assistance of shared entanglement between the qubit at the sender's end, and a qubit used to control the configuration of channels between the sender and each receiver.

such entanglement RRQC can only be achieved with quantum channels that preserve classical information [35].

Let us see how the protocol works. Initially, Alice and Charlie share a two-qubit maximally entangled state. Then, Alice converts it in to an $(N + 2)$-qubit GHZ state by applying CNOT gates on her qubit and N additional qubits in her laboratory. At this point, Alice sends N out of the $(N + 1)$ qubits from her part of the GHZ state to the N receivers. Crucially, the controlled channel preserves the GHZ state (see Appendix C for details). At this point, Charlie performs a Fourier measurement on his qubit and communicates the result to one of the N Bobs, who performs a local correction operation, leaving the remaining $N + 1$ qubits (one with the sender and N of them with Bob) in the GHZ state.

Also in this case, we prove that entanglement between control and target is strictly necessary for a perfect distribution of GHZ states. This result and its d-dimensional generalization are contained in the following theorem:

Theorem 4. Coherent control on the configuration of d orthogonal information-erasing channels enables perfect establishment of d-dimensional GHZ states between the sender and N spatially separated receivers if and only if the sender and controller initially share a d-dimensional maximally entangled state.

The details of the proof are provided in Appendix C.

In the subsequent section, we present a detailed comparison between our protocols and standard quantum communication schemes.

V. COMPARISON WITH STANDARD QUANTUM COMMUNICATION PROTOCOLS

It is interesting to compare our proposed communication scenario with the existing quantum information processing protocols. For instance, one may find mathematical similarities between the protocols for private classical communication and dense coding [63]. Note that dense coding achieves transmission of two bits using (1) a two-qubit entangled state *between sender and receiver*, and (2) *a perfectly noiseless* *communication channel*. In contrast, our protocol (1) uses entanglement between sender and controller, without requiring any initial entanglement between sender and receiver, and (2) allows for the transmission of one bit through channels that completely erase information. In short, the initial resources used in the two protocols are essentially different.

Regarding the performance difference between two bits in dense coding and one bit in our protocol, it is important to stress that, since our protocol does not require initial entanglement between the sender and the receiver, the receiver has to perform local decoding operations on a single qubit, instead of the two qubits in the dense coding protocol. This locality constraint implies that no more than one bit can be communicated, even in principle, and even if one had access to a noiseless channel. If we were to introduce the locality constraint in the original dense coding protocol, the number of bits would necessarily be reduced to one.

On the other hand, while the scenario of establishing entanglement is similar to that of quantum teleportation [64] at the mathematical level, our protocol demands its own importance in the conceptual point. The quantum correlations between sender and controller enable a perfect distribution of entanglement from the sender to the receiver, even if the original channels were information erasing.

Another important difference with teleportation or entanglement swapping is again that these protocols require entanglement between the sender and the receiver (or some intermediate receiver, in entanglement swapping), while our protocol provides a way to establish such entanglement through a noisy channel. Clearly, after entanglement has been established with our protocol, it can be used to perform quantum teleportation or entanglement swapping, but the point of our protocol is to show how this entanglement can be achieved even if the original channels are information erasing.

It is also worth noting that that the N-receiver version of our protocol achieves a task that is not achieved by normal teleportation or entanglement swapping: starting from a single ebit shared with Charlie, Alice can transfer an unknown quantum state to one of the N Bobs without knowing the identity of the actual receiver. Later, after the identity of the intended receiver is announced, the N Bobs can cooperate via LOCC operations in order to make the state accessible to the intended receiver. This protocol requires a ebit shared between Alice and Charlie, while a teleportation protocol would require an $(N+1)$-partite genuinely entangled state to be shared beforehand, to accomplish this task perfectly.

VI. CONCLUSION

In this work, we initiated the exploration of quantum networks whose configuration is entangled with the state of a control system. We focused on applications to quantum communication, identifying two tasks that can be perfectly achieved if and only if the sender and the controller initially share maximal entanglement.

Our first task, the transmission of classical messages without leakage to the controller of the network's configuration, highlights a fundamental difference between protocols where the configuration of the channels is coherently controlled, and protocols where the configuration is fixed and controlled operations are allowed before and after each channel: when the channels completely erase information, no protocol that uses them in a fixed configuration can achieve private communication between the sender and the receiver. Our second task highlights the benefits of sender-controller entanglement for establishing entanglement with one or more receivers.

While in this work we focused on quantum communication, we believe that protocols using quantum correlations with the configuration of quantum networks will have significant implications also in other quantum technology, likely including quantum metrology, thermodynamics, and computation. Such protocols are potentially within reach with existing photonic setups and would mark a new step in the development of a quantum technology of coherent control over the configurations of quantum networks.

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APPENDIX A: PRIVATE CLASSICAL COMMUNICATION

1. Proof of Theorem 1, *if* part

To prove the *if* part for Theorem 1, here we show that the sender, Alice, can convey $\log_2 d$ bit of classical information privately to the receiver Bob by encoding the classical information $x \in \{0, 1, \ldots, d-1\}$ with a local operation on one side of the shared maximally entangled state $|\Phi^+\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$.

Consider d orthogonal information-erasing channels $\{\mathcal{E}_j\}_{j=0}^{d-1}$ acting on the set of density matrices $\mathcal{D}(\mathcal{H}_d)$ over a d-dimensional Hilbert space $\mathcal{H}_d$. A set of Kraus for the channel $\mathcal{E}_j$ is

$$E_{i_j}^{(j)} = |j\rangle\langle i_j|, \quad i_j \in \{0, \ldots, d-1\}. \tag{A1}$$

We now add quantum control over the order of the d information-erasing channels, allowing a d-dimensional control system to select one out of d cyclic permutations. The resulting channel is [9,36,37]

$$\mathcal{S}(\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{d-1})(\rho_{\mathrm{AC}}) = \sum_{i_0, i_1, \ldots, i_{d-1}} S_{i_0, i_1, \ldots, i_{d-1}} \rho_{\mathrm{AC}} S_{i_0, i_1, \ldots, i_{d-1}}^{\dagger}, \tag{A2}$$

with Kraus operators

$$S_{i_0, i_1, \ldots, i_{d-1}} = \sum_{j=0}^{d-1} E_{i_j}^{(j)} E_{i_{j\oplus 1}}^{(j\oplus 1)} \cdots E_{i_{j\oplus(d-1)}}^{(j\oplus(d-1))} \otimes |j\rangle\langle j|, \tag{A3}$$

where $\oplus$ denotes the sum modulo d. Using Eq. (A1), we rewrite Eq. (A3) in the following compact form:

$$S_{i_0,i_1,\dots,i_{d-1}} = \sum_{j=0}^{d-1} s_j |j\rangle\langle i_{j\ominus 1}| \otimes |j\rangle\langle j|,$$

$$s_j := \prod_{l\neq j\ominus 1} \langle i_l | l\oplus 1\rangle, \tag{A4}$$

where $\ominus$ denotes subtraction modulo d.

At this point, there are three possible cases:

(1) $i_l = l\oplus 1$ for all $l\in\{0,\dots,d-1\}$;

(2) $i_l = l\oplus 1$ for all l except one, or equivalently, $i_{j\ominus 1} = j$ for all j except one;

(3) $i_l \neq l\oplus 1$ for two or more values of l.

In case 1, the Kraus operator is $S_{1,2,\dots,d-1,0} = \sum_{j=0}^{d-1} |j\rangle\langle j|\otimes |j\rangle\langle j| =: P_0$. In case 2, the Kraus operators are of the form $|j\rangle\langle i_{j\ominus 1}|\otimes |j\rangle\langle j|$, where j is the one index for which $i_j\neq j\ominus 1$. In case 3, the Kraus operator $S_{i_0,i_1,\dots,i_{d-1}}$ is zero. Summarizing, the controlled-order channel is given by

$$\mathcal{S}(\mathcal{E}_0,\mathcal{E}_1,\dots,\mathcal{E}_{d-1})(\rho_{\mathrm{AC}}) = P_0\rho_{\mathrm{AC}}P_0 + \sum_{j=0}^{d-1}\sum_{l\neq j} \langle l|\langle j|\rho_{\mathrm{AC}}|l\rangle|j\rangle |j\rangle\langle j|\otimes|j\rangle\langle j|. \tag{A5}$$

The same channel is obtained from a controlled choice of the information-erasing channels $\{\mathcal{E}_j\}_{j=0}^{d-1}$, provided that one adopts the extended channels $\{\widetilde{\mathcal{E}}_j\}_{j=0}^{d-1}$ with Kraus operators $\widetilde{E}^{(j)}_{i_j} = |j\rangle\langle i_j| + \langle i_j|j\rangle|\mathrm{triv}\rangle\langle\mathrm{triv}|$. Indeed, the controlled-choice channel is given by [4,5]

$$\mathcal{T}(\widetilde{\mathcal{E}}_0,\widetilde{\mathcal{E}}_1,\dots,\widetilde{\mathcal{E}}_{d-1})(\rho_{\mathrm{AC}}) = \sum_{i_0,i_1,\dots,i_{d-1}} T_{i_0,i_1,\dots,i_{d-1}}\rho_{\mathrm{AC}}T^\dagger_{i_0,i_1,\dots,i_{d-1}}, \tag{A6}$$

with Kraus operators

$$T_{i_0,i_1,\dots,i_{d-1}} = \sum_{j=0}^{d-1} t_j|j\rangle\langle i_j|\otimes|j\rangle\langle j|,\quad t_j := \prod_{l\neq j}\alpha^{(l)}_{i_l}, \tag{A7}$$

where $\alpha^{(l)}_{i_l}$ are the amplitudes associated with the lth channel. If we set $\alpha^{(l)}_{i_l} = \langle i_l|l\rangle$, then there are three possible cases:

(1) $i_l = l$ for every $l\in\{0,\dots,d-1\}$;

(2) $i_l = l$ for every l except one;

(3) $i_l\neq l$ for two or more values of l.

In case 1, the Kraus operator is $T_{0,1,\dots,d-1} = P_0$. In case 2, the Kraus operator is $|l\rangle\langle i_l|\otimes|l\rangle\langle l|$, where l is the one value such that $i_l\neq l$. In case 3, the Kraus operator $T_{i_0,i_1,\dots,i_{d-1}}$ is zero. Summarizing, we obtained the relation

$$\mathcal{S}(\mathcal{E}_0,\mathcal{E}_1,\dots,\mathcal{E}_{d-1}) = \mathcal{T}(\widetilde{\mathcal{E}}_0,\widetilde{\mathcal{E}}_1,\dots,\widetilde{\mathcal{E}}_{d-1}) =: \mathcal{K}, \tag{A8}$$

which proves Eq. (5) in the main text and generalizes it to $d\geqslant 2$. In the following we treat the controlled order and controlled choice in a unified way, referring to the channel $\mathcal{K}$.

Note that the channel $\mathcal{K}$ has a decoherence-free subspace spanned by the vectors $|j\rangle\otimes|j\rangle$, $j\in\{0,\dots,d-1\}$. Hence, it preserves the maximally entangled states

$$|\Phi_x\rangle_{\mathrm{AC}} = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{\frac{2\pi ijx}{d}}|j\rangle_{\mathrm{A}}\otimes|j\rangle_{\mathrm{C}},\quad x\in\{0,1,\dots,d-1\}.$$

Since all these states are maximally entangled, they are locally preparable from the canonical maximally entangled state $|\Phi^+\rangle_{\mathrm{AC}} = \sum_{j=0}^{d-1}|j\rangle_{\mathrm{A}}\otimes|j\rangle_{\mathrm{C}}/\sqrt{d}$ by means of suitable local unitary operations on Alice's side. Therefore, Alice can encode $\log_2 d$ bits by locally transforming the preshared $|\Phi^+\rangle_{\mathrm{AC}}$ one of these d maximally entangled states. Then, she can send her part of the state to Bob through the controlled quantum channels. After the transmission, Bob and Charlie share one of these d maximally entangled states.

The states $\{|\Phi_x\rangle_{\mathrm{BC}}\}$ can be perfectly discriminated under one-way LOCC. The protocol is simple: Charlie and Bob perform two independent measurements on the Fourier basis $\{|f_m\rangle = \sum_j e^{\frac{2\pi ijm}{d}}|j\rangle/\sqrt{d}\}_{m=0}^{d-1}$, and Charlie communicates his outcome to Bob. The joint probability distribution of their outcomes m_{B} and m_{C} is $p(m_{\mathrm{B}},m_{\mathrm{C}}) = \delta_{m_{\mathrm{B}}+m_{\mathrm{C}},x}/d$, and allows Bob to infer the value of the message x from his outcome m_{B} and from Charlie's m_{C}. At the same time, Charlie remains completely blind about the transmitted message becaus his measurement outcome alone contains no information about x.

2. Proof of Theorem 1, *only if* part

In the previous section we have shown that Alice can communicate $\log_2 d$ bits of classical information privately to Bob via d controlled pin maps, provided that she initially shares a d-dimensional maximally entangled state with the controller Charlie. We now prove that maximally entangled states are strictly necessary for this communication task. Precisely, we show that a perfect communication of $\log_2 d$ classical bits through coherently controlled information-erasing channels is possible only if Alice and Charlie initially share a bipartite state ρ^*_{AC} that can be locally converted into the d-dimensional maximally entangled state $|\Phi^+\rangle_{\mathrm{AC}}$.

The proof is rather complex and makes use of a series of lemmas, proved in the following. All throughout this section, we use the following notation: ρ^*_{AC} will be the state shared by Alice and Charlie at the beginning of the protocol, $\mathcal{A}_x$ be the local operation used by Alice to encode message x, $\rho_{x,\mathrm{AC}} := (\mathcal{A}_x\otimes\mathcal{I}_{\mathrm{C}})(\rho^*_{\mathrm{AC}})$ will be the joint state of Alice's and Charlie's systems right before transmission through the controlled channel, and $\rho'_{x,\mathrm{BC}}$ will be the state of Bob's and Charlie's systems right after transmission.

Lemma 1. Perfect communication of $\log_2 d$ bits through coherently controlled information-erasing channels is possible only if the final states $\{\rho'_{x,\mathrm{BC}}\}_{x=0}^{d-1}$ are pure, orthogonal, and maximally entangled.

Proof. Let $\mathcal{C}$ be either the controlled-order channel $\mathcal{S}(\mathcal{E}_0,\mathcal{E}_1,\dots,\mathcal{E}_{d-1})$ or the controlled-choice channel $\mathcal{T}(\widetilde{\mathcal{E}}_0,\widetilde{\mathcal{E}}_1,\dots,\widetilde{\mathcal{E}}_{d-1})$, so that $\rho'_{x,\mathrm{BC}} = \mathcal{C}(\rho_{x,\mathrm{AC}})$.

Now, $\mathcal{C}$ transforms every density matrix into a density matrix with support contained in the subspace $\mathcal{H}_0 := \mathsf{Span}(|j\rangle\otimes|j\rangle,\ j\in\{0,1,\dots,d-1\})$. This fact can be readily checked from Eq. (A4) and (A7) in the cases of controlled order and controlled choice, respectively.

Since the subspace $\mathcal{H}_0$ is d dimensional, the only way to achieve the perfect communication of $\log_2 d$ bits is that the states $\{\rho'_{x,\mathrm{BC}}\}$ are pure and orthogonal, say $\rho'_{x,\mathrm{BC}} = |\Phi_x\rangle\langle\Phi_x|_{\mathrm{BC}}$ where $\{|\Phi_x\rangle_{\mathrm{BC}}\}_{x=0}^{d-1}$ is an orthonormal basis for the subspace $\mathcal{H}_0$.

We now show that each state $|\Phi_x\rangle$ must be maximally entangled. By definition, we have

$$|\Phi_x\rangle\langle\Phi_x|_{\mathrm{BC}} = \mathcal{C}((\mathcal{A}_x \otimes \mathcal{I}_{\mathrm{C}})(\rho^*_{\mathrm{AC}})). \tag{A9}$$

Let us write $|\Phi_x\rangle_{\mathrm{BC}} = \sum_j c_{x,j}|j\rangle_{\mathrm{B}} \otimes |j\rangle_{\mathrm{C}}$. Multiplying both sides of Eq. (A9) by $I_{\mathrm{B}} \otimes |j\rangle\langle j|_{\mathrm{C}}$ on the left and on the right, we obtain

$$\begin{aligned}
&|c_{x,j}|^2|j\rangle\langle j|_{\mathrm{B}} \otimes |j\rangle\langle j|_{\mathrm{C}}\\
&\quad= (I_{\mathrm{B}} \otimes |j\rangle\langle j|_{\mathrm{C}})\mathcal{C}((\mathcal{A}_x \otimes \mathcal{I}_{\mathrm{C}})(\rho^*_{\mathrm{AC}}))(I_{\mathrm{B}} \otimes |j\rangle\langle j|_{\mathrm{C}})\\
&\quad= \mathcal{C}((I_{\mathrm{B}} \otimes |j\rangle\langle j|_{\mathrm{C}})(\mathcal{A}_x \otimes \mathcal{I}_{\mathrm{C}})(\rho^*_{\mathrm{AC}})(I_{\mathrm{B}} \otimes |j\rangle\langle j|_{\mathrm{C}}))\\
&\quad= \mathcal{C}(\mathcal{A}_x(\sigma_{j,\mathrm{A}}) \otimes |j\rangle\langle j|_{\mathrm{C}}),\\
&\sigma_{j,\mathrm{A}} := (I_{\mathrm{A}} \otimes \langle j|_{\mathrm{C}})\rho^*_{\mathrm{AC}}(I_{\mathrm{A}} \otimes |j\rangle_{\mathrm{C}})\\
&\quad= \mathcal{E}_j\mathcal{A}_x(\sigma_{j,\mathrm{A}}) \otimes |j\rangle\langle j|\\
&\quad= p_j|j\rangle\langle j|_{\mathrm{B}} \otimes |j\rangle\langle j|_{\mathrm{C}}, \quad p_j := \mathrm{Tr}[\sigma_{j,\mathrm{A}}],
\end{aligned} \tag{A10}$$

where the second equality follows from the expression of the Kraus operators of $\mathcal{K}$ [Eqs. (A4) and (A7) for the controlled order and controlled choice, respectively], the forth equality follows from the fact that $\mathcal{K}$ is a controlled information-erasing channel, and the fifth equation follows from the fact that $\mathcal{A}_x$ is trace preserving.

Since j and x are arbitrary, we conclude that $|c_{x,j}|^2 = p_j$ for every x and j. Now, recall that the vectors $\{|\Phi_x\rangle\}$ form an orthonormal basis for the subspace $\mathcal{H}_0$, and therefore $\sum_{x=0}^{d-1}|\Phi_x\rangle\langle\Phi_x| = \sum_{j=0}^{d-1}|j\rangle\langle j| \otimes |j\rangle\langle j|$. Multiplying both sides of this equation by $\langle j|_{\mathrm{C}}$ on the left and $|j\rangle_{\mathrm{C}}$ on the right, we obtain

$$\sum_{x=0}^{d-1}|c_{x,j}|^2|j\rangle\langle j| = |j\rangle\langle j|, \tag{A11}$$

which combined with the fact that $|c_{x,j}|^2$ is independent of x, implies $|c_{x,j}|^2 = 1/d$ for every j. In conclusion, the states $|\Phi_x\rangle$ are maximally entangled. ■

To continue the proof, we consider separately the cases of the controlled order and the controlled choice.

Proof for controlled order. The proof uses the following lemma:

Lemma 2. Perfect communication of $\log_2 d$ bits with d information-erasing channels in a controlled order is possible only if the initial state ρ^*_{AC} is locally convertible into a d-dimensional maximally entangled state.

Proof. By Lemma 1, perfect communication is possible only if the states $\rho'_{x,\mathrm{BC}}$ are pure, orthogonal, and maximally entangled. Then, one has $\rho'_{x,\mathrm{BC}} = |\Phi_x\rangle\langle\Phi_x|_{\mathrm{BC}} = \mathcal{S}(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{d-1})(\rho_{x,\mathrm{AC}})$.

A necessary condition for the state $\mathcal{S}(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{d-1})(\rho_{x,\mathrm{AC}})$ to be maximally entangled is that the separable terms in Eq. (A5) vanish, or equivalently, that $|\Phi_x\rangle\langle\Phi_x|_{\mathrm{BC}} = P_0\rho_{x,\mathrm{AC}}P_0$. Since P_0 is a projector (up to the inessential relabelling of the first space as B or A), the normalization of the state $P_0\rho_{x,\mathrm{AC}}P_0$ implies that $P_0\rho_{x,\mathrm{AC}}P_0 = \rho_{x,\mathrm{AC}}$, and therefore, $\rho_{x,\mathrm{AC}} = |\Phi_x\rangle\langle\Phi_x|_{\mathrm{AC}}$. In summary, all the states $\{\rho_{x,\mathrm{AC}}\}_{x=0}^{d-1}$ are maximally entangled. But these states are obtained by performing local operations on the initial state ρ^*_{AC}. Since local operations cannot increase entanglement, we conclude that ρ^*_{AC} must be locally convertible into a d-dimensional maximally entangled state. ■

Combining Lemmas 1 and 2, we obtain the desired necessity proof for the controlled order of information-erasing channels: perfect communication of $\log_2 d$ bits is possible only if the initial state shared by Alice and Charlie is (locally equivalent to) a d-dimensional maximally entangled state.

Proof for controlled choice. The proof is more subtle than the proof for controlled order, because there are infinitely many possible "controlled-choice channels," depending on which extensions $\widetilde{\mathcal{E}}_j$ are used. Our proof will hold for all possible choices.

To get started, we need a general fact on the controlled choice of d information-erasing channels:

Lemma 3. The controlled-choice channel $\mathcal{T}(\widetilde{\mathcal{E}}_0, \widetilde{\mathcal{E}}_1, \dots, \widetilde{\mathcal{E}}_{d-1})$ can be written as

$$\begin{aligned}
&\mathcal{T}(\widetilde{\mathcal{E}}_0, \widetilde{\mathcal{E}}_1, \dots, \widetilde{\mathcal{E}}_{d-1})(\rho_{\mathrm{AC}})\\
&\quad= T_{0,\dots,0}\rho_{\mathrm{AC}}T^\dagger_{0,\dots,0} + \sum_{j=0}^{d-1}\sum_{i_j\neq 0}\mathrm{Tr}[(I - |v_j\rangle\langle v_j|)_{\mathrm{A}}\\
&\qquad\otimes |j\rangle\langle j|_{\mathrm{C}}\rho_{\mathrm{AC}}]|j\rangle\langle j|_{\mathrm{B}} \otimes |j\rangle\langle j|_{\mathrm{C}},
\end{aligned} \tag{A12}$$

where $\{|v_j\rangle\}$ are suitable vectors satisfying $\||v_j\rangle\| \leqslant 1$ for every $j \in \{0, \dots, d-1\}$, and

$$T_{0,\dots,0} = \sum_{j=0}^{d-1}|j\rangle_{\mathrm{B}}\langle v_j|_{\mathrm{A}} \otimes |j\rangle_{\mathrm{C}}\langle j|_{\mathrm{C}}. \tag{A13}$$

Proof. The proof uses a property of extended channels proven in Ref. [7]: for every extended channel $\widetilde{\mathcal{E}}$ there exists a Kraus representation with operators of the form $\widetilde{E}_i = E_i + \alpha_i|\mathrm{triv}\rangle\langle\mathrm{triv}|$ such that $\alpha_0 = 1$ and $\alpha_i = 0$ for every $i > 0$. Applying this result to the channels $\widetilde{\mathcal{E}}_j$, we obtain Kraus representations

$$\widetilde{E}^{(j)}_{i_j} = |j\rangle\langle v^{(j)}_{i_j}| + \alpha^{(j)}_{i_j}|\mathrm{triv}\rangle\langle\mathrm{triv}|, \tag{A14}$$

where $(|v^{(j)}_{i_j}\rangle)_{i_j}$ are (possibly nonorthonormal) vectors satisfying the normalization condition $\sum_{i_j}|v^{(j)}_{i_j}\rangle\langle v^{(j)}_{i_j}| = I$ for every j. In this representation, the controlled-choice channel reads [4,5]

$$\mathcal{T}(\widetilde{\mathcal{E}}_0, \widetilde{\mathcal{E}}_1, \dots, \widetilde{\mathcal{E}}_{d-1})(\rho_{\mathrm{AC}}) = \sum_{i_0,i_1,\dots,i_{d-1}} T_{i_0,i_1,\dots,i_{d-1}}\rho_{\mathrm{AC}}T^\dagger_{i_0,i_1,\dots,i_{d-1}}, \tag{A15}$$

with Kraus operators

$$T_{i_0,i_1,\dots,i_{d-1}} = \sum_{j=0}^{d-1}t_j|j\rangle\langle v^{(j)}_{i_j}| \otimes |j\rangle\langle j|, \quad t_j := \prod_{l\neq j}\alpha^{(l)}_{i_l}. \tag{A16}$$

At this point, there are three possible cases:

(1) $i_j = 0$ for all j;
(2) $i_j = 0$ for all j except one;
(3) $i_j \neq 0$ for two or more values of j.

In case 1, the Kraus operator is $T_{0,\ldots,0}=\sum_{j=0}^{d-1}|j\rangle\langle v_0^{(j)}|\otimes|j\rangle\langle j|$. In case 2, the Kraus operators are of the form $|j\rangle\langle v_{i_j}^{(j)}|\otimes|j\rangle\langle j|$, where j is the one index for which $i_j\neq 0$. In case 3, the Kraus operator $T_{i_0,i_1,\ldots,i_{d-1}}$ is zero. Inserting these expressions into Eq. (A15), we obtain

$$\begin{aligned}\mathcal{T}(\widetilde{\mathcal{E}}_0,\widetilde{\mathcal{E}}_1,\ldots,\widetilde{\mathcal{E}}_{d-1})(\rho_{\mathrm{AC}}) &= T_{0,\ldots,0}\rho_{\mathrm{AC}}T_{0,\ldots,0}^{\dagger}+\sum_{j=0}^{d-1}\sum_{i_j\neq 0}\langle v_{i_j}^{(j)}|_{\mathrm{A}}\langle j|_{\mathrm{C}}\rho_{\mathrm{AC}}|v_{i_j}^{(j)}\rangle_{\mathrm{A}}|j\rangle_{\mathrm{C}}|j\rangle\langle j|_{\mathrm{B}}\otimes|j\rangle\langle j|_{\mathrm{C}}\\ &= T_{0,\ldots,0}\rho_{\mathrm{AC}}T_{0,\ldots,0}^{\dagger}+\sum_{j=0}^{d-1}\mathrm{Tr}\left[\left(I_A-|v_0^{(j)}\rangle\langle v_0^{(j)}|\right)\otimes|j\rangle\langle j|\rho_{\mathrm{AC}}\right]|j\rangle\langle j|_{\mathrm{B}}\otimes|j\rangle\langle j|_{\mathrm{C}},\end{aligned}\tag{A17}$$

the second equation following from the normalization condition $\sum_{i_j}|v_{i_j}^{(j)}\rangle\langle v_{i_j}^{(j)}|=I$ for every j. Defining $|v_j\rangle:=|v_0^{(j)}\rangle$ we then obtain Eq. (A12). ■

We now use the previous lemma to characterize the structure of the input states that give rise to orthogonal states in the output. To this purpose, recall Lemma 1, which states that the states $\{\rho'_{x,\mathrm{BC}}\}$ are orthogonal only if they are maximally entangled.

Lemma 4. If the state $\rho'_{x,\mathrm{BC}}$ is maximally entangled, then $\||v_j\rangle\|=1$ for every $j\in\{0,\ldots,d-1\}$, and the state $\rho_{x,\mathrm{AC}}$ has support contained in the subspace spanned by the vectors $\{|v_j\rangle\otimes|j\rangle\}_{j=0}^{d-1}$.

Proof. Recall that $\rho'_{x,\mathrm{BC}}=\mathcal{T}(\widetilde{\mathcal{E}}_0,\widetilde{\mathcal{E}}_1,\ldots,\widetilde{\mathcal{E}}_{d-1})(\rho_{x,\mathrm{AC}})$. For this state to be maximally entangled, the separable terms in Eq. (A12) must vanish. These terms vanish if and only if

$$\mathrm{Tr}[(I_{\mathrm{A}}-|v_j\rangle\langle v_j|)\otimes|j\rangle\langle j|\rho_{x,\mathrm{AC}}]=0\quad\forall\, j\in\{0,\ldots,d-1\}.\tag{A18}$$

This condition implies the relation $\rho_{x,\mathrm{AC}}(|v_j\rangle\langle v_j|_{\mathrm{A}}\otimes|j\rangle\langle j|_{\mathrm{C}})=0$ for every j such that $\|v_j\rangle\|<1$. In turn, this condition implies that the output state in Eq. (A15) becomes

$$\mathcal{T}(\widetilde{\mathcal{E}}_0,\widetilde{\mathcal{E}}_1,\ldots,\widetilde{\mathcal{E}}_{d-1})(\rho_{x,\mathrm{AC}})=T_{0,\ldots,0}\rho_{x,\mathrm{AC}}T_{0,\ldots,0}^{\dagger}=T_*\rho_{x,\mathrm{AC}}T_*^{\dagger},\tag{A19}$$

with

$$T_*=\sum_{j\in S_*}|j\rangle\langle v_j|\otimes|j\rangle\langle j|,\tag{A20}$$

S_* being the set of values of j such that $\||v_j\rangle\|=1$.

The normalization of the states in Eq. (A19) implies that the state $\rho_{x,\mathrm{AC}}$ has support contained in the vector space spanned by the vectors $\{|v_j\rangle\otimes|j\rangle\}_{j\in S_*}$. Moreover, the condition that the state $T_*\rho_{j,\mathrm{AC}}T_*^{\dagger}$ be maximally entangled implies that the set S_* must contain all values of j. Hence, the condition $\||v_j\rangle\|=1$ must be satisfied for every $j\in\{0,\ldots,d-1\}$. ■

We now show that the states sent by Alice and Charlie through the channel must be maximally entangled.

Lemma 5. If the states $\{\rho'_{x,\mathrm{BC}}\}_{x=0}^{d-1}$ are orthogonal and maximally entangled, then the states $\{\rho_{x,\mathrm{AC}}\}_{x=0}^{d-1}$ are maximally entangled.

Proof. Since the states $\{\rho'_{x,\mathrm{BC}}\}_{x=0}^{d-1}$ are obtained from the states $\{\rho_{x,\mathrm{AC}}\}_{x=0}^{d-1}$ through the action of a quantum channel, the former are orthogonal only if the latter are orthogonal.

By Lemma 4, the support of the states $\{\rho_{x,\mathrm{AC}}\}_{x=0}^{d-1}$ is contained in the d-dimensional subspace spanned by the vectors $\{|v_j\rangle\otimes|j\rangle\}_{j=0}^{d-1}$. Since the states $\{\rho_{x,\mathrm{AC}}\}_{x=0}^{d-1}$ are d orthogonal states in a d-dimensional subspace, they must be pure. Let us write them as $\rho_{x\mathrm{AC}}=|\Psi_x\rangle\langle\Psi_x|_{\mathrm{AC}}$, with

$$|\Psi_x\rangle_{\mathrm{AC}}=\sum_{j=0}^{d-1}\lambda_{x,j}|v_j\rangle_{\mathrm{A}}\otimes|j\rangle_{\mathrm{C}}.\tag{A21}$$

We now show that the orthogonal states $\{|\Psi_x\rangle_{\mathrm{AC}}\}_{x=0}^{d-1}$ must be maximally entangled. First, recall that one has $|\Psi_x\rangle\langle\Psi_x|_{\mathrm{AC}}=(\mathcal{A}_x\otimes\mathcal{I}_{\mathrm{C}})(\rho_{\mathrm{AC}}^*)$. Tracing out both sides on the equation with $I_{\mathrm{A}}\otimes|j\rangle\langle j|_{\mathrm{C}}$ we obtain

$$\begin{aligned}|\lambda_{x,j}|^2&=\mathrm{Tr}[(I_{\mathrm{A}}\otimes|j\rangle\langle j|_{\mathrm{C}})(\mathcal{A}_x\otimes\mathcal{I}_{\mathrm{C}})(\rho_{\mathrm{AC}}^*)]\\&=\mathrm{Tr}[(I_{\mathrm{A}}\otimes|j\rangle\langle j|_{\mathrm{C}})\rho_{\mathrm{AC}}^*]=:p_j.\end{aligned}\tag{A22}$$

In short, $|\lambda_{x,j}|$ is independent of x.

Moreover, since the states $\{|\Psi_x\rangle_{\mathrm{AC}}\}_{x=0}^{d-1}$ are orthogonal; that is, they are a basis for the subspace spanned by the vectors $\{|v_j\rangle_{\mathrm{A}}\otimes|j\rangle_{\mathrm{C}}\}_{j=0}^{d-1}$. Hence, we have

$$\sum_{x=0}^{d-1}|\Psi_x\rangle\langle\Psi_x|_{\mathrm{AC}}=\sum_{j=0}^{d-1}|v_j\rangle\langle v_j|_{\mathrm{A}}\otimes|j\rangle\langle j|_{\mathrm{C}}.\tag{A23}$$

Multiplying both sides of the equation by $\langle j|_{\mathrm{C}}$ on the left and $|j\rangle_{\mathrm{C}}$ on the right, we obtain

$$\sum_{x=0}^{d-1}|c_{xj}|^2|v_j\rangle\langle v_j|_{\mathrm{A}}=|v_j\rangle\langle v_j|_{\mathrm{A}},\tag{A24}$$

which implies $|c_{xj}|^2=1/d$ (recall that $\||v_j\rangle\|=1$ for every j and therefore $|v_j\rangle$ cannot be the zero vector).

Hence, the state $|\Psi_x\rangle_{\mathrm{AC}}$ can be rewritten as

$$|\Psi_x\rangle_{\mathrm{AC}}=\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{i\theta_{x,j}}|v_j\rangle_{\mathrm{A}}\otimes|j\rangle_{\mathrm{C}},\tag{A25}$$

for some suitable phases $\theta_{x,j}\in\mathbb{R}$.

To conclude that the vectors $|\Psi_x\rangle$ are maximally entangled, we show that the vectors $\{|v_j\rangle\}_{j=0}^{d-1}$ are mutually orthogonal. To this purpose, recall that all the states $|\Psi_x\rangle$ must have the same marginal on system C. The condition of equal marginals is

$$\begin{aligned}&\sum_{j,l}e^{i(\theta_{x,j}-\theta_{x,l})}\langle v_l|v_j\rangle|j\rangle\langle l|\\&\quad=\sum_{j,l}e^{i(\theta_{y,j}-\theta_{y,l})}\langle v_l|v_j\rangle|j\rangle\langle l|\quad\forall\, x,y\in\{0,\ldots,d-1\}.\end{aligned}\tag{A26}$$

The equality holds if and only if $e^{i(\theta_{x,j}-\theta_{x,l})} = e^{i(\theta_{y,j}-\theta_{y,l})}$ for every pair (j, l) such that $\langle v_l|v_j\rangle \neq 0$.

On the other hand, no such pair can exist. The proof is by contradiction: suppose that there existed a pair (j_1, j_2) such that $\langle v_{j_1}|v_{j_2}\rangle \neq 0$. Hence, there would exist a constant ω such that

$$e^{i\theta_{x,j_2}} = \omega e^{i\theta_{x,j_1}} \quad \forall\, x \in \{0, \dots, d-1\}. \tag{A27}$$

This condition would imply that two columns of the matrix $M = (e^{i\theta_{x,j}})$ are proportional to each other, and therefore $\det(M) = 0$. But this would be in contradiction with the fact that the states $\{|\Psi_x\rangle\}_{x=0}^{d-1}$ be orthogonal, which implies that the matrix M has full rank.

Hence, the condition $\langle v_l|v_j\rangle = 0$ must hold for every j and l. This implies that the vectors $\{|v_j\rangle\}$ form an orthonormal basis, and therefore the states $\{|\Psi_x\rangle_{\text{AC}}\}$ are maximally entangled. ■

Putting everything together, we obtain the desired result:

Lemma 6. Perfect communication of $\log_2 d$ bits with a controlled choice of d information-erasing channels is possible only if the initial state ρ^*_{AC} is locally convertible into a d-dimensional maximally entangled state.

Proof. By Lemma 1, perfect communication is possible only if the states $\rho'_{x,\text{BC}}$ are orthogonal and maximally entangled. Then, Lemma 5 implies that the states $\rho_{x,\text{AC}}$ must be maximally entangled. Since these states are obtained from the state ρ^*_{AC} by applying local operations, the state ρ^*_{AC} must be maximally entangled. ■

Together, Lemmas 2 and 6 conclude the proof of the *"Only if"* part of Theorem 1 in the main text.

APPENDIX B: ESTABLISHING ENTANGLEMENT WITH ONE RECEIVER

Here we consider the scenario to establish $\log_2 d$ ebits between Alice to Bob through d orthogonal information-erasing channels and a perfect side channel of quantum capacity $\log_2 d$. Importantly, Charlie, who has access on the side channel, is only allowed to communicate classically with the receiver Bob. This prevents Alice from bypassing the zero capacity channels via the perfect side channel.

1. Proof of Theorem 2, *if* part

Similar to the previous protocol, let us consider that Alice shares a maximally entangled state $|\Phi^+\rangle_{\text{AC}} \in \mathbb{C}_d^{\otimes 2}$ with Charlie, beforehand. Now to establish the maximal entanglement with Bob, she will first prepare a d-dimensional quantum state $|0\rangle_{\text{A}'}$ and apply a joint unitary $\mathbb{V}_{\text{AA}'}$ on the two qudits she has at her possession. The action of the joint unitary is $\mathbb{V}_{\text{AA}'}\,|k_A 0_{\text{A}'}\rangle = |k_A k_{\text{A}'}\rangle\, \forall\, k \in \{0, 1, \dots, (d-1)\}$, which can also be identified as the perfect cloning machine for the orthonormal basis $\{|k\rangle\}_{k=0}^{d-1}$. Hence, the final tripartite state among the two qudits at Alice's laboratory and a single qudit at Charlie's laboratory will be a genuinely entangled state given by

$$|\psi\rangle_{\text{AA}'\text{C}} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jjj\rangle_{\text{AA}'\text{C}}\,.$$

Now keeping the part A with her, Alice (and Charlie) will send the qudit A$'$ (and B) through the controlled quantum channels of d orthogonal information-erasing channels. The joint channel action hence can be depicted as

$$\begin{aligned} &I_{\text{A}} \otimes \mathcal{S}(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{d-1})(\rho_{\text{AA}'\text{C}}) \\ &\quad = I_{\text{A}} \otimes \mathcal{T}(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{d-1})(\rho_{\text{AA}'\text{C}}) \\ &\quad := \sum_{i_0, i_1, \dots, i_{d-1}} \widetilde{\mathcal{K}}_{i_0, i_1, \dots, i_{d-1}} \rho_{\text{AA}'\text{C}} \widetilde{\mathcal{K}}^\dagger_{i_0, i_1, \dots, i_{d-1}}, \end{aligned} \tag{B1}$$

where $\widetilde{\mathcal{K}}_{i_0, i_1, \dots, i_{d-1}} = I_{\text{A}} \otimes \mathcal{K}_{i_0, i_1, \dots, i_{d-1}}$ and $\mathcal{K}_{i_0, i_1, \dots, i_{d-1}}$ is same as in Eq. (A8). This, in turn, assures that the controlled operation (B1) maps any arbitrary three-qudit state to the subspace spanned by $|\psi\rangle \otimes |j\rangle \otimes |j\rangle\ \ \forall\ \psi$ and $\{j = 0, 1, \dots, d-1\}$. This directly follows from the structure of the Kraus operators in Eq. (B1), where there is no action on party A (I_{A}) and the operation on the A$'$C part ($\mathcal{K}_{i_0, i_1, \dots, i_{d-1}}$) has a decoherence-free subspace spanned by $\{|j\rangle \otimes |j\rangle\}$, with $\{j = 0, 1, \dots, d-1\}$. Also observing that the A$'$C marginal for the state $|\psi_{\text{AA}'\text{C}}\rangle$, i.e., $\rho_{\text{A}'\text{C}} = \text{Tr}_{\text{A}}(|\psi\rangle\langle\psi|_{\text{ATC}})$, is diagonal in the basis $|j\rangle \otimes |j\rangle$, $\{j = 0, 1, \dots, d-1\}$ we conclude

$$\begin{aligned} &I_{\text{A}} \otimes \mathcal{S}(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{d-1})|\psi\rangle\langle\psi|_{\text{AA}'\text{C}} \\ &\quad = I_{\text{A}} \otimes \mathcal{T}(\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{d-1})|\psi\rangle\langle\psi|_{\text{AA}'\text{C}} \\ &\quad = |\psi\rangle\langle\psi|_{\text{ABC}}. \end{aligned} \tag{B2}$$

Hence, at the end Alice, Bob, and Charlie share the same genuine entangled state among them. Now, considering the d-dimensional Fourier basis $\{|f_m\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\exp(i\frac{2\pi jm}{d})\,|j\rangle\}_{m=0}^{d-1}$ in Charlie's side, we can write

$$|\psi_{\text{ABC}}\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} |\Phi^{(m)}\rangle_{\text{AB}} \otimes |f_m\rangle_{\text{C}}\,,$$

$$\text{where} \quad |\Phi^{(m)}\rangle_{\text{AB}} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp\left(i\frac{2\pi jm}{d}\right) |jj\rangle_{\text{AB}}\,.$$

Evidently, Charlie, who has access on the control system, can perform a measurement in the $\{|f_m\rangle\}_{m=0}^{d-1}$ basis in his possession and communicate the outcome classically to Bob, who then applies a suitable unitary U_m on his qudit to get the state $|\Phi^+\rangle = |\Phi^0\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|jj\rangle$ between Alice and himself.

2. Proof of Theorem 2, *only if* part

With the help of the following lemmas we conclude that a preshared maximally entangled state between Alice and Charlie is necessary to establish $\log_2 d$ ebit between Alice and Bob, using d orthogonal qudit pin maps.

Let us first consider the following result for the state shared between Alice, Bob, and Charlie after the controlled quantum operation:

Lemma 7. The state shared between Alice, Bob, and Charlie after the controlled quantum operation must be three-qudit genuinely entangled GHZ state.

Proof. Let us consider the tripartite state produced after controlled quantum operation is ρ_{ABC}.

Now performing a measurement on his quantum system, Charlie will communicate the result to Bob. Depending upon which Bob will apply a local operation on his qudit to share a two-qudit maximally entangled state among Alice and himself.

Noting the fact that local operation and classical communication (LOCC) cannot increase entanglement, the state ρ_{ABC} should be maximally entangled in the A|BC bipartition. Therefore the marginal of A and at least one of B or C should be $\frac{\mathbb{I}}{d}$.

Also from Eqs. (A5) and (A17), the two-qudit marginal of the state ρ_{ABC} should be in the d-dimensional subspace spanned by $|j\rangle\otimes|j\rangle$, $j\in\{0,1,\dots,d-1\}$. These two conditions together imply

$$\sigma_{\mathrm{A}}:=\mathrm{Tr}_{\mathrm{BC}}[\rho_{\mathrm{ABC}}]=\frac{1}{d}\sum_{j=0}^{d-1}|\psi_j\rangle\langle\psi_j| \text{ and}$$

$$\sigma_{\mathrm{BC}}:=\mathrm{Tr}_{\mathrm{A}}[\rho_{\mathrm{ABC}}]=\frac{1}{d}\sum_{j=0}^{d-1}|jj\rangle\langle jj|,$$

where the states $\{|\psi_j\rangle\}_{j=0}^{d-1}$ are orthogonal to each other.

Now, the condition that the state ρ_{ABC} contains $\log_2 d$ ebit in A|BC bipartition implies that the state is pure and can be written in the Schmidt form,

$$|\psi\rangle_{\mathrm{ABC}}=\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{i\theta_j}\,|\psi_j\rangle_{\mathrm{A}}\otimes|jj\rangle_{\mathrm{BC}}\,. \tag{B3}$$

This completes the proof. ■

Now, the above lemma further helps us to conclude a corollary regarding the state just before the controlled quantum operation.

After sharing an arbitrary two-qudit state ρ_{AC} with Charlie, Alice prepares an ancillary system $\sigma_{\mathrm{A}'}$ and apply any possible quantum operation $\Lambda_{\mathrm{AA}'}$, which gives

$$\rho^*_{\mathrm{AA'C}}:=(\Lambda_{\mathrm{AA}'}\otimes I_{\mathrm{C}})\rho_{\mathrm{AC}}\otimes\sigma_{\mathrm{A}'}.$$

Corollary 1. The state $\rho^*_{\mathrm{AA'C}}$ must be maximally entangled in A|A′C bipartition.

Proof. After preparing, Alice sends the A′C subsystems of state $\rho^*_{\mathrm{AA'C}}$ through the controlled configuration of d orthogonal information-erasing channels, which maps A′C $\mapsto$ BC and produces the state $|\psi\rangle_{\mathrm{ABC}}$ [as in Eq. (B3)].

Since LOCC on any bipartition of a multipartite state cannot increase the entanglement, the state $\rho^*_{\mathrm{AA'C}}$ should be maximally entangled in A|A′C bipartition. ■

Now, with the help of Lemma 7 and Corollary 1, we will finally conclude that regarding the necessity of sharing maximal entanglement between Alice and Charlie, separately for the controlled-order and controlled-choice configuration.

Proof for the controlled order.

Lemma 8. To establish $\log_2 d$-bit entanglement between Alice and Bob after order-controlled configuration of d orthogonal pin maps, the state shared between Alice and Charlie must be maximally entanglement.

Proof. To preserve the maximal entanglement in A|A′C bipartition of the state $\rho^*_{\mathrm{AA'C}}$ under the controlled-order operation $I\otimes\mathcal{S}(\mathcal{E}_0,\mathcal{E}_1,\dots,\mathcal{E}_{d-1})$, the separable terms in Eq. (B1), i.e., in Eq. (A5), should vanish. This implies,

$$\sum_{l\neq j}(I_{\mathrm{A}}\otimes|j\rangle\langle l|_{\mathrm{A}'}\otimes|j\rangle\langle j|_{\mathrm{C}})\rho^*_{\mathrm{AA'C}}(I_{\mathrm{A}}\otimes|l\rangle\langle j|_{\mathrm{A}'}\otimes|j\rangle\langle j|_{\mathrm{C}})=0\quad\forall\, j\in\{0,1,\dots,d-1\}.$$

Therefore $\sigma^*_{\mathrm{A'C}}:=\mathrm{Tr}_{\mathrm{A}}(\rho^*_{\mathrm{AA'C}})$ will be orthogonal to the subspace spanned by $|l\rangle\otimes|j\rangle\ \ \forall\, l,j\in\{0,1,\dots,(d-1)\}$ and $l\neq j$.

This, along with Corollary 1, implies $\sigma^*_{\mathrm{A'C}}=\frac{1}{d}\sum_{j=0}^{d-1}|jj\rangle\langle jj|$ and hence the state $\rho^*_{\mathrm{AA'C}}$ is pure, which can be written as

$$|\psi^*\rangle_{\mathrm{AA'C}}=\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{i\phi_j}\,|\psi_j\rangle_{\mathrm{A}}\otimes|jj\rangle_{\mathrm{A'C}}\,,$$

$$\text{where}\quad\langle\psi_k|\psi_l\rangle=0\quad\forall\,k\neq l.$$

Note that, just by performing a measurement in d-dimensional Fourier basis of $\{|\psi_j\rangle\}$ on the subsystem A′, Alice can prepare a maximally entangled state between Charlie and herself. This, in turn, demands that the state ρ_{AC}, initially shared between Alice and Charlie, should be maximally entangled, otherwise Alice can increase entanglement only by performing local operations in her laboratory. ■

Proof for the controlled choice. Let us first consider the controlled-choice configuration of d orthogonal information-erasing channels acting on the three-qudit state $\rho^*_{\mathrm{AA'C}}$. Following from Eq. (A12) we can write,

$$\begin{aligned}&\widetilde{\mathcal{T}}(\widetilde{\mathcal{E}}_0,\widetilde{\mathcal{E}}_1,\dots,\widetilde{\mathcal{E}}_{d-1})(\rho_{\mathrm{AA'C}})\\&:=I_{\mathrm{A}}\otimes\mathcal{T}(\widetilde{\mathcal{E}}_0,\widetilde{\mathcal{E}}_1,\dots,\widetilde{\mathcal{E}}_{d-1})(\rho_{\mathrm{AA'C}})\\&=I_{\mathrm{A}}\otimes T_{0,0,\dots,0}\rho_{\mathrm{AA'C}}I_{\mathrm{A}}\otimes T^\dagger_{0,0,\dots,0}\\&\quad+\sum_{j=0}^{d-1}\mathrm{Tr}_{\mathrm{A'C}}[I_{\mathrm{A}}\otimes(I_{\mathrm{A}'}-|v_j\rangle\langle v_j|)\otimes|j\rangle\langle j|_{\mathrm{C}}\rho_{\mathrm{AA'C}}]\\&\quad\times|j\rangle\langle j|_{\mathrm{B}}\otimes|j\rangle\langle j|_{\mathrm{C}}\end{aligned}\tag{B4}$$

where $||v_j||\leqslant 1,\ \forall\, j\in\{0,1,\dots,d-1\}$ and $T_{0,0,\dots,0}$ is same as in Eq. (A13).

Keeping this in mind we now present the our main result.

Lemma 9. It is possible to obtain the state $|\psi\rangle_{\mathrm{ABC}}$ [as in Eq. (B3)] under controlled-choice configuration of d orthogonal pin maps only if Alice and Charlie share a maximally entangled state.

Proof. Following from Corollary 1, to preserve the maximal entanglement in the A|A′C bipartition of the state $\rho^*_{\mathrm{AA'C}}$ the separable terms in Eq. (B4) must vanish. Therefore,

$$\begin{aligned}&[I_{\mathrm{A}}\otimes(I_{\mathrm{A}'}-|v_j\rangle\langle v_j|)\otimes|j\rangle\langle j|_{\mathrm{C}}]\rho^*_{\mathrm{AA'C}}\\&=0,\ \forall\, j\in\{0,1,\dots,d-1\}.\end{aligned}$$

This further implies $(I_A \otimes |v_j\rangle\langle v_j|_{A'} \otimes |j\rangle\langle j|_C)\rho^*_{AA'C} = 0, \forall\, ||v_j|| < 1$, and hence we can rewrite Eq. (B4) as

$$\widetilde{\mathcal{T}}(\widetilde{\mathcal{E}}_0, \widetilde{\mathcal{E}}_1, \ldots, \widetilde{\mathcal{E}}_{d-1})\rho^*_{AA'C} = (I_A \otimes T_{0,0,\ldots,0})\rho^*_{AA'C}(I_A \otimes T^\dagger_{0,0,\ldots,0}),$$

where $T_{0,0,\ldots,0}$ is the same as in Eq. (A20). Also identifying the form of $\widetilde{\mathcal{T}}(\widetilde{\mathcal{E}}_0, \widetilde{\mathcal{E}}_1, \ldots, \widetilde{\mathcal{E}}_{d-1})\rho^*_{AA'C}$ with that of Eq. (B3), we can conclude that the A′C marginal of the state $\rho^*_{AA'C}$ has support contained in the subspace spanned by $|v_j\rangle \otimes |j\rangle$ and $T_{0,0,\ldots,0}$ contains every $||v_j|| = 1, \forall j \in \{0, 1, \ldots, d-1\}$.

Therefore, following from Corollary 1, the A′C subsystem of the state $\rho^*_{AA'C}$ should be maximally mixed in the d-dimensional subspace spanned by $|v_j\rangle \otimes |j\rangle$, i.e., $\sigma^*_{A'C} = \mathrm{Tr}_A(\rho^*_{AA'C}) = \frac{1}{d}\sum_{j=0}^{d-1}|v_j j\rangle\langle v_j j|$ and also maximally entangled in the A|A′C bipartition. Hence, the state $\rho^*_{AA'C}$ can be identified as a pure state given by

$$|\phi^*\rangle_{AA'C} = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{i\theta_j}\,|\psi_j\rangle_A \otimes |v_j\rangle_{A'} \otimes |j\rangle_C\,, \tag{B5}$$

where $\{|\psi_j\rangle\}_{j=0}^{d-1}$ is any orthonormal basis for the subsystem A. Therefore, applying a joint unitary $U_{AA'}$ Alice can transform the state

$$|\phi^*\rangle_{AA'C} \to |\xi^*\rangle_{AA'C} := \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{i\theta_j}\,|\psi_j\rangle_A \otimes |w_j\rangle_{A'} \otimes |j\rangle_C\,,$$

where $\{|w_j\rangle\}$ is an orthonormal basis for the subsystem A′, irrespective of the orthogonality condition for $\{|v_j\rangle\}$.

Now, performing a measurement on the subsystem A of the state $|\xi^*\rangle_{AA'C}$ Alice can establish maximal entanglement between Charlie and herself. This further demands that the state ρ_{AC}, initially shared between Alice and Charlie, should be maximally entangled. Otherwise, Alice will be able to increase entanglement only performing local operations at her possession.

This completes the proof. ■

APPENDIX C: ESTABLISHING MULTIPARTITE ENTANGLEMENT WITH MULTIPLE RECEIVERS

This section generalizes the results of the previous one from a single receiver to multiple, spatially separated receivers. In this case, the task is to establish a d-dimensional GHZ-state $\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle^{\otimes(N+1)}$ between Alice and N Bobs. This state represents a natural generalization of the canonical bipartite Bell state, and has applications in many quantum information processing tasks [57–62]. Moreover, the GHZ state is important in that it maximizes a distance-based measure of multipartite entanglement called the generalized geometric measure (GGM) [65–68], which for pure states admits the analytical expression

$$GGM(|\psi_n\rangle) = 1 - \max\{\lambda_{A:B}|A\cup B = \{1, 2, \ldots, n\},\ A\cap B = \emptyset\}, \tag{C1}$$

where $\lambda_{A:B}$ denotes the maximal Schmidt number in the $A:B$ bipartition, and the maximization is carried out over all possible bipartitions (note that, for mixed states, the computation of the GGM is generally hard [69]) In the case of the d-dimensional GHZ state, the GGM assumes the maximum value $(d-1)/d$.

1. The Kraus operators

Let us first consider the Kraus operators for controlled order of N noisy transmission lines for N spatially separated Bobs, each consisting of d orthogonal qudit information-erasing channels $\{\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{d-1}\}$, along with an identity channel on Alice's qudit.

$$\mathbb{I}_A \otimes \mathcal{S}^{(N)}(\mathcal{E}_0^{\otimes N}, \ldots, \mathcal{E}_{d-1}^{\otimes N})[\rho_{A\,A_1A_2\cdots A_NC}] = \sum_{I_0,I_1,\ldots,I_{d-1}} \widetilde{\mathcal{K}}_{I_0I_1\cdots I_{d-1}}[\rho_{A_1A_2\cdots A_NC}]\widetilde{\mathcal{K}}^\dagger_{I_1I_2\cdots I_{d-1}} \tag{C2}$$

where, I_k is a N-tuple consisting of the following set of numbers $\{i_{k,n}\}_{n=1}^N$. The individual Kraus operators can then be expressed as follows:

$$\begin{aligned}\widetilde{\mathcal{K}}_{I_0I_1\cdots I_{d-1}} &= \mathbb{I}_A \otimes \sum_{j=0}^{d-1}\left(\bigotimes_{n=1}^N E^{(j)}_{i_{j,n}}E^{(j\oplus 1)}_{i_{j\oplus 1},n}\cdots E^{(j\oplus(d-1))}_{i_{j\oplus(d-1)},n}\right)\otimes|j\rangle\langle j| \\ &= \mathbb{I}_A \otimes \sum_{j=0}^{d-1}\left(\bigotimes_{n=1}^N s_{j,n}|j\rangle_{B_n}\langle i_{j\ominus 1,n}|_{A_n}\right)\otimes|j\rangle\langle j|_C,\end{aligned} \tag{C3}$$

where $\forall\, n$, $s_{j,n} = \prod_{l\neq d\ominus 1}\langle i_{l,n}|l\oplus 1\rangle$. Again, one can identify $E^{(q)}_{i_{p,n}} = |q\rangle\langle i_p|\ \forall\ p, q \in \{0, 1, \ldots, (d-1)\}$ as the i_p^{th} Kraus operator of the qudit information-erasing channel $\mathcal{E}_q$ acting on the nth transmission line. Now $s_{j,n}$ is nonzero only when either one of the following two cases occur:

1. For all $n \in 1, 2, \ldots N$, $i_{l,n} = l\oplus 1$ for all $j \in 0, 1, 2, \ldots d-1$.

2. For some n values, say k of them $(1 \leqslant k \leqslant N)$, $i_{l,n} = l\oplus 1$ holds for all j except one. For the remaining $N-k$ cases, we have $i_{l,n} = l\oplus 1$ for all $j \in 0, 1, 2, \ldots d-1$.

The Kraus operator corresponding to case 1 is unique and is given by $\mathbb{I}\otimes\sum_{j=0}^{d-1}|j\rangle\langle j|^{\otimes N}\otimes|j\rangle\langle j| =: P_0^N$. The structure of the Kraus operators for case 2 is given by

$$\mathbb{I}\otimes|j\rangle^{\otimes N}\langle x_k(p)|\otimes|j\rangle\langle j|, \tag{C4}$$

where $|x_k(p)\rangle$ is a ket with N elements such that k of them are different from j. Clearly the elements of $|x_k(p)\rangle$ can be chosen in ${}^N\mathrm{C}_k = N!/[k!(N-k)!]$ ways, and p denotes the pth configuration. For example, for $N = 2$, k can take two values, namely, 1 and 2. Now for $k = 1$, we have $|x_1(1)\rangle = |jl\rangle$ and $|x_1(2)\rangle = |lj\rangle$. For $k = 2$, we get just $|x_2(1)\rangle = |l_1l_2\rangle$.

Lemma 10. The controlled operation of N noisy channels each consisting of d orthogonal information-erasing channels using a perfect side channel for the control qudit maps every $(N+1)$ qudit input state to the subspace spanned by $\{|0\rangle^{\otimes(N+1)}, |1\rangle^{\otimes(N+1)}, \ldots, |d-1\rangle^{\otimes(N+1)}\}$. This also constitutes the decoherence free subspace.

Proof. From the expression of the Kraus operators arising from cases 1 and 2, it is clear that the $(N+1)$ qudit $A_1A_2\cdots A_NC$ subsystem of any arbitrary state $\rho_{AA_1A_2\cdots A_NC}$ would be mapped into the subspace spanned by

$\{|j\rangle^{\otimes(N+1)}\}_{j=0}^{d-1}$. Furthermore, using these expressions, we can rewrite Eq. (C3) as

$$\begin{aligned}\mathbb{I}_A \otimes \mathcal{S}^{(N)}(\tilde{\mathcal{E}}_0,\dots,\tilde{\mathcal{E}}_{d-1})[\rho_{AA_1A_2\dots A_NC}] &= P_0^N \rho_{AA_1A_2\dots A_NC} P_0^N \\ &+ \sum_{k=1}^{N}\sum_{j=0}^{d-1}\sum_{l_1,l_2\dots l_k\neq j}\sum_{p=1}^{{}^NC_k}[\mathbb{I}_A \otimes \langle x_k^{\{l_i\}}(p)|\langle j|(\rho_{AA_1A_2\dots A_NC})\mathbb{I}_A \\ &\otimes |x_k^{\{l_i\}}(p)\rangle|j\rangle]|j\rangle\langle j|^{\otimes N} \otimes |j\rangle\langle j|. \end{aligned} \tag{C5}$$

Here the additional superscript $\{l_i\}$ in $|x_k^{\{l_i\}}(p)\rangle$ denotes the values of the k elements that are different from j.

The same channel is obtained from a controlled choice of the information-erasing channels $\{\mathcal{E}_j\}_{j=0}^{d-1}$, for each of the N noisy transmission lines when, as before, one considers the extended channels $\{\widetilde{\mathcal{E}}_j\}_{j=0}^{d-1}$ with Kraus operators $\widetilde{E}_{i_j}^{(j)} = |j\rangle\langle i_j| + \langle i_j|j\rangle|\mathrm{triv}\rangle\langle\mathrm{triv}|$. The controlled-choice channel is then given by [4,5]

$$\begin{aligned}\mathbb{I}_A \otimes \mathcal{T}(\widetilde{\mathcal{E}}_0^{\otimes N},\widetilde{\mathcal{E}}_1^{\otimes N},\dots,\widetilde{\mathcal{E}}_{d-1}^{\otimes N})(\rho_{AA_1A_2\dots A_NC}) \\ = \sum_{I_0,I_1,\dots,I_{d-1}} T_{I_0,I_1,\dots,I_{d-1}}\rho_{AA_1A_2\dots A_NC}T^\dagger_{I_0,I_1,\dots,I_{d-1}},\end{aligned} \tag{C6}$$

with Kraus operators

$$\begin{aligned}T_{I_0,I_1,\dots,I_{d-1}} &= \mathbb{I}_A \otimes \sum_{j=0}^{d-1}\left(\bigotimes_{n=1}^{N} t_{j,n}|j\rangle_{B_n}\langle i_{j,n}|_{A_n}\right)\otimes|j\rangle\langle j|_C, \\ t_{j,n} &:= \prod_{l\neq j}\alpha_{i_{l,n}}^{(l)},\end{aligned} \tag{C7}$$

where $\alpha_{i_{l,n}}^{(l)}$ are the amplitudes associated with the lth channel of the nth transmission line. If we set $\alpha_{i_{l,n}}^{(l)} = \langle i_{l,n}|l\rangle$, then there are two possible cases for which the Kraus operators become nonzero:

(1) $i_{l,n} = l$ for every $n \in \{1,2,\dots,N\}$ and for every $l \in \{0,1,\dots,d-1\}$.

(2) For some n values, say k of them ($1 \leqslant k \leqslant N$), $i_{l,n} = l$ holds for all except one value of l. For the remaining $N-k$ cases, $i_{l,n} = l$ for all $l \in \{0,1,\dots,d-1\}$.

For case 1, the Kraus operator is $T_{\mathbf{0},\mathbf{1},\dots,\mathbf{d-1}} = P_0^N$, where $\mathbf{k}$ is an n tuple with all elements equal to k. The Kraus operators for case 2 is of the form

$$\mathbb{I}\otimes|l\rangle^{\otimes N}\langle x_k(p)|\otimes|l\rangle\langle l|. \tag{C8}$$

Here $|x_k(p)\rangle$ is a ket with N elements such that k of them are different from l. As also mentioned after Eq. (C4), p denotes the pth configuration out of a total of ${}^NC_k$ configurations. We now arrive at the following relation:

$$\begin{aligned}\mathbb{I}\otimes\mathcal{S}^{(N)}(\mathcal{E}_0^{\otimes N},\mathcal{E}_1^{\otimes N},\dots,\mathcal{E}_{d-1}^{\otimes N}) \\ = \mathbb{I}\otimes\mathcal{T}^{(N)}(\widetilde{\mathcal{E}}_0^{\otimes N},\widetilde{\mathcal{E}}_1^{\otimes N},\dots,\widetilde{\mathcal{E}}_{d-1}^{\otimes N}) =: \mathcal{K}^{(N)}.\end{aligned} \tag{C9}$$

It generalizes Eq. (B2) for an arbitrary number of noisy transmission lines. In the following we treat the controlled order and controlled choice in a unified way, referring to channel $\mathcal{K}^{(N)}$.

Now if $\rho_{AA_1A_2\cdots A_NC}$ is so chosen such that the span of its $A_1A_2\cdots A_NC$ subsystem is $\{|j\rangle^{\otimes(N+1)}\}_{j=0}^{k}$, where $k \leqslant d-1$, the second term of Eq. (C5) vanishes identically. Hence its evolution is controlled by P_0^N which in turn keeps it unchanged. Since this is true for any $\rho_{A_1A_2\cdots A_NC}$ in the subspace $\{|j\rangle^{\otimes(N+1)}\}_{j=0}^{k}$ ($k \leqslant d-1$), we conclude that $\{|j\rangle^{\otimes(N+1)}\}_{j=0}^{d-1}$ is a decoherence-free subspace. ■

2. The *if* part of Theorem 3

Consider initially a maximally entangled state $\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i_A i_C\rangle$ shared between Alice and Charlie. Alice then makes local operations in her laboratory with additional ancillary qubits to extend her state to an $N+2$ qudit GHZ state $\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|j\rangle^{\otimes(N+2)}$ shared between A, A_1, $A_2,\dots,$ A_N, and C. Therefore we have $\rho_{AA_1A_2\cdots A_NC} = |GHZ\rangle_{N+2}$. Now using Eq. (C5) and Lemma 10, we have $\mathcal{K}^{(N)}(\tilde{\mathcal{E}}_0,\dots,\tilde{\mathcal{E}}_{d-1})(|GHZ\rangle_{N+2}) = |GHZ\rangle_{N+2}$. Therefore, the final state shared between Alice (A), the N spatially separated Bobs, (B_1, $B_2,\dots,$ B_N) and Charlie (C) is $|GHZ\rangle_{N+2}$. Now Charlie performs a measurement on his qudit in a d-dimensional Fourier basis $\{|f_m\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\exp(i\frac{2\pi jm}{d})|j\rangle\}_{m=0}^{d-1}$ and communicates the measurement outcome to the Bobs. They can now apply local unitaries to share an $(N+1)$ qudit $|GHZ\rangle_{N+1}$ among Alice and themselves. This completes the *if* part of the proof.

3. The *only if* part of Theorem 3

Here we prove the necessity of shared maximally entangled state between Alice and Charlie, with the help of the following lemmas:

Lemma 11. The state shared between Alice, all the N Bobs, and Charlie, after the controlled quantum operation, must be the $(N+2)$-qudit GHZ state.

Proof. Note that the lemma can be seen as a generalization of Lemma 7 and here we will use the same flow of arguments.

Suppose the state shared between Alice, N Bobs, and Charlie after controlled quantum operation is $\rho_{AB_1B_2\cdots B_NC}$. Performing a local measurement on his qudit, Charlie can communicate the result to the Bobs, who then able to apply proper local operations to obtain a $(N+1)$-qudit GHZ state among Alice and themselves.

Furthermore, it is possible to distill a two-qudit maximally entangled between Alice and exactly one Bob, if all N Bobs are allowed to perform only LOCC. Since, LOCC cannot increase entanglement, this implies that $\rho_{AB_1B_2\cdots B_NC}$ will be maximally entangled in the $A|B_1B_2\cdots B_NC$ bipartition. Hence, the marginal of the subsystem A and at least one among N Bobs and Charlie should be maximally mixed, I/d. This, along with Lemma 10, helps us to conclude that $\rho_{AB_1B_2\cdots B_NC}$ is pure and can be written in Schmidt form in the $A|B_1B_2\cdots B_NC$ bipartition,

$$|\psi^{(N)}\rangle_{AB_1B_2\cdots B_NC} = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{i\theta_j}|\psi_j\rangle_A\otimes|j\rangle^{\otimes(N+1)}_{B_1B_2\cdots B_NC}, \tag{C10}$$

where $\{|\psi_j\rangle\}_A$ is any arbitrary orthogonal basis for Alice's subsystem. One can identify the state (C10) as a generalization of Eq. (B3).

This completes the proof. ■

This helps us to conclude the following corollary (generalization of Corollary 1) regarding the entanglement content of the state $\rho^*_{\mathrm{AA_1A_2\cdots A}_N C}$ used as an input for the controlled quantum operations.

Corollary 2. The state $\rho^*_{\mathrm{AA_1A_2\cdots A}_N C}$ should be maximally entangled in $\mathrm{A|A_1A_2\cdots A}_N\mathrm{C}$ bipartition.

Proof. Similar to Corollary 1, the proof follows from the fact that operation performed on the $\mathrm{A_1A_2\cdots A}_N\mathrm{C}$ subsystem of the state $\rho^*_{\mathrm{AA_1A_2\cdots A}_N C}$ cannot increase the entanglement content in the $\mathrm{A|B_1B_2\cdots B}_N\mathrm{C}$ bipartition of the final state. Now, thanks to Eq. (C10) of Lemma C 3, the final state with $\log_2 d$ bits of entanglement completes the proof. ■

At this end, we will now prove the *only if* part of Theorem 3, both for the controlled-order and controlled-choice configuration, separately.

Proof for the controlled-order configuration.

Lemma 12. To establish a perfect GHZ state among Alice and all N Bobs after the controlled-order configuration of d orthogonal pin maps in each of N transmission lines, the state shared between Alice and Charlie must be maximally entangled.

Proof. The proof follows as a generalization of Lemma 8.

Note that we can conclude from Corollary 2 that the state $\rho^*_{\mathrm{AA_1A_2\cdots A}_N\mathrm{C}}$ is maximally entangled in the $\mathrm{A|B_1B_2\cdots B}_N\mathrm{C}$ bipartition, which can only be preserved after the controlled-order operation only if the separable contribution for each transmission lines from Eq. (C5) vanish and hence

$$\sum_{l_1,l_2,\ldots,l_k\neq j}\left(\mathbb{I}_{\mathrm{A}}\otimes\langle x_k^{\{l_i\}}(p)|\langle j|\right)\rho_{\mathrm{AA_1\cdots A}_N\mathrm{C}}\left(\mathbb{I}_{\mathrm{A}}\otimes|x_k^{\{l_i\}}(p)\rangle|j\rangle\right)=0$$

for every choice of ${}^N\mathrm{C}_k$ possibilities for each $k\in\{1,2,\ldots,N\}$ and for every $j\in\{0,1,\ldots,d-1\}$. This, in turn, claims that the state $\sigma^*_{\mathrm{A_1\cdots A}_N\mathrm{C}}:=\mathrm{Tr_A}[\rho^*_{\mathrm{AA_1\cdots A}_N\mathrm{C}}]$ is orthogonal to the subspace $\mathcal{S}_\perp$, where $\mathcal{S}_\perp$ is the subspace orthogonal to $|j\rangle^{\otimes N}\otimes|j\rangle\ \forall\ j\in\{0,1,\ldots,d-1\}$.

This, along with the result of Corollary 2, demands that the state $\rho^*_{\mathrm{AA_1\cdots A}_N\mathrm{C}}$ can be expressed as

$$|\psi^*\rangle_{\mathrm{AA_1\cdots A}_N\mathrm{C}}=\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{i\phi_j}|\psi_j\rangle_{\mathrm{A}}\otimes|j\rangle^{\otimes N}_{\mathrm{A_1\cdots A}_N}\otimes|j\rangle_{\mathrm{C}},$$
$$\text{where },\ \langle\psi_k|\psi_l\rangle=0\ \forall,\ k\neq l.$$

Now, performing consecutive Fourier measurement on all N qudits $\{\mathrm{A_1,A_2,\ldots,A}_N\}$ and performing local operations Alice can generate maximal entanglement between Charlie and herself, which once again confirms that the initial state shared between Alice and Charlie must have $\log_2 d$ bits of entanglement. ■

Proof for the controlled-choice configuration. Let us first consider the generalization of Eq. (B4) for N noisy transmissions lines, each with d orthogonal pin maps, in a controlled-choice configuration with the help of a perfect qudit side-channel. This reads

$$\begin{aligned}\mathbb{I}_{\mathrm{A}}\otimes\mathcal{T}^{(N)}\big(\widetilde{\mathcal{E}}_0^{\otimes N},\widetilde{\mathcal{E}}_1^{\otimes N},\ldots,\widetilde{\mathcal{E}}_{d-1}^{(N)}\big)[\rho_{\mathrm{AA_1A_2\cdots A}_N\mathrm{C}}]&=\mathbb{I}_{\mathrm{A}}\otimes T^{(N)}_{\mathbf{0,0,\ldots,0}}\rho_{\mathrm{AA_1A_2\cdots A}_N\mathrm{C}}\mathbb{I}_{\mathrm{A}}\otimes T^{(N)\dagger}_{\mathbf{0,0,\ldots,0}}\\&+\sum_{k=1}^{N}\sum_{p_k=1}^{{}^NC_k}\sum_{j=0}^{d-1}\mathrm{Tr}_{\mathrm{A_1\cdots A}_N\mathrm{C}}\left[\mathbb{I}_{\mathrm{A}}\bigotimes_{n\in p_k}\left(\mathbb{I}_{\mathrm{A}_n}-|v_j^n\rangle\langle v_j^n|\right)\bigotimes_{m\notin p_k}|j\rangle\langle v_j^m|\otimes|j\rangle\langle j|_{\mathrm{C}}\rho_{\mathrm{AA_1A_2\cdots A}_N\mathrm{C}}\right]|j\rangle\langle j|^{\otimes N}_{\mathrm{B_1\cdots B}_N}\otimes|j\rangle\langle j|_{\mathrm{C}},\end{aligned}\quad\text{(C11)}$$

where p_k is all possible choice of $k\in\{1,2,\ldots,N\}$ transmissions lines among N. We will now conclude the result finally with the following lemma:

Lemma 13. It is possible to obtain the state $|\psi^{(N)}\rangle_{\mathrm{AB_1B_2\cdots B}_N\mathrm{C}}$ of Eq. (C10) after controlled choice of N transmission lines, with d orthogonal pin maps in each, only if Alice and Charlie share a maximally entangled state initially.

Proof. To preserve the maximal entanglement in the $\mathrm{A|A_1A_2\cdots A}_N\mathrm{C}$ bipartition of the state $\rho^*_{\mathrm{AA_1A_2\cdots A}_N\mathrm{C}}$, every possible separable term in Eq. (C11) must vanish. In a similar argument as that of Lemma 9, the condition simply implies

$$T^{(N)}_{\mathbf{0,0,\ldots,0}}=\mathbb{I}_{\mathrm{A}}\otimes\sum_{j=0}^{d-1}\bigotimes_{n=1}^{N}|j\rangle\langle v_{j,n}|\otimes|j\rangle\langle j|_{\mathrm{C}},$$

where, for every $j\in\{0,1,\ldots,d-1\}$ and for every $n\in\{1,2,\ldots,N\}$, $||v_{j,n}||=1$. This, along with Corollary 2, demands a that the state $\rho^*_{\mathrm{AA_1A_2\cdots A}_N\mathrm{C}}$ is pure and can be written in the form

$$|\phi^{(N)*}\rangle_{\mathrm{AA_1\cdots A}_N\mathrm{C}}=\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{i\theta_j}|\psi_j\rangle_{\mathrm{A}}\bigotimes_{k=1}^{N}|v_{j,n}\rangle_{\mathrm{A}_k}\otimes|j\rangle_{\mathrm{C}},$$

where $\{|\psi_j\rangle\}$ is a orthonormal basis for the subsystem A and the state can be identified as the generalization of Eq. (B5).

Now, Alice is able to apply the joint unitary $U_{\mathrm{AA_1\cdots A}_N}$ on the $\mathrm{AA_1A_2\cdots A}_N$ subsystem which takes

$$U_{\mathrm{AA_1\cdots A}_N}|\psi_j\rangle_{\mathrm{A}}\bigotimes_{k=1}^{N}|v_{j,n}\rangle_{\mathrm{A}_k}\to|\psi_j\rangle_{\mathrm{A}}\bigotimes_{k=1}^{N}|w_{j,n}\rangle_{\mathrm{A}_k},$$

and hence

$$(U_{\mathrm{AA}_1\cdots\mathrm{A}_N}\otimes\mathbb{I}_{\mathrm{C}})|\phi^{(N)*}\rangle_{\mathrm{AA}_1\cdots\mathrm{A}_N\mathrm{C}}\to|\xi^{(N)*}\rangle_{\mathrm{AA}_1\cdots\mathrm{A}_N\mathrm{C}}:=\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}e^{i\theta_j}|\psi_j\rangle_{\mathrm{A}}\bigotimes_{k=1}^{N}|w_{j,n}\rangle_{\mathrm{A}_k}\otimes|j\rangle_{\mathrm{C}},$$

where $\{|w_{j,n}\rangle\}_{j=0}^{d-1}$ is an orthogonal basis for every $n\in\{1,2,\ldots,N\}$.

Now, just performing Fourier basis measurement on every A_k, $k\in\{1,2,\ldots,N\}$ and performing suitably chosen unitary on subsystem A, Alice can establish $\log_2 d$ bits of entanglement with Charlie. This, in turn, assures that the state initially shared between Alice and Charlie should contain $\log_2 d$ bits of entanglement, i.e., a maximally entangled two-qudit state. ■

[1] Y. Aharonov, J. Anandan, S. Popescu, and L. Vaidman, Superpositions of Time Evolutions of a Quantum System and a Quantum Time-Translation Machine, Phys. Rev. Lett. **64**, 2965 (1990).

[2] D. K. L. Oi, Interference of Quantum Channels, Phys. Rev. Lett. **91**, 067902 (2003).

[3] J. Åberg, Operations and single-particle interferometry, Phys. Rev. A **70**, 012103 (2004).

[4] A. A. Abbott, J. Wechs, D. Horsman, M. Mhalla, and C. Branciard, Communication through coherent control of quantum channels, Quantum **4**, 333 (2020).

[5] G. Chiribella and H. Kristjánsson, Quantum Shannon theory with superpositions of trajectories, Proc. R. Soc. London, Ser. A **475**, 20180903 (2019).

[6] Q. Dong, S. Nakayama, A. Soeda, and M. Murao, Controlled quantum operations and combs, and their applications to universal controllization of divisible unitary operations, arXiv:1911.01645.

[7] A. Vanrietvelde and G. Chiribella, Universal control of quantum processes using sector-preserving channels, Quantum Information and Computation, **21**, 1320 (2021).

[8] N. Gisin, N. Linden, S. Massar, and S. Popescu, Error filtration and entanglement purification for quantum communication, Phys. Rev. A **72**, 012338 (2005).

[9] N. Loizeau and A. Grinbaum, Channel capacity enhancement with indefinite causal order, Phys. Rev. A **101**, 012340 (2020).

[10] H. Kristjánsson, W. Mao, and G. Chiribella, Witnessing latent time correlations with a single quantum particle, Phys. Rev. Res. **3**, 043147 (2021).

[11] L.-P. Lamoureux, E. Brainis, N. J. Cerf, Ph. Emplit, M. Haelterman, and S. Massar, Experimental Error Filtration for Quantum Communication Over Highly Noisy Channels, Phys. Rev. Lett. **94**, 230501 (2005).

[12] G. Rubino, L. A. Rozema, D. Ebler, H. Kristjánsson, S. Salek, P. Allard Guérin, A. A. Abbott, C. Branciard, Č. Brukner, G. Chiribella, and P. Walther, Experimental quantum communication enhancement by superposing trajectories, Phys. Rev. Res. **3**, 013093 (2021).

[13] J. Thompson, K. Modi, V. Vedral, and M. Gu, Quantum plug n′ play: modular computation in the quantum regime, New J. Phys. **20**, 013004 (2018).

[14] F. Chapeau-Blondeau, Quantum parameter estimation on coherently superposed noisy channels, Phys. Rev. A **104**, 032214 (2021).

[15] G. Chiribella and X. Zhao, Heisenberg-limited metrology with coherent control on the probes' configuration, arXiv:2206.03052.

[16] L. Hardy, Towards quantum gravity: A framework for probabilistic theories with non-fixed causal structure, J. Phys. A: Math. Theor. **40**, 3081 (2007).

[17] L. Hardy, Quantum gravity computers: On the theory of computation with indefinite causal structure, in *The Western Ontario Series in Philosophy of Science* (Springer, Netherlands, 2009), pp. 379–401.

[18] W. C. Myrvold and J. Christian, *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle* (Springer, Netherlands, 2009).

[19] G. Chiribella, G. M. D′Ariano, P. Perinotti, and B. Valiron, Beyond quantum computers, Phys. Rev. A **88**, 022318 (2013).

[20] O. Oreshkov, F. Costa, and Č. Brukner, Quantum correlations with no causal order, Nat. Commun. **3**, 1092(2012).

[21] M. Araújo, C. Branciard, F. Costa, A. Feix, C. Giarmatzi, and Č. Brukner, Witnessing causal nonseparability, New J. Phys. **17**, 102001 (2015).

[22] L. M. Procopio, A. Moqanaki, M. Araújo, F. Costa, I. Alonso Calafell, E. G. Dowd, D. R. Hamel, L. A. Rozema, Č. Brukner, and P. Walther, Experimental superposition of orders of quantum gates, Nat. Commun. **6**, 7913 (2015).

[23] G. Rubino, L. A. Rozema, A. Feix, M. Araújo, J. M. Zeuner, L. M. Procopio, Č. Brukner, and P. Walther, Experimental verification of an indefinite causal order, Sci. Adv. **3**, e1602589(2017).

[24] K. Goswami, C. Giarmatzi, M. Kewming, F. Costa, C. Branciard, J. Romero, and A. G. White, Indefinite Causal Order in a Quantum Switch, Phys. Rev. Lett. **121**, 090503 (2018).

[25] K. Wei, N. Tischler, S.-R. Zhao, Y.-H. Li, J. M. Arrazola, Y. Liu, W. Zhang, H. Li, L. You, Z. Wang, Y.-A. Chen, B. C. Sanders, Q. Zhang, G. J. Pryde, F. Xu, and J.-W. Pan, Experimental Quantum Switching for Exponentially Superior Quantum Communication Complexity, Phys. Rev. Lett. **122**, 120504 (2019).

[26] Y. Guo, X.-M. Hu, Z.-B. Hou, H. Cao, J.-M. Cui, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, and G. Chiribella, Experimental Transmission of Quantum Information using a Superposition of Causal Orders, Phys. Rev. Lett. **124**, 030502 (2020).

[27] K. Goswami and J. Romero, Experiments on quantum causality, AVS Quantum Sci. **2**, 037101 (2020).

[28] G. Chiribella, Perfect discrimination of no-signalling channels via quantum superposition of causal structures, Phys. Rev. A **86**, 040301(R) (2012).

[29] M. Araújo, F. Costa, and Č. Brukner, Computational Advantage from Quantum-Controlled Ordering of Gates, Phys. Rev. Lett. **113**, 250402 (2014).

[30] P. A. Guérin, A. Feix, M. Araújo, and Č. Brukner, Exponential Communication Complexity Advantage from Quantum Superposition of the Direction of Communication, Phys. Rev. Lett. **117**, 100502 (2016).

[31] D. Ebler, S. Salek, and G. Chiribella, Enhanced Communication with the Assistance of Indefinite Causal Order, Phys. Rev. Lett. **120**, 120502 (2018).

[32] S. Salek, D. Ebler, and G. Chiribella, Quantum communication in a superposition of causal orders, arXiv:1809.06655.

[33] G. Chiribella, M. Banik, S. S. Bhattacharya, T. Guha, M. Alimuddin, A. Roy, S. Saha, S. Agrawal, and G. Kar, Indefinite causal order enables perfect quantum communication with zero capacity channels, New J. Phys. **23**, 033039 (2021).

[34] M. Caleffi and A. S. Cacciapuoti, Quantum switch for the quantum internet: Noiseless communications through noisy channels, IEEE J. Sel. Areas Commun. **38**, 575 (2020).

[35] S. S. Bhattacharya, A. G. Maity, T. Guha, G. Chiribella, and M. Banik, Random-receiver quantum communication, PRX Quantum **2**, 020350 (2021).

[36] S. Sazim, M. Sedlak, K. Singh, and A. K. Pati, Classical communication with indefinite causal order for N completely depolarizing channels, Phys. Rev. A **103**, 062610 (2021).

[37] G. Chiribella, M. Wilson, and H. F. Chau, Quantum and Classical Data Transmission Through Completely Depolarizing Channels in a Superposition of Cyclic Orders, Phys. Rev. Lett. **127**, 190502 (2021).

[38] X. Zhao, Y. Yang, and G. Chiribella, Quantum Metrology with Indefinite Causal Order, Phys. Rev. Lett. **124**, 190503 (2020).

[39] F. Chapeau-Blondeau, Noisy quantum metrology with the assistance of indefinite causal order, Phys. Rev. A **103**, 032615 (2021).

[40] D. Felce and V. Vedral, Quantum Refrigeration with Indefinite Causal Order, Phys. Rev. Lett. **125**, 070603 (2020).

[41] T. Guha, M. Alimuddin, and P. Parashar, Thermodynamic advancement in the causally inseparable occurrence of thermal maps, Phys. Rev. A **102**, 032215 (2020).

[42] K. Simonov, G. Francica, G. Guarnieri, and M. Paternostro, Work extraction from coherently activated maps via quantum switch, Phys. Rev. A **105**, 032217 (2022).

[43] T. Heinosaari and M. Ziman, *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement* (Cambridge University Press, Cambridge, UK, 2011).

[44] L. del Rio, J. Åberg, R. Renner, O. Dahlsten, and V. Vedral, The thermodynamic meaning of negative entropy, Nature (London) **474**, 61 (2011).

[45] P. Faist, F. Dupuis, J. Oppenheim, and R. Renner, The minimal work cost of information processing, Nat. Commun. **6**, 7669 (2015).

[46] C. H. Bennett and G. Brassard, Quantum cryptography: Public key distribution and coin tossing, in *Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing* (Bangalore, India, 1984), p. 175.

[47] A. K. Ekert, Quantum Cryptography Based on Bell's Theorem, Phys. Rev. Lett. **67**, 661 (1991).

[48] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Decoherence-Free Subspaces for Quantum Computation, Phys. Rev. Lett. **81**, 2594 (1998).

[49] P. G. Kwiat, A. J. Berglund, J. B. Altepeter, and A. G. White, Experimental verification of decoherence-free subspaces, Science **290**, 498 (2000).

[50] D. A. Lidar and K. B. Whaley, Decoherence-free subspaces and subsystems, in *Irreversible Quantum Dynamics* (Springer, Berlin, Heidelberg, 2003), pp. 83–120.

[51] D. A. Lidar, Review of decoherence-free subspaces, noiseless subsystems, and dynamical decoupling, in *Advances in Chemical Physics* (John Wiley & Sons, Inc., 2014), pp. 295–354.

[52] P. A. Guérin, G. Rubino, and Č. Brukner, Communication through quantum-controlled noise, Phys. Rev. A **99**, 062317 (2019).

[53] H. Kristjánsson, G. Chiribella, S. Salek, D. Ebler, and M. Wilson, Resource theories of communication, New J. Phys. **22**, 073014 (2020).

[54] C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin, Capacities of Quantum Erasure Channels, Phys. Rev. Lett. **78**, 3217 (1997).

[55] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Quantum Repeaters: The Role of Imperfect Local Operations in Quantum Communication, Phys. Rev. Lett. **81**, 5932 (1998).

[56] W. Dür, H.-J. Briegel, J. I. Cirac, and P. Zoller, Quantum repeaters based on entanglement purification, Phys. Rev. A **59**, 169 (1999).

[57] H. Buhrman, W. van Dam, P. Høyer, and A. Tapp, Multiparty quantum communication complexity, Phys. Rev. A **60**, 2737 (1999).

[58] G. Brassard, A. Broadbent, J. Fitzsimons, S. Gambs, and A. Tapp, Anonymous quantum communication, in *International Conference on the Theory and Application of Cryptology and Information Security* (Springer, 2007), pp. 460–473.

[59] A. Broadbent, P.-R. Chouha, and A. Tapp, The GHZ state in secret sharing and entanglement simulation, in *2009 Third International Conference on Quantum, Nano and Micro Technologies* (IEEE, 2009), pp. 59–62.

[60] P. Komar, E. M. Kessler, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, A quantum network of clocks, Nat. Phys. **10**, 582 (2014).

[61] Z. Eldredge, M. Foss-Feig, J. A. Gross, S. L. Rolston, and A. V. Gorshkov, Optimal and secure measurement protocols for quantum sensor networks, Phys. Rev. A **97**, 042337 (2018).

[62] K. Qian, Z. Eldredge, W. Ge, G. Pagano, C. Monroe, J. V. Porto, and A. V. Gorshkov, Heisenberg-scaling measurement protocol for analytic functions with quantum sensor networks, Phys. Rev. A **100**, 042304 (2019).

[63] C. H. Bennett and S. J. Wiesner, Communication Via One- and Two-Particle Operators on Einstein-Podolsky-Rosen States, Phys. Rev. Lett. **69**, 2881 (1992).

[64] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum State Via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).

[65] A. Sen De and U. Sen, Channel capacities versus entanglement measures in multiparty quantum states, Phys. Rev. A **81**, 012308 (2010).

[66] A. Shimony, Degree of entanglement, Ann. N. Y. Acad. Sci. **755**, 675 (1995).

[67] T.-C. Wei and P. M. Goldbart, Geometric measure of entanglement and applications to bipartite and multipartite quantum states, Phys. Rev. A **68**, 042307 (2003).

[68] M. Blasone, F. Dell'Anno, S. De Siena, and F. Illuminati, Hierarchies of geometric entanglement, Phys. Rev. A **77**, 062304 (2008).

[69] T. Das, S. S. Roy, S. Bagchi, A. Misra, A. Sen De, and U. Sen, Generalized geometric measure of entanglement for multiparty mixed states, Phys. Rev. A **94**, 022336 (2016).