LMI-based determination of the peak of the response of structured polytopic linear systems

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***Abstract*—This paper addresses the problem of determining the peak of the response to a linear time-invariant (LTI) signal of a linear system whose system matrices are rational functions of an uncertainty vector constrained into a convex bounded polytope. The uncertainty can be time-invariant, bounded-rate time-varying or arbitrarily time-varying. A novel approach based on linear matrix inequalities (LMIs) is proposed for obtaining upper bounds of the sought peak based on the construction of a structured polynomial Lyapunov function in the state and in the uncertainty. A priori and a posteriori conditions for establishing optimality of the obtained upper bounds are also provided. As shown by some numerical examples, which includes the model of an electric circuit, the proposed approach may have significant advantages with respect to the existing methods in terms of conservatism or computational burden.**

***Index Terms*—Output response; Peak; Uncertainty; LMI.**

I. Introduction

Establishing upper bounds of the peak of the response of a dynamical system is a fundamental problem in engineering. Indeed, it is important that the physical quantities (e.g., current, voltage, etc) of a real device remain within their operative ranges in order to avoid malfunctioning or even destruction, which means that upper bounds on the peak of the signals must be known when designing the real device. Also, it is important to be able to establish such upper bounds in the presence of uncertainties since real devices cannot be modeled exactly in general, due to impossibilities in measuring exactly the coefficients of the various components, or due to the fact that these coefficients may change. Moreover, it is important that these upper bounds are as less conservative as possible, since conservative upper bounds make more difficult the realization of real devices, and it is important that these upper bounds can be computed efficiently, in order to save computational time and be able to consider systems with larger dimension.

This problem is challenging due to various reasons. The first reason is that quadratic Lyapunov functions may be conservative in providing upper bounds of the peak of the response even for second-order linear systems without uncertainty, see for instance [3]. The second reason is that dynamical systems are often affected by uncertainties, see for instance [1], [2] for classical references, and [5], [20]–[23] for recent contributions in various areas such as 2D systems, event-triggered systems and networked systems. In such a case, a family of possible responses has to be considered, generally depending on the temporal nature of the uncertainty (time-invariant or time-varying) and on the way that the uncertainty affects the system matrices (e.g., linear, polynomial, etc). The third reason is that this family of possible responses cannot be considered via simulations in practice since the set of admissible uncertainties is continuous (and, hence, not finite).

This paper proposes a novel approach for this problem as follows. Firstly, the paper starts by considering the impulse response of a strictly proper single-input system, whose system matrices are polynomial functions of a time-invariant uncertain vector constrained in the simplex. A novel condition is proposed in terms of feasibility of a system of LMIs for establishing whether a chosen quantity is an upper bound of the sought peak based on the construction of a structured polynomial Lyapunov function in the state and in the uncertainty through the use of polynomials that can be expressed as sums of squares of polynomials and through the use of a projection operator. The proposed condition is sufficient for any chosen degree of this function, and also necessary for some finite degree whenever the system is robustly asymptotically stable. Secondly, it is shown that the proposed condition can be used to calculate upper bounds of the sought peak by solving a semidefinite program (SDP) obtained by augmenting the LMIs. Thirdly, a necessary and sufficient condition is proposed for establishing whether a calculated upper bound is tight through the determination of worst-case values of the uncertainty. Lastly, several generalizations of the proposed approach are presented, which include the extension to non-strictly proper systems, multi-input systems, uncertainty over convex bounded polytopes, response to LTI signals, time-varying uncertainties, and systems with rational dependence on the uncertainty.

It is useful to mention that the proposed approach is novel because for this problem:

- polynomial Lyapunov functions in the state and in the uncertainty have never been exploited;
- an LMI condition that is not only sufficient but also necessary has never been proposed;
- the best upper bound guaranteed for chosen degrees of the Lyapunov function can be obtained via a single LMI optimization rather than a sequence of LMI optimizations in a bisection algorithm as required by existing methods that exploit nonhomogeneous Lyapunov functions;

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- a necessary and sufficient condition for establishing tightness of the found upper bound has never been proposed.

This is also supported by the examples in Section VI, where it is shown that the proposed approach presents significant advantages with respect to the existing methods, in terms of conservatism (by providing the sought peak when the existing methods may return only a conservative upper bound of it or no upper bound at all) or in terms of computational burden (by allowing to find the sought peak with an SDP whose size may be half of that of the existing methods).

It is important to mention that methods for the determination of the peak of the response of dynamical systems have been developed since long time, in particular based on LMIs, since LMI methods can be solved with convex optimization and since they may be extended in some cases to the synthesis of feedback controllers, see [3] about LMIs. The main existing LMI methods and the advantages of the proposed approach with respect to such methods are as follows:

- the pioneering methods [3], [18]. These methods search for a quadratic Lyapunov function in linear systems without uncertainty, and can be used to search for a common quadratic Lyapunov function in linear systems depending linearly on polytopic uncertainty by repeating the LMIs at the vertices of the uncertainty set. Compared with the proposed approach, these methods have the advantages of an easier implementation and a smaller computational burden. On the other hand, the proposed approach may provide less conservative results by using polynomial Lyapunov functions, common or depending on the uncertainty;
- our previous work [8]. The proposed approach contains this previous work as a special case. Moreover, it has the following advantages: 1) it considers systems that depend not only linearly but also polynomially on the uncertainty; 2) it considers uncertainty that is not only arbitrarily time-varying but also time-invariant or bounded-rate time-varying; 3) it shows how an upper bound of the sought peak can be obtained with a single SDP rather than a sequence of LMI feasibility tests in a bisection search even when the Lyapunov function is not restricted to be homogeneous; 4) it proposes a necessary and sufficient condition for establishing tightness of a calculated upper bound. It is useful to mention that [8] includes our previous work [7] where only systems without uncertainty are considered, and where only the sufficiency of the LMI condition is proved;
- the recent method [16] based on occupation measures. Compared with the proposed approach, this method has the advantage of being more general, for instance because the dynamics can be not only linear but also nonlinear, and because the uncertainty set can be not only a polytope but also a semialgebraic set. On the other hand, the proposed approach may have two advantages: 1) it does not require knowledge of the region of interest of the trajectories, which is a region that should contain the response of the system and is typically unknown a priori in the problem considered in this paper; 2) the computational burden may be significantly smaller.

The paper is organized as follows. Section II reports the preliminaries. Section III provides the first part of the proposed approach, which is the condition for establishing upper bounds of the sought peak. Section IV provides the second part of the proposed approach, which investigates the nonconservatism and the computation of the upper bounds. Section V provides the generalizations. Lastly, Sections VI and VII present the numerical examples and conclusions.

A preliminary conference version of this paper was presented as reported in [9]. This preliminary conference version contains only the first part of the proposed approach (i.e., Section III) and some numerical examples.

II. Preliminaries

This section introduces the problem formulation and some preliminaries about polynomials.

A. Problem Formulation

The notation is as follows:

- $\mathbb{N}$, $\mathbb{R}$: sets of nonnegative integers and real numbers;
- 0_n (respectively, 1_n): $n \times 1$ vectors with all entries 0 (respectively, 1);
- I: identity matrix of size specified by the context;
- A': transpose of A;
- $A \otimes B$: Kronecker product of A with B;
- $A > 0$, $A \geq 0$: positive definite and positive semidefinite matrix A;
- $\operatorname{conv}\{\mathrm{A}_1, \ldots, \mathrm{A}_n\}$: convex hull of $A_1, \ldots, A_n$;
- $\nabla_x f(x,y)$: gradient of $f(x,y)$ with respect to x;
- $\deg_x f(x,y)$: degree of $f(x,y)$ in x;
- x^y, with $x, y \in \mathbb{R}^n$, is the quantity $\prod_{i=1}^n x_i^{y_i}$;
- x^y, with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, is the vector $(x_1^y, \ldots, x_n^y)'$;
- s.t.: subject to.

Let us start by considering the uncertain system[1]

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(\sigma)x(t) + B(\sigma)u(t) \\ y(t) & = & C(\sigma)x(t) \\ \sigma & \in & \mathcal{S} \end{array} \right. \tag{1}$$

where $t \in \mathbb{R}$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}^p$ is the output, $\sigma \in \mathbb{R}^r$ is the time-invariant uncertainty, $\mathcal{S}$ is the simplex defined by

$$\mathcal{S} = \{\sigma \in \mathbb{R}^r : \ 1_r'\sigma = 1, \ \sigma_i \geq 0 \ \forall i = 1, \ldots, r\} \tag{2}$$

and $A(\sigma)$, $B(\sigma)$ and $C(\sigma)$ are matrix polynomials.

Definition 1: An admissible impulse response of the system (1), denoted as $Y(t)$, is the solution $y(t)$ for some $\sigma \in \mathcal{S}$ and

$$\left\{ \begin{array}{rcl} x(0^-) & = & 0_n \\ u(t) & = & \delta(t) \end{array} \right. \tag{3}$$

where $\delta(t)$ is the Dirac distribution. □

[1] More general versions of this system will be considered in Section V.

This paper addresses two main problems, namely, establishing a desired upper bound on the largest peak of the admissible impulse responses of the system (1), and determining such a peak, which are formulated as follows.

Problem 1: Given $c \in (0, \infty)$, establish whether c is an upper bound of the largest peak of the admissible impulse responses of the system (1), i.e.,

$$\|Y(t)\|_\infty < c \quad \forall t \geq 0 \; \forall \sigma \in \mathcal{S}. \tag{4}$$

□

Problem 2: Determine the largest peak of the admissible impulse responses of the system (1), i.e.,

$$\rho = \inf_c c \quad \text{s.t. (4) holds.} \tag{5}$$

□

In the sequel of this paper, $C^{(i)}(\sigma)$ will denote the i-th row of $C(\sigma)$. Also, the dependence on the time t of the signals will be omitted for brevity unless specified otherwise.

B. Polynomials

Here we provide some preliminaries about a class of polynomials that will be exploited in the next sections. For $\sigma \in \mathbb{R}^r$ and $x \in \mathbb{R}^n$, let $w(\sigma, x)$ be a polynomial, i.e.

$$w(\sigma, x) = \sum_{a \in \mathbb{N}^r,\, b \in \mathbb{N}^n,\, 1_r' a + 1_n' b \leq 2d} \tau_{a,b}\, \sigma^a x^b \tag{6}$$

where $d \in \mathbb{N}$ defines the upper bound $2d$ on the degree of $w(\sigma, x)$, and $\tau_{a,b} \in \mathbb{R}$ is the coefficient of the monomial $\sigma^a x^b$. Let us gather all the coefficients $\tau_{a,b}$ into a vector τ. Then, $w(\sigma, x)$ can be expressed as

$$w(\sigma, x) = b(\sigma, x)' \left(W(\tau) + L(\alpha) \right) b(\sigma, x) \tag{7}$$

where $b(\sigma, x) \in \mathbb{R}^m$ is a vector whose entries are all the monomials in the variables σ and x of degree less than or equal to d, whose number is given by

$$m = \frac{(r+n+d)!}{(r+n)!d!}, \tag{8}$$

$W(\tau) \in \mathbb{R}^{m \times m}$ is a symmetric linear matrix function, and $L(\alpha) \in \mathbb{R}^{m \times m}$ is a symmetric linear matrix function that parameterizes the linear set

$$\mathcal{L} = \left\{ \tilde{L} = \tilde{L}' : \; b(\sigma, x)' \tilde{L} b(\sigma, x) = 0 \right\} \tag{9}$$

where $\alpha \in \mathbb{R}^q$ is a free vector with length equal to the dimension of $\mathcal{L}$ given by

$$q = \frac{1}{2} m(m+1) - \frac{(r+n+2d)!}{(r+n)!(2d)!}. \tag{10}$$

The representation (7) is known as Gram matrix method of $w(\sigma, x)$ with respect to $b(\sigma, x)$.

Definition 2: For $w(\sigma, x)$ polynomial, the notation

$$w(\sigma, x) \in \mathbb{S}(\sigma, x) \tag{11}$$

means that $w(\sigma, x)$ is a sum of squares of polynomials, i.e., there exist polynomials $w_i(\sigma, x)$, $i = 1, \ldots, k$, such that

$$w(\sigma, x) = \sum_{i=1}^{k} w_i(\sigma, x)^2. \tag{12}$$

□

The representation (7) is useful to establish if $w(\sigma, x) \in \mathbb{S}(\sigma, x)$. Indeed, $w(\sigma, x) \in \mathbb{S}(\sigma, x)$ if and only if the LMI

$$W(\tau) + L(\alpha) \geq 0 \tag{13}$$

is feasible for some α. Moreover, if τ depends affine linearly on some auxiliary variables, the above condition is still an LMI in such auxiliary variables and in α since $W(\tau)$ is linear. The reader is refereed to [10], [17] for more information about sums of squares of polynomials, and to [4], [6] for algorithms for the construction of the Gram matrices and for formulas about the complexity.

III. The Approach: Part I

In this section we address the solution of Problem 1. Let us start by introducing Definitions 3 and 4 which have the following goals:

- for Definition 3, to obtain an equivalent representation of a polynomial over an affine set through a homogeneous polynomial;
- for Definition 4, to impose that a polynomial is nonnegative over the simplex. This is done by imposing that the homogeneous polynomial obtained in Definition 3 for a suitable choice of the affine set is a sum of squares of polynomials after a suitable change of variables.

The first definition that needs to be introduced is as follows.

Definition 3: For a polynomial $w_1 : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$ and a vector $w_2 \in \mathbb{R}^r$, let us define

$$\Phi_\sigma(w_1(\sigma, x), w_2) = \sum_{a \in \mathbb{N}^r,\, 1_r' a \leq d_\sigma} \tau_a(x) \sigma^a (w_2' \sigma)^{d_\sigma - 1_r' a} \tag{14}$$

where $d_\sigma = \deg_\sigma(w_1(\sigma, x))$ and $\tau_a(x)$ is the coefficient of the monomial σ^a in $w_1(\sigma, x)$, i.e.,

$$w_1(\sigma, x) = \sum_{a \in \mathbb{N}^r,\, 1_r' a \leq d_\sigma} \tau_a(x) \sigma^a. \tag{15}$$

□

Definition 3 introduces the function $\Phi_\sigma(w_1(\sigma, x), w_2)$ which returns a polynomial homogeneous in σ that coincides with $w_1(\sigma, x)$ on the affine set $\{\sigma : \; w_2' \sigma = 1\}$. Hence, the function $\Phi_\sigma(w_1(\sigma, x), w_2)$ can be regarded as a projection operator, in particular, projecting a polynomial onto an affine set. This function has been exploited in different areas, see, e.g., [6] and references therein. Similarly, $\Phi_x(w_1(\sigma, x), w_3)$, $w_3 \in \mathbb{R}^n$, returns a polynomial homogeneous in x that coincides with $w_1(\sigma, x)$ on the affine set $\{x : \; w_3' x = 1\}$.

In order to clarify Definition 3, let us introduce the following simple numerical example.

Example 0. Consider $\sigma = (\sigma_1, \sigma_2)'$, $x = (x_1, x_2)'$, $w_1(\sigma, x) = 2x_1^4 + 2x_2^2 - \sigma_1 x_1^4 + \sigma_2 x_2^2$ and $w_2 = (1, 1)'$. Then, from Definition 3,

$$\Phi_\sigma(w_1(\sigma, x), w_2) = (\sigma_1 + 2\sigma_2)x_1^4 + (2\sigma_1 + 3\sigma_2)x_2^2$$

which is obtained by multiplying each monomial times a power of $\sigma_1 + \sigma_2$ in order to achieve a polynomial that is homogeneous in σ. □

The second definition that needs to be introduced is as follows.

Definition 4: For $w : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$ polynomial, the notation

$$w(\sigma, x) \in \mathbb{S}_+(\sigma, x) \tag{16}$$

stands for

$$w^{\#}(\sigma^2, x) \in \mathbb{S}(\sigma, x) \tag{17}$$

where

$$w^{\#}(\sigma, x) = \Phi_\sigma\left(w(\sigma, x), 1_r\right). \tag{18}$$

□

Definition 4 introduces the set $\mathbb{S}_+(\sigma, x)$. A polynomial $w(\sigma, x)$ belongs to this set if the polynomial obtained from $w^{\#}(\sigma, x)$ by squaring all entries of σ is a sum of squares of polynomials, where $w^{\#}(\sigma, x)$ is the homogeneous polynomial in σ that coincides with $w(\sigma, x)$ on the simplex and has the same degree of $w(\sigma, x)$. It should be mentioned that the parametrization of nonnegative polynomials over the simplex through the use of squared variables has been exploited in the context of robust analysis, see [6] and references therein.

Similarly to Definition 3, let us clarify Definition 4 through the following simple numerical example.

Example 0 (continued). For the polynomial $w_1(\sigma, x)$ previously seen, the condition $w_1(\sigma, x) \in \mathbb{S}_+(\sigma, x)$ introduced in Definition 4 stands for

$$(\sigma_1^2 + 2\sigma_2^2)x_1^4 + (2\sigma_1^2 + 3\sigma_2^2)x_2^2 \in \mathbb{S}(\sigma, x),$$

whose left hand side is obtained by replacing σ_1 and σ_2 with σ_1^2 and σ_2^2 in the polynomial $\Phi_\sigma(w_1(\sigma, x), w_2)$. □

Hereafter we formulate the first result of the paper, which provides a sufficient LMI condition for Problem 1. In order to reduce the conservatism of the existing methods [3], [8], [18], or to not require a priori information about the trajectories of the system (1) as done by the existing method [16], this result searches for a structured polynomial Lyapunov function in the state and in the uncertainty, whose sublevel sets are used to embed the admissible impulse responses of the system (1) and to evaluate their peak. To this aim, the projection operator introduced in Definition 3, the set of polynomials introduced in Definition 4, and sums of squares of polynomials are exploited to build the LMI condition.

Theorem 1: Let us define $\gamma = c^{-1}$ and $\xi = 1$. The condition (4) holds if there exist $\varepsilon > 0$ and a polynomial $v : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}$ of the form

$$v(\sigma, x) = \sum_{\substack{a \in \mathbb{N}^r,\ 1_r' a = d_\sigma \\ b \in \mathbb{N}^n,\ 2 \le 1_n' b \le 2d_x}} \zeta_{a,b}\, \sigma^a x^b, \tag{19}$$

with $d_\sigma, d_x \in \mathbb{N}$ and $\zeta_{a,b} \in \mathbb{R}$, such that

$$\begin{array}{c} e_{j,k}(\sigma), f(\sigma), g(\sigma, x), h_{j,k}(\sigma, x) \in \mathbb{S}_+(\sigma, x) \\ \forall j = 0, 1\ \forall k = 1, \ldots, p \end{array} \tag{20}$$

where

$$\left\{ \begin{array}{rcl} e_{j,k}(\sigma) & = & (1_r'\sigma)^{d_\sigma} + (-1)^j \gamma C^{(k)}(\sigma) B(\sigma) - \varepsilon \\ f(\sigma) & = & \xi - v(\sigma, \gamma B(\sigma)) \\ g(\sigma, x) & = & -\left(\nabla_x v(\sigma, x)\right)' A(\sigma) x \\ h_{j,k}(\sigma, x) & = & \Phi_x\left(v(\sigma, x) - \xi, (-1)^j C^{(k)}(\sigma)'\right) \\ & & -\varepsilon \|x\|_2^{2d_x}. \end{array} \right. \tag{21}$$

Before introducing the proof of this theorem, let us observe that the structure of the Lyapunov function sought in Theorem 1 is defined by (19), which imposes that the Lyapunov function is polynomial with monomials of a special form. In particular, these monomials are homogeneous of degree d_σ in the vector variable σ, and are locally quadratic (i.e., without constants or linear terms) of degree up to $2d_x$ in the vector variable x. This structure is novel and generalizes the structures exploited by the existing methods in the literature that consider only common quadratic Lyapunov functions or common polynomial Lyapunov functions by letting these functions depend on the uncertainty through homogeneous polynomials of arbitrary degree.

Also, let us observe that the condition (20) is equivalent to a system of LMIs because the polynomials $e_{j,k}(\sigma)$, $f(\sigma)$, $g(\sigma, x)$ and $h_{j,k}(\sigma, x)$ depend affine linearly on the decision variables ε and $v(\sigma, x)$, and because the condition that any of these polynomials is in the set $\mathbb{S}_+(\sigma, x)$ can be equivalently reformulated as an LMI in these decision variables and auxiliary variables as explained in Section II-B. The nonnegative integers d_σ and d_x define the degree and structure of $v(\sigma, x)$, and have to be chosen a priori in order to build the system of LMIs.

Lastly, as it will become clear in the proof, let us observe that:

- $e_{j,k}(\sigma)$ is introduced to impose the desired bound at time $t = 0$;
- $f(\sigma)$ is introduced to impose that the admissible impulse responses start inside the considered sublevel set;
- $g(\sigma, x)$ is introduced to impose that the considered sublevel set is invariant;
- $h_{j,k}(\sigma, x)$ is introduced to impose that the considered sublevel set does not intersect the states with the desired bound.

Proof. Suppose that (19)–(21) hold. Since $e_{j,k}(\sigma) \in \mathbb{S}_+(\sigma, x)$, it follows from Definition 4 that

$$e_{j,k}^{\#}(\sigma^2) \geq 0$$

where $e^{\#}_{j,k}(\sigma) = \Phi_\sigma(e_{j,k}(\sigma), 1_r)$. Since Definition 3 implies that $e^{\#}_{j,k}(\sigma) = e_{j,k}(\sigma)$ for all $\sigma \in \mathcal{S}$, and since $\{\sigma^2 : \sigma \in \mathbb{R}^r\} \supseteq \mathcal{S}$, it follows that

$$e_{j,k}(\sigma) \geq 0 \quad \forall \sigma \in \mathcal{S}.$$

Similarly, $f(\sigma)$, $g(\sigma, x)$ and $h_{j,k}(\sigma, x)$ are nonnegative for all $\sigma \in \mathcal{S}$ and for all $x \in \mathbb{R}^n$. Taking into account that $e_{j,k}(\sigma) \geq 0$ for all $\sigma \in \mathcal{S}$ and for all $j = 0, 1$ and $k = 1, \ldots, p$, the positivity of ε implies that

$$\|C(\sigma)\gamma B(\sigma)\|_\infty < 1 \quad \forall \sigma \in \mathcal{S}.$$

Let us observe that, for any admissible impulse response, the input u of the system (1) is the Dirac distribution, which has the effect of moving the initial condition at $t = 0$ from the origin to $B(\sigma)$. This implies that

$$\|Y(0)\|_\infty < \frac{1}{\gamma} \quad \forall \sigma \in \mathcal{S}.$$

Since $g(\sigma, x) \geq 0$ for all $\sigma \in \mathcal{S}$ and for all $x \in \mathbb{R}^n$, it follows that the time derivative of $v(\sigma, x)$ is nonpositive for $u = 0$. Hence, any trajectory of the system (1) that starts in the set

$$\mathcal{V}(\sigma) = \{x \in \mathbb{R}^n : \ v(\sigma, x) \leq \xi\}$$

remains in $\mathcal{V}(\sigma)$ for $u = 0$. Moreover, the condition that $f(\sigma) \geq 0$ for all $\sigma \in \mathcal{S}$ implies that

$$\gamma B(\sigma) \in \mathcal{V}(\sigma) \quad \forall \sigma \in \mathcal{S}.$$

Since $h_{j,k}(\sigma, x) \geq 0$ for all $\sigma \in \mathcal{S}$ and for all $x \in \mathbb{R}^n$, from Definition 3 it follows that

$$v(\sigma, x) - \xi - \varepsilon\|x\|_2^{2d_x} \geq 0 \quad \forall x \in \mathcal{T}_{j,k}(\sigma) \ \forall \sigma \in \mathcal{S}$$

where

$$\mathcal{T}_{j,k}(\sigma) = \left\{x \in \mathbb{R}^n : \ C^{(k)}(\sigma)x = (-1)^j\right\}.$$

Due to the positivity of ε, and since $\mathcal{T}_{j,k}(\sigma)$ does not contain the origin, the previous condition implies that

$$v(\sigma, x) > \xi \quad \forall x \in \mathcal{T}_{j,k}(\sigma) \ \forall \sigma \in \mathcal{S}.$$

Since the above condition holds for all $j = 0, 1$ and $k = 1, \ldots, p$, it follows that

$$\mathcal{V}(\sigma) \cap \mathcal{T}(\sigma) = \emptyset$$

where

$$\mathcal{T}(\sigma) = \{x \in \mathbb{R}^n : \ \|C(\sigma)x\|_\infty = 1\}.$$

Therefore, the trajectory of the system (1) starting at $\gamma B(\sigma)$ does not intersect the set of states for which the output has infinity norm equal to 1. Since $\gamma = c^{-1}$, and since the system (1) is linear, it follows that the trajectory starting at $B(\sigma)$ does not intersect the set of states for which the output has infinity norm equal to c. Taking into account that this trajectory is continuous and starts at a point for which the output has infinity norm less than c, it can be concluded that (4) holds. □

IV. The Approach: Part II

This section analyzes the nonconservatism of the proposed approach, shows how upper bounds of the sought peak can be obtained by solving SDPs, and proposes a tightness certificate for these upper bounds.

A. Nonconservatism Analysis

Here we investigate the nonconservatism of the sufficient condition provided by Theorem 1. The main steps for doing this are two: firstly, to show the existence of a suitable structured polynomial Lyapunov function in the state and in the uncertainty, whose level sets embeds the admissible impulse responses; secondly, to show that the condition of Theorem 1 holds with such a Lyapunov function after a suitable degree augmentation.

Theorem 2: Suppose that $A(\sigma)$ is Hurwitz for all $\sigma \in \mathcal{S}$. The condition (4) holds only if (19)–(21) hold for some $\varepsilon > 0$ and $v(\sigma, x)$. Moreover, $v(\sigma, x)$ can be chosen homogeneous also in x.

Proof. Suppose that (4) holds. From (21) define $w_1(\sigma, x) = e_{j,k}(\sigma)$, $w_2(\sigma, x) = f(\sigma)$, $w_3(\sigma, x) = g(\sigma, x)$ and $w_4(\sigma, x) = h_{j,k}(\sigma, x)$, which depend on $v(\sigma, x)$ and ε. Since $A(\sigma)$ is Hurwitz for all $\sigma \in \mathcal{S}$, from [8] it follows that there exist $\tilde{v}(\sigma, x)$, homogeneous polynomial in x with coefficients depending on σ, and $\varepsilon > 0$, such that, for all $i = 1, \ldots, 4$, $j = 0, 1$ and $k = 1, \ldots, p$,

$$\tilde{w}_i(\sigma, x) - \varepsilon\|x\|_2^{\deg_x(\tilde{w}_i(\sigma,x))} \in \mathbb{S}(\psi, x) \quad \forall \sigma \in \mathcal{S}$$

where ψ is an auxiliary variable introduced for considering in the condition above polynomials that are sums of squares of polynomials in x (but not necessarily in σ), and

$$\tilde{w}_i(\sigma, x) = w_i(\sigma, x)|_{v(\sigma,x)=\tilde{v}(\sigma,x), d_\sigma=0}.$$

Since $\tilde{w}_i(\sigma, x)$ is affine linear in $\tilde{v}(\sigma, x)$ and since $\tilde{w}_i(\sigma, x)|_{\tilde{v}(\sigma,x)=\tilde{v}(\psi,x)}$ is polynomial in σ, it follows that the condition above can be also satisfied with $\tilde{v}(\sigma, x)$ homogeneous polynomial in x with coefficients that are continuous functions of σ. Moreover, since $\mathcal{S}$ is compact, these coefficients can be approximated arbitrarily well over $\mathcal{S}$ by polynomial functions. This implies that there exists a function $\hat{v}(\sigma, x)$, homogeneous polynomial in x and polynomial in σ, such that

$$\hat{w}_i(\sigma, x) - \varepsilon\|x\|_2^{\deg_x(\hat{w}_i(\sigma,x))} \in \mathbb{S}(\psi, x) \quad \forall \sigma \in \mathcal{S}$$

where

$$\hat{w}_i(\sigma, x) = w_i(\sigma, x)|_{v(\sigma,x)=\hat{v}(\sigma,x), d_\sigma=\deg_\sigma(\hat{v}(\sigma,x))}.$$

Hence, $\hat{w}_i(\sigma, x)$ admits a positive definite Gram matrix $\hat{W}_i(\sigma)$ for all $\sigma \in \mathcal{S}$, i.e.,

$$\begin{cases} \hat{w}_i(\sigma, x) &= b_i(x)'\hat{W}_i(\sigma)b_i(x) \\ \hat{W}_i(\sigma) &> 0 \quad \forall \sigma \in \mathcal{S} \end{cases}$$

for some vector polynomial $b_i(x)$. Let us observe that $\hat{W}_i(\sigma)$ is a matrix polynomial. Since $1_r'\sigma = 1$ for all $\sigma \in \mathcal{S}$ and since $1_r'\sigma$ is linear, it follows that $\hat{v}(\sigma, x) = \bar{v}(\sigma, x)$,

$\hat{w}_i(\sigma,x) = \bar{w}_i(\sigma,x)$ and $\hat{W}_i(\sigma) = \bar{W}_i(\sigma)$ where $\bar{v}(\sigma,x)$, $\bar{w}_i(\sigma,x)$ and $\bar{W}_i(\sigma)$ are homogeneous also in σ. For $l \in \mathbb{N}$ define the Lyapunov function

$$v(\sigma,x) = (1_r'\sigma)^l \hat{v}(\sigma,x).$$

It follows that $w_i(\sigma,x)$ obtained from (21) with this $v(\sigma,x)$ satisfies

$$\begin{array}{rcl} u_i(\sigma,x) & = & \Phi_\sigma(w_i(\sigma,x),1_r) \\ & = & b_i(x)'(1_r'\sigma)^l \bar{W}_i(\sigma) b_i(x). \end{array}$$

Define $\delta_i = \deg_\sigma(w_i(\sigma,x))$. From Polya's theorem (see, e.g., [13]), it follows that there exists l such that

$$(1_r'\sigma)^l \bar{W}_i(\sigma) = \sum_{a\in\mathbb{N}^r,\, 1_r'a=\delta_i} U_{i,a}\sigma^a$$

where each $U_{i,a}$ is a positive definite matrix. For such value of l, it follows that

$$\begin{array}{rcl} u_i(\sigma^2,x) & = & b_i(x)' \left(\displaystyle\sum_{a\in\mathbb{N}^r,\, 1_r'a=\delta_i} U_{i,a}\sigma^{2a} \right) b_i(x) \\ & = & b_i(x)' \left(\tilde{b}_i(\sigma)\otimes I\right)' U_i \left(\tilde{b}_i(\sigma)\otimes I\right) b_i(x) \end{array}$$

where $\tilde{b}_i(\sigma)$ is a vector of monomials in σ and U_i is a block diagonal matrix whose diagonal blocks are the matrices $U_{i,a}$. This means that $U_i > 0$ and, hence,

$$u_i(\sigma^2,x) = \left\| U_i^{1/2}\left(\tilde{b}_i(\sigma)\otimes I\right) b_i(x)\right\|_2^2,$$

i.e., $u_i(\sigma^2,x) \in \mathbb{S}(\sigma,x)$. Hence, $u_i(\sigma,x) \in \mathbb{S}_+(\sigma,x)$, and $w_i(\sigma,x) \in \mathbb{S}_+(\sigma,x)$. From the definition of $w_i(\sigma,x)$, it is concluded that (19)–(21) hold. □

Theorem 2 states that the sufficient condition provided by Theorem 1 is also necessary under the assumption that $A(\sigma)$ is Hurwitz for all $\sigma \in \mathcal{S}$. This is a mild assumption because, if not satisfied, the admissible impulse responses of the system (1) can be unbounded.

B. Upper Bounds

Here we address the solution of Problem 2. Let us start by observing that a simple way to obtain an upper bound of ρ in (5) consists of adopting a bisection search where the condition of Theorem 1 is tested at each iteration for a fixed value of c. Indeed, bisection can be adopted since the infeasibility of this condition for $c = c_1$ implies the infeasibility for all $c = c_2$ with $c_2 < c_1$.

However, a bisection search may require to test the condition of Theorem 1 several times, which is undesirable, and which is also a problem of our previous work [8]. In [8], it is explained that the bisection search can be avoided by restricting the common polynomial Lyapunov function to be homogeneous. This can be done also for Theorem 1, in particular, by restricting the structured polynomial Lyapunov function to be homogeneous in the state. Although this restriction is not conservative as explained in [8], the degree required for establishing a sought upper bound may be quite larger, hence leading to a significant increment of the computational burden.

Hereafter we propose an alternative way to obtain an upper bound of ρ in (5), which requires neither a bisection search nor restricting $v(\sigma,x)$ to be homogeneous in x. The idea consists of letting γ in Theorem 1 be a decision variable, and removing the nonlinearity originated by the product of the Lyapunov function with γ by introducing an auxiliary variable and a polynomial in σ and this auxiliary variable.

Theorem 3: Let $\gamma_0 = 0$ and $\varrho_0 = \infty$. For $i = 1,2,\ldots$ one has

$$\rho \le \varrho_i \le \varrho_{i-1} \tag{22}$$

where

$$\varrho_i = \gamma_i^{-1} \tag{23}$$

and γ_i is the solution of the SDP

$$\begin{array}{l} \gamma_i = \displaystyle\sup_{\gamma>0,\xi>0,\varepsilon>0,v(\sigma,x)} \gamma \\ \text{s.t.} \left\{ \begin{array}{l} \text{(19) holds} \\ q_3(\sigma,z) \in \mathbb{S}_+(\sigma,z) \\ e_{j,k}(\sigma), g(\sigma,x), h_{j,k}(\sigma,x) \in \mathbb{S}_+(\sigma,x) \\ \forall j=0,1\ \forall k=1,\ldots,p \end{array}\right. \end{array} \tag{24}$$

where $z \in \mathbb{R}$ is an auxiliary quantity, and

$$\left\{ \begin{array}{rcl} q_1(\sigma,z) & = & \xi - v(\sigma, zB(\sigma)) \\ q_2(z) & = & (z-\gamma)(z-\gamma_{i-1})(1+z^2)^{d_x-1} \\ q_3(\sigma,z) & = & q_1(\sigma,z) + q_2(z). \end{array}\right. \tag{25}$$

Proof. Suppose that the constraints in (24) hold. Let us observe that

$$q_3(\sigma,\gamma) = q_3(\sigma,\gamma_{i-1}) = f(\sigma)$$

where $f(\sigma)$ is defined as in (21). Hence, the second constraint in (24) implies that

$$\xi - v(\sigma,\gamma B(\sigma)), \xi - v(\sigma,\gamma_{i-1}B(\sigma)) \in \mathbb{S}_+(\sigma,x).$$

Therefore, (19)–(21) hold, either as it is or by replacing γ with γ_{i-1}. This implies that $\gamma_i \ge \gamma_{i-1}$ and, hence, $\varrho_i \le \varrho_{i-1}$. Moreover, by repeating the proof of Theorem 1 and taking into account that $\xi > 0$, it follows that $\rho \le \varrho_i$. □

C. Tightness Certificate

Once that an upper bound of ρ in (5) has been found, a question arises: is this upper bound tight? This problem is particularly important for uncertain systems because attempting to establish tightness of the found upper bound by simulating an impulse response does require a candidate worst-case value of the uncertainty. In existing methods such as our previous work [8], this problem is not addressed.

Hereafter, we propose a necessary and sufficient condition for establishing if a computed upper bound ϱ_i is tight. Let $f^*(\sigma)$ and $e_{j,k}^*(\sigma)$ be $q_3(\sigma,\gamma_i)$ and $e_{j,k}(\sigma)$ evaluated for the optimal values of the decision variables in (24). Also, let F^*

and $E^*_{j,k}$ be the positive semidefinite matrices used to establish that $f^*(\sigma), e^*_{j,k}(\sigma) \in \mathbb{S}_+(\sigma, x)$, i.e.,

$$\left\{ \begin{array}{rcl} f^{\#}(\sigma^2) & = & b_0(\sigma)' F^* b_0(\sigma) \\ e^{\#}_{j,k}(\sigma^2) & = & b(\sigma)' E^*_{j,k} b(\sigma) \end{array} \right. \tag{26}$$

where $b_0(\sigma)$ and $b(\sigma)$ are homogeneous and

$$\left\{ \begin{array}{rcl} f^{\#}(\sigma) & = & \Phi_\sigma(f^*(\sigma), 1_r) \\ e^{\#}_{j,k}(\sigma) & = & \Phi_\sigma(e^*_{j,k}(\sigma), 1_r). \end{array} \right. \tag{27}$$

The following theorem is based on the determination of candidates for the worst-case values of the uncertainty, i.e., values of σ for which the peak of the impulse response is the computed upper bound. These candidates are obtained by investigating the zeros over $\mathcal{S}$ of $f^*(\sigma)$ and $e^*_{j,k}(\sigma)$, whose presence would imply that the sublevel set of the found Lyapunov function may not be shrunk, or the bound imposed at time $t = 0$ may not be reduced. Taking into account the expressions in (26)–(27) where F^* and $E^*_{j,k}$ are positive semidefinite, it follows that the set of sought candidates can be obtained by looking for values of σ such that $b_0(\sigma^{1/2})$ and $b(\sigma^{1/2})$ are in the null spaces of F^* and $E^*_{j,k}$. Hence, the set of candidates is

$$\mathcal{S}^* = \mathcal{S}^*_0 \cup \bigcup_{j=0,1,\ k=1,\ldots,p} \mathcal{S}^*_{j,k} \tag{28}$$

where

$$\left\{ \begin{array}{rcl} \mathcal{S}^*_0 & = & \left\{ w^2 \|w\|_2^{-2} :\ w \in \mathcal{W}_0 \setminus \{0_r\} \right\} \\ \mathcal{S}^*_{j,k} & = & \left\{ w^2 \|w\|_2^{-2} :\ w \in \mathcal{W}_{j,k} \setminus \{0_r\} \right\} \end{array} \right. \tag{29}$$

and $\mathcal{W}_0$ and $\mathcal{W}_{j,k}$ are the linear sets

$$\left\{ \begin{array}{rcl} \mathcal{W}_0 & = & \{ w \in \mathbb{R}^r :\ b_0(w) \in \ker(F^*) \} \\ \mathcal{W}_{j,k} & = & \left\{ w \in \mathbb{R}^r :\ b(w) \in \ker(E^*_{j,k}) \right\} \end{array} \right. \tag{30}$$

which can be determined through pivoting operations as explained in [6] and references therein.

Theorem 4: Suppose that $A(\sigma)$ is Hurwitz for all $\sigma \in \mathcal{S}$. Assume without loss of generality that $\varrho_i > 0$. Then,

$$\varrho_i = \rho \tag{31}$$

if and only if there exists $\sigma^* \in \mathcal{S}^*$ such that

$$\varrho_i = \sup_{t \geq 0} \left\| Y(t)|_{\sigma=\sigma^*} \right\|_\infty . \tag{32}$$

Proof. Let us start by supposing that (31) holds. From (5), there exists $\sigma^* \in \mathcal{S}$ be such that (32) holds because $\mathcal{S}$ is compact, the supremum is over the variable t, and $Y(t)$ is bounded due to the fact that $A(\sigma)$ is Hurwitz for all $\sigma \in \mathcal{S}$. Let us suppose for contradiction that $\sigma^* \notin \mathcal{S}^*$. This implies that $f^*(\sigma^*)$ and $e^*_{j,k}(\sigma^*)$ are positive. Indeed, let us consider $f^*(\sigma^*)$. It follows that:

$$f^*(\sigma^*) = f^{\#}(\sigma^*) = b_0((\sigma^*)^{1/2})' F^* b_0((\sigma^*)^{1/2}) > 0$$

since $F^* \geq 0$ and $b_0((\sigma^*)^{1/2}) \notin \ker(F^*)$ due to the fact that $\sigma^* \notin \mathcal{S}^*$ and $b_0(\sigma)$ is homogeneous. The same proof can be used to show that $e^*_{j,k}(\sigma^*) > 0$. Therefore, there exists $\tilde{\gamma}$ such that

$$\left\{ \begin{array}{rcl} \rho^{-1} & < & \tilde{\gamma} \\ 0 & \leq & \xi - v(\sigma^*, \tilde{\gamma} B(\sigma^*)) \\ 0 & < & 1 + (-1)^j \tilde{\gamma} C^{(k)}(\sigma^*) B(\sigma^*). \end{array} \right.$$

But this is impossible since it would imply that $\varrho_i < \rho$.

Next, let us suppose that there exists $\sigma^* \in \mathcal{S}^*$ such that (32) holds. Then, (31) directly follows from (5). □

V. The Approach: Part III

This section analyzes the generality of the proposed approach, explaining how it can be extended to deal with non-strictly proper systems, multi-input systems, uncertainty over convex bounded polytopes, response to LTI signals, time-varying uncertainties, and rational dependence on the uncertainty.

A. Non-Strictly Proper Systems

Let us start by saying that there is no loss of generality in considering that the system (1) is strictly proper (i.e., $y(t)$ does not depend directly on $u(t)$). Indeed, the impulse response of a system that is proper but not strictly proper is unbounded.

B. Multi-Input Systems

The system (1) is a single-input system. Multi-input systems can be considered by repeating Problems 1 and 2 for each input channel. Specifically, suppose that the differential equation in the system (1) is now replaced by

$$\dot{x}(t) = A(\sigma) x(t) + \tilde{B}(\sigma) \tilde{u}(t) \tag{33}$$

where $\tilde{u}(t) \in \mathbb{R}^{\tilde{q}}$ is the input and $\tilde{B}(\sigma)$ is a matrix polynomial. In this case, the impulse response is defined for a chosen input channel i, where $i = 1, \ldots, \tilde{q}$, by defining $B(\sigma)$ and $u(t)$ in the system (1) as the i-th column of $\tilde{B}(\sigma)$ and the i-th entry of $\tilde{u}(t)$. Hence, the results presented in Sections III–IV are applied for computing the peak of the impulse response with respect to the i-th input channel.

C. Uncertainty over Convex Bounded Polytopes

The set of admissible uncertainties in the system (1) is the simplex. Convex bounded polytopes can be analogously considered by introducing a parametrization through a linear function over the simplex. Specifically, the system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \tilde{A}(\tilde{\sigma}) x(t) + \tilde{B}(\tilde{\sigma}) u(t) \\ y(t) & = & \tilde{C}(\tilde{\sigma}) x(t) \\ \tilde{\sigma} & \in & \mathrm{conv}\{\mathrm{v}_1, \ldots, \mathrm{v}_\mathrm{q}\} \end{array} \right. \tag{34}$$

where $\tilde{\sigma} \in \mathbb{R}^{\tilde{r}}$ is the time-invariant uncertainty constrained in the convex bounded polytope defined by the convex hull of the vectors $v_1, \ldots, v_q \in \mathbb{R}^{\tilde{r}}$, and $\tilde{A}(\tilde{\sigma})$, $\tilde{B}(\tilde{\sigma})$ and $\tilde{C}(\tilde{\sigma})$ are matrix polynomials, can be transformed into the system (1) by introducing the transformation

$$\tilde{\sigma} = \left(\begin{array}{c} v_1, \ldots, v_q \end{array} \right) \sigma \tag{35}$$

where $\sigma \in \mathbb{R}^r$ with $r = q$. See also [12] among the first works to introduce convex bounded polytopes for the description of uncertain systems. It is useful to notice that convex bounded polytopes include multi-interval sets as a special case.

D. Response to LTI Signals

In the previous sections we have considered the impulse response of the system (1). Other responses can be considered, in particular, responses to LTI signals. Specifically, the response of the system (1) for the initial condition $x(0) = x_0$ to a signal obtainable as the impulse response of the LTI system

$$\begin{cases} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u} \\ \tilde{y}(t) &= \tilde{C}\tilde{x}(t) + \tilde{D}\tilde{u} \end{cases} \tag{36}$$

where $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$, $\tilde{u}(t) \in \mathbb{R}$ and $\tilde{y}(t) \in \mathbb{R}$, for some $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$, can be addressed by suitably redefining $A(\sigma)$, $B(\sigma)$ and $C(\sigma)$ through an augmentation that includes the signal model (36) into the system (1).

E. Time-Varying Uncertainty

The uncertainty considered in the system (1) is time-invariant. In this section we explain how the proposed approach can be extended to address the case of time-varying uncertainty. In particular, we consider the following two cases.

1) Case 1 (Arbitrarily Time-Varying Uncertainty): Here we suppose that the variation rate of the uncertainty σ in the system (1) is unbounded. In this case, Theorems 1 and 3 can be used with $d_\sigma = 0$, which corresponds to the case of a polynomial $v(\sigma, x)$ common to all uncertainties. Indeed, since $\dot{\sigma}$ is unbounded, $v(\sigma, x)$ has to be necessary independent on σ.

For $d_\sigma = 0$, the proposed approach contains the results in our previous work [8] as special cases. In particular, the sufficient condition provided by Theorem 1 is also necessary whenever σ is arbitrarily time-varying under the assumption that $A(\sigma)$ is robustly asymptotically stable for all $\sigma \in \mathcal{S}$.

2) Case 2 (Bounded-Rate Time-Varying Uncertainty): Here we suppose that a bound on the variation rate of the uncertainty σ in the system (1) is available. Such a bound is considered through the constraint adopted in the literature

$$\dot{\sigma} \in \mathrm{conv}\{\mathrm{D}_1, \ldots, \mathrm{D}_\mathrm{h}\} \tag{37}$$

where $D_1, \ldots, D_h \in \mathbb{R}^r$ are such that

$$1_h' D_l = 0 \quad \forall l = 1, \ldots, h. \tag{38}$$

In this case, Theorems 1 and 3 can be used by replacing the constraint $g(\sigma, x) \in \mathbb{S}_+(\sigma, x)$ with

$$g_l(\sigma, x) \in \mathbb{S}_+(\sigma, x) \quad \forall l = 1, \ldots, h \tag{39}$$

where

$$g_l(\sigma, x) = -\left(\nabla_x v(\sigma, x)\right)' A(\sigma)x - \left(\nabla_\sigma v(\sigma, x)\right)' D_l. \tag{40}$$

F. Rational Dependence

The system (1) is affected polynomially by the uncertainty. In this section we explain how the proposed approach can be extended to address the case of rational dependence on the uncertainty. In particular, we consider the more general system

$$\begin{cases} \dot{x}(t) &= \tilde{A}(\sigma)x(t) + \tilde{B}(\sigma)u(t) \\ y(t) &= \tilde{C}(\sigma)x(t) \\ \sigma &\in \mathcal{S} \end{cases} \tag{41}$$

where $\tilde{A}(\sigma)$, $\tilde{B}(\sigma)$ and $\tilde{C}(\sigma)$ are matrix rational functions expressed as

$$\tilde{A}(\sigma) = \frac{A(\sigma)}{\phi_A(\sigma)}, \quad \tilde{B}(\sigma) = \frac{B(\sigma)}{\phi_B(\sigma)}, \quad \tilde{C}(\sigma) = \frac{C(\sigma)}{\phi_C(\sigma)} \tag{42}$$

where $A(\sigma)$, $B(\sigma)$ and $C(\sigma)$ are matrix polynomials, and $\phi_A(\sigma)$, $\phi_B(\sigma)$ and $\phi_C(\sigma)$ are polynomials. In the sequel, $\tilde{C}^{(i)}(\sigma)$ will denote the i-th row of $\tilde{C}(\sigma)$.

First of all, it is assumed that the system (41) is well-posed, i.e., the polynomials $\phi_A(\sigma)$, $\phi_B(\sigma)$ and $\phi_C(\sigma)$ do not vanish over $\mathcal{S}$. Since $\mathcal{S}$ is connected and polynomials are continuous functions, it can be assumed that these polynomials are positive over $\mathcal{S}$ without loss of generality.

Hence, Theorem 1 can be used by replacing $e_{j,k}(\sigma)$, $f(\sigma)$ and $h_{j,k}(\sigma, x)$ with the following expressions:

$$\begin{cases} e_{j,k}(\sigma) &= \phi_B(\sigma)\phi_C(\sigma)\Big((1_r'\sigma)^{d_\sigma} + \\ & \quad +(-1)^j \gamma \tilde{C}^{(k)}(\sigma)\tilde{B}(\sigma) - \varepsilon\Big) \\ f(\sigma) &= \phi_B(\sigma)^{2d_x}\left(\xi - v(\sigma, \gamma\tilde{B}(\sigma))\right) \\ h_{j,k}(\sigma, x) &= \phi_C(\sigma)^{2d_x}\Big(\Phi_x(v(\sigma, x) - \xi, \ldots \\ & \quad (-1)^j \tilde{C}^{(k)}(\sigma)') - \varepsilon\|x\|_2^{2d_x}\Big). \end{cases} \tag{43}$$

This change ensures that the newly defined $e_{j,k}(\sigma)$, $f(\sigma)$ and $h_{j,k}(\sigma, x)$ are polynomials, and their nonnegativity is equivalent to the nonnegativity of the original $e_{j,k}(\sigma)$, $f(\sigma)$ and $h_{j,k}(\sigma, x)$ that would have been obtained by simply replacing $A(\sigma)$, $B(\sigma)$ and $C(\sigma)$ with $\tilde{A}(\sigma)$, $\tilde{B}(\sigma)$ and $\tilde{C}(\sigma)$. Analogous changes can be made for the other theorems presented in the paper and are omitted for brevity.

VI. Examples

In this section we present some numerical examples. The toolbox SeDuMi [19] for Matlab is adopted to solve the SDPs. The LMIs for constraining polynomials in the set $\mathbb{S}_+(\sigma, x)$ are built as explained in Section II-B using algorithms for the construction of the Gram matrices similar to those reported in [6] and simplified taking into account the symmetries as explained in [4]. For brevity, the examples show only the upper bound in (23) for $i = 1$, i.e., ϱ_1, which is obtained by solving the SDP (24). The computational burden is measured by the size of the problem solved by SeDuMi (denoted by SeDuMi size), which consists of the triplet [eqs,order,dim]. For the SDP (24), eqs is the number of scalar LMI variables plus one.

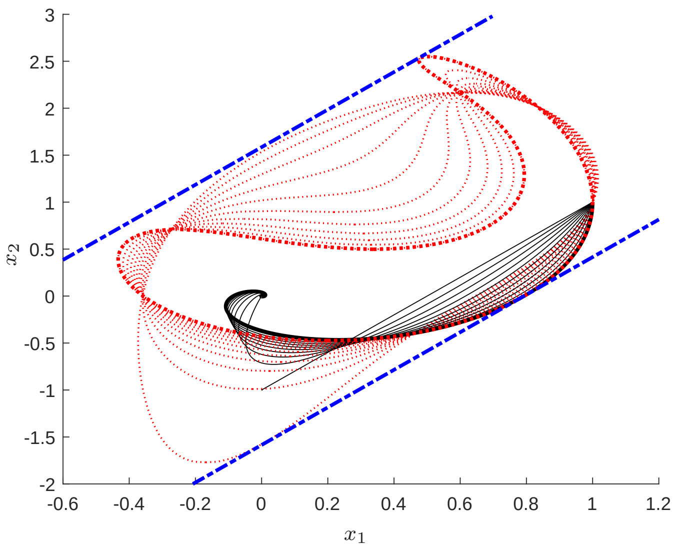

Fig. 1. Example 1: impulse response trajectories (black solid lines), level sets $v(\sigma, x) = 1$ (red dotted lines) and $\|C(\sigma)x\|_\infty = \rho$ (blue dash-dot line) for some values of $\sigma \in \mathcal{S}$. The thick lines denote the found worst-case value of the uncertainty $\sigma = \sigma^*$.

The proposed approach is compared with the existing methods discussed in Section I as follows:

- for [3], [18], the LMIs are repeated at the vertices of the uncertainty set;
- for [8], by letting the uncertainty be arbitrarily time-varying;
- for [16], by using the code available at `github.com/jarmill/peak` based on Gloptipoly3 [14] and YALMIP [15]. The initial condition and objective used are $B(\sigma)\|B(\sigma)\|_2^{-1}$ and $(C(\sigma)x\|B(\sigma)\|_2)^2$ (the term $\|B(\sigma)\|_2$ is used to remove the presence of σ from the initial condition in Examples 2 and 3). The region of interest is chosen as $[-2, 2]^n$ in the state space and $[0, 20]$ in the time domain.

A. Example 1

In the first example we consider a second-order system where the uncertainty affects one matrix only, specifically

$$\begin{cases} \dot{x}(t) &= \begin{pmatrix} -1 & 1-\theta \\ -2 & \theta - 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 2 & -1 \end{pmatrix} x(t) \end{cases}$$

where $\theta \in [0, 1]$ is the uncertainty. This system can be rewritten in the form of (1) by defining $r = 2$ and $\sigma_2 = \theta$.

Table I shows the upper bound ϱ_1 given by the SDP (24). It turns out that ϱ_1 for $d_\sigma = 1$ and $d_x = 2$ is tight, i.e., $\rho = 1.586$. This can be verified by exploiting Theorem 4. Indeed, we find $\mathcal{S}_0^* = \{(1, 0)'\}$, and (32) holds with $\sigma^* = (1, 0)'$ (which corresponds to $\theta = 0$).

Figure 1 shows the trajectories for some admissible impulse responses, together with the level sets $v(\sigma, x) = 1$ and $\|C(\sigma)x\|_\infty = \rho$.

Let us observe that the sought peak is obtained even if $A(\sigma)$ is not Hurwitz for all $\sigma \in \mathcal{S}$ in this example. Indeed, for $\sigma = (0, 1)'$ (which corresponds to $\theta = 1$), the eigenvalues of $A(\sigma)$ are -1 and 0. The fact that $A(\sigma)$ is not Hurwitz for all $\sigma \in \mathcal{S}$ can be also seen from Figure 1, which shows a trajectory that does not reach the origin and ends at the point $(0, -1)'$.

d_σ	d_x	ϱ_1	SeDuMi size [eqs,order,dim]
0	1	∞	$[7, 17, 40]$
0	2	2.219	$[25, 26, 157]$
1	1	1.674	$[14, 25, 103]$
1	**2**	**1.586**	$[\mathbf{68}, \mathbf{40}, \mathbf{406}]$

TABLE I
EXAMPLE 1: UPPER BOUND ϱ_1 GIVEN BY THE SDP (24).

d_σ	d_x	ϱ_1	SeDuMi size [eqs,order,dim]
0	1	∞	$[19, 27, 146]$
0	2	∞	$[145, 60, 923]$
1	1	0.666	$[30, 31, 205]$
1	2	0.603	$[204, 68, 1206]$
2	1	0.510	$[45, 35, 284]$
2	**2**	**0.460**	$[\mathbf{272}, \mathbf{76}, \mathbf{1566}]$

TABLE II
EXAMPLE 2: UPPER BOUND ϱ_1 GIVEN BY THE SDP (24).

For comparison, we also test some existing LMI methods:

- [3], [18] are infeasible;
- [8] provides, for common polynomial Lyapunov functions of degree $2, 4, 6$, the conservative upper bounds $\infty, 2.015, 1.679$;
- [16] provides the correct value of ρ by using order 2 LMI relaxations, for which the SeDuMi size is [eqs,order,dim]=[314,159,2686].

B. Example 2

In the second example we consider a second-order system where the uncertainty affects all matrices, specifically

$$\begin{cases} \dot{x}(t) &= \begin{pmatrix} 0 & , & 1 \\ 10\theta - 12 & , & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0.6 + 1.4\theta \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 2 - 1.6\theta & , & 0 \end{pmatrix} x(t) \end{cases}$$

where $\theta \in [0, 1]$ is the uncertainty. This system can be rewritten in the form of (1) by defining $r = 2$ and $\sigma_2 = \theta$.

Table II shows the upper bound ϱ_1 given by the SDP (24). It turns out that ϱ_1 for $d_\sigma = d_x = 2$ is tight, i.e., $\rho = 0.460$. This can be verified by exploiting Theorem 4. Indeed, we find $\mathcal{S}_0^* = \{(0.377, 0.623)'\}$, and (32) holds with $\sigma^* = (0.377, 0.623)'$ (which corresponds to $\theta = 0.623$).

Figure 2 shows the trajectories for some admissible impulse responses, together with the level sets $v(\sigma, x) = \xi$ and $\|C(\sigma)x\|_\infty = \rho$.

For comparison, we also test the existing LMI methods discussed in Section I:

- [3], [18] are infeasible;
- [8] is infeasible for any degree of the Lyapunov function;
- [16] provides the correct value of ρ by using order 3 LMI relaxations, for which the SeDuMi size is [eqs,order,dim]=[788,368,12937].

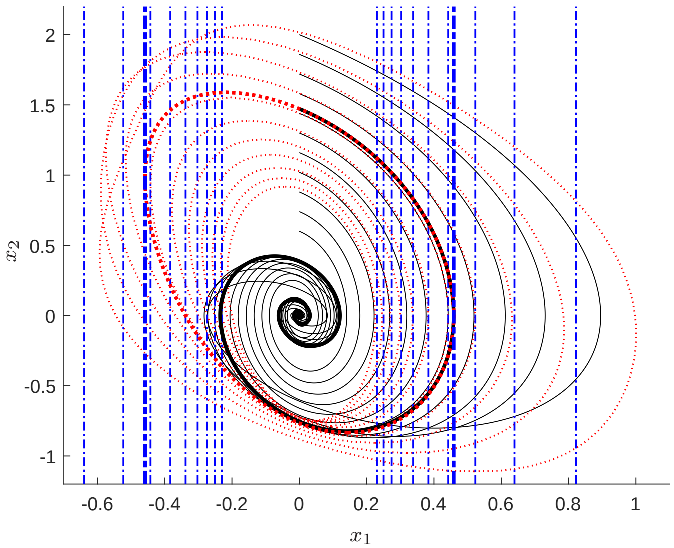

Fig. 2. Example 2: impulse response trajectories (black solid lines), level sets $v(\sigma, x) = \xi$ (red dotted lines) and $\|C(\sigma)x\|_\infty = \rho$ (blue dash-dot line) for some values of $\sigma \in \mathcal{S}$. The thick lines denote the found worst-case value of the uncertainty $\sigma = \sigma^*$.

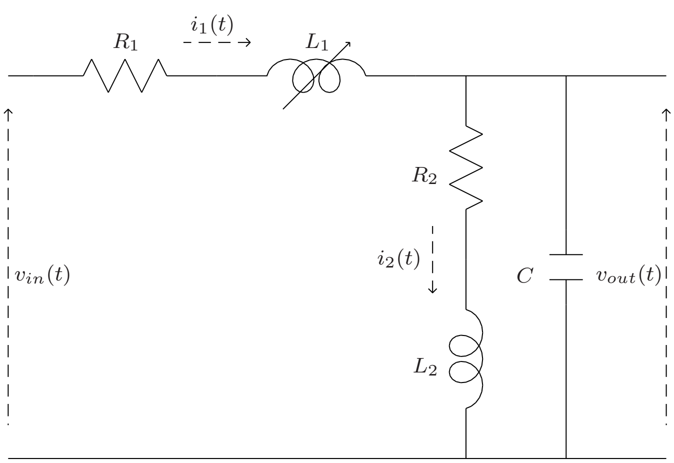

Fig. 3. Example 3: electric circuit with variable inductor.

C. Example 3

In the third example we consider a physical system in order to show a real application, specifically, the electric circuit in Figure 3 which is a third-order system. Let $i_1(t)$ and $i_2(t)$ be the currents in the inductors L_1 and L_2, and let $v_c(t)$ be the voltage of the capacitor C. By choosing the state variables

$$\begin{cases} x_1(t) = i_1(t), \quad x_2(t) = i_2(t), \quad x_3(t) = v_c(t) \\ u(t) = v_{in}(t), \quad y(t) = v_{out}(t), \end{cases}$$

d_σ	d_x	ϱ_1	SeDuMi size [eqs,order,dim]
0	1	2.272	$[14, 23, 107]$
0	2	1.970	$[116, 50, 740]$
1	1	1.221	$[33, 34, 253]$
1	**2**	**0.950**	$[\mathbf{273}, \mathbf{74}, \mathbf{1617}]$

TABLE III
EXAMPLE 3, SCENARIO 1: UPPER BOUND ϱ_1 GIVEN BY THE SDP (24).

this circuit can be modelled as

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} -\dfrac{R_1}{L_1}, & 0, & -\dfrac{1}{L_1} \\ 0, & -\dfrac{R_2}{L_2}, & \dfrac{1}{L_2} \\ \dfrac{1}{C}, & -\dfrac{1}{C}, & 0 \end{pmatrix} x(t) + \begin{pmatrix} \dfrac{1}{L_1} \\ 0 \\ 0 \end{pmatrix} u(t) \\ y(t) = \begin{pmatrix} 0, & 0, & 1 \end{pmatrix} x(t). \end{cases}$$

We consider the case where R_1, R_2, L_2 and C are constants, in particular given by the plausible values

$$R_1 = 1, \quad R_2 = 2, \quad L_2 = 0.7, \quad C = 0.5,$$

and L_1 is a variable, in particular according to

$$L_1 \in [0.5, 2].$$

This model can be rewritten in the form of (1) by defining $r = 2$ and $L_1^{-1} = 2 - 1.5\sigma_2$. We consider two scenarios as follows.

1) Scenario 1 (Time-Invariant Uncertainty): Here we suppose that the uncertainty is time-invariant. Table III shows the upper bound ϱ_1 given by the SDP (24). It turns out that ϱ_1 found for $d_\sigma = 1$ and $d_x = 2$ is tight, i.e., $\rho = 0.950$. This can be verified by exploiting Theorem 4. Indeed, we find $\mathcal{S}_0^* = \{(1, 0)'\}$, and (32) holds with $\sigma^* = (1, 0)'$ (which corresponds to $L_1 = 0.5$).

Figure 4 shows the trajectories for some admissible impulse responses, together with the level sets $v(\sigma^*, x) = \xi$ and $\|C(\sigma^*)x\|_\infty = \rho$.

For comparison, we also test the existing LMI methods discussed in Section I:

- [3], [18] provide the upper bound 2.272;
- [8] provides the conservative upper bound 1.970 via a sequence of LMI feasibility tests in a bisection search using Lyapunov functions of degree 4;
- [16] provides the correct value of ρ by using order 2 LMI relaxations, for which the SeDuMi size is [eqs,order,dim]=[657,255,6440].

2) Scenario 2 (Bounded-Rate Time-Varying Uncertainty): Here we suppose that the uncertainty is time-varying with a bounded variation rate according to

$$|\dot{L}_1| \le \nu$$

where $\nu \in [0, 10]$. This situation can be considered as explained in Section V-E2, in particular by defining

$$D_1 = \frac{8\nu}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad D_2 = -D_1.$$

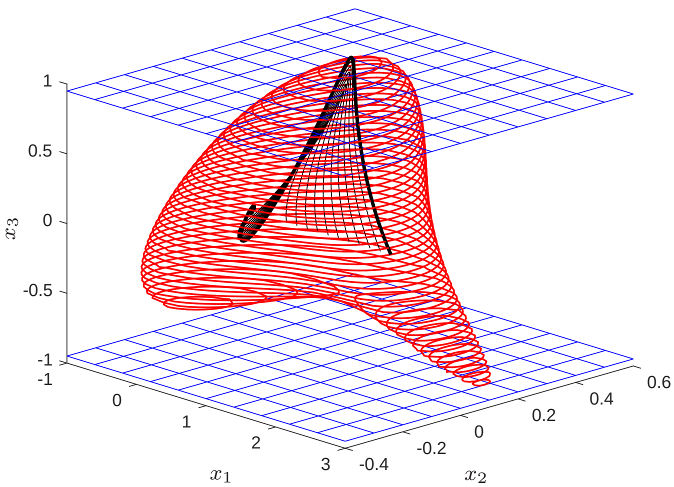

Fig. 4. Example 3, Scenario 1: impulse response trajectories (black solid lines), level sets $v(\sigma^*, x) = \xi$ (red dotted lines) and $\|C(\sigma^*)x\|_\infty = \rho$ (blue dash-dot line) for some values of $\sigma \in \mathcal{S}$. The thick lines denote the found worst-case value of the uncertainty $\sigma = \sigma^*$.

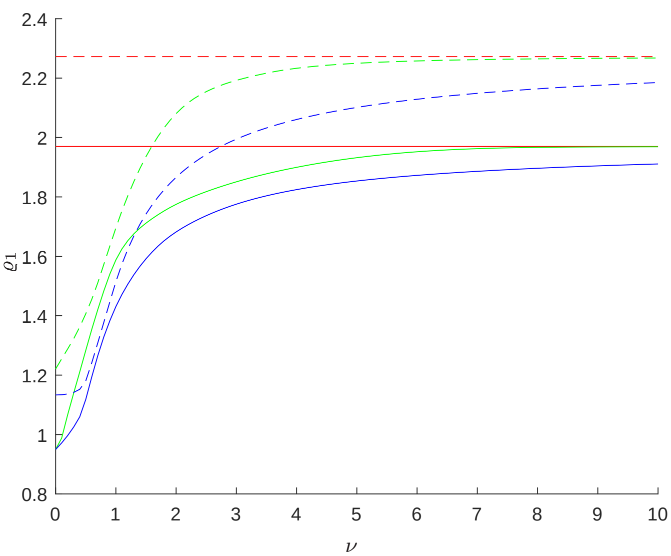

Fig. 5. Example 3, Scenario 2: upper bound ϱ_1 in (23) obtained for some values of ν and some degrees of $v(\sigma, x)$: $d_\sigma = 0$ (red line), $d_\sigma = 1$ (green line), $d_\sigma = 2$ (blue line), $d_x = 1$ (dashed line) and $d_x = 2$ (solid line).

Figure 5 shows the upper bound ϱ_1 in (23) obtained for some values of ν and some degrees of $v(\sigma, x)$.

D. *Example 4*

Lastly, we consider a higher-order system, specifically the fourth-order system

$$\begin{cases} \dot{x}(t) &= \begin{pmatrix} 0 & , 1 , & 0 & , 0 \\ -2-10\theta & , -1 , & 1 & , 0 \\ 0 & , 0 , & 0 & , 1 \\ -2 & , 0 , & 8\theta - 9 & , -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 , 0 , 2 , 0 \end{pmatrix} x(t) \end{cases}$$

where $\theta \in [0, 1]$ is the uncertainty. This system can represent two second-order systems connected in closed-loop, and can be rewritten in the form of (1) by defining $r = 2$ and $\sigma_2 = \theta$. Table IV shows the upper bound ϱ_1 given by the SDP (24). It turns out that ϱ_1 for $d_\sigma = d_x = 2$ is tight, i.e., $\rho = 0.978$. This can be verified by exploiting Theorem 4. Indeed, we find $\mathcal{S}_0^* = \{(0, 1)'\}$, and (32) holds with $\sigma^* = (0, 1)'$ (which corresponds to $\theta = 1$).

d_σ	d_x	ϱ_1	SeDuMi size [eqs,order,dim]
0	1	∞	[14, 23, 117]
0	2	∞	[190, 56, 1186]
1	1	1.304	[40, 37, 331]
1	2	0.985	[532, 93, 3135]
2	1	1.303	[102, 51, 685]
2	**2**	**0.978**	**[1083, 130, 6103]**

TABLE IV
EXAMPLE 4: UPPER BOUND ϱ_1 GIVEN BY THE SDP (24).

For comparison, we also test the existing LMI methods discussed in Section I:

- [3], [18] are infeasible;
- [8] is infeasible for any degree of the Lyapunov function;
- [16] provides the correct value of ρ by using order 3 LMI relaxations, for which the SeDuMi size is [eqs,order,dim]=[4388,1151,104904].

VII. CONCLUSIONS

This paper has proposed a novel LMI-based approach for determining the peak of the response to an LTI signal of a linear system whose system matrices are rational functions of an uncertainty vector constrained into a convex bounded polytope. The considered uncertainty can be time-variant, bounded-rate time-varying or arbitrarily time-varying. As shown by some numerical examples, the proposed approach may have significant advantages with respect to the existing LMI methods in terms of conservatism or computational burden.

Specifically, the proposed approach can be significantly less conservative than the existing LMI methods [3], [8], [18] based on the use of common quadratic Lyapunov functions or common polynomial Lyapunov functions, which are unable to provide the correct peak in all numerical examples presented in Section VI. Clearly, this advantage comes with a larger computational burden, which is expectable since the proposed approach exploits more sophisticated Lyapunov functions than these methods, in particular, polynomial Lyapunov functions polynomially parameterized by the uncertainty. Moreover, with respect to the existing LMI method [16] based on occupation measures, the proposed approach can present significant reductions of the computational burden as shown in all numerical examples presented in Section VI, where the computational burden (measured in terms of SeDuMi size) is, at least, halved.

The proposed approach can be applied to all real devices that can be modeled as a continuous-time linear system affected by structured uncertainty, which can be found in many areas of science and engineering. Besides the electric circuit presented in Example 3 in Section VI (where the peak of interest is on the output voltage), other possible examples are DC motors (where the peak of interest is on the rotor angle), loudspeakers (where the peak of interest is on the cone shift), read/write heads for disk drives (where the peak of interest is on the head position), etc. The reader is referred to books such as [11] for collections of such examples.

Several directions can be considered for future work. One of these directions is the derivation of upper bounds on the degree of the structured polynomial Lyapunov function in the state and in the uncertainty required to achieve nonconservatism. Another direction could consider the extension of the proposed approach for determining the peak-to-peak gain of uncertain systems. Also, it would be interesting to address the design of feedback controllers for ensuring desired upper bounds on the peak of the response of closed-loop uncertain systems. Lastly, one could explore the possibility of extending the proposed approach to the determination of the peak of the response of uncertain 2D systems.

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References

[1] B. R. Barmish. *New Tools for Robustness of Linear Systems*. Mcmillan Publishing Company, New York, 1993.

[2] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel. *Robust Control: The Parametric Approach*. Prentice Hall, NJ, 1995.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994.

[4] G. Chesi. On the complexity of SOS programming and applications in control systems. *Asian Journal of Control*, 20(5):2005–2013, 2018.

[5] G. Chesi. Stability test for complex matrices over the complex unit circumference via LMIs and applications in 2D systems. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 66(5):1960–1969, 2019.

[6] G. Chesi, A. Garulli, A. Tesi, and A. Vicino. *Homogeneous Polynomial Forms for Robustness Analysis of Uncertain Systems*, volume 390 of *Lecture Notes in Control and Information Sciences*. Springer, 2009.

[7] G. Chesi and T. Shen. On the computation of the peak of the impulse response of LTI systems. In *International Conference on Information Science and System*, pages 243–246, Tokyo, Japan, 2019.

[8] G. Chesi and T. Shen. Convergent upper bounds of peak response of LTI and polytopic LTV systems through LMIs. *Automatica*, 122(109260):1–12, 2020.

[9] G. Chesi and T. Shen. On the peak of the impulse response of structured polytopic linear systems. In *Asian Control Conference*, Jeju Island, South Korea, 2022.

[10] M. Choi, T. Lam, and B. Reznick. Sums of squares of real polynomials. In *Proc. Symposia in Pure Mathematics*, pages 103–126, 1995.

[11] G.F. Franklin, J. D. Powell, and A. Emami-Naeini. *Feedback Control of Dynamic Systems*. Prentice Hall, fifth edition, 2006.

[12] J. C. Geromel, P. L. D. Peres, and J. Bernussou. On a convex parameter space method for linear control design of uncertain systems. *SIAM Journal on Control and Optimization*, 29(2):381–402, 1991.

[13] G. Hardy, J. E. Littlewood, and G. Polya. *Inequalities*. Cambridge University Press, Cambridge, 1988.

[14] D. Henrion and J.-B. Lasserre. GloptiPoly: Global optimization over polynomials with Matlab and SeDuMi. *ACM Transactions on Mathematical Software*, 29(2):165–194, 2003.

[15] J. Lofberg. YALMIP : A toolbox for modeling and optimization in MATLAB. In *CACSD Conference*, pages 284–289, Taipei, Taiwan, 2004.

[16] J. Miller, D. Henrion, M. Sznaier, and M. Korda. Peak estimation for uncertain and switched systems. In *IEEE Conference on Decision and Control*, pages 3222–3228, Austin, USA, 2021.

[17] B. Reznick. Some concrete aspects of Hilbert's 17th problem. *Contemporary Mathematics*, 253:251–272, 2000.

[18] C. W. Scherer, P. Gahinet, and M. Chilali. Multiobjective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control*, 42(7):896–911, 1997.

[19] J. F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12:625–653, 1999.

[20] Y. Sun, J. Yu, and Z. Li. Event-triggered finite-time robust filtering for a class of state-dependent uncertain systems with network transmission delay. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 66(3):1076–1089, 2019.

[21] Y. Sun, J. Yu, X. Yu, and H. Gao. Decentralized adaptive event-triggered control for a class of uncertain systems with deception attacks and its application to electronic circuits. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 67(12):5405–5416, 2020.

[22] R. Yang and W. Zheng. Event-driven sliding mode control of discrete-time 2-D Roesser systems. In *IEEE International Symposium on Circuits & Systems*, Daegu, South Korea, 2021.

[23] X. Yao, L. Zhang, and W. Zheng. Uncertain disturbance rejection and attenuation for semi-Markov jump systems with application to 2-degree-freedom robot arm. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 68(9):3836–3845, 2021.

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