Profit optimization of public transit operators: Examining both interior and boundary solutions

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Abstract. We concern the modal choice of commuters in a transport system comprising a highway, which is only used by autos, in parallel to two separate transit lines, which are only used by buses. For the operation of the transit lines, two market structures are considered: monopolistic and duopolistic. Each transit operator sets its transit fare to maximize its own profit. The problem of optimizing the profit of each transit operator is formulated as an optimization model with equilibrium constraints. We theoretically prove that, to obtain both the interior and boundary solutions of the optimization model, it is sufficient to solve an alternative optimization model with equality constraints. Moreover, we prove that increasing the toll charge of private autos leads to an increase in the optimal profit of each transit operator. Based on the above two properties, for each market structure, we propose a period-to-period transit fare and auto toll scheme to locally or globally maximize the profit of each transit operator and to simultaneously make the profit of each transit operator more than a certain value (otherwise, the transit operator may leave the market due to unattractive profits or losses). Finally, by numerical examples, we show the effectiveness of the scheme in each market, the necessity of examining both the interior and boundary solutions of the optimization model, and the importance of designing a period-to-period transit fare and auto toll scheme.

Keywords: Day-to-day dynamics; period-to-period adjustment; monopoly; Bertrand-Nash duopoly; bus fare and auto toll scheme

1. Introduction

Multiple transport modes commonly exist in a city so as to provide substitutable

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transportation services for travelers. For example, travelers can choose either private cars or public transit to travel from their origins to their destinations. Here, the public transit refers to urban transit rather than extra-urban transit. The urban transit usually competes with other urban travel modes, e.g., private cars and taxis, and it mainly services commuters who regularly travel between home and work. Existing studies on multi-modal transport systems, including public transport systems, can be mainly categorized into two classes according to the nature of public transport provision: governmental provision of public transport and provision of public transport by a private firm.

Most of the existing studies on multi-modal transport systems belong to the first class. In some studies belonging to the first class (e.g., Huang, 2000, 2002; Kraus, 2003; Gonzales and Daganzo, 2012; Li et al., 2012; Tirachini and Hensher, 2012; David and Foucart, 2014; Wu and Huang, 2014; Gonzales, 2015; Liu et al., 2016), cutting down or minimizing the total social cost of a system is regarded as a target of planning and managing the system. In other studies belonging to the first class (e.g., Arnott and Yan, 2000; Pels and Verhoef, 2007; Ahn, 2009; van den Berg and Verhoef, 2014), maximizing user surplus or government surplus is regarded as a target. In the second class (e.g., Pels and Verhoef, 2007; Li et al., 2012; van der Weijde et al., 2013; Wu and Huang, 2014; Zhang et al., 2014; Li and Yang, 2016), researchers mainly concern how to improve or maximize the profit of the private firm of operating public transit lines.

In recent decades, there are public transport services provided by multiple private firms. As a result, various market structures have emerged and the level of competition between service operators differs greatly. In some countries, such as the Netherlands, one operator owns the exclusive rights to operate most or even all connections in the public transport network; in others, such as the UK, new operators can freely enter the market to offer new services, or directly compete for franchises to operate existing ones (van der Weijde et al., 2013). In Hong Kong, public transit services are mainly provided by five franchised bus operators, including Citybus, New World First Bus, Kowloon Motor Bus, Long Win Bus, and New Lantau Bus (see the homepage of the Transport Department, the Government of the Hong Kong Special Administrative Region).

In some multi-modal transport systems with the provision of public transit by private firms such as the UK, toll charging is also implemented in congested areas to manage travel demand. For this type of multi-modal transport system, two important questions should be duly considered. The first question is how operators operate their respective public transit lines to increase their own profit. The second question is how the government charges road users using private cars or equivalently subsidizes public transport operators to guarantee that the profit of each operator is not less than a certain value (otherwise operators may leave the market due to generating an unattractive profit or even running a loss). This paper answers the two questions on the basis of two proposed models for optimizing public transit operators' profits with general equilibrium constraints under both monopolistic and duopolistic market structures. The general equilibrium constraints herein mean that we examine both interior and boundary equilibrium solutions of traffic flow distributions.

In some existing studies on modal choice at an equilibrium state, e.g., Huang (2000), Arnott and Yan (2000), van der Weijde et al. (2013), Wu and Huang (2014), Zhang et al. (2014), and Liu et al. (2016), researchers generally only examine interior solutions, in which all modes are used between each origin-destination (OD) pair, and do not examine boundary solutions, in which at least one mode is not used between an OD pair. When only interior solutions are considered, the equilibrium conditions in those existing studies are formulated in the form of equalities. The resultant models with equality constraints can be investigated and analyzed more easily compared with the corresponding model with general equilibrium constraints. Unlike those existing studies, our objectives are to analyze the two models for optimizing operators' profits with general equilibrium constraints under different market structures and to examine both interior and boundary solutions of the two models.

There are at least three motivations for us to examine both interior and boundary solutions. First, the model for investigating interior solutions is just a special case of the model for investigating both interior and boundary solutions. Obtaining the results of a more general model is always a goal of scientific research. Second, a model for obtaining interior solutions is only applied to evaluate the performance efficiency of an existing transport mode (e.g., the profit of running the existing transport service). However, a model for obtaining both interior and boundary solutions can be applied to evaluate not only the efficiency of an existing transport mode but also whether it is necessary to open an extra transit line or cancel an existing transit line. The evaluation of the necessity is an important stage of designing a multi-modal transport system. Third, a model for optimizing operators' profits with general equilibrium constraints is similar to a model for transportation network design. Thus, investigating the model with general equilibrium constraints is helpful for developing an algorithm for solving the network design problem.

In this paper, we theoretically prove an interesting conclusion of the models for optimizing operators' profits with general equilibrium constraints, i.e., both the interior and boundary solutions of the models can be obtained by solving an optimization model with equality constraints. In addition, we prove that increasing the toll charge of private autos results in increases in the optimal profits of transit operators. Meanwhile, we show an application of the above two conclusions to develop a dynamic bus fare and car toll adjustment scheme, which relates to the discipline of dynamic adjustment processes of flows and control variables.

Recently, researchers begin to concern the dynamic adjustment process of flows and control variables in multi-modal transport systems. Indeed, in realistic multi-modal transport systems, commuters can adjust their transport modes, routes, or departure times from day to day based on their experiences or information provided by an advanced traffic information system, and the resultant traffic flows can evolve over days before reaching an equilibrium state. In addition, when a traffic control scheme is implemented in a multi-modal transport system at an equilibrium state, the control scheme may perturb the system and make the traffic flows fall into a disequilibrium state. As a result, the traffic flows may begin to adjust towards a new equilibrium state or may always oscillate from day to day.

The exploration of the day-to-day dynamics opens up another avenue for improving system utility, e.g., decreasing traffic congestion or total travel cost (Friesz et al., 2004; Xiao and Lo, 2015; Zhao et al., 2019). Readers may refer to a comprehensive review of the day-to-day dynamics of traffic flows and control variables by Watling and Cantarella (2013, 2015).

In the aspect of modeling the dynamics of flows and control variables in multi-modal transport systems, some advancements have been made. Cantarella et al. (2015) proposed a dynamical system to formulate the joint adjustment of modal choice and transit operation from day to day in a bi-modal transport system. In the system, the frequency of bus runs is prefixed to meet the demand with all the buses available or is daily updated to meet the demand with the minimum number of buses required to avoid oversaturation. They also showed the non-uniqueness of equilibrium by a numerical example. Li and Yang (2016) proposed a dynamical system model for formulating travelers' day-to-day modal choice in a bi-modal transportation system with responsive transit services. In their model, the frequency of bus runs is adjusted from period to period so that a given target profit of the transit operator is achieved at a stationary state.

Liu and Geroliminis (2017) modeled and controlled a multi-region and multi-modal transportation system, in which the travelers adjust their mode choices from day to day and the within-day traffic dynamics evolve over days. They developed an adaptive mechanism to update parking pricing from period to period so as to improve the system's efficiency. Liu et al. (2017) modeled the joint evolution of travelers' departure time and mode choices in a bi-modal transportation system by considering the impact of user inertia. They also analyzed the dynamic interactions between transport users and the traffic information provider. Guo and Szeto (2018) designed a control strategy to control the day-to-day modal choice of commuters in a bi-modal transportation system so as to simultaneously reduce the daily total travel cost of the transportation system and achieve a Pareto improvement or zero-sum revenue target at a stationary state. Moreover, they introduced new concepts of Pareto improvement and zero-sum revenue in a day-to-day dynamic setting and proposed the two targets' implementations in either a prior or a posterior form.

In this paper, we also focus on the dynamic adjustment process of the modal choice of commuters. However, different from Cantarella et al. (2015), Li and Yang (2016), Liu and Geroliminis (2017), Liu et al. (2017), and Guo and Szeto (2018), we concern a multi-modal transport system comprising two separate transit lines, which are only used by buses, in parallel to a highway, which is only used by autos. The two transit lines are owned by one or more private firms (private operators) and the highway is governed by the government (public operator). Two market structures are considered, i.e., one with a monopolistic public transport operator, which operates the two transit lines, and the other one, in which each of the two separate operators own one transit line. A transit operator can only set the bus fare on its transit line. The toll charge of private cars running on the highway is determined by the government. For each of the two market structures, we propose a dynamic bus fare and auto toll scheme to locally or globally maximize the profit of each operator and at the same time to ensure the profit of each private firm not less than a certain value (i.e., ensure that the profit is attractive to the operator to run the business). In each scheme, the bus fares and the auto toll are adjusted from period to period based on known information in the previous period, in which a period covers a number of successive days.

The contributions of this paper can be summarized as follows:

(1) We investigate the modal choice of commuters in a transport system comprising a highway, which is managed by the government (public operator), and two separate transit lines, which are operated by one or more private firms (private operators). We consider both monopolistic and duopolistic market structures for the transit operation.

(2) For each market structure, we analyze an optimization model with equilibrium constraints to optimize the profit of each transit operator.

(3) We theoretically prove that, to obtain both the interior and boundary solutions of each optimization model, it is sufficient to solve an alternative optimization model with equality

constraints.

(4) We prove that increasing the toll charge of private autos leads to an increase in the optimal profit of each transit operator.

(5) Based on the above two properties, we propose a period-to-period transit fare and auto toll scheme for each market structure to locally or globally maximize the profit of each transit operator and to simultaneously make the profit of each transit operator more than a certain value.

The remainder of this paper is organized as follows. In the next section, the optimization models for maximizing operators' profits and their properties are presented. In Section 3, we propose the dynamic bus fare and car toll schemes and their implementation processes. Several numerical examples are given to show the properties of the optimization models and the implementation effectiveness of the schemes in Section 4. Finally, some remarks and conclusions are provided in Section 5.

2. System description

2.1. Notations and assumptions

We consider a multi-modal transport network in Figure 1. In every morning, a fixed number d (>0) of commuters travel from an origin (O) to a destination (D). The assumption of fixed demand is acceptable when we consider work/school trips or morning peak hour trips in which the trips are always compulsory. The OD pair is connected by two transit lines in parallel to a highway. The three transport modes are separated. Commuters can choose to travel by bus running on one of the two transit lines or an auto running on the highway, i.e., they have three discrete choices. The number of bus users on transit line i ($=1,2$) is denoted by $x_{b,i}$ (≥ 0), the number of auto users on the highway is denoted by x_a (≥ 0), and $\mathbf{x}=(x_{b,1},x_{b,2},x_a)^{\mathrm{T}}$ is the corresponding vector. They then satisfy $x_{b,1}+x_{b,2}+x_a=d$. Let Ω be the feasible set of the numbers of bus users and auto users and it is denoted as

$$\Omega=\left\{\mathbf{x}\left|x_{b,1}+x_{b,2}+x_a=d,x_{b,1}\geq 0,x_{b,2}\geq 0,x_a\geq 0\right.\right\}. \tag{1}$$

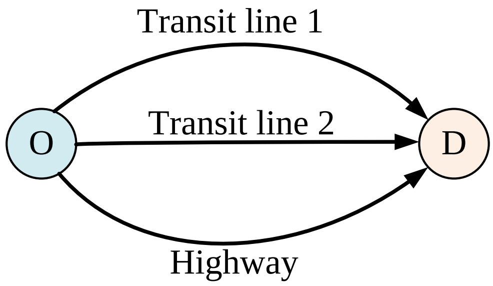


Figure 1. A multi-modal transport network.

We mainly investigate the effect of bus fares and the car toll on both the modal choice of commuters and the operating profit of each transit operator, and hence we assume that the frequency of bus runs on each transit line is constant. This assumption reflects the reality that the frequency of bus runs cannot be adjusted frequently and it is required to remain unchanged for a long period. Let $t_{b,i}$ (>0) stand for the average travel time cost of bus users on transit line i $(=1,2)$, including waiting time cost at a bus stop and in-vehicle time cost. $t_a(x_a)$ (>0) represents the average travel time cost of auto users on the highway, including both free flow travel cost and congestion cost (occurring on the road). The function t_a is continuously differentiable with respect to x_a. Moreover, it is supposed that $\mathrm{d}t_a(x_a)/\mathrm{d}x_a > 0$. This implies that a higher number of auto users generate more congestion for auto users because there are more autos on the highway. Hence, the function t_a is increasing in x_a.

The notation $g(x_{b,i})$ (≥ 0) denotes the average in-vehicle congestion cost of passengers on transit line i $(=1,2)$ and it reflects the discomfort generated by in-vehicle congestion, which has a significant effect on the choices of passengers between transit services and other transport modes (Huang, 2000, 2002; Huang et al., 2007; Li et al., 2012; van den Berg and Verhoef, 2014; Wu and Huang, 2014). The function g is continuously differentiable in $x_{b,i}$. It is assumed that $\mathrm{d}g(x_{b,i})/\mathrm{d}x_{b,i} > 0$, i.e., the in-vehicle congestion cost increases as the number of bus users increases. $p_{b,i}$ (≥ 0) stands for the transit fare (ticket price) charged from each bus user on transit line i $(=1,2)$, p_a (≥ 0) is the toll charge from each auto user on the highway, and $\mathbf{p} = (p_{b,1}, p_{b,2}, p_a)^{\mathrm{T}}$ is the corresponding vector. All those costs and prices, mentioned above, are measured in the monetary unit.

Based on the above notations, the travel costs $c_{b,i}(x_{b,i}, p_{b,i})$ and $c_a(x_a, p_a)$ of commuters using buses on transit line i $(=1,2)$ and using autos on the highway are respectively formulated as

$$c_{b,i}(x_{b,i}, p_{b,i}) = t_{b,i} + g(x_{b,i}) + p_{b,i} \quad \text{and} \tag{2}$$

$$c_a(x_a, p_a) = t_a(x_a) + p_a. \tag{3}$$

The perceived travel costs of commuters using buses on transit line i $(=1,2)$ and using autos on the highway are respectively denoted as $c_{b,i}(x_{b,i}, p_{b,i}) + \xi_{b,i}$ and $c_a(x_a, p_a) + \xi_a$, where $\xi_{b,1}$, $\xi_{b,2}$, and ξ_a are three random error terms which prescribe the difference among travel costs perceived by different commuters.

At the stochastic user equilibrium (SUE) state, no commuter can reduce his/her perceived travel cost by unilaterally altering his/her travel mode (Sheffi, 1985). The SUE conditions can be characterized as

$$x_{b,1} = d \Pr\left(c_{b,1}(x_{b,1}, p_{b,1}) + \xi_{b,1} \le \tilde{\mu}\right), \tag{4}$$

$$x_{b,2} = d \Pr\left(c_{b,2}(x_{b,2}, p_{b,2}) + \xi_{b,2} \le \tilde{\mu}\right), \text{ and} \tag{5}$$

$$x_a = d \Pr\left(c_a(x_a, p_a) + \xi_a \le \tilde{\mu}\right). \tag{6}$$

where

$$\tilde{\mu} = \min\left\{c_{b,1}(x_{b,1}, p_{b,1}) + \xi_{b,1}, c_{b,2}(x_{b,2}, p_{b,2}) + \xi_{b,2}, c_a(x_a, p_a) + \xi_a\right\}.$$

It is generally assumed that the random error terms can take any values that belong to $(-\infty, +\infty)$ in existing studies (Sheffi, 1985). For example, it is supposed that the random error terms are independently and identically distributed Gumbel variables or they are normally distributed variables. As a result, it is guaranteed that the traffic flows at the SUE state are in the interior of the feasible set of traffic flows and the SUE conditions can be equivalently written as a set of equalities.

Sometimes, in order to obtain some analytical results related to the performance evaluation of a policy/scheme in a multi-modal transport system, the implementation of the policy/scheme is analyzed and evaluated based on the assumption that the variances of the random error terms are zero in existing studies, e.g., Huang (2000), Pels and Verhoef (2007), Li et al. (2012), van der Weijde et al. (2013), and Liu et al. (2016). Under the assumption, the SUE conditions degenerate into the following deterministic user equilibrium (DUE) conditions:

$$x_{b,i}\left(c_{b,i}(x_{b,i}, p_{b,i}) - \bar{\mu}\right) = 0, \text{ for } i = 1,2, \quad x_a\left(c_a(x_a, p_a) - \bar{\mu}\right) = 0, \tag{7}$$

where $\bar{\mu}$ is the minimum travel cost among all modes, i.e.,

$$\bar{\mu} = \min\left\{c_{b,1}(x_{b,1}, p_{b,1}), c_{b,2}(x_{b,2}, p_{b,2}), c_a(x_a, p_a)\right\}. \tag{8}$$

Conditions (7) and (8) state that all used modes have equal travel cost, which is less than or equal to those of any of the unused modes. In this way, the analysis and evaluation can be simplified. Moreover, it can be verified by numerical simulations that some change trends and policy/scheme properties generated in the context of DUE are qualitatively identical with those obtained in the context of SUE. In subsequent analyses, we adopt the assumption that the variances of the random error terms are zero.

Let $k_{b,i}(x_{b,i})$ be the daily operating cost of transit line i ($=1,2$). The function $k_{b,i}$ is continuously differentiable with respect to $x_{b,i}$. It is supposed that $\mathrm{d}k_{b,i}(x_{b,i})/\mathrm{d}x_{b,i} \geq 0$ and this means that the function $k_{b,i}$ is non-decreasing in $x_{b,i}$. It is worth mentioning that there is a fact for the operation of public transit, namely, when the number $x_{b,i}$ of bus users on transit line i ($=1,2$) is large, the increasing rate of the operating cost $k_{b,i}(x_{b,i})$ of transit line i to $x_{b,i}$ may be slight or zero. The fact does not affect the conclusions in subsequent Sections 2.2 and 2.3, because the conclusions in subsequent Sections 2.2 and 2.3 are based on only a precondition related to the functions $k_{b,1}$ and $k_{b,2}$, i.e., the functions $k_{b,1}$ and $k_{b,2}$ are continuously differentiable with respect to $x_{b,1}$ and $x_{b,2}$, respectively.

2.2. The transit monopoly

For the market structure with a monopolistic transit operator, who operates both transit lines, the daily total profit of the transit operator is governed by

$$U = x_{b,1}p_{b,1} + x_{b,2}p_{b,2} - k_{b,1}(x_{b,1}) - k_{b,2}(x_{b,2}), \tag{9}$$

i.e., the daily total profit is equal to the difference between the revenue $x_{b,1}p_{b,1} + x_{b,2}p_{b,2}$ from transit fares and the operating cost $k_{b,1}(x_{b,1}) + k_{b,2}(x_{b,2})$.

For a given auto toll p_a, the operator maximizes the daily total profit at the equilibrium state through determining transit fares $p_{b,1}$ and $p_{b,2}$. The optimal monopolistic transit fares are obtained by solving the following optimization problem

$$\max_{(p_{b,1},p_{b,2})} U = x_{b,1}p_{b,1} + x_{b,2}p_{b,2} - k_{b,1}(x_{b,1}) - k_{b,2}(x_{b,2}), \tag{10}$$

where $(p_{b,1}, p_{b,2})$ is subject to

$$x_{b,i}\left(t_{b,i} + g(x_{b,i}) + p_{b,i} - \bar{\mu}\right) = 0, \text{ for } i = 1,2, \quad x_a\left(t_a(x_a) + p_a - \bar{\mu}\right) = 0, \tag{11}$$

$$p_{b,1} \geq 0, \quad p_{b,2} \geq 0, \text{ and } \mathbf{x} \in \Omega. \tag{12}$$

Condition (11) is the DUE condition, in which the minimum travel cost $\bar{\mu}$ among all modes is formulated in expression (8). The first two constraints in condition (12) are two non-negativity constraints for transit fares and the last one states the feasibility constraints for the numbers of bus and auto users.

Condition (11) indicates that the equilibrium user distribution among modes at the maximum point can be either in the interior of the feasible set Ω or on the boundary of Ω. Thus, compared with some existing studies, e.g., Huang (2000), Arnott and Yan (2000), van der Weijde et al. (2013), Wu and Huang (2014), Zhang et al. (2014), and Liu et al. (2016), a more general case is considered here.

The value of the objective function of the optimization problem (10) to (12) is determined by the variables $\mathbf{x}$ and $\mathbf{p}$. Once $\mathbf{p}$ is determined, a unique $\mathbf{x}$ at the equilibrium state is also determined because the functions g and t_a are increasing. Thus, $\mathbf{x}$ is a function of $\mathbf{p}$ and the value of the objective function is finally determined by $\mathbf{p}$.

In this paper, Assumption 1 is adopted in subsequent analyses, which is stated as follows.

Assumption 1. Let $\mathbf{x}^{\mathrm{UE}} = (x_{b,1}^{\mathrm{UE}}, x_{b,2}^{\mathrm{UE}}, x_a^{\mathrm{UE}})^{\mathrm{T}}$ be the flow distribution pattern at the DUE state without transit fares (i.e., $p_{b,1} = 0$ and $p_{b,2} = 0$) and it satisfies

$$x_{b,1}^{\mathrm{UE}} > 0,\ x_{b,2}^{\mathrm{UE}} > 0,\ x_a^{\mathrm{UE}} > 0, \text{ and } t_{b,i} + g\left(x_{b,i}^{\mathrm{UE}}\right) = t_a\left(x_a^{\mathrm{UE}}\right) + p_a, \text{ for } i = 1,2. \tag{13}$$

Assumption 1 means that, at the DUE state without transit fares, all modes are used, namely, the number of commuters using each mode is positive and the travel costs of all modes are equal. Then, we have the following property.

Property 1. Let $\left(p_{b,1}^*, p_{b,2}^*\right)$ be an optimal solution to the following optimization problem:

$$\max_{(p_{b,1}, p_{b,2})} U = x_{b,1} p_{b,1} + x_{b,2} p_{b,2} - k_{b,1}(x_{b,1}) - k_{b,2}(x_{b,2}), \tag{14}$$

where $(p_{b,1}, p_{b,2})$ is subject to

$$t_{b,i} + g(x_{b,i}) + p_{b,i} = t_a(x_a) + p_a, \text{ for } i = 1,2, \tag{15}$$

$$p_{b,1} \ge 0,\ p_{b,2} \ge 0, \text{ and } \mathbf{x} \in \Omega. \tag{16}$$

Then, under Assumption 1, $\left(p_{b,1}^*, p_{b,2}^*\right)$ is also an optimal solution to the optimization problem (10) to (12).

Property 1 is proved in Appendix A.1. In the proof, the feasible set of the optimization problem (10) to (12) is divided into several subsets, one of which satisfies conditions (15) and (16). To prove Property 1, it is shown that the objective function value at an optimal solution in the subset of satisfying conditions (15) and (16) is not less than the objective function value at an optimal solution in any one of the other subsets. Assumption 1 is adopted to guarantee that two optimal solutions in different subsets are comparable through an intermediary feasible solution.

Despite the fact that the DUE condition is expressed in the form of equalities in the optimization problem (14) to (16), an optimal transit fare vector, at which the equilibrium user distribution can be either in the interior or on the boundary of the feasible set Ω, can be obtained by solving the optimization problem. The set of transit fares satisfying constraints (15) and (16) is a subset of the set of transit fares satisfying constraints (11) and (12), and Property 1 means that the subset contains the optimal solutions to the optimization problem (10) to (12). Obviously, it is easier to analyze the problem (14) to (16) than the problem (10) to (12). In this way, the degree of difficulty for solving the general problem is reduced.

Once the transit fares and car toll $\mathbf{p}$ is determined, a unique user distribution pattern $\mathbf{x}$ of satisfying constraints (15) and (16) is also determined. Therefore, the variable U in (14) is a function of $\mathbf{p}$. The partial derivatives of the variable U with respect to $\mathbf{p}$ are formulated as

$$\frac{\partial U}{\partial p_{b,1}} = \frac{-\left(p_{b,1} - k'_{b,1}(x_{b,1})\right)\left(g'(x_{b,2}) + t'_a(x_a)\right) + \left(p_{b,2} - k'_{b,2}(x_{b,2})\right)t'_a(x_a)}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t'_a(x_a) + g'(x_{b,2})t'_a(x_a)} + x_{b,1}, \tag{17}$$

$$\frac{\partial U}{\partial p_{b,2}} = \frac{\left(p_{b,1} - k'_{b,1}(x_{b,1})\right)t'_a(x_a) - \left(p_{b,2} - k'_{b,2}(x_{b,2})\right)\left(g'(x_{b,1}) + t'_a(x_a)\right)}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t'_a(x_a) + g'(x_{b,2})t'_a(x_a)} + x_{b,2}, \text{ and} \tag{18}$$

$$\frac{\partial U}{\partial p_a} = \frac{\left(p_{b,1} - k'_{b,1}(x_{b,1})\right)g'(x_{b,2}) + \left(p_{b,2} - k'_{b,2}(x_{b,2})\right)g'(x_{b,1})}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t'_a(x_a) + g'(x_{b,2})t'_a(x_a)}. \tag{19}$$

The derivation of partial derivatives (17) to (19) is given in Appendix A.2.

When the bus fare on one of the two transit lines is relatively large so that no commuters choose to travel (by bus) on the transit line at the DUE state, the optimization problem (14) to (16) is simplified to

$$\max_{p_{b,i}} U = x_{b,i}p_{b,i} - k_{b,i}(x_{b,i}), \tag{20}$$

where $p_{b,i}$ is subject to

$$t_{b,i} + g(x_{b,i}) + p_{b,i} = t_a(x_a) + p_a, \tag{21}$$

$$x_{b,i} + x_a = d, \quad x_{b,i} \geq 0, \quad x_a \geq 0, \text{ and } \quad p_{b,i} \geq 0. \tag{22}$$

Here, $i = 1$ or 2. Similar to the derivation of partial derivatives (17) to (19), the partial derivatives of the variable U in (20) with respect to $p_{b,i}$ and p_a are formulated as

$$\frac{\partial U}{\partial p_{b,i}} = \left(p_{b,i} - k'_{b,i}(x_{b,i})\right)\frac{\partial x_{b,i}}{\partial p_{b,i}} + x_{b,i} = -\frac{p_{b,i} - k'_{b,i}(x_{b,i})}{g'(x_{b,i}) + t'_a(x_a)} + x_{b,i} \quad \text{and} \tag{23}$$

$$\frac{\partial U}{\partial p_a} = \left(p_{b,i} - k'_{b,i}(x_{b,i})\right)\frac{\partial x_{b,i}}{\partial p_a} = \frac{p_{b,i} - k'_{b,i}(x_{b,i})}{g'(x_{b,i}) + t'_a(x_a)}. \tag{24}$$

Formulae (17) to (19) indicate how the transit fares $p_{b,1}$ and $p_{b,2}$ and the auto toll p_a affect the profit U of the transit operator. In the transport system, the travel demand d is fixed and the functions t_a and g are increasing. Therefore, when the auto toll p_a increases and the bus fare vector $(p_{b,1}, p_{b,2})$ remains unchanged, the number of bus users ascends. However, this does not mean that the profit U of the transit operator also rises owing to the following reason. The profit of the transit operator is equal to the difference between the revenue from transit fares and the operating cost. We adopt a more general precondition with respect to the operating cost, i.e., the changing rate of the operating cost to the number of bus users is not specified. As a result, the marginal revenue may be less than the marginal operating cost even if the number of bus users increases. On the other hand, the optimal bus fare vector for optimizing the profit of the transit operator is adjusted as the auto toll increases. Thus, it is unclear (at least, it is not intuitively perceived) how the *optimal* profit of the transit operator changes as the auto toll increases.

Property 2 answers the question, which requires the following notations and definitions for clarity. Given an auto toll $\bar{p}_a$, let $(\bar{p}_{b,1}, \bar{p}_{b,2})$ be an optimal bus fare vector at the equilibrium state and $\bar{\mathbf{x}} = (\bar{x}_{b,1}, \bar{x}_{b,2}, \bar{x}_a)$ be the vector of the numbers of bus users and auto users at the optimal solution. The profit U of the transit operator is a function of $(\bar{p}_{b,1}, \bar{p}_{b,2})$ and $\bar{p}_a$, and hence the profit of the transit operator at the optimal solution is denoted as $U = \bar{U}(\bar{p}_{b,1}, \bar{p}_{b,2}, \bar{p}_a)$. Given another auto toll $\tilde{p}_a$, the notations $(\tilde{p}_{b,1}, \tilde{p}_{b,2})$, $\tilde{\mathbf{x}} = (\tilde{x}_{b,1}, \tilde{x}_{b,2}, \tilde{x}_a)$, and $U = \bar{U}(\tilde{p}_{b,1}, \tilde{p}_{b,2}, \tilde{p}_a)$ have similar meanings.

Property 2. If $\bar{p}_a < \tilde{p}_a$, then $\bar{U}(\bar{p}_{b,1}, \bar{p}_{b,2}, \bar{p}_a) < \bar{U}(\tilde{p}_{b,1}, \tilde{p}_{b,2}, \tilde{p}_a)$.

Property 2 is proved in Appendix A.3. This property gives a theoretical principle for the government to set the auto toll to make the optimal profit of the transit operator more than a certain value in the transit monopoly market. It indicates that, as the toll charge to car users ascends, the optimal profit of the operator also increases. Thus, to guarantee that the optimal profit of the operator is not less than a target value at the equilibrium state, the government should set a relatively higher toll charge to car users.

Property 2 is based on the precondition that, given an auto toll, an optimal transit fare vector can be exactly obtained. However, an optimal transit fare vector can only be obtained approximately in reality. When the precondition changes as “given an auto toll, only an approximately optimal transit fare vector can be obtained”, it can be seen from the proof of Property 2 in Appendix A.3 that Property 2 still holds so long as the approximately optimal profit at the larger auto toll $\tilde{p}_a$ satisfies inequality (A.34).

2.3. The transit Bertrand-Nash duopoly

When the two transit lines are owned by separate operators, the daily total profit of the operator of operating transit line j ($=1,2$) is governed by

$$V_j = x_{b,j} p_{b,j} - k_{b,j}(x_{b,j}), \tag{25}$$

i.e., the daily total profit is equal to the difference between the revenue $x_{b,j} p_{b,j}$ from transit fares and the operating cost $k_{b,j}(x_{b,j})$. The operator of transit line j determines the transit fare on transit line j to maximize its own profits at the DUE state (given the transit fare on the other transit line and the auto toll). The optimal transit fare is obtained by solving the following optimization problem:

$$\max_{p_{b,j}} V_j = x_{b,j} p_{b,j} - k_{b,j}(x_{b,j}), \tag{26}$$

where $p_{b,j}$ is subject to

$$x_{b,i}\left(t_{b,i} + g(x_{b,i}) + p_{b,i} - \bar{\mu}\right) = 0, \text{ for } i = 1,2, \quad x_a\left(t_a(x_a) + p_a - \bar{\mu}\right) = 0, \tag{27}$$

$$p_{b,j} \geq 0, \text{ and } \mathbf{x} \in \Omega. \tag{28}$$

Here, the minimum travel cost $\bar{\mu}$ among all three modes is formulated in expression (8).

For the optimization problem (26) to (28), the following property holds.

Property 3. Let $p_{b,j}^*$ ($j = 1,2$) be an optimal solution to the following optimization problem:

$$\max_{p_{b,j}} V_j = x_{b,j} p_{b,j} - k_{b,j}(x_{b,j}), \tag{29}$$

where $p_{b,j}$ is subject to

$$t_{b,i} + g(x_{b,i}) + p_{b,i} = t_a(x_a) + p_a, \text{ for } i = 1,2, \tag{30}$$

$$p_{b,j} \geq 0, \text{ and } \mathbf{x} \in \Omega. \tag{31}$$

Then, under Assumption 1, $p_{b,j}^*$ is also an optimal solution to the optimization problem (26) to (28).

Property 3 is proved in Appendix A.4. This property indicates that the feasible subset of satisfying constraints (30) and (31) contains the optimal solutions to the optimization problem

(26) to (28).

The variable V_j ($j=1,2$) in (29) is a function of the bus fares and auto toll $\mathbf{p}$. The partial derivatives of the variables V_1 and V_2 in (29) with respect to $\mathbf{p}$ are formulated as

$$\frac{\partial V_1}{\partial p_{b,1}} = -\frac{\left(p_{b,1} - k'_{b,1}(x_{b,1})\right)\left(g'(x_{b,2}) + t'_a(x_a)\right)}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t'_a(x_a) + g'(x_{b,2})t'_a(x_a)} + x_{b,1}, \tag{32}$$

$$\frac{\partial V_1}{\partial p_{b,2}} = \frac{\left(p_{b,1} - k'_{b,1}(x_{b,1})\right)t'_a(x_a)}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t'_a(x_a) + g'(x_{b,2})t'_a(x_a)}, \tag{33}$$

$$\frac{\partial V_1}{\partial p_a} = \frac{\left(p_{b,1} - k'_{b,1}(x_{b,1})\right)g'(x_{b,2})}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t'_a(x_a) + g'(x_{b,2})t'_a(x_a)}, \tag{34}$$

$$\frac{\partial V_2}{\partial p_{b,1}} = \frac{\left(p_{b,2} - k'_{b,2}(x_{b,2})\right)t'_a(x_a)}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t'_a(x_a) + g'(x_{b,2})t'_a(x_a)}, \tag{35}$$

$$\frac{\partial V_2}{\partial p_{b,2}} = -\frac{\left(p_{b,2} - k'_{b,2}(x_{b,2})\right)\left(g'(x_{b,1}) + t'_a(x_a)\right)}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t'_a(x_a) + g'(x_{b,2})t'_a(x_a)} + x_{b,2}, \text{ and} \tag{36}$$

$$\frac{\partial V_2}{\partial p_a} = \frac{\left(p_{b,2} - k'_{b,2}(x_{b,2})\right)g'(x_{b,1})}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t'_a(x_a) + g'(x_{b,2})t'_a(x_a)}. \tag{37}$$

The derivation of partial derivatives (32) to (37) is provided in Appendix A.5. These partial derivatives indicate how the transit fares $p_{b,1}$ and $p_{b,2}$ and the auto toll p_a affect the profit V_j of the operator of transit line j ($=1,2$). For example, when $p_{b,1} < k'_{b,1}(x_{b,1})$, the inequalities $\partial V_1/\partial p_{b,1} > 0$, $\partial V_1/\partial p_{b,2} < 0$, and $\partial V_1/\partial p_a < 0$ hold, since the functions g and t_a are increasing. Thus, increasing the bus fare $p_{b,1}$ on transit line 1 makes the profit V_1 of transit operator 1 ascend and increasing the bus fare $p_{b,2}$ on transit line 2 (or the auto toll p_a) leads to the descent of V_1.

Given an auto toll $\bar{p}_a$, let $\bar{p}_{b,1}$ and $\bar{p}_{b,2}$ be optimal bus fares at the equilibrium state for transit operators 1 and 2, respectively, and $\bar{\mathbf{x}} = (\bar{x}_{b,1}, \bar{x}_{b,2}, \bar{x}_a)$ be the vector of the numbers of bus users and auto users at the optimal solution. The profit V_j of transit operator j ($=1,2$) is a function of $\bar{p}_{b,j}$ and $\bar{p}_a$, and hence the profit of transit operator j at the optimal solution is denoted as $V_j = \bar{V}_j(\bar{p}_{b,j}, \bar{p}_a)$. Given another auto toll $\tilde{p}_a$, the notations $\tilde{p}_{b,1}$, $\tilde{p}_{b,2}$, $\tilde{\mathbf{x}} = (\tilde{x}_{b,1}, \tilde{x}_{b,2}, \tilde{x}_a)$, and $V_j = \bar{V}_j(\tilde{p}_{b,j}, \tilde{p}_a)$ ($j=1,2$) have similar meanings. The following property answers the question of how the *optimal* profit of transit operator j ($=1,2$) changes as the auto toll increases.

Property 4. If $\bar{p}_a < \tilde{p}_a$, then $\bar{V}_j(\bar{p}_{b,j}, \bar{p}_a) < \bar{V}_j(\tilde{p}_{b,j}, \tilde{p}_a)$, for $j=1,2$.

Property 4 is proved in Appendix A.6. Similar to Property 2 for the monopoly market, Property 4 gives a theoretical principle for the government to set the auto toll to make the optimal profit of each of the two transit operators more than a certain value in the transit Bertrand-Nash duopoly market. It indicates that, as the toll charge to car users ascends, the optimal profits of both transit operators also rise. Thus, to guarantee that the optimal profit of each transit operator is not less than a target value at the equilibrium state, the government should set a relatively high toll charge to car users.

3. Period-to-period operational schemes

In this section, we apply the above models and properties in Section 2 to propose a period-to-period transit fare and auto toll scheme for each market structure to maximize the profit of each transit operator and to simultaneously make the profit of each transit operator more than a certain value.

3.1. Dynamics of the transport system

In the transport system, on each day, commuters reconsider their transport modes based on known information (e.g., travel time costs and congestion costs on previous days and transit fares and the auto toll on that day) to minimize their own travel costs. On one hand, each public transit operator determines the transit fare(s) on its transit line(s) and adjusts the transit fare(s) from period to period according to user distribution among modes and the travel costs of all modes in the previous period (i.e., the transit fare of each line remains unchanged on all days during a period) so as to locally or globally maximize its profit from operating transit line(s). On the other hand, the government determines the auto toll for users on the highway from period to period based on the previous information to guarantee that the profit of each public transport operator is more than a certain value (otherwise, transit operators may leave the market due to unattractive profits or even losses). As a result, the decision-making of commuters, public transit operators, and the government interacts with each other. The interactive adjustment process is formulated as a dynamical system under a market structure. In the dynamical system, the traffic volumes for selecting the three transport modes are state variables and the transit fares for buses and the toll charge for autos are control variables. Under different market structures, the dynamics of the state variables are common and the control variables are determined by different schemes.

Here, we do not specify how commuters adjust their modal choices from day to day (i.e., the dynamics of the state variables). Similar to Yang et al. (2004), Han and Yang (2009), Yang et al. (2010), and Zhou et al. (2015), we adopt an assumption, i.e., the system gets to the equilibrium state at the end of a period given the bus fares and auto toll in the period. Day-to-day modal choice can be formulated by the models proposed by Smith (1984), Friesz et al. (1994), Zhang and Nagurney (1996), and He et al. (2010), or others. For instance, by applying the proportional swap rule of Smith (1984), the day-to-day adjustments of the numbers of bus users and auto users are formulated as

$$x_{b,j}^{(m+1)} = x_{b,j}^{(m)} + \delta F_{b,j}\left(\mathbf{x}^{(m)}, \mathbf{p}^{(m+1)}\right), \text{ for } j = 1,2, \text{ and} \tag{38}$$

$$x_{a}^{(m+1)} = x_{a}^{(m)} + \delta F_{a}\left(\mathbf{x}^{(m)}, \mathbf{p}^{(m+1)}\right), \tag{39}$$

for $m = 0,1,2,\cdots$. The superscript (m) refers to the mth day, e.g., $x_{b,j}^{(m)}$ represents the number of bus users on transit line j on day m. The adjustment parameter $\delta > 0$. The functions $F_{b,j}$ ($j = 1,2$) and F_a are expressed as

$$\begin{aligned} F_{b,1}(\mathbf{x},\mathbf{p}) = {} & x_{b,2}\left[c_{b,2}(x_{b,2}, p_{b,2}) - c_{b,1}(x_{b,1}, p_{b,1})\right]_+ + x_a\left[c_a(x_a, p_a) - c_{b,1}(x_{b,1}, p_{b,1})\right]_+ \\ & - x_{b,1}\left[c_{b,1}(x_{b,1}, p_{b,1}) - c_{b,2}(x_{b,2}, p_{b,2})\right]_+ - x_{b,1}\left[c_{b,1}(x_{b,1}, p_{b,1}) - c_a(x_a, p_a)\right]_+, \end{aligned} \tag{40}$$

$$\begin{aligned} F_{b,2}(\mathbf{x},\mathbf{p}) = {} & x_{b,1}\left[c_{b,1}(x_{b,1}, p_{b,1}) - c_{b,2}(x_{b,2}, p_{b,2})\right]_+ + x_a\left[c_a(x_a, p_a) - c_{b,2}(x_{b,2}, p_{b,2})\right]_+ \\ & - x_{b,2}\left[c_{b,2}(x_{b,2}, p_{b,2}) - c_{b,1}(x_{b,1}, p_{b,1})\right]_+ - x_{b,2}\left[c_{b,2}(x_{b,2}, p_{b,2}) - c_a(x_a, p_a)\right]_+, \text{ and} \end{aligned} \tag{41}$$

$$\begin{aligned} F_{a}(\mathbf{x},\mathbf{p}) = {} & x_{b,1}\left[c_{b,1}(x_{b,1}, p_{b,1}) - c_a(x_a, p_a)\right]_+ + x_{b,2}\left[c_{b,2}(x_{b,2}, p_{b,2}) - c_a(x_a, p_a)\right]_+ \\ & - x_a\left[c_a(x_a, p_a) - c_{b,1}(x_{b,1}, p_{b,1})\right]_+ - x_a\left[c_a(x_a, p_a) - c_{b,2}(x_{b,2}, p_{b,2})\right]_+, \end{aligned} \tag{42}$$

where the mapping $[\cdot] = \max\{\cdot, 0\}$.

Formulae (38) to (42) state that, on each day, commuters adjust their transport modes based on the previous day's travel costs and intraday transit fares and auto toll. A portion of commuters using a mode with a higher generalized travel cost will choose to travel by other modes with a lower generalized travel cost on the next day. The portion is proportional to both the number of commuters using the mode with a higher generalized travel cost and the generalized travel cost difference from other modes.

3.2. Operational scheme for the transit monopoly

Let $\mathbf{p}^{(n)}$ be the bus fares and the auto toll in period n and $\mathbf{x}^{(n)}$ be the numbers of bus users and auto users at the end of period n (namely, the numbers of bus users and auto users at the equilibrium state under the implementation of the bus fares and auto toll $\mathbf{p}^{(n)}$). For the monopoly market structure, in which the two transit lines are operated by an operator, the operator determines the transit fares $p_{b,1}^{(n+1)}$ and $p_{b,2}^{(n+1)}$ in period $n+1$ in three different cases.

First, when both transit lines are used at the end of period n (i.e., $x_{b,1}^{(n)}>0$ and $x_{b,2}^{(n)}>0$) or both transit lines are not used at the end of period n (i.e., $x_{b,1}^{(n)}=0$ and $x_{b,2}^{(n)}=0$), $p_{b,1}^{(n+1)}$ and $p_{b,2}^{(n+1)}$ are respectively formulated as

$$p_{b,1}^{(n+1)}=p_{b,1}^{(n)}+\theta\left(\frac{\left(p_{b,2}^{(n)}-k_{b,2}'\left(x_{b,2}^{(n)}\right)\right)t_a'\left(x_a^{(n)}\right)-\left(p_{b,1}^{(n)}-k_{b,1}'\left(x_{b,1}^{(n)}\right)\right)\left(g'\left(x_{b,2}^{(n)}\right)+t_a'\left(x_a^{(n)}\right)\right)}{g'\left(x_{b,1}^{(n)}\right)g'\left(x_{b,2}^{(n)}\right)+g'\left(x_{b,1}^{(n)}\right)t_a'\left(x_a^{(n)}\right)+g'\left(x_{b,2}^{(n)}\right)t_a'\left(x_a^{(n)}\right)}+x_{b,1}^{(n)}\right) \tag{43}$$

and

$$p_{b,2}^{(n+1)}=p_{b,2}^{(n)}+\theta\left(\frac{\left(p_{b,1}^{(n)}-k_{b,1}'\left(x_{b,1}^{(n)}\right)\right)t_a'\left(x_a^{(n)}\right)-\left(p_{b,2}^{(n)}-k_{b,2}'\left(x_{b,2}^{(n)}\right)\right)\left(g'\left(x_{b,1}^{(n)}\right)+t_a'\left(x_a^{(n)}\right)\right)}{g'\left(x_{b,1}^{(n)}\right)g'\left(x_{b,2}^{(n)}\right)+g'\left(x_{b,1}^{(n)}\right)t_a'\left(x_a^{(n)}\right)+g'\left(x_{b,2}^{(n)}\right)t_a'\left(x_a^{(n)}\right)}+x_{b,2}^{(n)}\right). \tag{44}$$

Second, when $x_{b,1}^{(n)}=0$ and $x_{b,2}^{(n)}>0$ (i.e., transit line 1 is not used and transit line 2 is used), $p_{b,1}^{(n+1)}$ and $p_{b,2}^{(n+1)}$ are respectively governed by

$$p_{b,1}^{(n+1)}=\widehat{p}_{b,1} \quad \text{and} \tag{45}$$

$$p_{b,2}^{(n+1)}=p_{b,2}^{(n)}+\theta\left(\frac{k_{b,2}'\left(x_{b,2}^{(n)}\right)-p_{b,2}^{(n)}}{g'\left(x_{b,2}^{(n)}\right)+t_a'\left(x_a^{(n)}\right)}+x_{b,2}^{(n)}\right). \tag{46}$$

Third, when $x_{b,1}^{(n)}>0$ and $x_{b,2}^{(n)}=0$ (i.e., transit line 1 is used and transit line 2 is not used), $p_{b,1}^{(n+1)}$ and $p_{b,2}^{(n+1)}$ are respectively given by

$$p_{b,1}^{(n+1)}=p_{b,1}^{(n)}+\theta\left(\frac{k_{b,1}'\left(x_{b,1}^{(n)}\right)-p_{b,1}^{(n)}}{g'\left(x_{b,1}^{(n)}\right)+t_a'\left(x_a^{(n)}\right)}+x_{b,1}^{(n)}\right) \quad \text{and} \tag{47}$$

$$p_{b,2}^{(n+1)}=\widehat{p}_{b,2}\,. \tag{48}$$

Here, θ (>0) is a sensitivity parameter and it determines the rate of change of the transit fares. Formulae (43) and (44) describe that the bus fares are updated in the positive gradient direction of the objective function in expression (14), where the gradients are given by equations (17) and (18). $\widehat{p}_{b1}$ ($\widehat{p}_{b2}$) is a relatively large value to make transit line 1 (transit line 2) is not used in the next period yet. In this way, the bus fares are updated in the positive gradient direction of the objective function in expression (20), as described by formulae (45) to (48). Formulae (43) to (48) indicate that the transit fares in each period are calculated according to known information in the previous period.

Given a fixed auto toll p_a, the period-to-period adjustment process of bus fares is summarized as follows.

Step 1. Set the initial bus fares $\left(p_{b,1}^{(1)}, p_{b,2}^{(1)}\right)$ in period 1 and compute the numbers of bus users and auto users $\mathbf{x}^{(1)}$ at the end of period 1 according to an adjustment rule of modal choice, e.g., that formulated by formulae (38) to (42). Set $n=1$.

Step 2. If $t_{b,1}+g\left(x_{b,1}^{(n)}\right)+p_{b,1}^{(n)}>t_a\left(x_a^{(n)}\right)+p_a$, then set $p_{b,1}^{(n)}=t_a\left(x_a^{(n)}\right)+p_a-t_{b,1}-g\left(x_{b,1}^{(n)}\right)$; if $t_{b,2}+g\left(x_{b,2}^{(n)}\right)+p_{b,2}^{(n)}>t_a\left(x_a^{(n)}\right)+p_a$, then set $p_{b,2}^{(n)}=t_a\left(x_a^{(n)}\right)+p_a-t_{b,2}-g\left(x_{b,2}^{(n)}\right)$.

Step 3. If $n<N$, then go to *Step 4*; otherwise, stop.

Step 4. Compute the bus fares $\left(p_{b,1}^{(n+1)}, p_{b,2}^{(n+1)}\right)$ in period $n+1$ according to formulae (43) and (44). If $x_{b,1}^{(n)}=0$ and $x_{b,2}^{(n)}>0$, then re-compute $\left(p_{b,1}^{(n+1)}, p_{b,2}^{(n+1)}\right)$ according to formulae (45) and (46); otherwise if $x_{b,1}^{(n)}>0$ and $x_{b,2}^{(n)}=0$, then re-compute $\left(p_{b,1}^{(n+1)}, p_{b,2}^{(n+1)}\right)$ according to formulae (47) and (48). Then, compute the numbers of bus users and auto users $\mathbf{x}^{(n+1)}$ at the end of period $n+1$ according to the adjustment rule of modal choice. Set $n=n+1$ and go to *Step 2*.

We call the above period-to-period adjustment process of bus fares BFPAP1. In the BFPAP1, N is the maximum number of periods. Given the bus fares $\left(p_{b,1}^{(n)}, p_{b,2}^{(n)}\right)$ at the beginning of period n, the numbers of bus users and auto users $\mathbf{x}^{(n)}$ at the end of period n are determined. $\left(p_{b,1}^{(n)}, p_{b,2}^{(n)}\right)$ and $\mathbf{x}^{(n)}$ satisfy DUE condition (11); however, they may not satisfy equality constraint (15). In the BFPAP1, Step 2 is used to guarantee that the variables $\left(p_{b,1}^{(n)}, p_{b,2}^{(n)}\right)$ and $\mathbf{x}^{(n)}$ are projected onto the feasible set of satisfying constraints (15) and (16) in each period. (i.e., this step is used to guarantee that they always adjust in the feasible set in the BFPAP1). Obviously, after the projection onto the feasible set, the value of the objective function in formula (14) remains unchanged.

Step 3 is used to check whether the stopping criterion is met (when the current period is the last one). In Step 4, formulae (43) and (44) are used to update the bus fares to make the value of the objective function increase. They may not take action for a point on the boundary of the feasible set of satisfying constraints (15) and (16). In the case that the point is on the boundary of the feasible set in a period, the bus fares need to be updated in a direction along the boundary of the feasible set that makes the value of the objective function increase. That is to say, when the point is on the boundary, the bus fares should be updated by formulae (45) to (48). As seen from expression (23), formulae (45) to (48) describe that the bus fares are updated in the positive gradient direction of the objective function in (20). In this way, under the precondition that the sensitivity parameter θ takes a reasonable value, the value of the objective function always increases as the number of periods increases, unless the trajectory of the system gets to a stationary state. The increasing degree is determined by the initial state (i.e., the bus fares in period 1). If the initial state is in the attraction domain of a stationary point, then the trajectory converges to the stationary point. The attraction domain of a stationary point defines the collection of all states that will evolve towards the stationary point over time (Bie and Lo, 2010).

In addition, it is worth mentioning that the BFPAP1 is designed based on Property 1, a precondition of which is Assumption 1. However, Assumption 1 is not necessary for the implementation of the BFPAP1 in a realistic multi-mode transportation system. In fact, before the BFPAP1 is applied to a multi-mode transportation system, which does not satisfy Assumption 1, a transit fare/subsidy vector $(\hat{p}_{b,1}, \hat{p}_{b,2})$ is implemented in the transportation system so that all modes are used at the equilibrium state under the implementation of the transit fare/subsidy vector $(\hat{p}_{b,1}, \hat{p}_{b,2})$. The travel costs of commuters using buses on transit lines are formulated as

$$c_{b,i}(x_{b,i}, p_{b,i}) = t_{b,i} + g(x_{b,i}) + \hat{p}_{b,i} + p_{b,i}\text{, for } i = 1,2.$$

In each period, the bus fare vector $\left(p_{b,1}^{(n)}, p_{b,2}^{(n)}\right)$ is computed based on the above travel cost formulations (the actual bus fares charged from bus users are $\hat{p}_{b,i} + p_{b,i}^{(n)}$ for $i = 1,2$). In this way, the multi-mode transportation system, which does not satisfy Assumption 1, is converted into one, which satisfies Assumption 1, and the BFPAP1 can be implemented in the multi-mode transportation system.

To make the profit of the transit operator at the stationary state more than a certain value, the government can adjust the auto toll from period to period according to the following process.

Step 1. Set an auto toll p_a.

Step 2. Under the implementation of the auto toll, an optimal bus fare vector $(p_{b,1}, p_{b,2})$ at the equilibrium state and the numbers $\mathbf{x}$ of bus users and auto users at the optimal solution are observed using the BFPAP1 mentioned in this section earlier.

Step 3. Compute the profit of the transit operator at the stationary state. If the profit is less than a target value, then set $p_a = p_a + \gamma$ and go to *Step* 2; otherwise, stop.

We call the above period-to-period adjustment process of auto toll ATPAP1. In Step 2 of the ATPAP1, the BFPAP1 is adopted to optimize the profit of the transit operator given an auto toll. This implies that the ATPAP1 is one overall algorithm that iteratively updates both auto toll and bus fares. In Step 3, γ is a small positive number. By Property 2, it is known that Step 3 takes effect in increasing the profit of the transit operator. As a result, the BFPAP1 is effective for maximizing the profit of the transit operator and for simultaneously making the profit of the transit operator more than a certain value.

3.3. Operational scheme for the transit Bertrand-Nash duopoly

For the duopoly market structure, in which each transit line is operated by an operator, each transit operator determines the bus fare on its transit line to maximize its profit. The operator of operating transit line 1 calculates the transit fare $p_{b,1}^{(n+1)}$ on transit line 1 in period $n+1$ as follows:

$$p_{b,1}^{(n+1)} = p_{b,1}^{(n)} + \rho_1 \left(\frac{\left(k'_{b,1}\left(x_{b,1}^{(n)}\right) - p_{b,1}^{(n)}\right)\left(g'\left(x_{b,2}^{(n)}\right) + t'_a\left(x_a^{(n)}\right)\right)}{g'\left(x_{b,1}^{(n)}\right)g'\left(x_{b,2}^{(n)}\right) + g'\left(x_{b,1}^{(n)}\right)t'_a\left(x_a^{(n)}\right) + g'\left(x_{b,2}^{(n)}\right)t'_3\left(x_a^{(n)}\right)} + x_{b,1}^{(n)} \right), \tag{49}$$

where ρ_1 (>0) is a sensitivity parameter and determines the rate of change of the transit fare. The operator of operating transit line 2 calculates the transit fare $p_{b,2}^{(n+1)}$ on transit line 2 in period $n+1$ by the following formula:

$$p_{b,2}^{(n+1)} = p_{b,2}^{(n)} + \rho_2 \left(\frac{\left(k'_{b,2}\left(x_{b,2}^{(n)}\right) - p_{b,2}^{(n)}\right)\left(g'\left(x_{b,1}^{(n)}\right) + t'_a\left(x_a^{(n)}\right)\right)}{g'\left(x_{b,1}^{(n)}\right)g'\left(x_{b,2}^{(n)}\right) + g'\left(x_{b,1}^{(n)}\right)t'_a\left(x_a^{(n)}\right) + g'\left(x_{b,2}^{(n)}\right)t'_a\left(x_a^{(n)}\right)} + x_{b,2}^{(n)} \right), \tag{50}$$

where ρ_2 (>0) is a sensitivity parameter and determines the rate of change of the transit fare. Formulae (49) and (50) show that each operator calculates the transit fare on its transit line in each period according to known information in the previous period. Moreover, the bus fare on each transit line is updated in the gradient direction of the objective function in expression (29).

Given a fixed auto toll p_a, the period-to-period adjustment process of bus fares is summarized as follows.

Step 1. Set the initial bus fares $p_{b,1}^{(1)}$ and $p_{b,2}^{(1)}$ in period 1 and compute the numbers of bus users and auto users $\mathbf{x}^{(1)}$ at the end of period 1 according to an adjustment rule of modal choice. Set $n=1$.

Step 2. If $t_{b,1} + g\left(x_{b,1}^{(n)}\right) + p_{b,1}^{(n)} > t_a\left(x_a^{(n)}\right) + p_a$, then set $p_{b,1}^{(n)} = t_a\left(x_a^{(n)}\right) + p_a - t_{b,1} - g\left(x_{b,1}^{(n)}\right)$; if $t_{b,2} + g\left(x_{b,2}^{(n)}\right) + p_{b,2}^{(n)} > t_a\left(x_a^{(n)}\right) + p_a$, then set $p_{b,2}^{(n)} = t_a\left(x_a^{(n)}\right) + p_a - t_{b,2} - g\left(x_{b,2}^{(n)}\right)$.

Step 3. If $n < N$, then go to *Step 4*; otherwise, stop.

Step 4. Compute the bus fares $p_{b,1}^{(n+1)}$ and $p_{b,2}^{(n+1)}$ in period $n+1$ according to formulae (49) and (50). Then, compute the numbers of bus users and auto users $\mathbf{x}^{(n+1)}$ at the end of period $n+1$ according to the adjustment rule of modal choice. Set $n=n+1$ and go to *Step 2*.

We call the above period-to-period adjustment process of bus fares BFPAP2. The first three steps in the BFPAP2 take a similar effect as those in the BFPAP1. It can be seen from expressions (32) and (36) that formulae (49) and (50) describe that each operator updates its bus fare in an ascent direction (or the positive gradient direction) of its profit function in each period. Thus, Step 4 makes the profit of each operator increase until a stationary state is reached (under the precondition that the sensitivity parameters ρ_1 and ρ_2 take reasonable values).

To make the profit of each transit operator at the stationary state more than a certain value, the government can adjust the auto toll from period to period according to the following process.

Step 1. Set an auto toll p_a.

Step 2. Under the implementation of the auto toll, optimal bus fares $p_{b,1}$ and $p_{b,2}$ at the equilibrium state and the numbers $\mathbf{x}$ of bus users and auto users at the optimal solution are observed using the BFPAP2 mentioned in this section earlier.

Step 3. Compute the profits of both transit operators at the stationary state. If the profit of at least a transit operator is less than a target value, then set $p_a = p_a + \gamma$ and go to *Step 2*; otherwise, stop.

We call the above period-to-period adjustment process of auto toll ATPAP2. The ATPAP2 is one overall algorithm that iteratively updates both auto toll and bus fares and it can take effect in maximizing the profit of each transit operator and in simultaneously making the profit of each transit operator more than a certain value.

4. Numerical examples

In this section, we give a set of numerical examples to show the properties of the two implementation processes. The total number of commuters is $d = 9000$. The average travel time costs of bus users on transit lines 1 and 2 are $t_{b,1} = 1.1$ and $t_{b,2} = 1.5$, respectively. The average travel time cost of auto users on the highway is formulated as

$$t_a(x_a) = 0.9 \times \left(\frac{x_a}{4300}\right)^4 + 1.0. \tag{51}$$

The average in-vehicle congestion costs of passengers on transit lines 1 and 2 are expressed as

$$g(x_{b,i}) = 1.1 \times \left(\frac{x_{b,i}}{4600}\right)^3, \quad i = 1, 2. \tag{52}$$

The operating costs of transit lines 1 and 2 are governed by

$$k_{b,1}(x_{b,1}) = 400 + 10\sqrt{x_{b,1}+1} \quad \text{and} \quad k_{b,2}(x_{b,2}) = 380 + 8\sqrt{x_{b,2}+1}. \tag{53}$$

The day-to-day adjustment of modal choice is formulated by formulae (38) to (42) and the adjustment parameter δ in formulae (38) and (39) takes 0.01.

4.1. The transit monopoly case

We first investigate the operational scheme for the transit monopoly in Section 3.2. The sensitivity parameter $\theta = 0.0004$ in formulae (43), (44), (46), and (47). The parameters $\widehat{p}_{b,1} = 20$ and $\widehat{p}_{b,2} = 20$ in formulae (45) and (48), respectively. When the transit fares and auto toll $(p_{b,1}, p_{b,2}, p_a) = (0,0,0)$, the numbers of bus users and auto users at the DUE state

$$(x_{b,1}^{\mathrm{UE}}, x_{b,2}^{\mathrm{UE}}, x_a^{\mathrm{UE}}) = (3439.28, 1742.10, 3818.62).$$

Therefore, Assumption 1 holds for the case of $p_a = 0$.

Figure 2 depicts the evolutionary trajectories of the transit fares $(p_{b,1}, p_{b,2})$ and the daily total profit U from period 1 to 50 when the initial transit fares $\left(p_{b,1}^{(1)}, p_{b,2}^{(1)}\right) = (0, 0.80)$ and $(1.40, 0)$ in period 1 and the auto toll $p_a = 0$ in all periods. Table 1 shows the transit fares $\left(p_{b,1}^{(1)}, p_{b,2}^{(1)}\right)$ and $\left(p_{b,1}^{(50)}, p_{b,2}^{(50)}\right)$ in periods 1 and 50, the numbers of bus users and auto users $\mathbf{x}^{(50)}$ at the end of period 50, and the daily total profit $U^{(50)}$ of the transit operator at the end of period 50 in the case of $p_a = 0$.

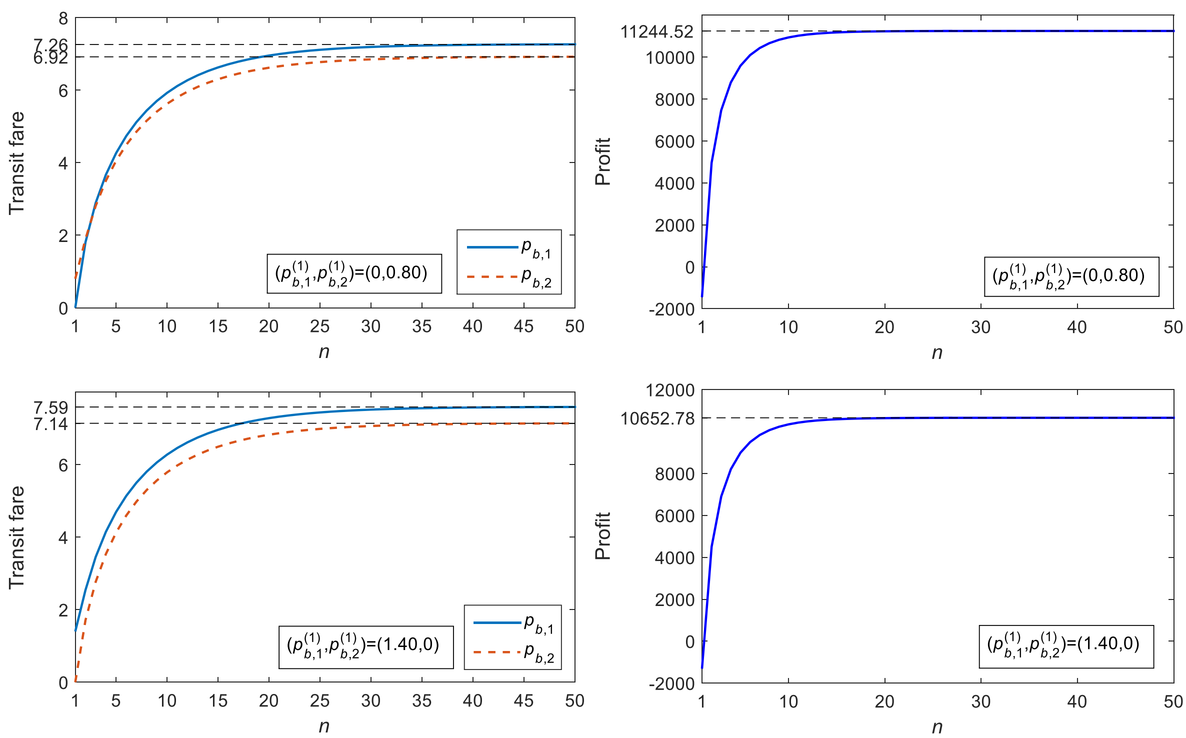


Figure 2. The evolutionary trajectories of the transit fares $(p_{b,1}, p_{b,2})$ and the daily total profit U from period 1 to 50 when the operational scheme for the transit monopoly in Section 3.2 is applied.

Table 1. The auto toll p_a in all periods, the transit fares $\left(p_{b,1}^{(1)}, p_{b,2}^{(1)}\right)$ and $\left(p_{b,1}^{(50)}, p_{b,2}^{(50)}\right)$ in periods 1 and 50, the numbers $\left(x_{b,1}^{(50)}, x_{b,2}^{(50)}, x_a^{(50)}\right)$ of bus users and auto users at the end of period 50, and the daily total profit $U^{(50)}$ of the transit operator at the end of period 50 when the operational scheme for the transit monopoly in Section 3.2 is applied.

p_a	$\left(p_{b,1}^{(1)}, p_{b,2}^{(1)}\right)$	$\left(p_{b,1}^{(50)}, p_{b,2}^{(50)}\right)$	$\left(x_{b,1}^{(50)}, x_{b,2}^{(50)}, x_a^{(50)}\right)$	$U^{(50)}$
0	(0,0.80)	(7.26, 6.92)	(1713.29,0.00,7286.71)	11244.52
0	(1.40,0)	(7.59, 7.14)	(0.00,1648.19,7351.81)	10652.78

One can see that the trajectories of the transit fares $(p_{b,1}, p_{b,2})$ get to a stationary state as the number of periods increases from 1 to 50. Under the implementation of the operational scheme, the daily total profit of the operator ascends and gets to a stationary value as the number of periods increases. Therefore, the operational scheme is effective for locally or globally maximizing the profit of the operator.

The initial transit fares significantly affect the transit fares and user distribution at the stationary state. The trajectories of user distribution adjust to two stationary points located at the boundary of the feasible set Ω of the numbers of bus users and auto users. The optimal solutions occur at the boundary of the set Ω and only transit line 1 is used at the optimal solution. This result indicates that it is essential to examine optimal solutions at the boundary of the feasible set of user distribution. When both transit lines are operated by a monopolistic operator, the number of travelers using one of the two transit lines is zero at the optimal solution, and hence the operator can choose to operate one of the two transit lines to maximize its profit.

Figure 3 shows the changing trend of the maximum (optimal) total profit U of the transit operator as the auto toll p_a varies from 0 to 1 with an interval of 0.2. Assumption 1 holds for all these p_a-values. It can be seen that, when the auto toll p_a increases, the maximum total profit U of the transit operator also increases. Therefore, to guarantee that the profit of the transit operator is more than a certain value, the government can increase the toll charge to private car users.

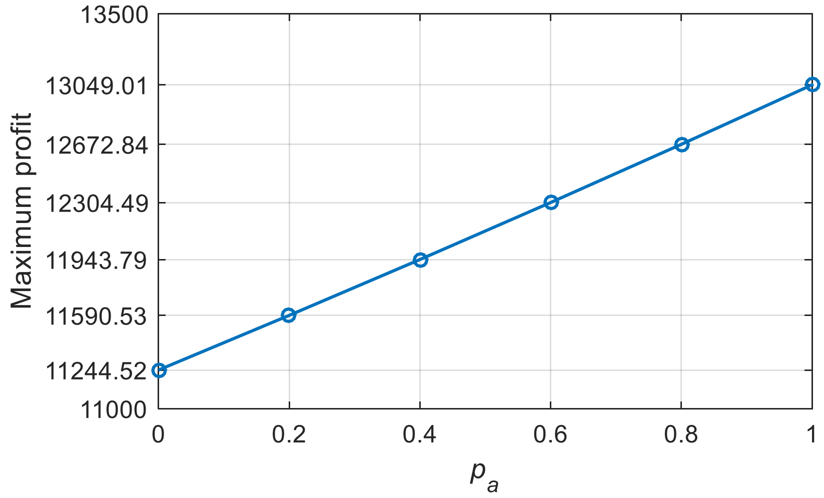
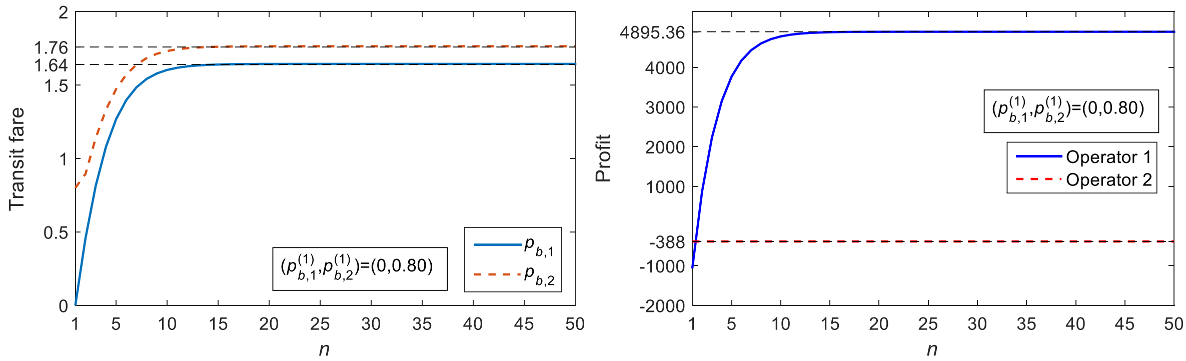


Figure 3. The trend of the maximum (optimal) total profit U of the transit operator as the auto toll p_a varies from 0 to 1 with an interval of 0.2.

4.2. The transit Bertrand-Nash duopoly case

We then examine the operational scheme for the transit Bertrand-Nash duopoly in Section 3.3. The sensitivity parameters $\rho_1 = 0.0001$ and $\rho_2 = 0.0001$ in formulae (49) and (50), respectively. Figure 4 describes the evolutionary trajectories of the transit fares $p_{b,1}$ and $p_{b,2}$ and the daily total profits V_1 and V_2 from period 1 to 50 when the initial transit fares $\left(p_{b,1}^{(1)}, p_{b,2}^{(1)}\right) = (0, 0.80)$, $(0, 0.40)$, $(0, 0)$, $(0.70, 0)$, and $(1.40, 0)$ in period 1 and the auto toll $p_a = 0$ in all periods. Table 2 records the transit fares $\left(p_{b,1}^{(1)}, p_{b,2}^{(1)}\right)$ and $\left(p_{b,1}^{(50)}, p_{b,2}^{(50)}\right)$ in periods 1 and 50, the numbers of bus users and auto users $\mathbf{x}^{(50)}$ at the end of period 50, and the daily total profits $V_1^{(50)}$ and $V_2^{(50)}$ of the two transit operators at the end of period 50 in the case of $p_a = 0$.



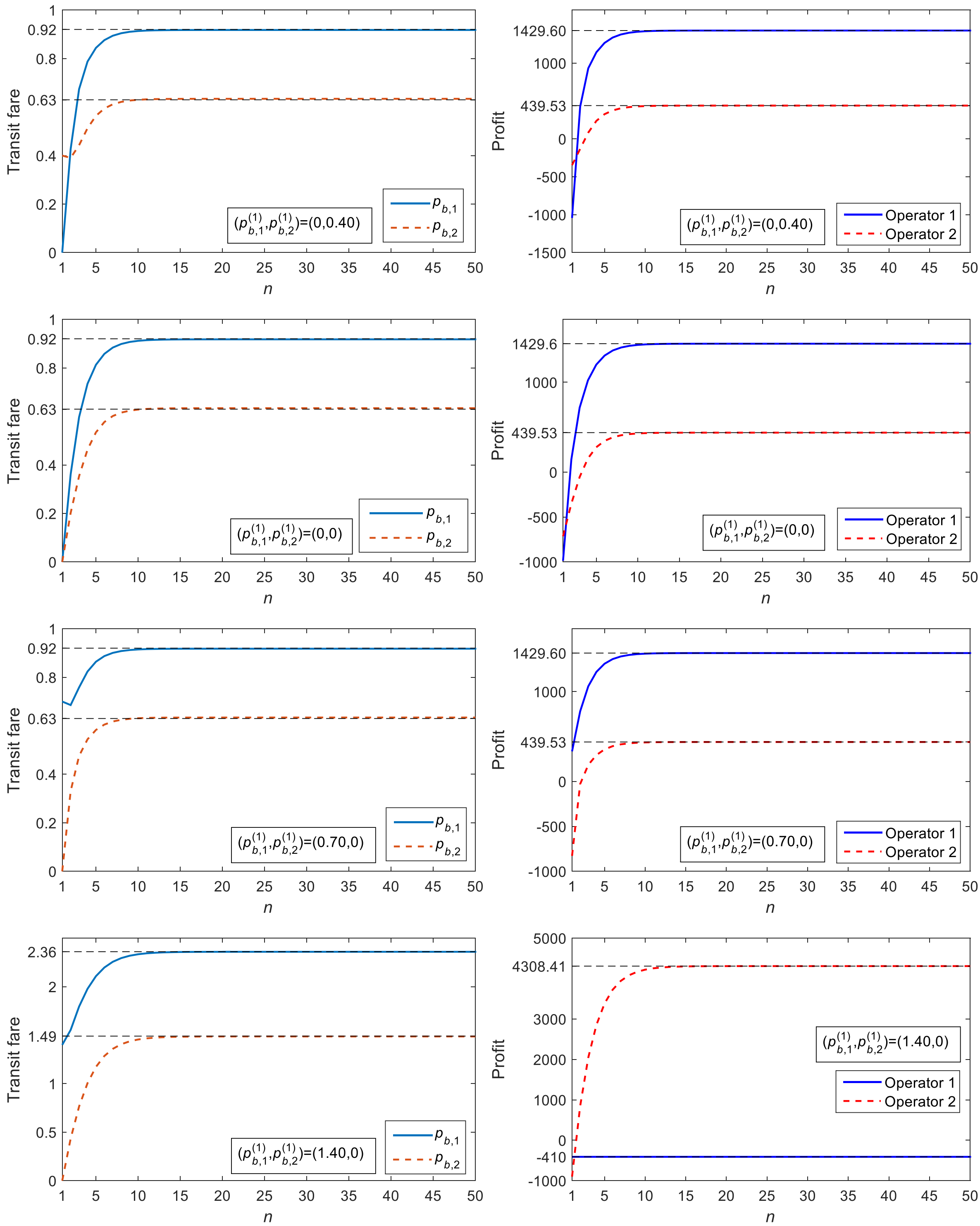


Figure 4. The evolutionary trajectories of the transit fares $p_{b,1}$ and $p_{b,2}$ and the daily total profits V_1 and V_2 from period 1 to 50 when the operational scheme for the transit Bertrand-Nash duopoly in Section 3.3 is applied.

Table 2. The auto toll p_a in all periods, the transit fares $\left(p_{b,1}^{(1)}, p_{b,2}^{(1)}\right)$ and $\left(p_{b,1}^{(50)}, p_{b,2}^{(50)}\right)$ in periods 1 and 50, the numbers $\left(x_{b,1}^{(50)}, x_{b,2}^{(50)}, x_a^{(50)}\right)$ of bus users and auto users at the end of

period 50, and the daily total profits $V_1^{(50)}$ and $V_2^{(50)}$ of the two transit operators at the end of period 50 when the operational scheme for the transit Bertrand-Nash duopoly in Section 3.3 is applied.

p_a	$\left(p_{b,1}^{(1)}, p_{b,2}^{(1)}\right)$	$\left(p_{b,1}^{(50)}, p_{b,2}^{(50)}\right)$	$\left(x_{b,1}^{(50)}, x_{b,2}^{(50)}, x_a^{(50)}\right)$	$V_1^{(50)}$	$V_2^{(50)}$
0	(0,0.80)	(1.64,1.76)	(3584.15,0.00,5415.85)	4895.36	–388
0	(0,0.40)	(0.92,0.63)	(2543.08,1832.76,4624.16)	1429.60	439.53
0	(0,0)	(0.92,0.63)	(2543.08,1832.76,4.62416)	1429.60	439.53
0	(0.70,0)	(0.92,0.63)	(2543.08,1.832.76,4624.16)	1429.60	439.53
0	(1.40,0)	(2.36,1.49)	(0.00,3471.55,5528.45)	–410	4308.41

Several phenomena can be seen in Figure 4 and Table 2. First, the trajectories from different initial points evolve to three different stationary points, two of which are located at the boundary of the feasible set of satisfying conditions (30) and (31) and one of which is located in the interior of the feasible set.

Second, initial bus fares on the two transit lines significantly affect the profits of the two operators. A high initial bus fare on a transit line (at the same time, the initial bus fare on the other transit line is zero) can make all commuters give up choosing to use that transit line at the stationary state. As a result, the operator of that transit line becomes loss-making and the operator of the other transit line becomes profitable at the stationary state. This means that, in the transit Bertrand-Nash duopoly, the profit of a transit operator is affected by not only its decision-making but also the decision-making of the other transit operator. In the dynamic process, when a transit operator sets the initial bus fare on its transit line as the optimal one at the stationary state with the consideration of the Bertrand-Nash duopoly, the profit of the operator may not be optimal. For example, when transit operator 1 sets the initial bus fare $p_{b,1}^{(1)}$ on its transit line as the optimal value 1.64 (i.e., $p_{b,1}^{(50)}$), operator 1 will become loss-making if transit operator 2 sets the initial bus fare $p_{b,2}^{(1)}$ as zero. Thus, a static optimal transit fare cannot be directly applied to a multi-modal system in a non-stationary state.

Third, although the profit of a transit operator is maximum at a stationary point located at the boundary of the feasible set, the stationary point is not realizable. In fact, the profit of a transit operator is also affected by the decision-making of the other transit operator, and hence each operator is not willing to set a high initial transit fare on its transit line to obtain an optimal profit. As a result, both operators set a small initial transit fare and the trajectories of the system evolve to a stationary point in the interior of the feasible set. That is to say, the stationary point in the interior of the feasible set is a Nash equilibrium point. At that equilibrium point, the profits of both operators are not globally optimal. This also results in a low transit fare for commuters and commuters pay a lower transit fare in a Bertrand-Nash duopoly market than in a monopoly market. At the Nash equilibrium point, more commuters choose to use a transit line with lower travel time cost and the profit of the operator of a transit line with lower travel time is higher compared with the other operator.

5. Conclusions

In this paper, we concern about a multi-modal transport system comprising two separate public transit lines and a highway. The highway is only used by autos and the two transit lines are only used by buses. In the system, the two public transit lines are owned by one or more private firms and the highway is managed by the government. Two market structures are involved, i.e., one with a monopolistic public transport operator, which operates both transit lines, and the other one, in which separate operators own one transit line. For each market structure, each operator can only set the bus fare on its transit line to maximize its profit at the equilibrium state. This problem can be formulated as an optimization model with general equilibrium constraints. We theoretically prove that, to obtain the optimal solutions of the model not only in the interior of the feasible set of user distributions but also at the boundary of the feasible set, it is sufficient to solve an optimization model with equality constraints.

In the transport system, when the toll charge of private autos increases and the fares for bus users remain unchanged, the number of bus users increases. However, this does not mean that the profit of each transit operator also increases, because the marginal revenue may be less than the marginal operating cost even if the number of bus users increases. On the other hand, the optimal bus fare for optimizing the profit of each transit operator is adjusted as the auto toll increases. Thus, it is unclear how the optimal profit of each transit operator changes as the auto toll increases. In this paper, we answer the question and prove that increasing the auto toll makes the optimal profit of each transit operator increase.

Based on the above models and properties, for each of the two market structures, we propose a period-to-period bus fare and auto toll scheme to locally or globally maximize the profit of each operator and to simultaneously make the profit of each transit operator not less than a certain value (otherwise, the transit operator may leave the market due to unattractive profits or losses).

Finally, by numerical examples, we examine the two proposed bus fare and auto toll schemes and obtain the following conclusions. For the transit monopoly, (*i*) the operational scheme is effective for locally or globally maximizing the profit of the operator; (*ii*) the operator can choose to operate one of the two transit lines to maximize its profit, and hence it is essential to examine the optimal solution at the boundary of the feasible set of user distribution; (*iii*) to make the profit of the transit operator more than a certain value, the government can improve the toll charge to private car users. For the transit Bertrand-Nash duopoly, (*i*) under the implementation of the operational scheme, the trajectory of user distributions can evolve to either a stationary point located at the boundary of the feasible set of user distributions or a stationary point located in the interior of the feasible set; (*ii*) when both transit operators are willing to set a small initial transit fare, the trajectory of user distributions evolves to a stationary point in the interior of the feasible set, and the profits of both transit operators are not globally optimal at the interior stationary point; (*iii*) a static optimal transit fare scheme cannot be directly applied to a multi-modal system in a non-stationary state.

The analyses and results presented in this paper are based on the assumption of fixed demand, i.e., each commuter must travel every morning. The assumption is involved in the proof of Property 1 (or Property 3) to compare the operator profits at two different equilibrium points and it is not involved in the proof of Property 2 (or Property 4). This means that Property 1 (or Property 3) and its proof may not be directly applied to the case that the assumption of fixed demand is replaced with another one, e.g., the travel demand is elastic, i.e., commuters can choose either to travel or not every morning; but Property 2 (or Property 4) and its proof may be valid for the case of elastic demand.

The period-to-period transit fare and auto toll schemes in Section 3 are developed based on the assumption that the system gets to the equilibrium state at the end of a period given the bus fares and auto toll in the period. However, in reality, traffic flows may not be in equilibrium at any arbitrary time. By relaxing the assumption that day-to-day traffic flows must reach an equilibrium state at the end of each period, Ye et al. (2015) extended the trial-and-error method of Yang et al. (2004) and proposed a period-to-period toll charge scheme to achieve system optimum target. Similar to the idea of Ye et al. (2015), it is possible to relax the assumption of the period-to-period transit fare and auto toll schemes in future work.

Acknowledgments

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Appendix A. Property proofs and formula derivations

A.1. Proof of Property 1

Proof. Equilibrium condition (11) can be further deduced into seven different sub-conditions (A.1) to (A.7):

$$t_{b,i} + g(x_{b,i}) + p_{b,i} = t_a(x_a) + p_a \text{, for } i = 1,2 \text{,} \tag{A.1}$$

$$x_{b,1} = 0 \text{ and } t_{b,1} + g(x_{b,1}) + p_{b,1} > t_{b,2} + g(x_{b,2}) + p_{b,2} = t_a(x_a) + p_a \text{,} \tag{A.2}$$

$$x_{b,2} = 0 \text{ and } t_{b,2} + g(x_{b,2}) + p_{b,2} > t_{b,1} + g(x_{b,1}) + p_{b,1} = t_a(x_a) + p_a \text{,} \tag{A.3}$$

$$x_a = 0 \text{ and } t_a(x_a) + p_a > t_{b,1} + g(x_{b,1}) + p_{b,1} = t_{b,2} + g(x_{b,2}) + p_{b,2} \text{,} \tag{A.4}$$

$$x_{b,1} = 0 \text{, } x_{b,2} = 0 \text{, } t_{b,1} + g(x_{b,1}) + p_{b,1} > t_a(x_a) + p_a \text{,}$$
$$\text{and } t_{b,2} + g(x_{b,2}) + p_{b,2} > t_a(x_a) + p_a \text{,} \tag{A.5}$$

$$x_{b,1} = 0 \text{, } x_a = 0 \text{, } t_{b,1} + g(x_{b,1}) + p_{b,1} > t_{b,2} + g(x_{b,2}) + p_{b,2} \text{,}$$
$$\text{and } t_a(x_a) + p_a > t_{b,2} + g(x_{b,2}) + p_{b,2} \text{, and} \tag{A.6}$$

$$x_{b,2} = 0 \text{, } x_a = 0 \text{, } t_{b,2} + g(x_{b,2}) + p_{b,2} > t_{b,1} + g(x_{b,1}) + p_{b,1} \text{,}$$
$$\text{and } t_a(x_a) + p_a > t_{b,1} + g(x_{b,1}) + p_{b,1} \text{.} \tag{A.7}$$

Therefore, the feasible set of decision variables of the optimization problem (10) to (12) can be subdivided into seven subsets respectively satisfying the seven different sub-conditions.

Let $\left(p_{b,1}^j, p_{b,2}^j\right)$ ($j = 1, 2, \cdots, 7$) be an optimal solution to the optimization problem

$$\max_{(p_{b,1}, p_{b,2})} U = x_{b,1} p_{b,1} + x_{b,2} p_{b,2} - k_{b,1}(x_{b,1}) - k_{b,2}(x_{b,2}) \text{,} \tag{A.8}$$

where $(p_{b,1}, p_{b,2})$ is subject to sub-condition (A.*j*) and constraint (12). Let $\mathbf{x}^j = (x_{b,1}^j, x_{b,2}^j, x_a^j)^{\mathrm{T}}$ be the unique user distribution pattern at the DUE state determined by $\left(p_{b,1}^j, p_{b,2}^j\right)$ ($j = 1, 2, \cdots, 7$). Therefore, to prove Property 1, it is sufficient to prove

$$U\left(p_{b,1}^1, p_{b,2}^1\right) \geq U\left(p_{b,1}^j, p_{b,2}^j\right) \text{, for } j = 2, 3, \cdots, 7 \text{.} \tag{A.9}$$

First, we prove

$$U\left(p_{b,1}^1, p_{b,2}^1\right) \geq U\left(p_{b,1}^2, p_{b,2}^2\right) \text{.} \tag{A.10}$$

It is supposed that $x_a^{\mathrm{UE}} \geq x_a^2$ holds ($\mathbf{x}^{\mathrm{UE}}$ is the user distribution pattern at the DUE state without transit fares). Associated with the fact $x_{b,1}^{\mathrm{UE}} > 0 = x_{b,1}^2$, it is generated that

$$x_{b,2}^{\mathrm{UE}} = d - x_{b,1}^{\mathrm{UE}} - x_a^{\mathrm{UE}} < d - x_{b,1}^2 - x_a^2 = x_{b,2}^2 . \tag{A.11}$$

Thus, we have

$$t_{b,2} + g\left(x_{b,2}^{\mathrm{UE}}\right) < t_{b,2} + g\left(x_{b,2}^2\right) \leq t_{b,2} + g\left(x_{b,2}^2\right) + p_{b,2}^2 = t_a\left(x_a^2\right) + p_a \leq t_a\left(x_a^{\mathrm{UE}}\right) + p_a . \tag{A.12}$$

The first inequality in expression (A.12) follows the precondition that the function g is increasing. The equality results from the condition that $\left(p_{b,1}^2, p_{b,2}^2\right)$ and $\mathbf{x}^2$ satisfy sub-condition (A.2). The third (or last) inequality is obtained by the assumption that the function t_a is increasing. Expression (A.12) contradicts condition (13). Therefore, $x_a^{\mathrm{UE}} < x_a^2$ holds. It immediately follows that

$$t_{b,1} + g\left(x_{b,1}^2\right) < t_{b,1} + g\left(x_{b,1}^{\mathrm{UE}}\right) = t_a\left(x_a^{\mathrm{UE}}\right) + p_a < t_a\left(x_a^2\right) + p_a . \tag{A.13}$$

In addition, sub-condition (A.2) shows that

$$t_{b,1} + g\left(x_{b,1}^2\right) + p_{b,1}^2 > t_a\left(x_a^2\right) + p_a . \tag{A.14}$$

Therefore, there is a $\overline{p}_{b,1}^2 = t_a\left(x_a^2\right) + p_a - t_{b,1} - g\left(x_{b,1}^2\right) \in \left(0, p_{b,1}^2\right)$ so that

$$t_{b,1} + g\left(x_{b,1}^2\right) + \overline{p}_{b,1}^2 = t_{b,2} + g\left(x_{b,2}^2\right) + p_{b,2}^2 = t_a\left(x_a^2\right) + p_a , \tag{A.15}$$

i.e., $\left(\overline{p}_{b,1}^2, p_{b,2}^2\right)$ and $\mathbf{x}^2$ satisfy sub-condition (A.1). Thus, it is obtained that

$$U\left(p_{b,1}^1, p_{b,2}^1\right) \geq U\left(\overline{p}_{b,1}^2, p_{b,2}^2\right) = U\left(p_{b,1}^2, p_{b,2}^2\right). \tag{A.16}$$

The equality in equation (A.16) follows from the fact $x_{b,1}^2 = 0$. Thus, inequality (A.10) is generated.

Second, similar to the proof of inequality (A.10), it can be generated that

$$U\left(p_{b,1}^1, p_{b,2}^1\right) \geq U\left(p_{b,1}^3, p_{b,2}^3\right). \tag{A.17}$$

Third, we prove

$$U\left(p_{b,1}^1, p_{b,2}^1\right) > U\left(p_{b,1}^4, p_{b,2}^4\right). \tag{A.18}$$

Let

$$\overline{p}_{b,1}^4 = t_a\left(x_a^4\right) + p_a - t_{b,1} - g\left(x_{b,1}^4\right) \text{ and } \overline{p}_{b,2}^4 = t_a\left(x_a^4\right) + p_a - t_{b,2} - g\left(x_{b,2}^4\right). \tag{A.19}$$

Then, $\left(\overline{p}_{b,1}^4, \overline{p}_{b,2}^4\right)$ and $\mathbf{x}^4$ satisfy sub-condition (A.1), and hence

$$U\left(p_{b,1}^1, p_{b,2}^1\right) \geq U\left(\overline{p}_{b,1}^4, \overline{p}_{b,2}^4\right). \tag{A.20}$$

By the definition of U, we have

$$U\left(\bar{p}_{b,1}^{4},\bar{p}_{b,2}^{4}\right)=x_{b,1}^{4}\left(t_a\left(x_a^4\right)+p_a-t_{b,1}-g\left(x_{b,1}^{4}\right)\right)$$
$$+x_{b,2}^{4}\left(t_a\left(x_a^4\right)+p_a-t_{b,2}-g\left(x_{b,2}^{4}\right)\right)-k_{b,1}\left(x_{b,1}^{4}\right)-k_{b,2}\left(x_{b,2}^{4}\right)$$
$$>x_{b,1}^{4}p_{b,1}^{4}+x_{b,2}^{4}p_{b,2}^{4}-k_{b,1}\left(x_{b,1}^{4}\right)-k_{b,2}\left(x_{b,2}^{4}\right)=U\left(p_{b,1}^{4},p_{b,2}^{4}\right). \tag{A.21}$$

The inequality is generated by the precondition $x_{b,1}^{4}>0$ (or $x_{b,2}^{4}>0$) and sub-condition (A.4). Combining (A.20) and (A.21) leads to inequality (A.18).

Fourth, we prove

$$U\left(p_{b,1}^{1},p_{b,2}^{1}\right)\geq U\left(p_{b,1}^{5},p_{b,2}^{5}\right). \tag{A.22}$$

On one hand, $x_{b,1}^{5}=0<x_{b,1}^{\mathrm{UE}}$, $x_{b,2}^{5}=0<x_{b,2}^{\mathrm{UE}}$, and $x_a^5=d>x_a^{\mathrm{UE}}$ hold and also the functions g and t_a are increasing. Thus, associated with condition (13), it is obtained that

$$t_{b,1}+g\left(x_{b,1}^{5}\right)<t_{b,1}+g\left(x_{b,1}^{\mathrm{UE}}\right)=t_a\left(x_a^{\mathrm{UE}}\right)+p_a<t_a\left(x_a^5\right)+p_a \quad\text{and} \tag{A.23}$$

$$t_{b,2}+g\left(x_{b,2}^{5}\right)<t_{b,2}+g\left(x_{b,2}^{\mathrm{UE}}\right)=t_a\left(x_a^{\mathrm{UE}}\right)+p_a<t_a\left(x_a^5\right)+p_a. \tag{A.24}$$

On the other hand, it holds that

$$t_{b,1}+g\left(x_{b,1}^{5}\right)+p_{b,1}^{5}>t_a\left(x_a^5\right)+p_a \quad\text{and}\quad t_{b,2}+g\left(x_{b,2}^{5}\right)+p_{b,2}^{5}>t_a\left(x_a^5\right)+p_a. \tag{A.25}$$

Therefore, there are

$$\bar{p}_{b,1}^{5}=t_a\left(x_a^5\right)+p_a-t_{b,1}-g\left(x_{b,1}^{5}\right)\in\left(0,p_{b,1}^{5}\right) \quad\text{and}\quad \bar{p}_{\mathrm{b},2}^{5}=t_a\left(x_a^5\right)+p_a-t_{b,2}-g\left(x_{b,2}^{5}\right)\in\left(0,p_{b,2}^{5}\right)$$

so that

$$t_{b,1}+g\left(x_{b,1}^{5}\right)+\bar{p}_{b,1}^{5}=t_{b,2}+g\left(x_{b,2}^{5}\right)+\bar{p}_{b,2}^{5}=t_a\left(x_a^5\right)+p_a, \tag{A.26}$$

i.e., $\mathbf{x}^5$ and $\left(\bar{p}_{b,1}^{5},\bar{p}_{b,2}^{5}\right)$ satisfy sub-condition (A.1). Thus, it is obtained that

$$U\left(p_{b,1}^{1},p_{b,2}^{1}\right)\geq U\left(\bar{p}_{b,1}^{5},\bar{p}_{b,2}^{5}\right)=U\left(p_{b,1}^{5},p_{b,2}^{5}\right). \tag{A.27}$$

The equality in equation (A.27) follows from the conditions $x_{b,1}^{5}=0$ and $x_{b,2}^{5}=0$. Thus, inequality (A.22) holds.

Fifth, we prove

$$U\left(p_{b,1}^{1},p_{b,2}^{1}\right)>U\left(p_{b,1}^{6},p_{b,2}^{6}\right). \tag{A.28}$$

The proof needs to be illustrated in three cases. When

$$t_{b,1}+g\left(x_{b,1}^{6}\right)+p_{b,1}^{6}>t_a\left(x_a^6\right)+p_a,$$

we take $\bar{p}_{b,2}^{6}=t_a\left(x_a^6\right)+p_a-t_{b,2}-g\left(x_{b,2}^{6}\right)$ and then $\left(p_{b,1}^{6},\bar{p}_{b,2}^{6}\right)$ and $\mathbf{x}^6$ satisfy sub-condition (A.2). As a result, we have

$$U\left(p_{b,1}^{2}, p_{b,2}^{2}\right) \geq U\left(p_{b,1}^{6}, \bar{p}_{b,2}^{6}\right)$$
$$= x_{b,1}^{6} p_{b,1}^{6} + x_{b,2}^{6}\left(t_a\left(x_a^{6}\right) + p_a - t_{b,2} - g\left(x_{b,2}^{6}\right)\right) - k_{b,1}\left(x_{b,1}^{6}\right) - k_{b,2}\left(x_{b,2}^{6}\right)$$
$$> x_{b,1}^{6} p_{b,1}^{6} + x_{b,2}^{6} p_{b,2}^{6} - k_{b,1}\left(x_{b,1}^{6}\right) - k_{b,2}\left(x_{b,2}^{6}\right) = U\left(p_{b,1}^{6}, p_{b,2}^{6}\right). \quad \text{(A.29)}$$

Combining inequality (A.29) with inequality (A.10), inequality (A.28) is obtained. By a similar deduction, when

$$t_{b,1} + g\left(x_{b,1}^{6}\right) + p_{b,1}^{6} = t_a\left(x_a^{6}\right) + p_a \quad \text{and} \quad t_{b,1} + g\left(x_{b,1}^{6}\right) + p_{b,1}^{6} < t_a\left(x_a^{6}\right) + p_a,$$

inequality (A.28) also holds.

Sixth, similar to the proof of inequality (A.28), it can be obtained that

$$U\left(p_{b,1}^{1}, p_{b,2}^{1}\right) > U\left(p_{b,1}^{7}, p_{b,2}^{7}\right). \quad \text{(A.30)}$$

Thus, by inequalities (A.10), (A.17), (A.18), (A.22), (A.28), and (A.30), we can conclude that $\left(p_{b,1}^{1}, p_{b,2}^{1}\right)$ is an optimal solution to the optimization problem (10) to (12). ■

A.2. Derivation of partial derivatives (17) to (19)

Proof. Differentiating both sides of equation (15) and the user conservation condition with respect to $\mathbf{p}$ generates

$$g'(x_{b,1})\frac{\partial x_{b,1}}{\partial p_{b,1}} + 1 = t_a'(x_a)\frac{\partial x_a}{\partial p_{b,1}}, \quad g'(x_{b,2})\frac{\partial x_{b,2}}{\partial p_{b,1}} = t_a'(x_a)\frac{\partial x_a}{\partial p_{b,1}}, \quad \frac{\partial x_{b,1}}{\partial p_{b,1}} + \frac{\partial x_{b,2}}{\partial p_{b,1}} + \frac{\partial x_a}{\partial p_{b,1}} = 0, \quad \text{(A.31)}$$

$$g'(x_{b,1})\frac{\partial x_{b,1}}{\partial p_{b,2}} = t_a'(x_a)\frac{\partial x_a}{\partial p_{b,2}}, \quad g'(x_{b,2})\frac{\partial x_{b,2}}{\partial p_{b,2}} + 1 = t_a'(x_a)\frac{\partial x_a}{\partial p_{b,2}}, \quad \frac{\partial x_{b,1}}{\partial p_{b,2}} + \frac{\partial x_{b,2}}{\partial p_{b,2}} + \frac{\partial x_a}{\partial p_{b,2}} = 0, \quad \text{(A.32)}$$

$$g'(x_{b,1})\frac{\partial x_{b,1}}{\partial p_a} = t_a'(x_a)\frac{\partial x_a}{\partial p_a} + 1, \quad g'(x_{b,2})\frac{\partial x_{b,2}}{\partial p_a} = t_a'(x_a)\frac{\partial x_a}{\partial p_a} + 1, \quad \frac{\partial x_{b,1}}{\partial p_a} + \frac{\partial x_{b,2}}{\partial p_a} + \frac{\partial x_a}{\partial p_a} = 0. \quad \text{(A.33)}$$

Solving the above equations results in

$$\frac{\partial x_{b,1}}{\partial p_{b,1}} = -\frac{g'(x_{b,2}) + t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t_a'(x_a) + g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,2}}{\partial p_{b,1}} = \frac{t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t_a'(x_a) + g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_a}{\partial p_{b,1}} = \frac{g'(x_{b,2})}{g'(x_{b,1})g'(x_{b,2}) + g'(x_{b,1})t_a'(x_a) + g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,1}}{\partial p_{b,2}}=\frac{t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,2}}{\partial p_{b,2}}=-\frac{g'(x_{b,1})+t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_a}{\partial p_{b,2}}=\frac{g'(x_{b,1})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,1}}{\partial p_a}=\frac{g'(x_{b,2})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,2}}{\partial p_a}=\frac{g'(x_{b,1})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}, \text{ and}$$

$$\frac{\partial x_a}{\partial p_a}=-\frac{g'(x_{b,1})+g'(x_{b,2})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}.$$

Thus, we have

$$\frac{\partial U}{\partial p_{b,1}}=\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)\frac{\partial x_{b,1}}{\partial p_{b,1}}+\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)\frac{\partial x_{b,2}}{\partial p_{b,1}}+x_{b,1}$$

$$=\frac{-\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)\left(g'(x_{b,2})+t_a'(x_a)\right)+\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}+x_{b,1},$$

$$\frac{\partial U}{\partial p_{b,2}}=\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)\frac{\partial x_{b,1}}{\partial p_{b,2}}+\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)\frac{\partial x_{b,2}}{\partial p_{b,2}}+x_{b,2}$$

$$=\frac{\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)t_a'(x_a)-\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)\left(g'(x_{b,1})+t_a'(x_a)\right)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}+x_{b,2}, \text{ and}$$

$$\frac{\partial U}{\partial p_a}=\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)\frac{\partial x_{b,1}}{\partial p_a}+\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)\frac{\partial x_{b,2}}{\partial p_a}$$

$$=\frac{\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)g'(x_{b,2})+\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)g'(x_{b,1})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}.$$

This derivation is completed. ■

A.3. Proof of Property 2

Proof. We take

$$\hat{p}_{b,1}=t_a\left(\bar{x}_a\right)+\tilde{p}_a-t_{b,1}-g\left(\bar{x}_{b,1}\right) \text{ and } \hat{p}_{b,2}=t_a\left(\bar{x}_a\right)+\tilde{p}_a-t_{b,2}-g\left(\bar{x}_{b,2}\right).$$

On one hand, it follows from $\bar{p}_a < \tilde{p}_a$ that

$$\hat{p}_{b,1} > t_a\left(\bar{x}_a\right) + \bar{p}_a - t_{b,1} - g\left(\bar{x}_{b,1}\right) = \bar{p}_{b,1} \geq 0 \quad \text{and}$$

$$\hat{p}_{b,2} > t_a\left(\bar{x}_a\right) + \bar{p}_a - t_{b,2} - g\left(\bar{x}_{b,2}\right) = \bar{p}_{b,2} \geq 0 .$$

On the other hand,

$$t_{b,1} + g\left(\bar{x}_{b,1}\right) + \hat{p}_{b,1} = t_a\left(\bar{x}_a\right) + \tilde{p}_a \quad \text{and} \quad t_{b,2} + g\left(\bar{x}_{b,2}\right) + \hat{p}_{b,2} = t_a\left(\bar{x}_a\right) + \tilde{p}_a .$$

Thus, $\bar{\mathbf{x}}$ and $(\hat{p}_{b,1}, \hat{p}_{b,2}, \tilde{p}_a)$ satisfy constraints (15) and (16). As a result, we have

$$\bar{U}(\hat{p}_{b,1}, \hat{p}_{b,2}, \tilde{p}_a) \leq \bar{U}(\tilde{p}_{b,1}, \tilde{p}_{b,2}, \tilde{p}_a) . \tag{A.34}$$

In addition,

$$\bar{U}(\hat{p}_{b,1}, \hat{p}_{b,2}, \tilde{p}_a) = \bar{x}_{b,1}\hat{p}_{b,1} + \bar{x}_{b,2}\hat{p}_{b,2} - k_{b,1}(\bar{x}_{b,1}) - k_{b,2}(\bar{x}_{b,2})$$

$$> \bar{x}_{b,1}\bar{p}_{b,1} + \bar{x}_{b,2}\bar{p}_{b,2} - k_{b,1}(\bar{x}_{b,1}) - k_{b,2}(\bar{x}_{b,2}) = \bar{U}(\bar{p}_{b,1}, \bar{p}_{b,2}, \bar{p}_a) . \tag{A.35}$$

Thus, combining inequalities (A.34) and (A.35) generates

$$\bar{U}(\bar{p}_{b,1}, \bar{p}_{b,2}, \bar{p}_a) < \bar{U}(\tilde{p}_{b,1}, \tilde{p}_{b,2}, \tilde{p}_a) .$$

This proof is completed. ■

A.4. Proof of Property 3

Proof. Equilibrium condition (27) can be further deduced into seven different sub-conditions (A.1) to (A.7), and hence the feasible set of decision variables of the optimization problem (26) to (28) can be subdivided into seven subsets of respectively satisfying the seven different sub-conditions. Let $p_{b,i}^j$ ($i = 1,2$ and $j = 1,2,\cdots,7$) be an optimal solution to the optimization problem

$$\max_{p_{b,i}} V_i = x_{b,i}p_{b,i} - k_{b,i}(x_{b,i}) , \tag{A.36}$$

where $p_{b,i}$ is subject to sub-condition (A.*j*) and constraint (28). Let $\mathbf{x}^j$ be the unique user distribution pattern at the DUE state determined by $\left(p_{b,1}^j, p_{b,2}^j\right)$ ($j = 1,2,\cdots,7$). Therefore, to prove Property 3, it is sufficient to prove

$$V_i\left(p_{b,i}^1\right) \geq V_i\left(p_{b,i}^j\right), \text{ for } i = 1,2 \text{ and } j = 2,3,\cdots,7 . \tag{A.37}$$

First, we prove

$$V_i\left(p_{b,i}^1\right) \geq V_i\left(p_{b,i}^2\right), \text{ for } i = 1,2 . \tag{A.38}$$

It is supposed that $x_a^{\text{UE}} \geq x_a^2$ ($\mathbf{x}^{\text{UE}}$ is the user distribution pattern at the DUE state without transit fares). Associated with the condition $x_{b,1}^{\text{UE}} > 0 = x_{b,1}^2$, it is generated that

$$x_{b,2}^{\mathrm{UE}} = d - x_{b,1}^{\mathrm{UE}} - x_a^{\mathrm{UE}} < d - x_{b,1}^2 - x_a^2 = x_{b,2}^2 . \tag{A.39}$$

Thus, we have

$$t_{b,2} + g\left(x_{b,2}^{\mathrm{UE}}\right) < t_{b,2} + g\left(x_{b,2}^2\right) \le t_{b,2} + g\left(x_{b,2}^2\right) + p_{b,2}^2 = t_a\left(x_a^2\right) + p_a \le t_a\left(x_a^{\mathrm{UE}}\right) + p_a . \tag{A.40}$$

The first inequality in expression (A.40) follows the precondition that the function g is increasing. The equality results from the condition that $\left(p_{b,1}^2, p_{b,2}^2\right)$ and $\mathbf{x}^2$ satisfy sub-condition (A.2). The third (or last) inequality is obtained by the assumption that the function t_a is increasing. Expression (A.40) contradicts condition (13). Therefore, $x_a^{\mathrm{UE}} < x_a^2$ holds. It immediately follows that

$$t_{b,1} + g\left(x_{b,1}^2\right) < t_{b,1} + g\left(x_{b,1}^{\mathrm{UE}}\right) = t_a\left(x_a^{\mathrm{UE}}\right) + p_a < t_a\left(x_a^2\right) + p_a . \tag{A.41}$$

In addition, it follows from condition (A.2) that

$$t_{b,1} + g\left(x_{b,1}^2\right) + p_{b,1}^2 > t_a\left(x_a^2\right) + p_a . \tag{A.42}$$

Therefore, there is a $\bar{p}_{b,1}^2 = t_a\left(x_a^2\right) + p_a - t_{b,1} - g\left(x_{b,1}^2\right) \in \left(0, p_{b,1}^2\right)$ so that

$$t_{b,1} + g\left(x_{b,1}^2\right) + \bar{p}_{b,1}^2 = t_a\left(x_a^2\right) + p_a = t_{b,2} + g\left(x_{b,2}^2\right) + p_{b,2}^2 , \tag{A.43}$$

i.e., $\left(\bar{p}_{b,1}^2, p_{b,2}^2\right)$ and $\mathbf{x}^2$ satisfy sub-condition (A.1). Thus, it is obtained that

$$V_1\left(p_{b,1}^1\right) \ge V_1\left(\bar{p}_{b,1}^2\right) = -k_{b,1}\left(x_{b,1}^2\right) = V_1\left(p_{b,1}^2\right) \text{ and} \tag{A.44}$$

$$V_2\left(p_{b,2}^1\right) \ge V_2\left(p_{b,2}^2\right), \tag{A.45}$$

Thus, inequality (A.38) is generated.

Second, similar to the proof of inequality (A.38), it can be obtained that

$$V_i\left(p_{b,i}^1\right) \ge V_i\left(p_{b,i}^3\right), \text{ for } i = 1, 2 . \tag{A.46}$$

Third, we prove

$$V_i\left(p_{b,i}^1\right) \ge V_i\left(p_{b,i}^4\right), \text{ for } i = 1, 2 . \tag{A.47}$$

Let

$$\bar{p}_{b,1}^4 = t_a\left(x_a^4\right) + p_a - t_{b,1} - g\left(x_{b,1}^4\right) \text{ and } \bar{p}_{b,2}^4 = t_a\left(x_a^4\right) + p_a - t_{b,2} - g\left(x_{b,2}^4\right). \tag{A.48}$$

Then, $\left(\bar{p}_{b,1}^4, \bar{p}_{b,2}^4\right)$ and $\mathbf{x}^4$ satisfy sub-condition (A.1), and hence

$$V_i\left(p_{b,i}^1\right) \ge V_i\left(\bar{p}_{b,i}^4\right), \text{ for } i = 1, 2 . \tag{A.49}$$

By the definition of V_i ($i = 1, 2$), we have

$$V_i\left(\bar{p}_{b,i}^4\right) = x_{b,i}^4\left(t_a\left(x_a^4\right) + p_a - t_{b,i} - g\left(x_{b,i}^4\right)\right) - k_{b,i}\left(x_{b,i}^4\right) \ge x_{b,i}^4 p_{b,i}^4 - k_{b,i}\left(x_{b,i}^4\right) = V_i\left(p_{b,i}^4\right). \tag{A.50}$$

The inequality in expression (A.50) is generated by the fact $x_{b,i}^4 \ge 0$ and sub-condition (A.4).

Combining (A.49) and (A.50) leads to inequality (A.47).

Fourth, we prove

$$V_i\left(p_{b,i}^1\right) \geq V_i\left(p_{b,i}^5\right), \text{ for } i=1,2 . \tag{A.51}$$

On one hand, $x_{b,1}^5 = 0 < x_{b,1}^{\mathrm{UE}}$, $x_{b,2}^5 = 0 < x_{b,2}^{\mathrm{UE}}$, and $x_a^5 = d > x_a^{\mathrm{UE}}$ hold and also the functions g and t_a are increasing. Thus, associated with condition (13), it is obtained that

$$t_{b,i} + g\left(x_{b,i}^5\right) < t_{b,i} + g\left(x_{b,i}^{\mathrm{UE}}\right) = t_a\left(x_a^{\mathrm{UE}}\right) + p_a < t_a\left(x_a^5\right) + p_a, \text{ for } i=1,2 . \tag{A.52}$$

On the other hand, it holds that

$$t_{b,i} + g\left(x_{b,i}^5\right) + p_{b,i}^5 > t_a\left(x_a^5\right) + p_a, \text{ for } i=1,2 . \tag{A.53}$$

Therefore, there are

$$\overline{p}_{b,i}^5 = t_a\left(x_a^5\right) + p_a - t_{b,i} - g\left(x_{b,i}^5\right) \in \left(0, p_{b,i}^5\right), \text{ for } i=1,2 ,$$

so that

$$t_{b,i} + g\left(x_{b,i}^5\right) + \overline{p}_{b,i}^5 = t_a\left(x_a^5\right) + p_a, \text{ for } i=1,2 , \tag{A.54}$$

i.e., $\left(\overline{p}_{b,1}^5, \overline{p}_{b,2}^5\right)$ and $\mathbf{x}^5$ satisfy sub-condition (A.1). Thus, it is obtained that

$$V_i\left(p_{b,i}^1\right) \geq V_i\left(\overline{p}_{b,i}^5\right) = -k_{b,i}\left(x_{b,i}^5\right) = V_i\left(p_{b,i}^5\right), \text{ for } i=1,2 . \tag{A.55}$$

The equalities in expression (A.55) follow from the condition $x_{b,i}^5 = 0$. Thus, inequality (A.51) holds.

Fifth, we prove

$$V_i\left(p_{b,i}^1\right) \geq V_i\left(p_{b,i}^6\right), \text{ for } i=1,2 . \tag{A.56}$$

The proof needs to be illustrated in three cases. When

$$t_{b,1} + g\left(x_{b,1}^6\right) + p_{b,1}^6 > t_a\left(x_a^6\right) + p_a ,$$

we take $\overline{p}_{b,2}^6 = t_a\left(x_a^6\right) + p_a - t_{b,2} - g\left(x_{b,2}^6\right)$ and then $\left(p_{b,1}^6, \overline{p}_{b,2}^6\right)$ and $\mathbf{x}^6$ satisfy sub-condition (A.2). As a result, we have

$$V_1\left(p_{b,1}^2\right) \geq V_1\left(p_{b,1}^6\right) \text{ and}$$

$$V_2\left(p_{b,2}^2\right) \geq V_2\left(\overline{p}_{b,2}^6\right) = x_{b,2}^6\left(t_a\left(x_a^6\right) + p_a - t_{b,2} - g\left(x_{b,2}^6\right)\right) - k_{b,2}\left(x_{b,2}^6\right)$$

$$> x_{b,2}^6 p_{b,2}^6 - k_{b,2}\left(x_{b,2}^6\right) = V_2\left(p_{b,2}^6\right).$$

Associated with relation (A.38), inequality (A.56) is obtained. By a similar deduction, when

$$t_{b,1} + g\left(x_{b,1}^6\right) + p_{b,1}^6 = t_a\left(x_a^6\right) + p_a \text{ and } t_{b,1} + g\left(x_{b,1}^6\right) + p_{b,1}^6 < t_a\left(x_a^6\right) + p_a ,$$

inequality (A.56) also holds.

Sixth, similar to the proof of inequality (A.56), it can be proved that

$$V_i\left(p_{b,i}^1\right) \geq V_i\left(p_{b,i}^7\right), \text{ for } i=1,2. \tag{A.57}$$

Thus, by inequalities (A.38), (A.46), (A.47), (A.51), (A.56), and (A.57), we can conclude that $p_{b,i}^1$ ($i=1,2$) is an optimal solution to optimization problem (26) to (28). ■

A.5. Derivation of partial derivatives (32) to (37)

Proof. Differentiating both sides of equation (30) and the user conservation condition with respect to $\mathbf{p}$ generates

$$g'(x_{b,1})\frac{\partial x_{b,1}}{\partial p_{b,1}}+1=t_a'(x_a)\frac{\partial x_a}{\partial p_{b,1}},\quad g'(x_{b,2})\frac{\partial x_{b,2}}{\partial p_{b,1}}=t_a'(x_a)\frac{\partial x_a}{\partial p_{b,1}},\quad \frac{\partial x_{b,1}}{\partial p_{b,1}}+\frac{\partial x_{b,2}}{\partial p_{b,1}}+\frac{\partial x_a}{\partial p_{b,1}}=0, \tag{A.58}$$

$$g'(x_{b,1})\frac{\partial x_{b,1}}{\partial p_{b,2}}=t_a'(x_a)\frac{\partial x_a}{\partial p_{b,2}},\quad g'(x_{b,2})\frac{\partial x_{b,2}}{\partial p_{b,2}}+1=t_a'(x_a)\frac{\partial x_a}{\partial p_{b,2}},\quad \frac{\partial x_{b,1}}{\partial p_{b,2}}+\frac{\partial x_{b,2}}{\partial p_{b,2}}+\frac{\partial x_a}{\partial p_{b,2}}=0, \tag{A.59}$$

$$g'(x_{b,1})\frac{\partial x_{b,1}}{\partial p_a}=t_a'(x_a)\frac{\partial x_a}{\partial p_a}+1,\quad g'(x_{b,2})\frac{\partial x_{b,2}}{\partial p_a}=t_a'(x_a)\frac{\partial x_a}{\partial p_a}+1,\quad \frac{\partial x_{b,1}}{\partial p_a}+\frac{\partial x_{b,2}}{\partial p_a}+\frac{\partial x_a}{\partial p_a}=0. \tag{A.60}$$

Solving the above equations results in

$$\frac{\partial x_{b,1}}{\partial p_{b,1}}=-\frac{g'(x_{b,2})+t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,2}}{\partial p_{b,1}}=\frac{t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_a}{\partial p_{b,1}}=\frac{g'(x_{b,2})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,1}}{\partial p_{b,2}}=\frac{t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,2}}{\partial p_{b,2}}=-\frac{g'(x_{b,1})+t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_a}{\partial p_{b,2}}=\frac{g'(x_{b,1})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,1}}{\partial p_a}=\frac{g'(x_{b,2})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial x_{b,2}}{\partial p_a}=\frac{g'(x_{b,1})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}, \text{ and}$$

$$\frac{\mathrm{d}x_a}{\mathrm{d}p_a}=-\frac{g'(x_{b,1})+g'(x_{b,2})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}.$$

Thus, we have

$$\frac{\partial V_1}{\partial p_{b,1}}=\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)\frac{\partial x_{b,1}}{\partial p_{b,1}}+x_{b,1}=-\frac{\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)\left(g'(x_{b,2})+t_a'(x_a)\right)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}+x_{b,1},$$

$$\frac{\partial V_1}{\partial p_{b,2}}=\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)\frac{\partial x_{b,1}}{\partial p_{b,2}}=\frac{\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial V_1}{\partial p_a}=\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)\frac{\partial x_{b,1}}{\partial p_a}=\frac{\left(p_{b,1}-k_{b,1}'(x_{b,1})\right)g'(x_{b,2})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial V_2}{\partial p_{b,1}}=\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)\frac{\partial x_{b,2}}{\partial p_{b,1}}=\frac{\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)t_a'(x_a)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)},$$

$$\frac{\partial V_2}{\partial p_{b,2}}=\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)\frac{\partial x_{b,2}}{\partial p_{b,2}}+x_{b,2}=-\frac{\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)\left(g'(x_{b,1})+t_a'(x_a)\right)}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}+x_{b,2},$$

and

$$\frac{\partial V_2}{\partial p_a}=\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)\frac{\mathrm{d}x_{b,2}}{\mathrm{d}p_a}=\frac{\left(p_{b,2}-k_{b,2}'(x_{b,2})\right)g'(x_{b,1})}{g'(x_{b,1})g'(x_{b,2})+g'(x_{b,1})t_a'(x_a)+g'(x_{b,2})t_a'(x_a)}.$$

This derivation is completed. ■

A.6. Proof of Property 4

Proof. We take

$$\hat{p}_{b,1}=t_a(\overline{x}_a)+\tilde{p}_a-t_{b,1}-g(\overline{x}_{b,1}) \text{ and } \hat{p}_{b,2}=t_a(\overline{x}_a)+\tilde{p}_a-t_{b,2}-g(\overline{x}_{b,2}).$$

On one hand, it is obtained that

$$\hat{p}_{b,1}>t_a(\overline{x}_a)+\overline{p}_a-t_{b,1}-g(\overline{x}_{b,1})=\overline{p}_{b,1}\geq 0 \text{ and}$$

$$\hat{p}_{b,2}>t_a(\overline{x}_a)+\overline{p}_a-t_{b,2}-g(\overline{x}_{b,2})=\overline{p}_{b,2}\geq 0.$$

On the other hand, it holds that

$$t_{b,1}+g(\overline{x}_{b,1})+\hat{p}_{b,1}=t_a(\overline{x}_a)+\tilde{p}_a \text{ and } t_{b,2}+g(\overline{x}_{b,2})+\hat{p}_{b,2}=t_a(\overline{x}_a)+\tilde{p}_a.$$

Thus, $\hat{p}_{b,1}$, $\hat{p}_{b,2}$, $\tilde{p}_a$, and $\overline{\mathbf{x}}$ satisfy constraints (30) and (31). As a result, we have

$$\bar{V}_j(\hat{p}_{b,j}, \tilde{p}_a) < \bar{V}_j(\tilde{p}_{b,j}, \tilde{p}_a), \text{ for } \quad j = 1, 2. \tag{A.61}$$

In addition,

$$\bar{V}_j(\hat{p}_{b,j}, \tilde{p}_a) = \bar{x}_{b,j}\hat{p}_{b,j} - k_{b,j}(\bar{x}_{b,j}) > \bar{x}_{b,j}\bar{p}_{b,j} - k_{b,j}(\bar{x}_{b,j}) = \bar{V}_j(\bar{p}_{b,j}, \bar{p}_a), \text{ for } \quad j = 1, 2. \tag{A.62}$$

Thus, by inequalities (A.61) and (A.62), it immediately follows that

$$\bar{V}_j(\bar{p}_{b,j}, \bar{p}_a) < \bar{V}_j(\tilde{p}_{b,j}, \tilde{p}_a), \text{ for } \quad j = 1, 2.$$

This proof is completed. ■

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