Dynamic Event-Triggered Intermittent Control for Stabilization of Delayed Dynamical Systems ⋆

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Abstract

This paper studies the dynamic event-triggered intermittent control (D-ETIC) for the stabilization of delayed dynamical systems (DDS). The stabilization of DDS via general intermittent control is formulated as a problem of delay-dependent minimal activation time rate (MATR). A D-ETIC scheme with input delay is proposed to stabilize DDS. The delay-dependent MATR is estimated. And the maximal input delay allowed in D-ETIC is estimated for quasi-linear DDS. Both theoretical and numerical comparisons are given among D-ETIC, static ETIC (S-ETIC), and the recently developed static intermittent control schemes, including time-triggered intermittent control and event-triggered aperiodic intermittent control. It is shown that the proposed D-ETIC achieves the lowest MATR. It is also shown that larger delays, including the time delay in DDS and the input delay in D-ETIC, may lead to more control activation time for the stabilization.

Key words: Stabilization; delayed dynamical systems; event-triggered intermittent control (ETIC); activation time; time delay.

1 Introduction

Intermittent control is attracting more and more interest in the field of control (e.g., [1-4]). In some practical problems, such as vehicle control, orbital adjustment of space shuttles, and management of smart grids with distributed generations and energy storage units, the control of these systems is intermittent. There is no need to continuously control these systems at all the times like continuous control. And it is not possible to execute and complete every control in an instant like impulsive control (e.g., [5-8]). Hence, in these practical applications, it is meaningful to use intermittent control.

Recently, two types of intermittent control schemes have been proposed, including periodic intermittent control (PIC) and aperiodic intermittent control (AIC). In the PIC scheme, the control width and the non-control width all keep unchanged. So the entire control is periodic (e.g., [2,11-13]). This control method is easy to be executed, but as noted in [14-16], the restriction on periodicity might be conservative and unnecessary in some practical problems, e.g., vehicle control. The AIC scheme is more flexible and can relax the periodicity restriction in PIC. This makes AIC more general and versatile than PIC, see, e.g., [14-18]. However, the reported AIC is often time-triggered and is designed via Lyapunov stability conditions or dwell-time conditions ([19,20]). The sufficient stabilization conditions are independent of states and may be conservative. It may lead to redundant and sometimes unnecessary control activation time. This problem was formulated as the minimum activation time rate (MATR) of AIC in [17].

For the stabilization via AIC, although it is difficult to find the MATR, it is meaningful to design an AIC with a relatively lower MATR. By integrating the event-triggered control ([21-26]) and the event-triggered impulsive control ([9,10]) into AIC, the event-triggered aperiodic intermittent control (E-AIC) was proposed ([17,18]). It was shown that E-AIC could achieve smaller MATR than the time-triggered intermittent controls (TTIC) (including time-triggered PIC and AIC). However, there are still some shortcomings in the designed E-AIC. The E-AIC by [17,18] is not fully event-triggered, and for the MATR problem, the control width that is not based on the state may result in excess control time.

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Thus, the activation time rate of E-AIC may still be high. Moreover, the E-AIC in [17,18] is designed only for systems without time delays. It is clearly not suitable for systems with time delays. Hence, an open question is how to design event-triggered intermittent control (ETIC) with lower MATR for delayed dynamical systems (DDS). And the questions on how to estimate the MATR of ETIC, and how to estimate the effects of time delays on ETIC and MATR are also challenging.

Note that a scheme of dynamic event-triggered control (D-ETC) was developed in [27]. Compared with static ETC, D-ETC has extra internal dynamics in the event-trigger condition. By [27], the triggering times of D-ETC are less than that of the static ETC. Thus D-ETC can save more system resources. Recently, many developments on D-ETC have been obtained, e.g., [28-31].

Motived by the above observations, in this paper, we study dynamic event-triggered intermittent control (D-ETIC) with input delay for stabilization of DDS. First, we formulate the stabilization via general intermittent control as a problem of delay-dependent MATR. Then, we propose a scheme of D-ETIC with input delay, where a dynamics is designed into ETIC. Both the control starting time and the control width are state-dependent and determined by event conditions. Moreover, the time delay in DDS is taken into the design of events. It is shown that the stabilization of DDS is achieved by the designed D-ETIC with input delay. Comparisons on MATR of D-ETIC, TTIC, E-AIC by [17,18], and static ETIC (S-ETIC), are given via both theoretical analysis and numerical simulations.

The main contributions of the paper include: an event-triggered and delay-dependent D-ETIC with input delay is proposed, which is a clear improvement over static and delay-independent E-AIC without input delay by recent [17,18]; the dynamics introduced into D-ETIC make D-ETIC have lower MATR than S-ETIC; both theoretical comparisons and numerical simulations are derived among MATR of D-ETIC, TTIC, E-AIC, and S-ETIC, and D-ETIC achieves the lowest MATR; the effect of time delay on MATR is analyzed, and it is shown that a larger delay including time delay in DDS and input delay in D-ETIC leads to more control activation time for stabilization, and the maximal input delay allowed in D-ETIC is derived for quasi-linear DDS.

The rest of this paper is organized as follows. In Section 2, we present preliminaries. In Section 3, we design the D-ETIC scheme, give comparisons on MATR of D-ETIC, TTIC, E-AIC, and S-ETIC, analyze the effect of the time delay on MATR, and extend the D-ETIC to the sampling-based D-ETIC. In Section 4, a numerical example is given to illustrate the correctness of the obtained results. Section 5 provides some conclusions.

2 Preliminaries

In the sequel, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{N} = \{0, 1, 2, \cdots\}$. Given a matrix $A \in \mathbb{R}^{n\times n}$, let $\|A\| = [\lambda_{\max}(A^TA)]^{\frac{1}{2}}$, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of the matrix $(\cdot)$. And let $\rho(A)$ denote the spectral radius of the matrix A and I_n denote the $n \times n$ unity matrix. For $h^* \in \mathbb{R}_+$, let $C([-h^*, 0]; \mathbb{R}^n)$ be the set of continuous functions defined on $[-h^*, 0]$, and define $\|\phi\|_{h^*} \triangleq \sup_{-h^*\le s\le 0}\{\|\phi(s)\|\}, \forall \phi \in C([-h^*, 0]; \mathbb{R}^n)$.

Consider a delayed dynamical system (DDS) as

$$\dot{x}(t) = f(x(t), x(t-h(t)), u(t)),\ t \ge t_0, \tag{1}$$

where $x \in \mathbb{R}^n$; f is continuous with $f(0, 0, 0) = 0$; u is the control input; and $h(t)$ is the time delay satisfying $0 \le h(t) \le h_0^* < \infty$ with the maximal time delay h_0^*. For any $\tau^* \ge h_0^*$, assume the solution $x(t) \triangleq x(t, t_0, \phi)$ of (1) exists uniquely for any initial condition (t_0, ϕ) with $x_0 = \phi \in C([t_0 - \tau^*, t_0], \mathbb{R}^n)$ and is forward complete.

Assumption 2.1. There exists a Lyapunov-like function $V : \mathbb{R}^n \to \mathbb{R}_+$, satisfying:
(i) for constants $c_1 > 0, c_2 > 0, r > 0$,

$$c_1\|x\|^r \le V(x) \le c_2\|x\|^r,\ \forall x \in \mathbb{R}^n; \tag{2}$$

(ii) the Dini derivative D^+V along (1) with $u = 0$ satisfies: for some constants $a_0 \ge 0$, $b_0 \ge 0$,

$$D^+V(x(t))\big|_{u=0} \le a_0V(x(t)) + b_0\bar{V}(x(t)); \tag{3}$$

(iii) there is a control law, for continuous function $\psi(\cdot)$,

$$u = \psi(x), \tag{4}$$

such that for some input delay $q^* \ge 0$ and some constants a and b with $a > b \ge 0$,

$$D^+V(x(t))\big|_{u(t)=\psi(x(t-q(t)))} \le -aV(x(t)) + b\bar{V}(x(t)), \tag{5}$$

holds for all continuous input delay $q(t)$ satisfying $0 \le q(t) \le q^*$, where $\bar{V}(x(t)) = \max_{t-h^*\le s\le t}\{V(x(s))\}$ with $h^* \triangleq \max\{2q^*, h_0^* + q^*\}$.

Remark 2.1. Assumption 2.1 (ii) means DDS (1) with $u = 0$ is unstable while DDS (1) is stabilized by the control law (4) (see Lemma 2.2). Here, the maximum delay h^* is set as $h^* = \max\{2q^*, h_0^* + q^*\}$. This is based on the fact that under the input delay of u, the transformation $x(t - q(t)) = x(t) + (x(t - q(t)) - x(t)) = x(t) - \int_{-q(t)}^{0} f(x(t+s), x(t+s-h(t+s)), \psi(x(t+s-q(t+s))))ds$ is often required in the calculation of $D^+V(x(t))$.

In this paper, by Assumption 2.1, we consider an intermittent control denoted by $(u, \{t_i\}, \{\tau_i\})$ with $\tau_0 = 0$ as:

$$u(t) = \begin{cases} \psi(x(t-q(t))),\ t \in (t_i, t_i+\tau_i],\ i \geq 1, \\ 0,\ t \in (t_i+\tau_i, t_{i+1}],\ i \geq 0, \end{cases} \tag{6}$$

where t_i and $t_i + \tau_i$ are respectively the starting and the completion time of u on the interval $(t_i, t_{i+1}]$ and satisfy

$$t_0 < t_1 \leq t_1 + \tau_1 < t_2 \leq \cdots < t_i \leq t_i + \tau_i < t_{i+1} \leq \cdots \tag{7}$$

Here, τ_i is called the i-th *control-width* or *active time* and $s_i = t_{i+1} - t_i - \tau_i$ is the i-th *non-control-width*.
Under the control law (6), DDS (1) becomes:

$$\begin{cases} \dot{x} = f(x, x(t-h(t)), \psi(x(t-q(t)))),\ t \in (t_i, t_i+\tau_i], \\ \dot{x} = f(x, x(t-h(t)), 0),\ t \in (t_i+\tau_i, t_{i+1}],\ i \in \mathbb{N}. \end{cases} \tag{8}$$

Remark 2.2. In some literature, the intermittent control might be inactively forced due to the uncertainty or disturbances which can lead to that control can be implemented intermittently. Like many other works (see [11-18] for examples), this paper considers active intermittent control problems. The meaning of considering this active intermittent control is that in some physical systems, such as smart grids and spacecraft, the active intermittent control should be used once the state of the system runs into unsafe and unstable areas.

Definition 2.1. The DDS (1) is said to be exponentially stabilized by the intermittent control (6) if the system (8) is globally exponentially stable, i.e., $\forall (t_0, \phi)$, and for some $\alpha > 0$, $K > 0$, $\|x(t)\| \leq K e^{-\alpha(t-t_0)} \|\phi\|_{h^*}$, $\forall t \geq t_0$.

Definition 2.2. The intermittent control $(u, \{t_i\}, \{\tau_i\})$ is called *non-Zeno* (NZ) if $t_{i+1} > t_i$, $\forall i \in \mathbb{N}$, $\lim_{i\to\infty} t_i = \infty$. Further, the intermittent control $(u, \{t_i\}, \{\tau_i\})$ is called *non-trivial* (NT) if $0 < \tau_{\max} \triangleq \sup_{i\in\mathbb{N}}\{\tau_i : \tau_i > 0\} < \infty$.

MATR Problem: Let Assumption 2.1 be satisfied. We formulate the stabilization issue of (1) via intermittent control $(u, \{t_i\}, \{\tau_i\})$ as a problem of delay-dependent *minimal activation time rate* (MATR):

$$\mathcal{R}_{\min}(h^*) \triangleq \min_{\{t_i\},\{\tau_i\}} \Big\{ \liminf_{i\to\infty} \frac{\sum_{j=0}^{i} \tau_j}{t_{i+1} - t_0} \Big\},$$

s.t. (1) is stabilized by $(u, \{t_i\}, \{\tau_i\})$ satisfying NZ-NT.

In this paper, for the MATR problem, we design a dynamic event-triggered intermittent control law (D-ETIC), estimate the MATR, and analyze the impact of the time delay on the MATR $\mathcal{R}_{\min}(h^*)$ and the D-ETIC.

Remark 2.3. (i) In [17], the delay-independent $\mathcal{R}_{\min}$ is used for the MATR problem. Here, note that the control time sequence $\{t_i\}$ and the control width sequence $\{\tau_i\}$ are dependent on the time delay. Hence, in the MATR problem, we use $\mathcal{R}_{\min}(h^*)$ to replace $\mathcal{R}_{\min}$ to reflect the effect on MATR from the time delay.
(ii) Generally, it is difficult to solve the MATR problem. Note that $\mathcal{R}_{\min}(h^*) = \min_{\{t_i\},\{\tau_i\}} \big\{ \liminf_{i\to\infty} \mathcal{R}(h^*, i) \big\}$, where $\mathcal{R}(h^*, i) = \frac{\sum_{j=0}^{i} \tau_j}{\sum_{j=0}^{i} \tau_j + \sum_{j=0}^{i} s_j}$ is defined as the i-th *activation time rate*.

Lemma 2.1. ([32]) Assume $v \in C([t_0 - h^*, +\infty), \mathbb{R}_+)$ satisfies the following differential delayed inequality (DDI): $D^+ v(t) \leq a_0 v(t) + b_0 \bar{v}(t)$, $t \geq t_0 \geq 0$, where $a_0 \in \mathbb{R}$ and $b_0 \in \mathbb{R}_+$ satisfy $a_0 + b_0 \geq 0$, and $\bar{v}(t) = \max_{t-h^* \leq s \leq t}\{v(s)\}$. Then, $v(t) \leq (1 + b_0 h^*) e^{(a_0+b_0)(t-t_0)} \bar{v}(t_0)$, $t \geq t_0$.

Lemma 2.2. (Halanay Lemma ([32])) Let $v \in C([t_0 - h^*, +\infty), \mathbb{R}_+)$ satisfy the Halanay inequality: $D^+ v(t) \leq -a v(t) + b \bar{v}(t)$, $t \geq t_0 \geq 0$, where $a > 0$ and $b \in \mathbb{R}_+$ satisfy $-a + b \leq 0$, and $\bar{v}(t) = \max_{t-h^* \leq s \leq t}\{v(s)\}$. Then, $v(t) \leq e^{-\lambda(t-t_0)} \bar{v}(t_0)$, $t \geq t_0$, where $\lambda > 0$ is the unique root of $\lambda - a + b e^{\lambda h^*} = 0$.

3 Main Results

In this section, we give the D-ETIC scheme. Then, we derive the stabilization of DDS (1) via D-ETIC and estimate the MATR of D-ETIC. And we make comparisons on MATR of D-ETIC, TTIC, E-AIC, and S-ETIC.

Let Assumption 2.1 hold with parameters $a_0 > 0, b_0 \geq 0, a > b \geq 0$. Let $g_1(h^*) > 0$ be the unique root of

$$s - a + b e^{s h^*} = 0. \tag{9}$$

Note that for every $h^* \geq 0$, there exists a unique root $g_1(h^*) > 0$ satisfying (9). For some constants $\sigma_{\max} > 1$ and $\sigma_{\min} < 1$, define g_0, δ_0, δ_1 as: $g_0 \triangleq a_0 + b_0, \delta_0 \triangleq \frac{\ln \sigma_{\max} - \ln(1 + b_0 h^*)}{g_0}, \delta_1 \triangleq h^* - \frac{\ln \sigma_{\min}}{g_1(h^*)}$.

For DDS (1), we define a dynamic system as:

$$\begin{cases} \dot{\nu}(t) = -\xi \nu(t) + \eta V(x(t)),\ t \geq t_0, \\ \nu(t_0) = \bar{V}(x_0), \end{cases} \tag{10}$$

where $\xi > 0$ and $\eta \geq 0$ are some constants.
Given $V(x)$ and $\nu(t)$, constants $\sigma_{\max} > 1$, $\gamma \geq 0$, define:

$$\mathcal{C}(s, t] \triangleq \{\theta : s < \theta \leq t, V(x(\theta)) \geq \gamma \nu(\theta) + \sigma_{\max} \bar{V}(x(s))\}.$$

D-ETIC for DDS (1): The D-ETIC scheme is based on the dynamics (10) and three indices: the threshold-value $\sigma_{\max} > 1$, the control-goal index $\sigma_{\min} < 1$, and the check-period $\Delta > 0$, which satisfy:

$$\Delta > \max\{\delta_0, \delta_1, h^*\}, \quad 1 + b_0 h^* < \sigma_{\max} < \sigma_{\min}^{-1}. \quad (11)$$

Set $\tau_0 = 0$. The $(i+1)$-th D-ETIC $(u(t), t_{i+1}, \tau_{i+1})$ is:

$$t_{i+1} = \begin{cases} \min\{t : t \in \mathcal{C}(t_i + \tau_i, t_i + \tau_i + \Delta]\}, \\ \quad \text{if } \mathcal{C}(t_i + \tau_i, t_i + \tau_i + \Delta] \neq \emptyset; \\ t_i + \tau_i + \Delta, \text{ if } \mathcal{C}(t_i + \tau_i, t_i + \tau_i + \Delta] = \emptyset; \end{cases} \quad (12)$$

$$\tau_{i+1} = \min\{\tau : \bar{V}(x(t_{i+1} + \tau)) \leq \sigma_{\min} \bar{V}(x(t_{i+1}))\}; (13)$$

$$u(t) = \begin{cases} \psi(x(t - q(t))), \quad t \in [t_{i+1}, t_{i+1} + \tau_{i+1}), \\ 0, \quad t \in [t_{i+1} + \tau_{i+1}, t_{i+2}). \end{cases} \quad (14)$$

S-ETIC for DDS (1): Specifically, in (12), if $\gamma = 0$, then the ETIC is independent of the dynamics (10), and thus, it is static. We call it *static* ETIC (S-ETIC).

Remark 3.1. The D-ETIC (10)-(14) is based on the Lyapunov-like function V on the delay intervals, the dynamics $\nu(t)$, and the three indices: $\sigma_{\max}$, $\sigma_{\min}$, and Δ. The threshold-value $\sigma_{\max}$ and the check-period Δ are used to get the control starting time sequence $\{t_i\}$, while the control-goal index $\sigma_{\min}$ is used to check the control goal, i.e., $\bar{V}(x(t + \tau)) \leq \sigma_{\min} \bar{V}(x(t))$, and determines the control width sequence $\{\tau_i\}$. Compared with time-triggered intermittent control (TTIC) ([2,11-16]) and event-triggered aperiodic intermittent control (E-AIC) ([17,18]), there are some basic differences among D-ETIC, TTIC, and E-AIC. Both TTIC and E-AIC are static. Here, in (12), only if $\gamma = 0$, then the ETIC is static, i.e., S-ETIC. In TTIC, both the control starting time t_i and the control width τ_i are determined only by Lyapunov stability conditions and are state-independent. In E-AIC, the information of time delay, including the input delay $q(t)$, is not taken into consideration in u. In E-AIC, every τ_i is determined by the stabilization condition and is state-independent. While in D-ETIC, it is fully state-dependent, i.e., all τ_i and t_i are determined by the event-trigger conditions. Moreover, in E-AIC, $\tau_i > 0$ is required for all $i \geq 1$. While in D-ETIC (10)-(14), $\tau_i = 0$ in the case that the control goal is already satisfied at some control starting instants t_i.

Note that $s_i = t_{i+1} - t_i - \tau_i$. Define the i-th *average control width* and *average non-control-width* respectively as: $\bar{\tau}_i \triangleq \frac{\sum_{j=0}^{i} \tau_j}{i}$, $\bar{s}_i \triangleq \frac{\sum_{j=0}^{i} s_j}{i}$. Then, $\mathcal{R}(h^*, i) = \frac{\bar{\tau}_i}{\bar{\tau}_i + \bar{s}_i}$. Define the maximal/minimal average control width as: $\bar{\tau}_{\max} \triangleq \sup_{i \in \mathbb{N}}\{\bar{\tau}_i\}$, $\bar{\tau}_{\min} \triangleq \inf_{i \in \mathbb{N}}\{\bar{\tau}_i\}$.

Theorem 3.1. Let Assumption 2.1 be satisfied. Let the parameters γ, ξ, and η in D-ETIC (10)-(14) satisfy $\gamma \geq 0$, $\xi > \gamma\eta \geq 0$ and $\rho(\Theta) < 1$, where

$$\Theta \triangleq \begin{pmatrix} \sigma_{\min}\sigma_{\max} + \gamma\left(\frac{\eta}{\xi} + \frac{\eta\sigma_{\min}\sigma_{\max}}{\xi - \gamma\eta}\right) & \gamma e^{(\xi - \gamma\eta)(h^* - \delta_0)} \\ \frac{\eta}{\xi} + \frac{\eta\sigma_{\min}\sigma_{\max}}{\xi - \gamma\eta} & e^{-(\xi - \gamma\eta)\delta_0} \end{pmatrix}.$$

Then, D-ETIC (10)-(14) is NZ-NT and DDS (1) is exponentially stabilized by D-ETIC (10)-(14) with

$$\mathcal{R}_{\min}(h^*) \leq \frac{\bar{\tau}_{\max}}{\delta_0 + \bar{\tau}_{\min}}, \quad \bar{\tau}_{\max} \leq \tau_{\max} \leq \delta_1. \quad (15)$$

Proof. First, we show that

$$\tau_{\max} = \sup\{\tau_i\} \leq \delta_1. \quad (16)$$

For $\forall t \in [t_i, t_i + \tau_i)$, since the D-ETIC is input into (1) on $[t_i, t_i + \tau_i)$, i.e., $u(t) = \psi(x(t))$, it follows from (5) and (14) that $D^+V(x(t)) \leq -aV(x(t)) + b\bar{V}(x(t))$ for $\forall t \in [t_i, t_i + \tau_i)$. By Lemma 2.2, we get that for $\tau_i > 0$,

$$V(x(t)) \leq e^{-g_1(h^*)(t - t_i)} \bar{V}(x(t_i)), \quad t \in [t_i, t_i + \tau_i). \quad (17)$$

From (17), we get that

$$\bar{V}(x(t_i + \tau_i)) \leq e^{g_1(h^*)h^*} e^{-g_1(h^*)\tau_i} \bar{V}(x(t_i)). \quad (18)$$

Moreover, by the continuity of V, we have $\bar{V}(x(t_i + \tau_i)) = \sigma_{\min}\bar{V}(x(t_i))$. It yields that $\sigma_{\min} \leq e^{g_1(h^*)h^*} e^{-g_1(h^*)\tau_i}$. Thus, $\tau_i \leq h^* - \frac{\ln \sigma_{\min}}{g_1(h^*)} = \delta_1$. Hence, (16) holds.

Now, for the non-Zeno (NZ) of D-ETIC (10)-(14), we show that

$$0 < \delta_0 \leq \min\{s_i\} \leq \Delta. \quad (19)$$

By (11), we get $\delta_0 > 0$. And by (16) and (11), we have $\tau_i \leq \tau_{\max} \leq \delta_1 < \Delta$ for all $i \in \mathbb{N}$. Since there is no control input on $[t_i + \tau_i, t_{i+1})$ for (1), i.e., $u(t) = 0$, $\forall t \in [t_i + \tau_i, t_{i+1})$, by (3), we have $D^+V(x(t)) \leq a_0 V(x(t)) + b_0 \bar{V}(x(t))$ on $[t_i + \tau_i, t_{i+1})$. Thus, by Lemma 2.1, we get

$$V(x(t)) \leq \xi_0 e^{g_0(t - t_i - \tau_i)} \bar{V}(x(t_i + \tau_i)), \quad (20)$$

where $\xi_0 = 1 + b_0 h^*$ and $g_0 = a_0 + b_0 > 0$. Thus, by (20) and the continuity of V, we get

$$V(x(t_{i+1})) \leq \xi_0 e^{g_0(t_{i+1} - t_i - \tau_i)} \bar{V}(x(t_i + \tau_i)). \quad (21)$$

If $\mathcal{C}(t_i + \tau_i, t_i + \tau_i + \Delta] \neq \emptyset$, then $t_{i+1} \leq t_i + \tau_i + \Delta$ and by the continuity of $V(x(t))$, we have $V(x(t_{i+1})) = \sigma_{\max}\bar{V}(x(t_i + \tau_i)) + \gamma\nu(t_{i+1})$. It follows from (21) that $\sigma_{\max} \leq \xi_0 e^{g_0(t_{i+1} - t_i - \tau_i)}$. Thus, we get

$$\delta_0 \leq s_i = t_{i+1} - t_i - \tau_i \leq \Delta. \quad (22)$$

If $\mathcal{C}(t_i + \tau_i, t_i + \tau_i + \Delta] = \emptyset$, then $t_{i+1} = t_i + \tau_i + \Delta$. By (11), we get that

$$s_i = t_{i+1} - t_i - \tau_i = \Delta > \delta_0 \quad (23)$$

Hence, by (22)-(23), we get that (19) holds.

It follows from (19) that $t_{i+1} - t_i = s_i + \tau_i \geq \delta_0 > 0$. Thus, D-ETIC (10)-(14) is non-Zeno satisfying (19).

By using the estimates of (16) and (19) and the definition of $\bar{\tau}_{\max}$, we get that $\mathcal{R}_{\min}(h^*) \leq \frac{\bar{\tau}_{\max}}{\delta_0 + \bar{\tau}_{\min}}$. Clearly, $\bar{\tau}_{\max} \leq \tau_{\max} \leq \delta_1$. Hence, we get that (15) is satisfied.

Now, we show DDS (1) is stabilized by the D-ETIC. For any $t \in [t_i, t_i + \tau_i]$, under the control input $u(t) = \psi(x(t))$ and by (13) and the continuity of V, we get that

$$\bar{V}(x(t_i + \tau_i)) = \sigma_{\min} \bar{V}(x(t_i)). \tag{24}$$

Moreover, from (12), we have, $\forall t \in [t_i + \tau_i, t_{i+1}]$,

$$V(x(t)) \leq \sigma_{\max} \bar{V}(x(t_i + \tau_i)) + \gamma \nu(t), \tag{25}$$

which implies that

$$\bar{V}(x(t_{i+1})) \leq \sigma_{\max} \bar{V}(x(t_i + \tau_i)) + \gamma \bar{\nu}(t_{i+1}). \tag{26}$$

For the function $\nu(t)$, $\forall t \in [t_i + \tau_i, t_{i+1}]$, by (10), (12), and (24)-(25), we get that $\dot{\nu}(t) \leq -(\xi - \gamma\eta)\nu(t) + \eta\sigma_{\min}\sigma_{\max}\bar{V}(x(t_i))$. Thus, $\forall t \in [t_i + \tau_i, t_{i+1}]$, we get that

$$\nu(t) \leq e^{-(\xi-\gamma\eta)(t-t_i-\tau_i)}\nu(t_i + \tau_i) + \frac{\eta\sigma_{\min}\sigma_{\max}}{\xi - \gamma\eta}\bar{V}(x(t_i)). \tag{27}$$

And for $t \in [t_i, t_i + \tau_i]$, from (25) and (17), we get that

$$\nu(t) \leq e^{-\xi(t-t_i)}\nu(t_i) + \frac{\eta}{\xi}\bar{V}(x(t_i)). \tag{28}$$

It follows from (27)-(28) that, $\forall t \in [t_i + \tau_i, t_{i+1}]$,

$$\nu(t) \leq e^{-(\xi-\gamma\eta)(t-t_i)}\nu(t_i) + \theta_{21}\bar{V}(x(t_i)), \tag{29}$$

where $\theta_{21} = \frac{\eta}{\xi} + \frac{\eta\sigma_{\min}\sigma_{\max}}{\xi-\gamma\eta}$. From (29), (19), we get that

$$\nu(t_{i+1}) \leq \theta_{22}\nu(t_i) + \theta_{21}\bar{V}(x(t_i)), \tag{30}$$
$$\bar{\nu}(t_{i+1}) \leq e^{(\xi-\gamma\eta)(h^*-\delta_0)}\nu(t_i) + \theta_{21}\bar{V}(x(t_i)), \tag{31}$$

where $\theta_{22} = e^{-(\xi-\gamma\eta)\delta_0}$. By (26), (24), and (31), we get

$$\bar{V}(x(t_{i+1})) \leq \theta_{11}\bar{V}(x(t_i)) + \theta_{12}\nu(t_i), \tag{32}$$

where $\theta_{11} = \sigma_{\min}\sigma_{\max} + \gamma\theta_{21}$, $\theta_{12} = \gamma e^{(\xi-\gamma\eta)(h^*-\delta_0)}$.

Denote $y(i) = (\bar{V}(x(t_i)), \nu(t_i))^T$ for all $i \in \mathbb{N}$. Noting $\Theta = (\theta_{ij})_{2\times 2}$, and by (30) and (32), we get that $\forall i \in \mathbb{N}$,

$$y(i+1) \leq \Theta y(i), \ \forall i \in \mathbb{N}. \tag{33}$$

By $\rho(\Theta) < 1$, we get that the discrete-time system (33) is exponentially stable. Thus, for some $\alpha > 0$ and $K > 0$,

$$\|y(i)\| \leq K e^{-\alpha i}\|y(0)\|, \ i \in \mathbb{N}. \tag{34}$$

Thus, $\forall t \in [t_i, t_i + \tau_i]$, by (34) and (17), we get that

$$V(x(t)) \leq \bar{V}(x(t_i)) \leq K e^{-\alpha i}\|y(0)\|. \tag{35}$$

By (34), (24)-(25), and (29), $\forall t \in [t_i + \tau_i, t_{i+1}]$, we get

$$V(x(t)) \leq \gamma\nu(t_i) + \theta_{11}\bar{V}(x(t_i)) \leq K e^{-\alpha i}(\gamma + \theta_{11})\|y(0)\|. \tag{36}$$

Thus, for all $t \in [t_i, t_{i+1}]$, from (35)-(36) and noting $\nu(t_0) = \bar{V}(x_0)$, we have

$$V(x(t)) \leq \sqrt{2}K \max\{1, \gamma + \theta_{11}\}e^{-\alpha i}\bar{V}(x_0). \tag{37}$$

By (2), (37), and (19), we get

$$\|x(t)\| \leq \tilde{K}e^{-\tilde{\alpha}(t-t_0)}\|\phi\|_{h^*}, \tag{38}$$

where $\tilde{K} = \left(\frac{\mu_{\max}\sqrt{2}K\max\{1,\gamma+\theta_{11}\}}{\mu_{\min}}\right)^{1/r}$, $\tilde{\alpha} = \frac{\alpha}{r(\Delta+\tau_{\max})} > 0$. Hence, DDS (1) is exponentially stabilized. □

Remark 3.2. (i) By (15) of Theorem 3.1, the MATR $\mathcal{R}_{\min}(h^*)$ is dependent on h^* and $g_1(h^*)$. Note that $g_1(h^*)$ is strictly decreasing w.r.t. h^*. By (15), a larger (smaller) maximal time delay h^* in DDS (1) makes D-ETIC (10)-(14) have more (less) control activation time.

On the other hand, for a fixed h^*, if we use stronger (weaker) control function $\psi(x)$ such that a in (5) is bigger (smaller), then $g_1(h^*)$ in (9) is bigger (smaller). By (15), a stronger (weaker) control leads to less (more) control activation time in D-ETIC (10)-(14).

(ii) Noting for any $\sigma_{\min}$ and $\sigma_{\max}$ satisfying $\sigma_{\min}\sigma_{\max} < 1$, there always exist parameters $\gamma \geq 0$, $\xi > \gamma\eta \geq 0$ satisfying $\rho(\Theta) < 1$. Hence, there always exists a dynamics (10) satisfying the condition of Theorem 3.1.

Corollary 3.1. Let Assumption 2.1 be satisfied and let the parameters γ, ξ, and η in (10)-(14) satisfy $\gamma\eta = 0$ and $\xi > 0$. Then D-ETIC (10)-(14) is NZ-NT, and DDS (1) is exponentially stabilized by D-ETIC (10)-(14), satisfying the estimates in (15).

Proof. It is derived by Theorem 3.1 with $\gamma\eta = 0$. □

Now, we give comparisons on MATR of D-ETIC, TTIC, E-AIC, and S-ETIC, and analyze the effect of the delay. Let $(u(t), \{t_i^{(1)}\}, \{\tau_i^{(1)}\})$ and $(u(t), \{t_i^{(2)}\}, \{\tau_i^{(2)}\})$ denote TTIC and E-AIC, respectively, where $\tau_0^{(1)} = \tau_0^{(2)} = \tau_0 = 0$, and $u(t)$ satisfies Assumption 2.1. For the comparison, we use the same gain function $\psi(x)$. Thus, for $(u(t), \{t_i^{(j)}\}, \{\tau_i^{(j)}\})$, $j = 1, 2$, $i \geq 1$, $u(t)$ is in form of

$$u(t) = \begin{cases} \psi(x(t - q(t))), \ t \in (t_i^{(j)}, t_i^{(j)} + \tau_i^{(j)}], \\ 0, \ t \in (t_i^{(j)} + \tau_i^{(j)}, t_{i+1}^{(j)}]. \end{cases} \tag{39}$$

Here, we also use the same Lyapunov-like function V and the same parameters $\sigma_{\max}$, $\sigma_{\min}$, and Δ satisfying (11) to give TTIC and E-AIC. And the following stabilization conditions are used to design TTIC and E-AIC:

$$\bar{V}(x(t_i^{(j)} + \tau_i^{(j)})) \leq \sigma_{\min}\bar{V}(x(t_i^{(j)})),\ j = 1, 2, \quad (40)$$
$$\bar{V}(x(t_{i+1}^{(j)})) \leq \sigma_{\min}\sigma_{\max}\bar{V}(x(t_i^{(j)})),\ j = 1, 2,\ i \in \mathbb{N}. \quad (41)$$

Remark 3.3. By (10)-(14) and Theorem 3.1, S-ETIC, i.e., D-ETIC with $\gamma = 0$, also satisfies (40)-(41).

Define $c(h^*) = \frac{g_0 h^* g_1(h^*)}{-\ln\sigma_{\min}}$, $\tilde{g}_1(h^*) = \frac{g_1(h^*)\ln\frac{\sigma_{\max}}{1+b_0 h^*}}{-\ln\sigma_{\min}}$, and $\mathcal{R}^*(h^*) = \frac{g_0 + c(h^*)}{g_0 + \tilde{g}_1(h^*) + c(h^*)}$. Let $\mathcal{R}_{\min,\gamma>0}(h^*)$, $\mathcal{R}_{\min,\gamma=0}(h^*)$, $\mathcal{R}^{(1)}_{\min}(h^*)$, and $\mathcal{R}^{(2)}_{\min}(h^*)$ denote MATR of D-ETIC with $\gamma > 0$, D-ETIC with $\gamma = 0$ (S-ETIC), TTIC, E-AIC, respectively.

Theorem 3.2. Let Assumption 2.1 be satisfied. For the stabilization of DDS (1) via the above TTIC, E-AIC, S-ETIC, and D-ETIC, suppose that (40)-(41) hold. Then, (i) D-ETIC (10)-(14) with $\gamma > 0$ achieves the smallest MATR than that of S-ETIC, E-AIC, and TTIC, and

$$\mathcal{R}_{\min,\gamma>0}(h^*) \leq \mathcal{R}_{\min,\gamma=0}(h^*) < \mathcal{R}^{(2)}_{\min}(h^*) \leq \mathcal{R}^{(1)}_{\min}(h^*) = \mathcal{R}^*(h^*). \quad (42)$$

(ii) $\mathcal{R}^*(h^*)$ is strictly increasing w.r.t. h^* and satisfies: for any $0 < h_1^* < h_2^*$,

$$\frac{g_0}{g_0 + g_1(0)} < \mathcal{R}^*(0) < \mathcal{R}^*(h_1^*) < \mathcal{R}^*(h_2^*) < 1, \quad (43)$$
$$\lim_{h^* \to \infty} \mathcal{R}^*(h^*) = 1. \quad (44)$$

Proof. See Appendix.

Remark 3.4. (i) From Theorem 3.2, D-ETIC achieves the lowest MATR than that of TTIC, E-AIC, and S-ETIC, while TTIC has the highest MATR. (ii) By Theorem 3.2 and (43)-(44), a larger time delay h^* leads to more control activation time. Specifically, if $h^* \to \infty$, then by (42)-(43), we get that the control activation time of TTIC will reach the full time. In this case, there is no basic difference between TTIC and the continuous feedback control $u(t) = \psi(x(t))$ for all $t \geq t_0$. Another specific case is $h^* = 0$ or $h^* \to 0$. By (42), we get $\mathcal{R}_{\min,\gamma>0}(0) \leq \mathcal{R}_{\min,\gamma=0}(0) \leq \mathcal{R}^{(2)}_{\min}(0) < \mathcal{R}^{(1)}_{\min}(0) = \mathcal{R}^*(0) = \frac{g_0}{g_0 + g_1(0)\frac{\ln\sigma_{\max}}{-\ln\sigma_{\min}}}$. Note that if let $\lambda = \sigma_{\max}\sigma_{\min} \to 1^-$, then, we get that $\mathcal{R}_{\min,\gamma>0}(0)\big|_{\lambda\to 1^-} \leq \mathcal{R}_{\min,\gamma=0}(0)\big|_{\lambda\to 1^-} \leq \mathcal{R}^{(2)}_{\min}(0)\big|_{\lambda\to 1^-} \leq \frac{g_0}{g_0+g_1(0)} = \mathcal{R}^{(1)}_{\min}(0)\big|_{\lambda\to 1^-}$. Hence, if no time delay exists in (1) and D-ETIC or the time delay h^* is sufficiently small, then we can choose $\sigma_{\max}$ and $\sigma_{\min}$ to satisfy $\sigma_{\max}\sigma_{\min} \to 1^-$, the MATR of both D-ETIC (10)-(14) and E-AIC will be less than $\mathcal{R}(0) = \frac{g_0}{g_0+g_1(0)}$, while the MATR of TTIC equals to $\mathcal{R}(0) = \frac{g_0}{g_0+g_1(0)} = \frac{a_0+b_0}{(a_0+b_0)+(a-b)}$. This result is consistent with the results in [17,18] without delays.

In the following, we extend the D-ETIC to a sampling-based D-ETIC to reduce the computation in the D-ETIC. Here, assume Assumption 2.1 is satisfied.

Sampling instants: Assume that N is the sampling number during a check-period $\Delta > 0$. The sequence of triggering instants $\{t_i\}$ satisfies: $t_{i+1} = t_i + \tau_i + \frac{m_i}{N}\Delta$ for some integer m_i satisfying $1 \leq m_i \leq N$. By (11) and (16), the control width τ_i satisfies $\tau_i < \Delta$. Thus in the case of sampling-based D-ETIC, assume $\tau_i = \frac{n_i}{N}\Delta$ for some integer n_i satisfying $0 \leq n_i < N$. Hence, all t_i and $t_i + \tau_i$ are the sampling instants. In addition, from $h^* < \Delta$, assume that the maximal time delay h^* satisfies $h^* = \frac{n_{h^*}}{N}\Delta$ for some integer n_{h^*} satisfying $0 < n_{h^*} \leq N$, otherwise, we use $\max\{\frac{n_{h^*}}{N}\Delta \leq h^*\}$ to replace h^* without loss of the generality.

Calculations: We may only calculate the values of $V(x(t))$ and $\bar{V}(x(t))$ at the sampling instants as: $t_i - \frac{n_{h^*}}{N}\Delta = t_i - h^*, \cdots, t_i - \frac{1}{N}\Delta,\ t_i,\ t_i + \frac{1}{N}\Delta, \cdots,\ t_i + \frac{n_i}{N}\Delta = t_i + \tau_i,\ t_i + \tau_i + \frac{1}{N}\Delta,\ \cdots,\ t_i + \tau_i + \frac{m_i}{N}\Delta = t_{i+1}$.

Sampling-based D-ETIC: Choose three indices $\sigma_{\max}$, $\sigma_{\min}$, and Δ, which satisfy (11). Let N be the sampling number during a check-period Δ, and $t_0 = \tau_0 = 0$. The $(i+1)$-th D-ETIC $(u(t), t_{i+1}, \tau_{i+1})$ is set by

$$t_{i+1} = \begin{cases} \min\{t_i + \tau_i + \frac{m_i}{N}\Delta \in \mathcal{C}(t_i + \tau_i, t_i + \tau_i + \Delta]\}, \\ \text{if } m_i = \min\big\{m \in \mathbb{N} : 1 \leq m \leq N, \\ \quad V(x(t_i + \tau_i + \frac{m}{N}\Delta)) \geq \gamma\nu(t_i + \tau_i + \frac{m}{N}\Delta) \\ \quad +\sigma_{\max}\bar{V}(x(t_i + \tau_i))\big\} \neq \emptyset; \\ t_i + \tau_i + \Delta, \text{ otherwise.} \end{cases} \quad (45)$$

$$\tau_{i+1} = \frac{n_i}{N}\Delta, \text{ where } n_i = \min\{l \in \mathbb{N} : 1 \leq l \leq N-1, \ \bar{V}(x(t_{i+1} + \frac{l}{N}\Delta)) \leq \sigma_{\min}\bar{V}(x(t_{i+1}))\}, \quad (46)$$

where $u(t)$ is set by (14) and the nonnegative constants γ, ξ, and η are to be determined.

Theorem 3.3. Let Assumption 2.1 hold. Let the constants γ, ξ, η, and the sampling number N in (51)-(46) satisfy $\gamma \geq 0$, $\xi > \gamma\eta \geq 0$, and $\rho(\Theta_s) < 1$, where $\Theta_s \triangleq \begin{pmatrix} \sigma_{\min}\sigma_{\max}e^{g_0\frac{\Delta}{N}} + \gamma\big(\frac{\eta}{\xi} + \frac{\eta\sigma_{\min}\sigma_{\max}e^{g_0\frac{\Delta}{N}}}{\xi-\gamma\eta}\big) & \gamma e^{(\xi-\gamma\eta)(h^*-\delta_0)} \\ \frac{\eta}{\xi} + \frac{\eta\sigma_{\min}\sigma_{\max}e^{g_0\frac{\Delta}{N}}}{\xi-\gamma\eta} & e^{-(\xi-\gamma\eta)\delta_0} \end{pmatrix}$. Then the sampling-based D-ETIC satisfying (10)-(11), (14), and (45)-(46) is NZ-NT and DDS (1) is exponentially stabilized by such a sampling-based D-ETIC.

Proof. By the same proof of Theorem 3.1, we get that the sampling-based D-ETIC is NZ-NT satisfying (19). Note that in the sampling-based D-ETIC, by the triggering condition (45) for $\{t_i\}$, we get that $V(x(t)) \le \gamma\nu(t) + \sigma_{\max} e^{g_0 \frac{\Delta}{N}} \bar{V}(x(t_i + \tau_i))$ for all $t \in [t_i + \tau_i, t_{i+1}]$. Thus, by using $\sigma_{\max} e^{g_0 \frac{\Delta}{N}}$ to replace the parameter $\sigma_{\max}$ in D-ETIC (10)-(14), the left proof follows the similar process of Theorem 3.1. The details are omitted here. □

For $s \in \mathbb{R}$, let $[s]$ be the minimal integer larger than s.

Remark 3.5. In (10)-(11) and (45)-(46), suppose $\gamma\eta = 0$, and the sampling number N satisfies:

$$N \ge 1 + \Big[\frac{g_0 \Delta}{-\ln(\sigma_{\max}\sigma_{\min})}\Big]. \tag{47}$$

Then, by Theorem 3.3, DDS (1) is exponentially stabilized by such a sampling-based D-ETIC. The condition (47) implies $\sigma_{\max}\sigma_{\min} < e^{-g_0 \frac{\Delta}{N}} < 1$. Compared to $\sigma_{\max}\sigma_{\min} < 1$ in D-ETIC (10)-(14), we can see the sampling brings the constraint in the design of D-ETIC.

At the end of the section, we consider a quasi-linear DDS:

$$\dot{x}(t) = Ax(t) + Bx(t - h(t)) + \varphi(x(t)) + u(t), \tag{48}$$

where $x \in \mathbb{R}^n$; $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n\times n}$; φ satisfies $\varphi(0) = 0$ and for some $L = (l_{ij})$ with $l_{ij} \ge 0$, $|\varphi(x)| \le L|x|$, $\forall x \in \mathbb{R}^n$, with $|x| = (|x_1|, \cdots, |x_n|)^T$.

Here, for the stabilization of (48) under D-ETIC, let $\psi(x) = Kx$ and $V(x) = \max_{1\le i\le n}\{|x_i|\}$.
Let $K = (k_{ij}), A_K = A + K$, $D = KA = (d_{ij}), F = KB = (f_{ij})$, $G = K^2 = (g_{ij})$, $C = KL = (c_{ij})$. And let $a_0 = \max_{1\le i\le n}\{a_{ii} + l_{ii}\}$, $b_0 = \max_{1\le i\le n}\big\{\sum_{j\ne i}^n |a_{ij}| + \sum_{j=1}^n |b_{ij}|\big\}$, $a_1 = -\max_{1\le i\le n}\{a_{ii} + k_{ii} + l_{ii}\}$, $b_1 = \max_{1\le i\le n}\big\{|b_{ii}| + \sum_{j\ne i}^n (|a_{ij} + k_{ij}| + |b_{ij}|)\big\}$, and $d_1 = \max_{1\le i\le n}\big\{\sum_{j=1}^n (|d_{ij}| + |g_{ij}| + |f_{ij}| + |c_{ij}|)\big\}$.

Theorem 3.4. For DDS (48), suppose $a_0 \ge 0$ and $b_0 \ge 0$ and the matrix K is chosen to satisfy:

$$a_1 - b_1 > 0. \tag{49}$$

Then, (i) DDS (48) is exponentially stabilized by D-ETIC (10)-(14), where the parameters γ, ξ, η satisfy $\gamma \ge 0, \xi > \gamma\eta \ge 0$ and $\rho(\Theta) < 1$, and the maximal input delay q^* in (14) satisfies

$$q^* < \frac{a_1 - b_1}{d_1}. \tag{50}$$

(ii) DDS (48) is exponentially stabilized by the sampling-based D-ETIC satisfying (10)-(11), (14), and (51)-(46), where γ, ξ, η, and N satisfy $\gamma \ge 0$, $\xi > \gamma\eta \ge 0$ and $\rho(\Theta_s) < 1$, and the maximal input delay q^* satisfies (50).

Proof. We first prove that Assumption 2.1 holds. For $V(x) = \max_{1\le i\le n}\{|x_i|\}$, Assumption 2.1 (i) holds for $r = 1/2$, $c_2 = 1$, and $c_1 = 1/\sqrt{n}$.

Letting $u = 0$, by (48), we get $\dot{x}(t) = Ax(t) + Bx(t - h(t)) + \varphi(x(t))$. It is easy to get that $D^+V(x(t))|_{u=0} \le a_0 V(x(t)) + b_0 \bar{V}(x(t))$. Thus, Assumption 2.1 (ii) holds.

Note that $d_1 > 0$ and there exists a positive constant q^* satisfying (50). Thus, for the control u with input delay $q(t)$ satisfying $0 \le q(t) \le q^*$, by (48), we get $\dot{x}(t) = A_K x(t) + Bx(t-h(t)) + \varphi(x(t)) - \int_{-q(t)}^0 \big(Dx(t+\theta) + Gx(t+\theta-q(t+\theta)) + Fx(t+\theta-h(t+\theta)) + K\varphi(x(t+\theta))\big)d\theta$. It follows that $D^+V(x(t))|_{u(t)=Kx(t-q(t))} \le -a_1 V(x(t)) + (b_1 + d_1 \cdot q(t))\bar{V}(x(t))$, where $\bar{V}(x(t)) = \max_{t-h^*\le s\le t}\{V(x(s))\}$ with $h^* = \max\{2q^*, h_0^* + q^*\}$. Thus, Assumption 2.1 (iii) is satisfied with $a = a_1$ and $b = b_1 + d_1 q^*$. Therefore, Assumption 2.1 is satisfied.

By Assumption 2.1, the results (i)-(ii) are derived directly from Theorems 3.1 and 3.3, respectively. □

Remark 3.6. Theorem 3.4 is still true if $V(x) = x^T Px$ and $\psi(x) = Kx$ for some matrices $P > 0$ and K. In this case, the parameters a_0, b_0, a_1, b_1, d_1 in Theorem 3.4 are obtained by the LMI technique.

4 Examples

In this section, we give one example for illustrations. Consider a delayed Chua's system as:

$$\dot{x}(t) = Ax(t)+Bx(t-h(t))+\varphi(x(t))+u(t),\ t \ge 0, \tag{51}$$

where $x \in \mathbb{R}^3$, $A = \begin{pmatrix} -\alpha(1+r_2) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}$, $B = 0.015 I_3$, $\varphi(x) = (\varphi_1(x), 0, 0)^T$, $\varphi_1(x) = \frac{-\alpha(r_1-r_2)(|x_1+1|-|x_1-1|)}{2}$, $\alpha = 0.9216, \beta = 0.15995, r_1 = -1.2495, r_2 = -0.75735$.
Let $V(x) = \max_{1\le i\le n}\{|x_i|\}$, $\psi(x) = Kx, K = -3.25 I_3$. Solving Assumption 2.1, we get that $a_0 = 0.2299, b_0 = 2.150, a = a_1 = 4.250, b = b_1 + d_1 q^*, b_1 = 2.150, d_1 = 20.800$. Hence, by Theorem 3.4 and Remark 3.2(ii), there exists a dynamics (10) such that the unstable system (51) is stabilized by the D-ETIC (10)-(14) with the maximal input delay q^* satisfying $q^* \le \frac{a_1 - b_1}{d_1} = 0.1010$.

Let $h(t) = h_0^* = 0.4$, $q^* = 0.1$. Then $h^* = \max\{2q^*, q^* + h_0^*\} = 0.5$. And $\delta_0(h^*) = 0.9520$, $g_0 = a_0 + b_0 = 2.3799$, and $g_1(h^*) = 0.8918$. In the simulation, let $x_0(s) = (\sin(-h_0^*), -e^{-h_0^*}, \cos(-h_0^*))^T$ for all $s \in [-h^*, 0]$.

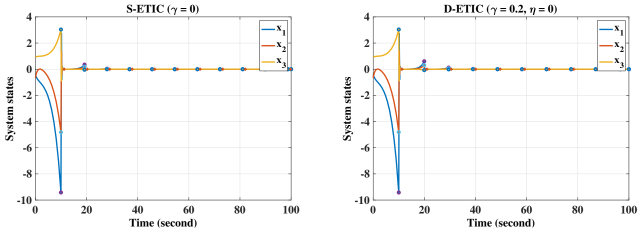

Fig. 1. (left) Stabilization via S-ETIC with input delay $q(t) = q^* = 0.1$. (right) Stabilization via D-ETIC with $\gamma = 0.2$, $\eta = 0$, and $q(t) = q^* = 0.1$. Here, "o" is the starting instant and "+" is the ending instant of each control.

For D-ETIC (10)-(14), choose indices $\sigma_{\max}, \sigma_{\min}, \Delta$ as: $\sigma_{\max} = 20, \sigma_{\min} = 0.045, \Delta = 10$. Then, $\sigma_{\max}\sigma_{\min} = 0.9 < 1$ and (11) is satisfies. And $c(h^*) = \frac{g_0 g_1 h^*}{-\ln \sigma_{\min}} = 0.3422$, $\tilde{g}_1(h^*) = (g_1(h^*) \ln \frac{\sigma_{\max}}{1+b_0 h^*})/(-\ln \sigma_{\min}) = 0.6516$. Thus, by Theorem 3.2, the maximal activation time rate: $\mathcal{R}^*(h^* = 0.5) = \frac{g_0 + c(h^*)}{g_0 + \tilde{g}_1(h^*) + c(h^*)} = 80.69\%$.

Case-I. Stabilization via S-ETIC and D-ETIC.

(i) S-ETIC, i.e., $\gamma = 0$. The condition of Theorem 3.1 is satisfied since $\sigma_{\max}\sigma_{\min} < 1$. By Theorem 3.1, DDS (51) is exponentially stabilized by such a S-ETIC. The simulation is given in Fig. 1 (left). By the simulation, the estimate of the activation time rate of S-DETIC is

$$\mathcal{R}_{\gamma=0}(h^* = 0.5) = \frac{\sum \tau_i}{\sum \tau_i + \sum s_i} \approx 13.06\%. \quad (52)$$

(ii) D-ETIC with $\gamma > 0$ and $\eta = 0$. Here, set $\gamma = 0.2$ and $\xi = 0.1$, and use the same $h^* = 0.5$. Then, $\Theta = \begin{pmatrix} 0.9 & 0.1912 \\ 0 & 0.9092 \end{pmatrix}$. And $\rho(\Theta) = 0.9092 < 1$. Then, by Theorem 3.4, the system (51) is stabilized by the D-ETIC. The simulation is given in Fig. 1 (right). Here, the activation time rate of D-ETIC is:

$$\mathcal{R}_{\gamma=0.2,\eta=0}(h^* = 0.5) \approx 12.18\%. \quad (53)$$

(iii) D-ETIC with $\gamma > 0$ and $\eta > 0$. Here, set $\eta = 0.002$, and use the same γ and ξ, and h^* as in the above (ii), i.e., $\gamma = 0.2$, $\xi = 0.1$, $h^* = 0.5$. Then, the matrix $\Theta = \begin{pmatrix} 0.9076 & 0.1912 \\ 0.0381 & 0.9095 \end{pmatrix}$. And $\rho(\Theta) = 0.9939 < 1$. Then, by Theorem 3.4, DDS (51) is stabilized by such a D-ETIC. The simulation is given in Fig. 2 (left). By the simulation, the activation time rate of D-ETIC is:

$$\mathcal{R}_{\gamma=0.2,\eta=0.002}(h^* = 0.5) \approx 11.61\%. \quad (54)$$

(iv) D-ETIC with $\gamma > 0$, $\eta > 0$, and bigger h^*. Here, set $h^* = 1.0$, i.e., $h_0^* = 0.9$. The parameters γ, ξ, and η are the same as in (iii). i.e., $\gamma = 0.2, \xi = 0.1, \eta = 0.002$. Then, by Theorem 3.4, DDS (51) is stabilized by such a D-ETIC. The simulation is given in Fig. 2 (right). By the simulation, the activation time rate of D-ETIC is:

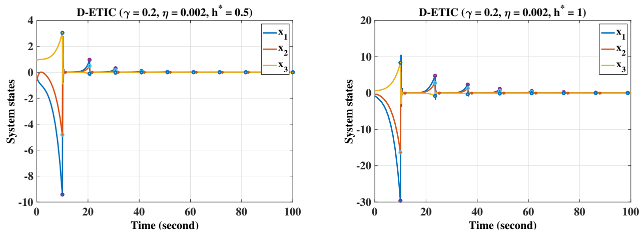

Fig. 2. (left) Stabilization via D-ETIC with $\gamma = 0.2$, $\eta = 0.002$, $q(t) = q^* = 0.1$, $h^* = 0.5$. (right) Stabilization via D-ETIC with $\gamma = 0.2$, $\eta = 0.002$, $q(t) = 0.1$, $h^* = 1$. Here, "o" and "+" are the same meanings as in Fig. 1.

$$\mathcal{R}_{\gamma=0.2,\eta=0.002}(h^* = 1) \approx 14.63\%. \quad (55)$$

(v) No input delay: $q(t) \equiv 0$. Consider S-ETIC ($\gamma = 0$) and D-ETIC and use the same $h^* = 0.5$ but $q(t) \equiv 0$. In D-ETIC, the parameters γ, ξ, and η are the same as in (iii). i.e., $\gamma = 0.2, \xi = 0.1, \eta = 0.002$. By the simulation, the activation time rates of S-ETIC and D-ETIC are:

$$\mathcal{R}_{\gamma=0,q(t)\equiv 0} \approx 10.31\%, \ \ \mathcal{R}_{\gamma=0.2,q(t)\equiv 0} \approx 9.89\%. \quad (56)$$

Summary-I: From **Case-I**, we conclude: by (52)-(54), under the same time delay h^*, the activation time rate of D-ETIC is less than that of S-ETIC; by (53)-(54), under the same h^*, the activation time rate of D-ETIC with bigger dynamics ($\eta > 0$) is less than that of D-ETIC with smaller dynamics ($\eta = 0$); by (54)-(55), the activation time rate of D-ETIC under bigger time delay is larger than that of D-ETIC under smaller time delay; by (56) and (52)-(54), the input delay lets the activation time rate of both S-ETIC and D-ETIC be bigger.

Case-II. Comparisons on MATR of TTIC, E-AIC, S-ETIC, and D-ETIC.

Here, for the comparison, we use the same control matrix $K = -3.25 I_3$, the same time delay $h^* = 0.5$, and the same stabilization conditions (40)-(41) to design TTIC and E-AIC. Note: S-ETIC also satisfies (40)-(41).

Stabilization via TTIC $(u(t), \{t_i^{(1)}\}, \{\tau_i^{(1)}\})$: Let TTIC $(u(t), \{t_i^{(1)}\}, \{\tau_i^{(1)}\})$ with $t_0^{(1)} = \tau_0^{(1)} = 0$ satisfy: $t_{i+1}^{(1)} = t_i^{(1)} + \tau_i^{(1)} + s_i^{(1)}$, where by the proof of Theorem 3.2 (see (60)-(61) in Appendix), $\tau_i^{(1)} = h^* - \frac{\ln \sigma_{\min}}{g_1} = 3.9772$, $s_0^{(1)} = s_i^{(1)} = \frac{\ln \sigma_{\max} - \ln(1 + b_0 h^*)}{g_0} = 0.9520$. And $u(t)$ is in the form of (39) with $\psi(x) = Kx$. Then, by

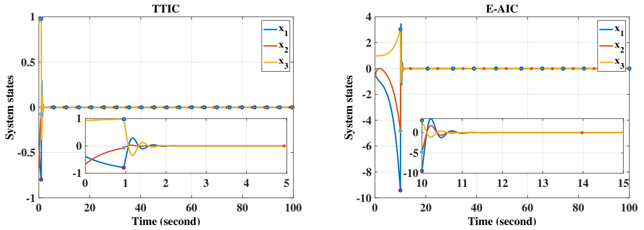

Fig. 3. (left) Stabilization via TTIC, (right) stabilization via E-AIC, where $q(t) = 0.1$, $h^* = 0.5$. Here, "o" and "+" are the same meanings as in Fig. 1.

the stabilization conditions (40)-(41), DDS (51) is stabilized by such a TTIC. The simulation is given in Fig. 3 (left). For the activation time rate of this TTIC, it is

$$\mathcal{R}^{(1)}(h^* = 0.5) = \mathcal{R}^*(h^* = 0.5) = 80.69\%. \tag{57}$$

Stabilization via E-AIC $(u(t), \{t_i^{(2)}\}, \{\tau_i^{(2)}\})$: By [17,18], let E-AIC $(u(t), \{t_i^{(2)}\}, \{\tau_i^{(2)}\})$ with $t_0^{(2)} = \tau_0^{(2)} = 0$ set as: $\forall i \in \mathbb{N}$, $\tau_i^{(2)} = \tau_i^{(1)} = 3.9772, s_i^{(2)} \geq s_i^{(1)} = 0.9520$; $\{t_i^{(2)}\}$ satisfies (12) with $\gamma = 0$; and $u(t)$ is in form of (39) with $\psi(x) = Kx$. Then, by (40)-(41), DDS (51) is stabilized by such an E-AIC. The simulation is given in Fig. 3 (right) with the activation time rate:

$$\mathcal{R}^{(2)}(h^* = 0.5) = \frac{\sum \tau_i^{(2)}}{\sum \tau_i^{(2)} + \sum s_i^{(2)}} \approx 44.19\%. \tag{58}$$

Summary-II: Under the same time delay h^*, from (52)-(54) and (57)-(58), the activation time rate of D-ETIC is the lowest than that of TTIC, E-AIC, and S-ETIC.

5 Conclusions

In this paper, the dynamic event-triggered intermittent control (D-ETIC) with input delay has been proposed for the stabilization of delayed dynamical systems (DDS). The stabilization of DDS via general intermittent control was formulated as a problem of delay-dependent minimal activation time rate (MATR). Then, a D-ETIC scheme was designed. It was shown that the stabilization of DDS was achieved by D-ETIC, and the MATR of D-ETIC was estimated. And the maximal input delay was derived for quasi-linear DDS. Moreover, the comparisons have been given among D-ETIC, static ETIC (S-ETIC), the time-triggered intermittent control (TTIC), and event-triggered aperiodic intermittent control (E-AIC). From the theoretical analysis and numerical simulations, we conclude that: the dynamics let the activation time rate of D-ETIC be lower than that of S-ETIC; D-ETIC has the lowest MATR than TTIC, E-AIC, and S-ETIC; a larger time delay, including input delay, leads to more control activation time for the stabilization.

Appendix. Proof of Theorem 3.2. Note that TTIC, E-AIC, and S-ETIC (10)-(14) with $\gamma = 0$, all have the common stabilization conditions (40)-(41). Letting $y^{(j)}(i) = \bar{V}(x(t_i^{(j)}))$ for $j = 1, 2$ and $i \in \mathbb{N}$, by similar proof of (33), we get that

$$y^{(j)}(i+1) \leq \sigma_{\max}\sigma_{\min} y^{(j)}(i), \ i \in \mathbb{N}. \tag{59}$$

From (41) with $\sigma_{\max}\sigma_{\min} < 1$ and by the similar proof of (35)-(38) of Theorem 3.1, DDS (1) is exponentially stabilized by the designed TTIC and E-AIC, respectively. (i) Now, we show the inequality (42) holds.

Firstly, consider the case of TTIC $(u(t), \{t_i^{(1)}\}, \{\tau_i^{(1)}\})$. For $t \in (t_i^{(1)}, t_i^{(1)} + \tau_i^{(1)}]$, by Lemma 2.2, we get that $V(x(t)) \leq e^{-g_1(h^*)(t-t_i^{(1)})}\bar{V}(x(t_i^{(1)}))$. Thus, by (41), the control width $\tau_i^{(1)}$ in TTIC satisfies:

$$\tau_i^{(1)} \geq h^* - \frac{\ln \sigma_{\min}}{g_1(h^*)}. \tag{60}$$

For $t \in (t_i^{(1)} + \tau_i^{(1)}, t_{i+1}^{(1)}]$, by Lemma 2.1, we get that $V(x(t)) \leq \xi_0 e^{g_0(t-t_i^{(1)}-\tau_i^{(1)})}\bar{V}(x(t_i^{(1)}+\tau_i^{(1)}))$. By (42), the non-control width $s_i^{(1)}$ in TTIC satisfies:

$$s_i^{(1)} \leq \frac{\ln \sigma_{\max} - \ln \xi_0}{g_0} = \frac{\ln \sigma_{\max} - \ln(1 + b_0 h^*)}{g_0}. \tag{61}$$

Therefore, from (60)-(61), we get that

$$\mathcal{R}_{\min}^{(1)}(h^*) = \frac{g_0 + c(h^*)}{g_0 + \tilde{g}_1(h^*) + c(h^*)} = \mathcal{R}^*(h^*). \tag{62}$$

Secondly, consider the case of E-AIC $(u(t), \{t_i^{(2)}\}, \{\tau_i^{(2)}\})$. By [17,18], in E-AIC, the triggering time sequence $\{t_i^{(2)}\}$ is determined by the event condition (12) while the control width sequence $\{\tau_i^{(2)}\}$ is determined by the stabilization condition (41). Thus, by (12) and Lemma 2.1, we get that $\sigma_{\max}\bar{V}(x(t_i^{(2)} + \tau_i^{(2)})) = \bar{V}(x(t_{i+1}^{(2)})) \leq (1 + b_0 h^*)e^{g_0 s_i^{(2)}}\bar{V}(x(t_i^{(2)} + \tau_i^{(2)}))$, which implies that

$$s_i^{(2)} \geq \frac{\ln \sigma_{\max} - \ln(1 + b_0 h^*)}{g_0}, \ i \geq 1. \tag{63}$$

The control width $\tau_i^{(2)}$ satisfies $e^{h^* g_1(h^*) - g_1(h^*)\tau_i^{(2)}} \leq \sigma_{\min}$. Thus, from (12) and (42), we get that

$$\tau_i^{(2)} \geq h^* - \frac{\ln \sigma_{\min}}{g_1(h^*)}. \tag{64}$$

Therefore, for the MATR problem, by (63)-(64), we get

$$\mathcal{R}_{\min}^{(2)}(h^*) < \frac{g_0 + c(h^*)}{g_0 + \tilde{g}_1(h^*) + c(h^*)} = \mathcal{R}^*(h^*). \tag{65}$$

Thirdly, consider S-ETIC $(u, \{t_i\}, \{\tau_i\})$: $\gamma = 0$.

Compared S-ETIC and E-AIC ([17,18]), the difference is that every control width τ_i in S-ETIC is state-dependent, satisfying the event-trigger condition (13), while $\tau_i^{(2)}$ in E-AIC by [17] satisfies the stabilization condition (41). By comparing (13) and (41), we have $\tau_i \leq \tau_i^{(2)}$. It follows that the average control widths satisfy:

$$\bar{\tau}_i = \sum_{j=1}^{i} \tau_j / i \leq \bar{\tau}_i^{(2)} = \sum_{j=1}^{i} \tau_j^{(2)} / i, \quad \forall i \geq 1. \tag{66}$$

For the non-control width, both S-ETIC and E-AIC use the same event condition (12) with the same parameters. So we have $\sum_{j=1}^{\infty} s_j = \sum_{j=1}^{\infty} s_j^{(2)}$. Thus, for a sufficiently large $i \geq 1$, we get $\bar{s}_i = \sum_{j=1}^{i} s_j / i \approx \bar{s}_i^{(2)} = \sum_{j=1}^{i} s_j^{(2)} / i$. Note that $R(r) = \frac{r}{r+\bar{s}_i}$ is strictly increasing. Hence, by (66), for the sufficiently large i, $R(\bar{\tau}_i) = \frac{\bar{\tau}_i}{\bar{\tau}_i + \bar{s}_i} \leq R(\bar{\tau}_i^{(2)}) = \frac{\bar{\tau}_i^{(2)}}{\bar{\tau}_i^{(2)} + \bar{s}_i} \approx \frac{\bar{\tau}_i^{(2)}}{\bar{\tau}_i^{(2)} + \bar{s}_i^{(2)}}$,, which implies that

$$\mathcal{R}_{\min,\gamma=0}(h^*) \leq \mathcal{R}_{\min}^{(2)}(h^*). \tag{67}$$

Hence, by (67), (65), and (62), S-ETIC achieves the lowest MATR than that of TTIC and E-AIC, i.e.,

$$\mathcal{R}_{\min,\gamma=0}(h^*) \leq \mathcal{R}_{\min}^{(2)}(h^*) \leq \mathcal{R}_{\min}^{(1)}(h^*). \tag{68}$$

Finally, compare S-ETIC ($\gamma = 0$) and D-ETIC ($\gamma > 0$). Clearly, it should wait for longer to trigger the event condition with $\gamma > 0$ than the case of $\gamma = 0$. Thus, the total non-control width for D-ETIC (10)-(14) with $\gamma > 0$ will be bigger than S-ETIC. For the control width, both cases have the same triggering condition. Thus, the total control width for both cases is the same. Thus, we have

$$\mathcal{R}_{\min,\gamma>0}(h^*) \leq \mathcal{R}_{\min,\gamma=0}(h^*). \tag{69}$$

Hence, the inequality (42) is derived from (68)-(69).

(ii) The inequalities (43)-(44) is derived by the monotonicities of the functions $g_1(h^*)$, $c(h^*)$, $\tilde{g}_1(h^*)$, and $\mathcal{R}^*(h^*)$. The details are omitted here. □

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