Optimal Contract under Double Moral Hazard and Limited Liability

Jiajia Cong and Wen Zhou*

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Abstract

This paper investigates optimal contracts between risk-neutral parties when both exert efforts and the agent faces limited liability. We identify a sufficient and necessary condition for any contract to implement the second-best outcome, i.e., the best possible outcome in double moral hazard even when the agent faces unlimited liability. It is shown that a simple share-or-nothing with bonus contract (SonBo for short) is optimal and implements the second-best outcome when the condition holds. SonBo contracts have one degree of freedom, which is very useful in dealing with heterogeneous circumstances while still maintaining consistency in contracting. SonBo admits as special cases the option-like and step bonus contracts, which are widely used in dealing with limited liability. Nevertheless, we demonstrate that a step bonus contract is more powerful because an option-like contract can be problematic in some situations. The paper also discusses the performance of SonBo when the principal also faces liability constraint and investigates the optimal contract when the second-best outcome is not achievable.

Keywords: double moral hazard, limited liability, optimal contract

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*Cong (corresponding author): Department of Industrial Economics, School of Management, Fudan University, Shanghai, China. Email: jjcong@fudan.edu.cn. Zhou: Faculty of Business and Economics, The University of Hong Kong, Pokfulam Road, Hong Kong. Email: wzhou@hku.hk. This research is sponsored by National Natural Science Foundation of China (Project 72003035) and Shanghai Pujiang Program (2019PJC007). We would like to thank Wing Suen, Steve Xu, Yujing Xu, and seminar participants at the University of Hong Kong and Nanjing Audit University for helpful comments and suggestions.

1 Introduction

Many contractual relationships involve economic activities that require efforts from both the principal and the agent. Typical examples include franchise contracting, author-publisher relationship, venture capital and entrepreneur, and sharecropping. This paper investigates optimal contracts in double moral hazard (i.e., both the principal and the agent exert effort) when both parties are risk-neutral and the agent faces limited liability. In the literature, contracting between risk-neutral principal and agent has been investigated in standard, one-sided moral hazard (i.e., only the agent exerts effort) with limited liability (Park, 1995; Kim, 1997; Poblete and Spulber, 2012), or double moral hazard without liability constraint (Kim and Wang, 1998), but not the combination studied here. Given the ubiquity of double moral hazard and limited liability in real life, a thorough investigation of the theoretical issue is warranted.

The solution arising from this research is a share-or-nothing with bonus contract, or *SonBo* for short. If the output is below some threshold, the agent's compensation is zero; beyond the threshold, it is the sum of a lump-sum bonus and a fixed share of the extra output. We identify a sufficient and necessary condition under which SonBo implements the second-best outcome (i.e., the best possible outcome in double moral hazard) even when the agent faces unlimited liability. This condition is also sufficient and necessary for the existence for any optimal contract that implements the second-best outcome. Therefore, a principal can safely focus on SonBo for its simplicity without worrying about missing out any other contracts that might be more powerful.

Out of the three instruments of SonBo, namely the bonus, output threshold, and output share, one can be freely chosen (within the feasibility set), after which the other two will be determined uniquely. In other words, the principal has the freedom to choose among a continuum of SonBo contracts without compromising its performance. Contracting flexibility is desirable because real life contracts often need to address some additional concerns such as government regulations or tax policies, which may make the implementation of an otherwise optimal contract very costly or even impossible. For example, franchising contracts in the U.S. are subject to heterogeneous state regulations. It is common that franchisors maintain flexibility in specifying certain fees in franchising contracts (Bhattacharyya and Lafontaine, 1995).[1] Another reason for contracting flexibility is that some terms in the con-

[1] Franchise laws in most states are primarily concerned with termination, renewal and transfer of franchise rights, but Washington, Michigan, Indiana, and Iowa go well beyond these requirements. States may differ substantially even in the definition of key elements such as franchise fees (Pitegoff and Garner, 2008). In response, franchisors headquartered in states that restrict termination or renew rights usually charge significantly higher royalty rates and lower franchise fees than franchisors in other states (Brickley, 2002).

tract are determined at higher levels of a business hierarchy, such as headquarter or business associations (Griva and Vettas, 2015). Lower ranked principals face rigid format and therefore need some flexibility in adjusting the remaining components to circumstances.

Arising from its flexibility in the contract design, SonBo admits as special cases two contracts that are widely used in dealing with limited liability. If the bonus is set to zero, SonBo degenerates into an option-like contract; if the agent's output share is set to zero, SonBo becomes a step bonus contract. Option-like contracts have been shown to be optimal in one-sided moral hazard (Innes, 1990; Jewitt, Kadan, and Swinkels, 2008; Poblete and Spulber, 2012; Kadan and Swinkels, 2013). In double moral hazard, however, such contract may have to give the agent an output share that is greater than one, which is undesirable because it invalidates the first order approach and causes an ex-post moral hazard problem by the principal (Innes, 1990; Poblete and Spulber, 2012). On the other hand, an optimal step bonus contract always exists as long as the second-best outcome is implementable. This suggests that although both contracts pay the agent zero unless the output is sufficiently large, bonus contracts are more powerful than option-like contracts in solving double moral hazard problems. The principal should therefore be cautious when attempting to remove the bonus component or downgrade its value.

If the principal also faces limited liability (which would be the case in, for example, most start-up companies), SonBo still achieves second-best outcome under additional, mild conditions. In other words, the optimality of SonBo can be invariant to additional liability constraints. This is because an optimal contract in double moral hazard has to take care of the incentives of both the principal and the agent. This, in turn, tends to require both parties to share the marginal output, making it no more difficult to satisfy two-sided liability than one-sided liability. When the sufficient and necessary condition of implementing the second-best outcome fails, the principal faces a trade-off between inducing the optimal efforts and giving the agent a limited liability rent. In some situations, the principal cares more about inducing the optimal efforts, and SonBo continues to implement the second-best efforts. In some other situations, the principle would like to implement a third-best outcome, and SonBo is still useful as an optimal contract format under additional conditions.

Many studies of agent limited liability focus on one-sided moral hazard. Park (1995) and Kim (1997) find that a bonus contract can achieve the first-best outcome. The bonus component is indispensable in SonBo, and our condition is comparable to theirs except a difference to incorporate features of double moral hazard. When both parties face limited liabilities, Innes (1990) proves the optimal contract is either a debt contract or live-or-die contract depending on whether the principal's payment is required to be monotonic in firm profits. The live-or-die contract is similar to SonBo;[2] the debt contract (a truncated linear contract which gives the agent all marginal output beyond a threshold) is option-like and is therefore also similar to SonBo with zero bonus.

Optimal contract for double moral hazard has also been studied by Kim and Wang (1998), but for a risk-averse agent. They argue that the agent's wage must be bounded from both below and above, and arrive at a truncated, non-linear contract. We focus on a risk-neutral agent, which gives rise naturally to a linear contract that is further enriched by a truncation and a discrete jump in the compensation.

The remainder of the paper is organized as follows. After setting up the model in Section 2, we first show in Section 3 how to calculate second-best effort levels and why the linear sharing contract is optimal if liability is unlimited. Section 4 introduces SonBo and establishes its general optimality. Section 5 discusses the major properties of SonBo including the two special cases of option-like and step bonus contracts, its optimality when the principal also faces liability constraint, the optimal contract when the second-best outcome is not feasible, and some comparative statics. Section 6 concludes.

2 Model Setting

A principal hires an agent, both risk-neutral, to finish a project which requires efforts from both parties. Denote the principal's effort by $a \in R_+$ and the agent's effort by $e \in R_+$. A composite effort is then constructed as $h = h(a, e)$, where $h(a, e) : R_+^2 \to R$ is continuous and differentiable with $h_a(a, e) > 0$, $h_{aa}(a, e) \leq 0$, $h_e(a, e) > 0$ and $h_{ee}(a, e) \leq 0$. The project generates an output x, which is a random variable following a distribution with p.d.f. $f(x|h(a, e)) > 0$ and c.d.f. $F(x|h(a, e))$ on $[\underline{x}, \overline{x}]$ given any composite effort,[3] where $\underline{x}$ can be negative infinity and $\overline{x}$ can be positive infinity. Both $f(x|h(a, e))$ and $F(x|h(a, e))$ are continuous and twice differentiable with respect to x and h. Costs of the two parties' efforts, $v(a)$ and $c(e)$, are strictly convex:[4] $v'(a) > 0$, $v''(a) > 0$, $v'(0) = 0$; $c'(e) > 0$, $c''(e) > 0$, $c'(0) = 0$.

The game unfolds in the following sequence. The principal proposes to the agent a contract $w(x)$, which specifies the agent's compensation w as a function of the realized

[2] In a live-or-die contract, the agent's compensation is zero when the output is below a threshold, jumps at the threshold by an amount that equals the threshold output level, and increases with output one-for-one. In SonBo, the agent's compensation is zero when the output is below a threshold, but the jump at the threshold and the slope beyond the threshold are endogenously and jointly determined.

[3] Compared with the more general specification of $f(x|a, e)$, the introduction of composite effort simplifies the analysis. Most studies including Romano (1994), Kim and Wang (1998), Wang and Zhu (2005), and Suzuki (2007) adopt the composite effort.

[4] To simplify notations, we use h_e, h_a, $f(x|h)$, $f_h(x|h)$, $F(x|h)$, $F_h(x|h)$, $v(a)$, $v'(a)$, $c(e)$, $c'(e)$ to refer to functions evaluated at general values a and e, and will be specific when a or e takes the second-best level.

output level, x, with the constraint that $w(x)$ is non-negative for any x. Conditional on the realization of x, the contract gives an ex post utility of $w(x) - c(e)$ to the agent, and $x - w(x) - v(a)$ to the principal. If the agent accepts the contract, both parties choose their efforts non-cooperatively and simultaneously, after which x is realized and the contract is executed. If the agent rejects the contract, the project is terminated, in which case the agent receives his outside option $\overline{U}_e$.

We make the following three standard assumptions:

Assumption 1: The expected output $\int_{\underline{x}}^{\overline{x}} xf(x|h(a,e))dx$ is concave and strictly increasing separately in a and e.

Assumption 2: $\forall h \in R$, $\frac{\partial}{\partial x}\left(\frac{f_h(x|h)}{f(x|h)}\right) > 0$ for any $x \in [\underline{x}, \overline{x}]$.

Assumption 3: $\forall x \in [\underline{x}, \overline{x}]$, $F(x|h)$ is convex in h.

Assumption 1, along with strict convexity of the cost functions, guarantee that the optimal effort choices are unique. Assumptions 2 and 3 are sufficient for the first order approach (Milgrom, 1981; Rogerson, 1985; Jewitt, 1988). Given that the first order approach is valid, the principal's problem with agent's limited liability is:

$$[P(LL)] \qquad \max_{w(x),a} \quad \int_{\underline{x}}^{\overline{x}} (x - w(x))f(x|h(a,e))dx - v(a) \tag{1}$$

$$s.t.$$

$$\int_{\underline{x}}^{\overline{x}} w(x)f(x|h(a,e))dx - c(e) \geq \overline{U}_e \tag{2}$$

$$\int_{\underline{x}}^{\overline{x}} w(x)f_h(x|h(a,e))h_e dx = c'(e) \tag{3}$$

$$\int_{\underline{x}}^{\overline{x}} (x - w(x))f_h(x|h(a,e))h_a dx = v'(a) \tag{4}$$

$$w(x) \geq 0 \tag{5}$$

Constraint (2) is the agent's participation constraint (*agent-PC* hereafter). Given that both parties choose efforts simultaneously, the incentive compatibility constraints of the agent (*agent-IC*) and the principal (*principal-IC*), captured by (3) and (4) respectively, must be satisfied. Finally, (5) reflects the agent's limited liability.

3 Unlimited Liability: the Second-Best Outcome

To analyze the original program $[P(LL)]$, we first solve a relaxed program $[P(UL)]$ in which the constraint (5) is removed. Without the agent's limited liability, the principal's problem becomes:

$$[P(UL)] \qquad \max_{w(x),a} \int_{\underline{x}}^{\overline{x}} (x - w(x)) f(x|h(a,e))\, dx - v(a)$$
$$s.t. \quad (2), (3), (4).$$

Kim and Wang (1998) and Wang and Zhu (2005) have proved that the optimal contract for $[P(UL)]$ should make the agent-PC (2) binding, so $\int_{\underline{x}}^{\overline{x}} w(x) f(x|h(a,e))\, dx = c(e) + \overline{U}_e$. Then the objective function (1) becomes $\int_{\underline{x}}^{\overline{x}} x f(x|h(a,e))\, dx - v(a) - c(e) - \overline{U}_e$. Since agent-PC always binds for any choice of optimal contracts, maximizing the principal's own expected payoff is equivalent to maximizing the two parties' expected joint payoff, which is independent of $w(x)$. Combining agent-IC (3) and principal-IC (4) to get rid of $w(x)$, we arrive at the constraint (6) below. To solve $[P(UL)]$, therefore, we will first solve the following further-relaxed problem $[P(E)]$.

$$[P(E)] \qquad \max_{(a,e)} \int_{\underline{x}}^{\overline{x}} x f(x|h(a,e))\, dx - v(a) - c(e) - \overline{U}_e$$
$$s.t.$$
$$\int_{\underline{x}}^{\overline{x}} x f_h(x|h(a,e))\, dx = \frac{c'(e)}{h_e(a,e)} + \frac{v'(a)}{h_a(a,e)}. \tag{6}$$

Denote the solution of $[P(E)]$ by (a^*, e^*). Notice that (a^*, e^*) is independent of any specific contract and is referred to as the second-best efforts (Kim and Wang, 1998; Wang and Zhu, 2005). It is not the first-best because from (6) we can derive:

$$\int_{\underline{x}}^{\overline{x}} x f_h(x|h(a^*, e^*)) h_e(a^*, e^*)\, dx > c'(e^*), \tag{7}$$

meaning that given a^*, the expected marginal revenue from e^* is strictly larger than its marginal cost. Similarly, a^* is not the first-best effort choice, either.[5]

Kim and Wang (1998) and Wang and Zhu (2005) have proved that the linear sharing contract $w^\star(x) = \gamma^\star(x - \widehat{x}^\star)$, in which

$$\gamma^\star = \frac{c'(e^*)}{\int_{\underline{x}}^{\overline{x}} x f_h(x|h(a^*, e^*)) h_e(a^*, e^*)\, dx} \text{ and } \widehat{x}^\star = \int_{\underline{x}}^{\overline{x}} x f(x|h(a^*, e^*))\, dx - \frac{c(e^*) + \overline{U}_e}{\gamma^\star},$$

can always achieve the second-best outcome, which implements the second-best efforts (a^*, e^*) and makes the agent-PC binding. In other words, the linear sharing contract $w^\star(x)$

[5] The reason why (a^*, e^*) is not the first-best can be understood from the perspective of team production. Double moral hazard problem is essentially team production, with one team member being chosen as the principal and residual claimant (Suzuki, 2007). The impossibility of achieving the first-best outcome is related to balancing-budget problem studied by Hölmstrom (1982). No matter what the output is, the sum of the principal's payment and the agent's payment is always equal to the whole output. As Hölmstrom (1982) shows, team production with balancing-budget condition cannot achieve the first-best outcome.

is a solution to the program $[P(UL)]$. The intuition is as follows. The principal needs to accomplish two tasks with the contract. First, the contract will induce the agent to choose e^* (given e^*, the principal's best response would be a^* as long as the agent-PC is binding);[6] second, when the agent exerts effort e^* and expects to receive compensation according to the contract, he is just indifferent between accepting and rejecting the contract. The first task (agent-IC) is achieved solely by $\gamma^\star$: Given inequality (7), there must exist a unique constant $\gamma^\star \in (0,1)$ to make the following equality hold

$$\gamma^\star \int_{\underline{x}}^{\overline{x}} x f_h(x|h(a^*,e^*))h_e(a^*,e^*)\,dx = c'(e^*).$$

Once $\gamma^\star$ is determined, $\widehat{x}^\star$ will be chosen to accomplish the second task, i.e., to make the agent-PC binding.

4 Limited Liability: the SonBo Contract

We now return to the original problem $[P(LL)]$ and study whether the principal can still implement the second-best outcome when the agent faces limited liability. To make our investigation non-trivial (otherwise the linear sharing contract $w^\star(x)$ is optimal), we make the following assumption throughout the remainder of the paper.

Assumption 4: $\gamma^\star(\underline{x} - \widehat{x}^\star) < 0$.

We propose the following Share-or-Nothing with Bonus (*SonBo* for short) contract:

$$w^*(x) = \begin{cases} 0, & \text{if } x < \widehat{x} \\ \gamma^*(x - \widehat{x}) + B, & \text{if } x \geq \widehat{x} \end{cases} \tag{8}$$

as a solution to $[P(LL)]$, where $B \geq 0$. Under SonBo, the agent receives zero compensation if the output is below a threshold $\widehat{x}$; once the threshold is reached, the agent's compensation equals a lump-sum bonus plus a fixed share of the extra output beyond the threshold.

There are three parameters to be determined in $w^*(x)$: the bonus B, the threshold output $\widehat{x}$, and the agent's share γ^*. The optimal agent share is:

$$\gamma^* = \frac{\frac{c'(e^*)}{h_e(a^*,e^*)} + B \cdot F_h(\widehat{x}|h(a^*,e^*))}{\int_{\widehat{x}}^{\overline{x}} (x - \widehat{x}) f_h(x|h(a^*,e^*))\,dx}, \tag{9}$$

which is a function of both B and $\widehat{x}$. Given a bonus B, SonBo always satisfies the agent-IC condition at the second-best outcome by construction. Moreover, at second-best effort levels, *agent-IC* holds implies principal-IC holds as long as the agent-PC is binding. Therefore,

[6] The intuition is that (a^*,e^*) satisfies equation (6), which is the sum of agent-IC and principal-IC. This means that if a contract gives the agent a part of the output such that the agent would choose e^*, the remaining output must induce the principal to choose a^*.

given a bonus B, there is a continuum of $\widehat{x}$ to make both IC constraints hold. The value of $\widehat{x}$ is determined by the binding agent-PC:

$$H(\widehat{x}) + B \cdot L(\widehat{x}) = c(e^*) + \overline{U}_e. \tag{10}$$

The right-hand-side of (10) is the sum of agent's cost of exerting e^* and outside option. The left-hand-side is the agent's expected compensation under SonBo, which consists of a bonus part weighted by $L(\widehat{x})$ and a non-bonus part $H(\widehat{x})$:

$$H(\widehat{x}) = \frac{c'(e^*)}{\int_{\widehat{x}}^{\overline{x}} (x-\widehat{x}) f_h(x|h(a^*,e^*)) h_e(a^*,e^*)\, dx} \int_{\widehat{x}}^{\overline{x}} (x-\widehat{x}) f(x|h(a^*,e^*))\, dx,$$

$$L(\widehat{x}) = \frac{F_h(\widehat{x}|h(a^*,e^*))}{\int_{\widehat{x}}^{\overline{x}} (x-\widehat{x}) f_h(x|h(a^*,e^*))\, dx} \int_{\widehat{x}}^{\overline{x}} (x-\widehat{x}) f(x|h(a^*,e^*))\, dx + \int_{\widehat{x}}^{\overline{x}} f(x|h(a^*,e^*))\, dx.$$

Thus, given a bonus B, $\widehat{x}$ is determined by (10), and γ^* is determined by (9).

Out of the three components in SonBo, i.e., the bonus B, output threshold $\widehat{x}$, and agent's share γ^*, one can be freely chosen, after which the other two components will be determined uniquely from the model parameters. The reason for the one degree of freedom is as follows. The principal maximizes her own payoff. When the agent-PC binds, maximizing her own payoff is equivalent to maximizing the total expected payoff, which is how the second-best outcome is derived. At that outcome, the marginal expected output from both parties' efforts equals the sum of their marginal costs (see condition (6)). If the agent-IC holds, i.e., agent's marginal cost is compensated by a portion of marginal output, the remaining marginal output must exactly compensate the principal's marginal cost. Therefore, when agent-PC binds, agent-IC implies principal-IC and vice versa, so one of the constraints is redundant.

When designing the contract, therefore, the principal only needs to care about agent-PC and one of the two IC constraints at (a^*, e^*). The key question is what is the condition for a binding agent-PC, i.e., under what conditions will equality (10) always have a solution in $[\underline{x}, \overline{x}]$. It turns out that, given Assumption 4 holds, the sufficient and necessary condition for a binding agent-PC is

$$\frac{f_h(\overline{x}|h(a^*,e^*)) h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} \geq \frac{c'(e^*)}{c(e^*) + \overline{U}_e}. \tag{11}$$

Proposition 1. *When the agent has limited liability, SonBo contract $w^*(x)$ implements the second-best outcome (a^*, e^*) if and only if inequality (11) holds.*

Proposition 1 specifies inequality (11) as a sufficient and necessary condition for SonBo to be the optimal contract. The intuition is the following. Given any B, Assumption 4 makes sure that when $\widehat{x} = \underline{x}$, the agent's expected compensation from $w^*(x)$ is larger than $c(e^*) + \overline{U}_e$; condition (11) implies that when $\widehat{x} = \overline{x}$, the agent's expected compensation is smaller than

$c(e^*) + \overline{U}_e$. Therefore, there must exist an $\widehat{x}$ in $[\underline{x}, \overline{x}]$ to make agent's expected compensation equal to $c(e^*) + \overline{U}_e$.

A crucial question is whether there exists any other contract that can achieve the same outcome under more relaxed condition. The answer is no.

Proposition 2. *A contract achieves the second-best outcome in problem* $[P(LL)]$ *if and only if inequality (11) holds.*

Proposition 2 states that inequality (11) is the sufficient and necessary condition for *any* contract to achieve the second-best outcome when the agent has limited liability. The reason is the following. If a contract implements the second-best outcome when the agent has limited liability, the following condition (derived from agent-IC and agent-PC) must hold:

$$\frac{\int_{\underline{x}}^{\overline{x}} w(x) f_h(x|h(a^*, e^*)) h_e(a^*, e^*)\, dx}{\int_{\underline{x}}^{\overline{x}} w(x) f(x|h(a^*, e^*))\, dx} = \frac{c'(e^*)}{c(e^*) + \overline{U}_e}. \tag{12}$$

The left hand side of (12) is the agent's marginal expected compensation rescaled by his total expected compensation, or as Park (1995) puts it, "the rate of change in agent's expected utility from extending optimal effort e^*." The right hand side is his marginal cost rescaled by his total cost, or the rate of change in the agent's disutility from extending optimal effort e^*. Due to MLRP and non-negative wage, the left hand side of (12) is not greater than $\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))}$, which turns out to be the left hand side of condition (11). So if (11) fails, which implies (12) cannot hold, the maximum rate of change in the agent's expected utility from extending e^* is smaller than his rate of change in disutility from extending e^*. Then under this contract, the agent's incentive-compatibility and participation constraints cannot be satisfied simultaneously, implying the second-best outcome cannot be implemented by this contract.

Propositions 1 and 2 suggest that no other contract can do better (i.e., achieving the second-best outcome under strictly more relaxed conditions) than SonBo. Therefore, the principal can limit her attention to SonBo contracts without worrying about losing anything useful.

5 Discussion

In this section, we will discuss two special cases of SonBo, explore SonBo's optimality when the principal also faces limited liability, study SonBo's performance when the second-best outcome is infeasible, and highlight comparative static properties.

5.1 Option or bonus?

SonBo combines features of an option-like contract and a bonus contract, both of which have been proved effective in dealing with limited liability problems (Park, 1995; Kim, 1997; Jewitt, Kadan, and Swinkels, 2008; Kadan and Swinkels, 2013). Because it has one degree of freedom in the contract design, SonBo can degenerate to an option-like contract (i.e., $B = 0$) or a step bonus contract (i.e., $\gamma^* = 0$). We now demonstrate that the bonus contract is generally optimal, while the option-like contract can cause some problems. In other words, between the two instruments of SonBo, the bonus component is more important than the output share.

We first show that, as a special case of SonBo, the bonus contract always exists. Based on binding agent-PC, B can be written as a function of $\widehat{x}$:

$$B(\widehat{x}) = \frac{c(e^*) + \overline{U}_e - H(\widehat{x})}{L(\widehat{x})}. \tag{13}$$

The agent's share γ^*, as defined by (9) can then be expressed as a function of $\widehat{x}$ alone:

$$\gamma^*(\widehat{x}) = \frac{\frac{c'(e^*)}{h_e(a^*,e^*)} + \frac{c(e^*)+\overline{U}_e - H(\widehat{x})}{L(\widehat{x})} \cdot F_h(\widehat{x}|h(a^*,e^*))}{-\int_{\widehat{x}}^{\overline{x}} F_h(x|h(a^*,e^*))\,dx}. \tag{14}$$

When $B = 0$, denote the solution to equation (10) as $\widehat{x}_L$. $\gamma^*(\widehat{x})$ is a continuous function with domain $[\widehat{x}_L, \overline{x})$. Although the monotonicity of $\gamma^*(\widehat{x})$ cannot be verified, we can nevertheless prove that $\gamma^*(\widehat{x}_L) > 0$ and $\lim_{\widehat{x}\to\overline{x}} \gamma^*(\widehat{x}) = -\infty$.[7] As a result, there must exist a $\widehat{x}_B \in (\widehat{x}_L, \overline{x})$ such that $\gamma^*(\widehat{x}_B) = 0$. In that case, the SonBo contract $w^*(x)$ degenerates to a step bonus contract:

$$w_B^*(x) = \begin{cases} 0, & \text{if } x < \widehat{x}_B \\ B, & \text{if } x \geq \widehat{x}_B. \end{cases} \tag{15}$$

To ensure the existence of step bonus contract, condition (11) is needed as it is the sufficient and necessary for any optimal contract.

Now look at the option-like contract as a special case of SonBo. Setting the bonus as zero, define

$$\gamma_L^* \equiv \gamma^*(\widehat{x}_L) = \frac{c'(e^*)}{-\int_{\widehat{x}_L}^{\overline{x}} F_h(x|h(a^*,e^*))h_e(a^*,e^*)\,dx}.$$

SonBo degenerates into an option-like contract $w_L^*(x)$:

$$w_L^*(x) = \begin{cases} 0, & \text{if } x < \widehat{x}_L, \\ \gamma_L^*(x - \widehat{x}_L), & \text{if } x \geq \widehat{x}_L. \end{cases} \tag{16}$$

[7] Refer to Section 4 of online appendix for the proof of $\lim_{\widehat{x}\to\overline{x}} \gamma^*(\widehat{x}) = -\infty$. This conclusion serves as a necessary technical condition to prove Proposition 3.

The two instruments $\widehat{x}_L$ and γ_L^* are uniquely determined. One problem with option-like contract, however, is that the agent's output share is not necessarily less than one. In particular, $\gamma_L^* \leq 1$ if and only if:

$$c'(e^*) \leq -\int_{\widehat{x}_L}^{\overline{x}} F_h(x|h(a^*,e^*))h_e(a^*,e^*)\,dx. \tag{17}$$

If $\gamma_L^* > 1$, it means that the principal's payment decreases with output. This is undesirable, as the principal would have an incentive to under-report or sabotage the output (Innes, 1990; Poblete and Spulber, 2012). Moreover, the second-order condition for the principal's effort choice may fail, which invalidates the first-order approach.

Proposition 3. *When inequality (11) holds, there always exists a step bonus contract (15) to achieve the second-best outcome (a^*, e^*), but the option-like contract (16) may be problematic, and it implements the second-best outcome (a^*, e^*) if and only if an additional condition (17) holds.*

We use the following two examples to illustrate how SonBo may degenerate into the two special cases. Condition (17) fails in Example 2, in which case the agent's share in the option-like contract $w_L^*(x)$ is larger than one.

Example 1: $f(x|h(a,e)) = 1 + \frac{1-2x}{h(a,e)+1}$, $x \in [0,1]$, $h(a,e) = a+e$, $v(a) = \frac{1}{2}a^2$, $c(e) = \frac{1}{2}e^2$, $\overline{U}_e = 0.1$. Then $(a^*, e^*) \approx (0.07, 0.07)$. When $\gamma^* = 0$, the step bonus contract $w_B^*(x)$ is determined by $\widehat{x}_B = 0.47$ and $B = 0.33$. When $B = 0$, the option-like contract $w_L^*(x)$ is determined by $\widehat{x}_L = 0.21$ and $\gamma_L^* = 0.56$.

Example 2: $f(x|h(a,e)) = 1 + \frac{1}{2}(1-2h(a,e))(1-2x)$, $x \in [0,1]$, $h(a,e) = a + \frac{5}{2}e$, $v(a) = \frac{1}{2}a^2$, $c(e) = \frac{1}{2}e^2$, $\overline{U}_e = \frac{1}{4}$. Then $(a^*, e^*) \approx (0.02, 0.36)$. When $\gamma^* = 0$, the step bonus contract $w_B^*(x)$ is determined by $\widehat{x}_B = 0.57$ and $B = 0.59$. When $B = 0$, the option-like contract $w_L^*(x)$ is determined by $\widehat{x}_L = 0.35$ and $\gamma_L^* = 1.20$.

In both examples, the agent's share γ^* decreases with bonus B. In Example 2, to ensure the agent's share is not larger than one, the bonus must satisfy $B \geq 0.21$. Therefore, the principal may not have the freedom to get rid of the bonus component or set a very low value for it.

5.2 SonBo and two-sided liability constraints

Principals in many double moral hazard problems also face limited liability. For example, the owner of a start-up company is constrained by financial resources and can endure limited losses. In this section, we explore whether SonBo contract can still achieve the second-best outcome when the principal also faces limited liability, i.e. when the constraint

(5) in $[P(LL)]$ is replaced by

$$0 \leq w(x) \leq x.$$

Note that the lower bound of output, $\underline{x}$, should not be negative, otherwise it is impossible for both parties to receive non-negative payments if the output x turns out to be negative. Throughout this section we assume $\underline{x} \geq 0$.

Under SonBo, the principal's payment is

$$x - w^*(x) = \begin{cases} x, & \text{if } x < \widehat{x} \\ x - \gamma^*(x - \widehat{x}) - B, & \text{if } x \geq \widehat{x}. \end{cases}$$

If $\gamma^* \in [0, 1]$ and $\widehat{x} \geq B$, then SonBo satisfies two-sided limited liability.

Let us first analyze the condition for $\gamma^* \in [0, 1]$. Given $\gamma^*(\widehat{x})$ is a continuous function on $[\widehat{x}_L, \overline{x})$ with $\gamma^*(\widehat{x}_L) > 0$ and $\lim_{\widehat{x} \to \overline{x}} \gamma^*(\widehat{x}) = -\infty$, we immediately conclude that there must exist an interval of $\widehat{x}$ in which agent's share $\gamma^*(\widehat{x}) \in [0, 1]$. Define the set $\widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]} = \{\widehat{x} | \gamma^*(\widehat{x}) \in [0, 1]\}$, which is non-empty. If (17) holds, $\gamma^*(\widehat{x}_L) \leq 1$. There must exist a $\widehat{x}'$ such that $\gamma^*(\widehat{x}) \in [0, 1]$ holds for any $\widehat{x} \in [\widehat{x}_L, \widehat{x}']$. The left panel of Figure 1 shows the case when (17) fails.

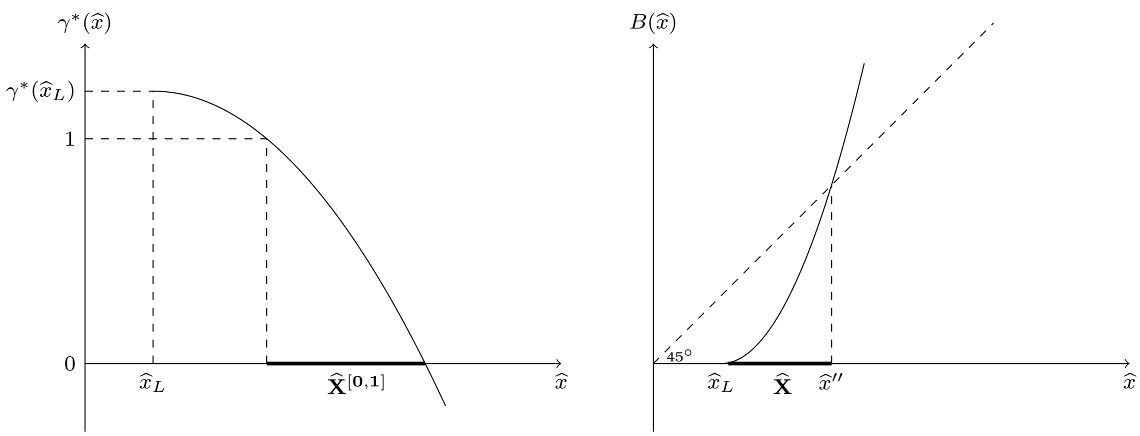

Figure 1: An illustration of $\widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]}$ and $\widehat{\mathbf{X}}$

We now turn to the analysis of $\widehat{x} \geq B$. According to equation (13), $\widehat{x} \geq B$ is equivalent to

$$\widehat{x} \geq B(\widehat{x}) = \frac{c(e^*) + \overline{U}_e - H(\widehat{x})}{L(\widehat{x})}, \quad \widehat{x} \in [\widehat{x}_L, \overline{x}). \tag{18}$$

The function $B(\widehat{x})$ is continuous, with $B(\widehat{x}_L) = 0$, $\lim_{\widehat{x} \to \overline{x}} B(\widehat{x}) = +\infty$, and $B(\widehat{x}) > 0$ when $\widehat{x} \in (\widehat{x}_L, \overline{x})$. So some parts of the $B(\widehat{x})$ curve, at least when $\widehat{x}$ is slightly larger than $\widehat{x}_L$, must lie below the 45 degree line, i.e., $\widehat{x} \geq B(\widehat{x})$. Define $\widehat{\mathbf{X}}$ as the set of all $\widehat{x}$ where $\widehat{x} \geq B(\widehat{x})$ hold. We immediately conclude that there must exist a $\widehat{x}''$ larger than $\widehat{x}_L$ such that the interval $[\widehat{x}_L, \widehat{x}'']$ belongs to $\widehat{\mathbf{X}}$. The right panel of Figure 1 illustrates an example of $B(\widehat{x})$ function and the set $\widehat{\mathbf{X}}$ when $B(\widehat{x})$ is an increasing function.

It remains to check whether there is any $\widehat{x}$ to make $\gamma^* \in [0,1]$ and $\widehat{x} \geq B$ hold at the same time. We already know that when inequality (17) holds, the interval $[\widehat{x}_L, \widehat{x}']$ is a subset of $\widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]}$ and the interval $[\widehat{x}_L, \widehat{x}']$ is a subset of $\widehat{\mathbf{X}}$. So $\widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]} \cap \widehat{\mathbf{X}}$ must be non-empty. In this case, the principal can always design a continuum of SonBo contracts, including the option-like contract, to achieve the second-best outcome when the principal and the agent both face limited liabilities. When inequality (17) does not hold, there exists $\widehat{x}$ to make $\gamma^* \in [0,1]$ and $\widehat{x} \geq B$ hold simultaneously if and only if $\widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]} \cap \widehat{\mathbf{X}}$ is non-empty. Although the non-emptiness cannot be established more generally, it can nevertheless be verified given any specific setting.

Proposition 4. *Suppose inequality (11) holds and the principal and the agent both face limited liabilities. There exists (at least) one SonBo contract $w^*(x)$ to achieve the second-best outcome when*

(I) condition (17) holds; or

(II) condition (17) does not hold, but $\widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]} \cap \widehat{\mathbf{X}}$ is non-empty.

An immediate corollary of Proposition 4 is that as long as the option-like contract $w_L^*(x)$ is optimal under agent's limited liability, i.e., condition (17) holds, it must be an optimal contract under two-sided limited liability. The following examples illustrate how SonBo contract can achieve the second-best outcome when both parties have limited liability. Condition (17) holds only in Example 1.

Example 1': The setting is identical to Example 1 from Section 5.1. When $\widehat{x} \in \widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]} = [0.21, 0.47]$, $\gamma^*(\widehat{x}) \in [0, 0.56]$. When $\widehat{x} \in \widehat{\mathbf{X}} = [0.21, 0.54]$, $\widehat{x} \geq B(\widehat{x})$ holds. Then $\widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]} \cap \widehat{\mathbf{X}} = [0.21, 0.47]$. By choosing $\widehat{x} \in [0.21, 0.47]$, the principal can design a continuum of SonBo contracts, including the option-like contract $w_L^*(x)$ and the step bonus contract $w_B^*(x)$, to achieve the second-best outcome when both parties have limited liability.

Example 2': The setting is identical to Example 2 from Section 5.1. When $\widehat{x} \in \widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]} = [0.47, 0.57]$, $\gamma^*(\widehat{x}) \in [0,1]$. When $\widehat{x} \in \widehat{\mathbf{X}} = [0.35, 0.56]$, $\widehat{x} \geq B(\widehat{x})$ holds. Then $\widehat{\mathbf{X}}^{[\mathbf{0},\mathbf{1}]} \cap \widehat{\mathbf{X}} = [0.47, 0.56]$. By choosing $\widehat{x} \in [0.47, 0.56]$, the principal can design a continuum of SonBo contracts to achieve the second-best when both parties have limited liability, but neither an option-like contract nor a step bonus contract will work.

5.3 The optimal contract when (11) fails

When condition (11) fails, the second-best outcome can no longer be implemented. Notice that the second-best outcome is characterized by two features: the second-best efforts (a^*, e^*) and binding agent-PC. Now the principal faces a trade-off between desirable efforts and giving the agent a larger rent arising from his limited liability.

To solve this problem, we introduce an additional instrument, which is the rent that the principal is willing to give the agent, denoted as R. We will first treat R as an exogenous parameter, and then determine its optimal value.

First, define a threshold $\widehat{R}$ by

$$\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} = \frac{c'(e^*)}{c(e^*)+\overline{U}_e+\widehat{R}}.$$

If $R \geq \widehat{R}$, then condition $\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} \geq \frac{c'(e^*)}{c(e^*)+\overline{U}_e+R}$ holds. The second-best efforts (a^*, e^*) are still implementable by an adjusted SonBo contract $w_R^*(x)$. The contract $w_R^*(x)$ has the same format as (8), except that the value $\widehat{x}$ is determined by $H(\widehat{x})+B\cdot L(\widehat{x}) = c(e^*)+\overline{U}_e+R$, and the agent's ex ante expected payoff is equal to $\overline{U}_e + R$. The intuition is straightforward: by giving the a agent a large rent, condition (11) is restored, and the second-best efforts can be induced by SonBo. This scenario applies to a principal who is primarily concerned about implementing the desirable efforts but does not mind the payoff distribution. Many public-private partnerships in which the government acts as the principal fit well into this scenario, as the government cares mainly about how to induce the socially optimal efforts in the project, and the rent to the private partner does not affect social welfare.

If $R < \widehat{R}$, the principal faces the problem $[P(LL)]$ with one more constraint: $\int_{\underline{x}}^{\overline{x}} w(x)f(x|h(a,e))dx - c(e) \leq \overline{U}_e + R$. Suppose the new problem's optimal contract is $w_R^\star(x)$ and the corresponding efforts are $(a^\star, e^\star)$, referred to as "R-induced third-best efforts". We can prove that $(a^\star, e^\star) \neq (a^*, e^*)$.[8] Define $R_b = \int_{\underline{x}}^{\overline{x}} w_R^\star(x)f(x|h(a^\star,e^\star))dx - c(e^\star) - \overline{U}_e$, which measures how much rent the principal needs to give up in order to implement $(a^\star, e^\star)$ with binding agent-PC. It is straightforward to see that $0 \leq R_b \leq R$. In the adjusted SonBo contract $w_R^*(x)$, the variable $\widehat{x}$ is determined by $H(\widehat{x}) + B \cdot L(\widehat{x}) = c(e^\star) + \overline{U}_e + R_b$, and the agent's share γ^* is the same as (9) except the second-best efforts (a^*, e^*) are replaced by $(a^\star, e^\star)$.

Consider the following condition:

$$\frac{f_h(\overline{x}|h(a^\star,e^\star))h_e(a^\star,e^\star)}{f(\overline{x}|h(a^\star,e^\star))} \geq \frac{c'(e^\star)}{c(e^\star)+\overline{U}_e+R_b}. \tag{19}$$

Then we have the following result, which implies that, for any rent $R < \widehat{R}$, the adjusted SonBo contract $w_R^*(x)$ is as good as any other contract because they require the same sufficient and necessary condition.[9]

[8]Suppose (a^*, e^*) are optimal for the new problem. Based on agent-PC and agent-IC, we have

$$\frac{\int_{\underline{x}}^{\overline{x}} w(x)f_h(x|h(a^*,e^*))h_e(a^*,e^*)\,dx}{\int_{\underline{x}}^{\overline{x}} w(x)f(x|h(a^*,e^*))\,dx} \geq \frac{c'(e^*)}{c(e^*)+\overline{U}_e+R}.$$

Since the left hand side is not greater than $\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))}$, we have $\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} \geq \frac{c'(e^*)}{c(e^*)+\overline{U}_e+R}$, which violates the fact that $\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} < \frac{c'(e^*)}{c(e^*)+\overline{U}_e+R}$ for $R < \widehat{R}$.

[9]Refer to Section 6 of the Online Appendix for the proof of Proposition 5.

Proposition 5. *Suppose the principal has to give the agent an ex ante limited liability rent of $R < \widehat{R}$. Then the R-induced third-best efforts $(a^\star, e^\star)$ can be implemented by*

(i) the adjusted SonBo contract $w_R^*(x)$ if and only if (19) holds;

(ii) any contract if and only if (19) holds.

So far we have treated R as an exogenous parameter. The last step is to determine the optimal R, which should maximize the principal's ex ante expected payoff. When $R \geq \widehat{R}$, the principal's largest ex ante expected payoff is $EU(\widehat{R}) = \int_{\underline{x}}^{\overline{x}} xf(x|h(a^*, e^*))dx - c(e^*) - v(a^*) - \overline{U}_e - \widehat{R}$. For each $R < \widehat{R}$, the principal's ex ante expected payoff is $EU(R) = \int_{\underline{x}}^{\overline{x}} xf(x|h(a^\star, e^\star))dx - v(a^\star) - c(e^\star) - \overline{U}_e - R_b$, in which $a^\star$, $e^\star$ and R_b are functions of R. Let $\widetilde{R} = \arg\max_{R<\widehat{R}} EU(R)$. Then the optimal rent R is $\widehat{R}$ if and only if $EU(\widehat{R}) \geq EU(\widetilde{R})$. Otherwise, the optimal rent is $\widetilde{R}$.

5.4 Comparative statics

SonBo has three instruments with one degree of freedom. How does the value of one instrument affects the other two? Given (9), we have $\frac{\partial \gamma^*}{\partial B} = \frac{F_h(\widehat{x}|h(a^*,e^*))}{-\int_{\widehat{x}}^{\overline{x}} F_h(x|h(a^*,e^*))\,dx} < 0$ because $F_h(\widehat{x}|h(a^*, e^*)) < 0$ holds for any $\widehat{x} \in (\underline{x}, \overline{x})$.[10] This means that the agent's share and the bonus are strategic substitutes in SonBo contract. On the other hand, the signs of $\frac{\partial B}{\partial \widehat{x}}$ and $\frac{\partial \gamma^*}{\partial \widehat{x}}$ are in general indeterminate. In the two numerical examples introduced in section 5.1, the bonus B increases with $\widehat{x}$, and agent's share γ^* decreases with $\widehat{x}$.

The following example demonstrates how the effort and cost parameters affect the SonBo contract.

Example 1": The functions from Example 1 are generalized as follows: $f(x|h(a, e)) = 1 + \frac{1-2x}{h(a,e)+1}$, $x \in [0, 1]$, $h(a, e) = \mu_a a + \mu_e e$, $v(a) = \theta_a a^2$, $c(e) = \theta_e e^2$, $\overline{U}_e = 0.1$. Parameters μ_a and μ_e capture the two parties' marginal contributions to the composite effort, and parameters θ_a and θ_e indicate their marginal costs of effort. We find that a^* decreases with θ_a and μ_e, and increases with θ_e and μ_a; it is the opposite for e^*. That is, the principal's effeort is larger when it contributes more to the output or less costly to exert, etc. The impacts of θ is easy to understand. As for the impact of μ, note that the two parties' second-best efforts are substitutes given that $h(a, e) = \mu_a a + \mu_e e$. If μ_a is larger, the principal's effort contributes more to the output. Conversely, if μ_e is larger, the agent exerts more effort, which induces the principal to exert less effort.

For any given $B \geq 0$, the output threshold $\widehat{x}$ and the agent's share γ^* both increase with μ_e and θ_a and decrease with μ_a and θ_e. That is, when the agent is more productive,

[10] Refer to Section 5 of the Online Appendix for the proof.

the production relies more on the agent's effort. The principal would then set a higher threshold and higher share parameter for the agent.

6 Conclusion

This paper has studied optimal contracts in double moral hazard problem when both parties are risk-neutral and the agent faces limited liability. It is shown that SonBo contract can implement the second-best outcome and is therefore optimal. We have also identified a sufficient and necessary condition for SonBo to achieve the second-best outcome, and shown that this condition is sufficient and necessary for any optimal contract to implement the second-best outcome. The SonBo contract enjoys one degree of freedom, but the principal should be cautious when removing the bonus component or downgrading its value. Furthermore, when the principal and agent both face limited liabilities, SonBo contract may still implement the second-best outcome. Even when the second-best outcome is infeasible, SonBo is still optimal under additional conditions.

In closing, we would like to highlight two potential directions for future research. Firstly, when (11) fails, the SonBo contract is optimal under additional conditions, and the optimal contract needs further characterization when the second-best outcome is infeasible. Secondly, although the SonBo contract can achieve second-best under two-sided limited liability, it is still not a perfect solution because there may be some situations in which SonBo cannot achieve the second-best outcome. This deserves further investigations.

Appendix

This appendix collects the proofs of Proposition 1 and Proposition 2 in the paper. The proof of Proposition 5 is similar with the proof of Proposition 1 and Proposition 2 if we treat the agent's outside option as $\overline{U}_e + R_b$. Its proof is in the Section 6 of the Online Appendix.

A1 Proof of Proposition 1

Name $\eta^* = B - \gamma^*\widehat{x}$. Then $w^*(x) = \gamma^* x + \eta^*$ when $x \geq \widehat{x}$. Next, we prove the SonBo contract $w^*(x)$ can make agent-IC, principal-IC and agent-PC hold at the second-best effort levels (a^*, e^*).

Step 1: Agent-IC holds

In this section, we prove the contract $w^*(x)$ can make agent-IC hold. Given the principal chooses a^*, the marginal expected compensation of agent choosing effort $e \in R_+$ divided by $h_e(a^*, e)$ is:

$$\begin{aligned}
&\int_{\widehat{x}}^{\overline{x}} (\gamma^* x + \eta^*) f_h(x|h(a^*, e))\, dx \\
&= (\gamma^* x + \eta^*) F_h(x|h(a^*, e))\Big|_{\widehat{x}}^{\overline{x}} - \int_{\widehat{x}}^{\overline{x}} \gamma^* F_h(x|h(a^*, e))\, dx \\
&= -B \cdot F_h(\widehat{x}|h(a^*, e)) - \frac{\frac{c'(e^*)}{h_e(a^*,e^*)} + B \cdot F_h(\widehat{x}|h(a^*, e^*))}{-\int_{\widehat{x}}^{\overline{x}} F_h(x|h(a^*, e^*))\, dx} \int_{\widehat{x}}^{\overline{x}} F_h(x|h(a^*, e))\, dx.
\end{aligned}$$

The above equations hold because $F_h(\overline{x}|h) = 0$ and $\gamma^*\widehat{x} + \eta^* = B$. When $e = e^*$, the above expression is equal to $\frac{c'(e^*)}{h_e(a^*,e^*)}$, implying agent-IC holds.

Step 2: Principal-IC holds

Given the agent chooses e^*, the marginal expected compensation of principal choosing $a \in R_+$ divided by $h_a(a, e^*)$ is:

$$\begin{aligned}
&\int_{\underline{x}}^{\widehat{x}} x f_h(x|h(a, e^*))\, dx + \int_{\widehat{x}}^{\overline{x}} [(1-\gamma^*)x - \eta^*] f_h(x|h(a, e^*))\, dx \\
&= \int_{\underline{x}}^{\overline{x}} x f_h(x|h(a, e^*))\, dx + \frac{\frac{c'(e^*)}{h_e(a^*,e^*)} + B \cdot F_h(\widehat{x}|h(a^*, e^*))}{\int_{\widehat{x}}^{\overline{x}} F_h(x|h(a^*, e^*))\, dx} \cdot \int_{\widehat{x}}^{\overline{x}} x f_h(x|h(a, e^*))\, dx - \int_{\widehat{x}}^{\overline{x}} \eta^* f_h(x|h(a, e^*))\, dx.
\end{aligned} \tag{20}$$

To prove $w^*(x)$ induces principal to choose a^*, we need to prove equation (20) is equal to $\frac{v'(a^*)}{h_a(a^*,e^*)}$ when principal chooses $a = a^*$. Since

$$\int_{\underline{x}}^{\overline{x}} x f_h(x|h(a^*, e^*)) dx = \frac{v'(a^*)}{h_a(a^*, e^*)} + \frac{c'(e^*)}{h_e(a^*, e^*)},$$

to prove equation (20) is equal to $\frac{v'(a^*)}{h_a(a^*,e^*)}$, we need to prove:

$$\frac{\frac{c'(e^*)}{h_e(a^*,e^*)} + B \cdot F_h(\widehat{x}|h(a^*,e^*))}{\int_{\widehat{x}}^{\overline{x}} F_h(x|h(a^*,e^*))\,dx} \cdot \int_{\widehat{x}}^{\overline{x}} x f_h(x|h(a^*,e^*))\,dx - \int_{\widehat{x}}^{\overline{x}} \eta^* f_h(x|h(a^*,e^*))\,dx = -\frac{c'(e^*)}{h_e(a^*,e^*)}$$

$$\iff \gamma^* \int_{\widehat{x}}^{\overline{x}} x f_h(x|h(a^*,e^*))\,dx + \int_{\widehat{x}}^{\overline{x}} \eta^* f_h(x|h(a^*,e^*))\,dx = \frac{c'(e^*)}{h_e(a^*,e^*)} \tag{21}$$

Since $w^*(x)$ can induce the agent to choose e^* when the principal chooses a^*, we can easily get:

$$\gamma^* \int_{\widehat{x}}^{\overline{x}} x f_h(x|h(a^*,e^*))\,dx + \eta^* \int_{\widehat{x}}^{\overline{x}} f_h(x|h(a^*,e^*))\,dx = \frac{c'(e^*)}{h_e(a^*,e^*)}.$$

So equation (21) is true, which implies that the proposed $w^*(x)$ can induce the principal to choose a^* given the agent chooses e^*.

Thus we have proved, given there exists $w^*(x)$ to make agent-PC binding, the $w^*(x)$ can implement second-best effort choices.

Step 3: Conditions for binding Agent-PC

Under SonBo $w^*(x)$, agent's expected compensation, denoted as $G(B,\widehat{x})$, is:

$$G(B,\widehat{x}) = \gamma^* \int_{\widehat{x}}^{\overline{x}} (x-\widehat{x}) f(x|h(a^*,e^*))\,dx + B \int_{\widehat{x}}^{\overline{x}} f(x|h(a^*,e^*))\,dx.$$

$G(B,\widehat{x})$ can be factored into two parts: $G(B,\widehat{x}) = H(\widehat{x}) + B \cdot L(\widehat{x})$.

The function $H(\widehat{x})$ is strictly decreasing on $[\underline{x},\overline{x})$. The function $L(\widehat{x})$ is the weight on bonus B. The domain of $H(\widehat{x})$ and $L(\widehat{x})$ is $[\underline{x},\overline{x})$. We extend the domain of these two functions to $[\underline{x},\overline{x}]$ by defining: $H(\overline{x}) = \lim_{\widehat{x}\to\overline{x}} H(\widehat{x})$ and $L(\overline{x}) = \lim_{\widehat{x}\to\overline{x}} L(\widehat{x})$. These extensions make $H(\widehat{x})$ and $L(\widehat{x})$ continuous on $[\underline{x},\overline{x}]$.

Step 3.1

Define $\eta^\star = -\gamma^\star \widehat{x}^\star$. Given a feasible B, $G(B,\underline{x}) = H(\underline{x}) + B$ because $L(\underline{x}) = 1$, and

$$\begin{aligned}
H(\underline{x}) &= \gamma^\star \int_{\underline{x}}^{\overline{x}} (x-\underline{x}) f(x|h(a^*,e^*))\,dx \\
&= \gamma^\star \int_{\underline{x}}^{\overline{x}} x f(x|h(a^*,e^*))\,dx + \eta^\star \int_{\underline{x}}^{\overline{x}} f(x|h(a^*,e^*))\,dx - \eta^\star \int_{\underline{x}}^{\overline{x}} f(x|h(a^*,e^*))\,dx - \gamma^\star \underline{x} \int_{\underline{x}}^{\overline{x}} f(x|h(a^*,e^*))\,dx \\
&= \int_{\underline{x}}^{\overline{x}} [\gamma^\star x + \eta^\star] f(x|h(a^*,e^*))\,dx - \eta^\star - \gamma^\star \underline{x}.
\end{aligned}$$

We have proved that without limited liability, the optimal contract $w^\star(x) = \gamma^\star x + \eta^\star$ can make the agent-PC binding, that is $\int_{\underline{x}}^{\overline{x}} [\gamma^\star x + \eta^\star] f(x|h(a^*,e^*))\,dx = c(e^*) + \overline{U}_e$. So

$$G(B,\underline{x}) = c(e^*) + \overline{U}_e - \eta^\star - \gamma^\star \underline{x} + B.$$

Given Assumption 4 that the linear sharing contract is not feasible, i.e. $\gamma^\star \underline{x} + \eta^\star < 0$,

$$G(B,\underline{x}) - c(e^*) - \overline{U}_e = B - (\gamma^\star \underline{x} + \eta^\star) > 0.$$

So $G(B,\underline{x}) > c(e^*) + \overline{U}_e$.

Step 3.2

Next, we prove $G(B,\overline{x}) \le c(e^*) + \overline{U}_e$ if and only if $\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} \ge \frac{c'(e^*)}{c(e^*)+\overline{U}_e}$. Given a feasible B, $G(B,\overline{x}) = H(\overline{x})$ because $L(\overline{x}) = 0$. We will prove $H(\overline{x}) \le c(e^*) + \overline{U}_e$ is equivalent to $\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} \ge \frac{c'(e^*)}{c(e^*)+\overline{U}_e}$.

$H(\overline{x})$ is solved from the following limit problem.

$$H(\overline{x}) = \lim_{\widehat{x}\to\overline{x}} H(\widehat{x}) = \lim_{\widehat{x}\to\overline{x}} \frac{\int_{\widehat{x}}^{\overline{x}}(x-\widehat{x})f(x|h(a^*,e^*))\,dx}{\int_{\widehat{x}}^{\overline{x}} F_h(x|h(a^*,e^*))\,dx}\frac{c'(e^*)}{h_e(a^*,e^*)} = \frac{f(\overline{x}|h(a^*,e^*))}{f_h(\overline{x}|h(a^*,e^*))}\frac{c'(e^*)}{h_e(a^*,e^*)}.$$

The above process uses L'Hospital's rule and $F_h(\overline{x}|h(a^*,e^*)) = 0$. It should be noticed that $f_h(\overline{x}|h(a^*,e^*)) > 0$.[11]

So

$$H(\overline{x}) \le c(e^*) + \overline{U}_e \iff \frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} \ge \frac{c'(e^*)}{c(e^*)+\overline{U}_e}.$$

Combining step 3.1 and step 3.2, there must exist one $\widehat{x} \in [\underline{x},\overline{x}]$ to make agent-PC hold.

Step 3.3

Next, we prove given a feasible B, if there exists one $\widehat{x} \in [\underline{x},\overline{x}]$ to make $G(B,\widehat{x}) = c(e^*) + \overline{U}_e$ hold, the condition $\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} \ge \frac{c'(e^*)}{c(e^*)+\overline{U}_e}$ must hold.

If there exists such $\widehat{x}$, SonBo achieves the second-best outcome, so agent-IC holds and agent-PC binds. The binding agent-PC and agent-IC are

$$\int_{\widehat{x}}^{\overline{x}}(\gamma^*x+\eta^*)f_h(x|h(a^*,e^*))h_e(a^*,e^*)\,dx = c'(e^*),$$
$$\int_{\widehat{x}}^{\overline{x}}(\gamma^*x+\eta^*)f(x|h(a^*,e^*))\,dx = c(e^*) + \overline{U}_e.$$

Combing these two equations,

$$\frac{\int_{\widehat{x}}^{\overline{x}}(\gamma^*x+\eta^*)f_h(x|h(a^*,e^*))h_e(a^*,e^*)\,dx}{\int_{\widehat{x}}^{\overline{x}}(\gamma^*x+\eta^*)f(x|h(a^*,e^*))\,dx} = \frac{c'(e^*)}{c(e^*)+\overline{U}_e}.$$

Define

$$\psi(z) = \frac{\int_{z}^{\overline{x}}(\gamma^*x+\eta^*)f_h(x|h(a^*,e^*))h_e(a^*,e^*)\,dx}{\int_{z}^{\overline{x}}(\gamma^*x+\eta^*)f(x|h(a^*,e^*))\,dx},\ z \in [\widehat{x},\overline{x}].$$

Let $h^* = h(a^*,e^*)$ and $h_e^* = h_e(a^*,e^*)$.

$$\psi'(z) = \frac{-(\gamma^*z+\eta^*)f_h(z|h^*)h_e^*\cdot\int_z^{\overline{x}}(\gamma^*x+\eta^*)f(x|h^*)\,dx + \int_z^{\overline{x}}(\gamma^*x+\eta^*)f_h(x|h^*)h_e^*\,dx\cdot(\gamma^*z+\eta^*)f(z|h^*)}{[\int_z^{\overline{x}}(\gamma^*x+\eta^*)f(x|h(a^*,e^*))\,dx]^2}.$$

[11] This is because $\int_{\underline{x}}^{\overline{x}} f(x|h(a^*,e^*))\,dx = 1$, we can have $\int_{\underline{x}}^{\overline{x}} \frac{f_h(x|h(a^*,e^*))}{f(x|h(a^*,e^*))}f(x|h(a^*,e^*))\,dx = 0$. Due to MLRP, we conclude that there exists a $x_0 \in (\underline{x},\overline{x})$ such that $\frac{f_h(x|h(a^*,e^*))}{f(x|h(a^*,e^*))} < 0$ for $x < x_0$, $\frac{f_h(x_0|h(a^*,e^*))}{f(x_0|h(a^*,e^*))} = 0$, and $\frac{f_h(x|h(a^*,e^*))}{f(x|h(a^*,e^*))} > 0$ for $x > x_0$. So $f_h(\overline{x}|h(a^*,e^*)) > 0$ and the result of the limit problem is valid.

SonBo achieves second-best outcome under agent's limited liability, so it gives the agent non-negative payment at any possible output. Thus, given a feasible B, when $z \geq \widehat{x}$, $\gamma^* z + \eta^* > 0$, so $\psi'(z) \geq 0$ is equivalent to

$$\frac{\int_z^{\overline{x}} (\gamma^* x + \eta^*) f_h(x|h^*)\, dx}{\int_z^{\overline{x}} (\gamma^* x + \eta^*) f(x|h^*)\, dx} \geq \frac{(\gamma^* z + \eta^*) f_h(z|h^*)}{(\gamma^* z + \eta^*) f(z|h^*)}.$$

Based on MLRP, for any $z_1 \geq z_2 \geq z$,

$$\frac{(\gamma^* z_1 + \eta^*) f_h(z_1|h^*)}{(\gamma^* z_1 + \eta^*) f(z_1|h^*)} \geq \frac{(\gamma^* z_2 + \eta^*) f_h(z_2|h^*)}{(\gamma^* z_2 + \eta^*) f(z_2|h^*)}$$

$$\iff (\gamma^* z_1 + \eta^*) f_h(z_1|h^*)(\gamma^* z_2 + \eta^*) f(z_2|h^*) \geq (\gamma^* z_2 + \eta^*) f_h(z_2|h^*)(\gamma^* z_1 + \eta^*) f(z_1|h^*).$$

Since z_1 and z_2 are arbitrary, integrate z_1 from z_2 to $\overline{x}$ and let $z_2 = z$:

$$\int_z^{\overline{x}} (\gamma^* z_1 + \eta^*) f_h(z_1|h^*)\, dz_1 \cdot (\gamma^* z + \eta^*) f(z|h^*) \geq (\gamma^* z + \eta^*) f_h(z|h^*) \cdot \int_z^{\overline{x}} (\gamma^* z_1 + \eta^*) f(z_1|h^*)\, dz_1$$

$$\iff \frac{\int_z^{\overline{x}} (\gamma^* z_1 + \eta^*) f_h(z_1|h^*)\, dz_1}{\int_z^{\overline{x}} (\gamma^* z_1 + \eta^*) f(z_1|h^*)\, dz_1} \geq \frac{(\gamma^* z + \eta^*) f_h(z|h^*)}{(\gamma^* z + \eta^*) f(z|h^*)}$$

$$\iff \frac{\int_z^{\overline{x}} (\gamma^* x + \eta^*) f_h(x|h^*)\, dx}{\int_z^{\overline{x}} (\gamma^* x + \eta^*) f(x|h^*)\, dx} \geq \frac{(\gamma^* z + \eta^*) f_h(z|h^*)}{(\gamma^* z + \eta^*) f(z|h^*)}.$$

Thus, we have proved $\psi'(z) \geq 0$ for $z \in [\widehat{x}, \overline{x}]$. Since the function $\psi(z)$ is increasing, $\lim_{z \to \overline{x}} \psi(z) \geq \psi(\widehat{x}) = \frac{c'(e^*)}{c(e^*) + \overline{U}_e}$.

Notice that

$$\lim_{z \to \overline{x}} \psi(z) = \frac{(\gamma^* \overline{x} + \eta^*) f_h(\overline{x}|h(a^*, e^*)) h_e(a^*, e^*)\, dx}{(\gamma^* \overline{x} + \eta^*) f(\overline{x}|h(a^*, e^*))\, dx} = \frac{f_h(\overline{x}|h(a^*, e^*)) h_e(a^*, e^*)\, dx}{f(\overline{x}|h(a^*, e^*))\, dx}.$$

So if there exists any $\widehat{x} \in [\underline{x}, \overline{x}]$ to make agent-PC binding, we should have

$$\frac{f_h(\overline{x}|h(a^*, e^*)) h_e(a^*, e^*)}{f(\overline{x}|h(a^*, e^*))} \geq \frac{c'(e^*)}{c(e^*) + \overline{U}_e}.$$

A2 Proof of Proposition 2

Suppose contract $w(x)$ is an optimal solution to $[P(LL)]$. Next, we prove this contract must satisfy condition (11). Contract $w(x)$ achieves second-best outcome means that it can make agent-PC binding and agent-IC and principal-IC hold when effort levels are (a^*, e^*). Based on the binding agent-PC and agent-IC:

$$\int_{\underline{x}}^{\overline{x}} w(x) f(x|h(a^*, e^*))\, dx = c(e^*) + \overline{U}_e,$$

$$\int_{\underline{x}}^{\overline{x}} w(x) f_h(x|h(a^*, e^*)) h_e(a^*, e^*)\, dx = c'(e^*),$$

which implies

$$\frac{\int_{\underline{x}}^{\overline{x}} w(x) f_h(x|h(a^*, e^*)) h_e(a^*, e^*)\, dx}{\int_{\underline{x}}^{\overline{x}} w(x) f(x|h(a^*, e^*))\, dx} = \frac{c'(e^*)}{c(e^*) + \overline{U}_e}. \tag{22}$$

Step 1

If $w(x)$ is an optimal solution of $[P(LL)]$, (22) should hold. Next we prove if (22) holds, condition (11) should hold.

Define $V(x_c) = \frac{\int_{x_c}^{\overline{x}} w(x) f_h(x|h(a,e)) h_e(a,e)\, dx}{\int_{x_c}^{\overline{x}} w(x) f(x|h(a,e))\, dx}$. The effort choices (a, e) in $V(x_c)$ can be any feasible effort choices including (a^*, e^*). Notice that the left-hand side of (22) is equal to $V(\underline{x})$ when $(a, e) = (a^*, e^*)$. We write $f(x|h(a,e))$ as $f(x|h)$ and $h_e(a,e)$ as h_e for short. In this step, we prove $V(x_c)$ is an increasing function of x_c for any pair of (a, e) , i.e., $\frac{dV(x_c)}{dx_c} \geq 0$ for any (a, e). We have

$$\frac{dV(x_c)}{dx_c} = \frac{w(x_c) h_e \big[f(x_c|h) \int_{x_c}^{\overline{x}} w(x) f_h(x|h)\, dx - f_h(x_c|h) \int_{x_c}^{\overline{x}} w(x) f(x|h)\, dx \big]}{\big[\int_{x_c}^{\overline{x}} w(x) f(x|h)\, dx \big]^2}.$$

When $w(x_c) = 0$, $\frac{dV(x_c)}{dx_c} = 0$.

When $w(x_c) > 0$, since $h_e > 0$, $\frac{dV(x_c)}{dx_c} \geq 0$ is equivalent to:

$$\frac{\int_{x_c}^{\overline{x}} w(x) f_h(x|h)\, dx}{\int_{x_c}^{\overline{x}} w(x) f(x|h)\, dx} \geq \frac{f_h(x_c|h)}{f(x_c|h)}. \tag{23}$$

Notice that when $w(x_c) > 0$, $\frac{f_h(x_c|h)}{f(x_c|h)} = \frac{w(x_c) f_h(x_c|h)}{w(x_c) f(x_c|h)}$. Then (23) is equivalent to:

$$\frac{\int_{x_c}^{\overline{x}} w(x) f_h(x|h)\, dx}{\int_{x_c}^{\overline{x}} w(x) f(x|h)\, dx} \geq \frac{w(x_c) f_h(x_c|h)}{w(x_c) f(x_c|h)}. \tag{24}$$

Next, we prove (24) holds. Based on MLRP, the function $\frac{f_h(x|h)}{f(x|h)}$ is increasing with $x \geq x_c$ when $w(x) \neq 0$. Pick x_1 and x_2 such that $x_1 \geq x_2 \geq x_c$. If $w(x_1) \neq 0$ and $w(x_2) \neq 0$,

$$\frac{w(x_1) f_h(x_1|h)}{w(x_1) f(x_1|h)} \geq \frac{w(x_2) f_h(x_2|h)}{w(x_2) f(x_2|h)}$$
$$\Longleftrightarrow w(x_1) f_h(x_1|h) w(x_2) f(x_2|h) \geq w(x_2) f_h(x_2|h) w(x_1) f(x_1|h).$$

If $w(x_1) = 0$ or $w(x_2) = 0$ or both,

$$w(x_1) f_h(x_1|h) w(x_2) f(x_2|h) \geq w(x_2) f_h(x_2|h) w(x_1) f(x_1|h).$$

So for $x_1 \geq x_2 \geq x_c$,

$$w(x_1) f_h(x_1|h) w(x_2) f(x_2|h) \geq w(x_2) f_h(x_2|h) w(x_1) f(x_1|h).$$

Since x_1 and x_2 are arbitrary, integrate x_1 from x_2 to $\overline{x}$ and let $x_2 = x_c$. Then

$$\int_{x_c}^{\overline{x}} w(x_1) f_h(x_1|h)\, dx_1 \cdot w(x_c) f(x_c|h) \geq w(x_c) f_h(x_c|h) \cdot \int_{x_c}^{\overline{x}} w(x_1) f(x_1|h)\, dx_1$$
$$\Longleftrightarrow \frac{\int_{x_c}^{\overline{x}} w(x) f_h(x|h)\, dx}{\int_{x_c}^{\overline{x}} w(x) f(x|h)\, dx} \geq \frac{w(x_c) f_h(x_c|h)}{w(x_c) f(x_c|h)}.$$

Thus (24) is true. So the function $\frac{dV(x_c)}{dx_c} \geq 0$ when $w(x_c) > 0$.

So far we have proved that given the contract $w(x) \geq 0$ for any $x \geq x_c$, $\frac{dV(x_c)}{dx_c} \geq 0$.

Step 2

The (a, e) in $V(x_c)$ is arbitrary. Now let $(a, e) = (a^*, e^*)$. Since $V(x_c)$ is increasing, $V(x_c) \geq V(\underline{x})$ for $x_c > \underline{x}$. Condition (22) implies that $V(x_c) \geq \frac{c'(e^*)}{c(e^*)+\overline{U}_e}$ for $x_c > \underline{x}$. Furthermore, we should have

$$\lim_{x_c \to \overline{x}} V(x_c) \geq \frac{c'(e^*)}{c(e^*) + \overline{U}_e}.$$

Based on L'Hospital's rule,

$$\lim_{x_c \to \overline{x}} V(x_c) = \lim_{x_c \to \overline{x}} \frac{w(x_c) f_h(x_c|h(a^*, e^*)) h_e(a^*, e^*)}{w(x_c) f(x_c|h(a^*, e^*))} = \lim_{x_c \to \overline{x}} \frac{f_h(x_c|h(a^*, e^*)) h_e(a^*, e^*)}{f(x_c|h(a^*, e^*))} = \frac{f_h(\overline{x}|h(a^*, e^*)) h_e(a^*, e^*)}{f(\overline{x}|h(a^*, e^*))}.$$

So given the contract $w(x)$ implements the second-best outcome, i.e. condition (22) holds, $\frac{f_h(\overline{x}|h(a^*,e^*))h_e(a^*,e^*)}{f(\overline{x}|h(a^*,e^*))} \geq \frac{c'(e^*)}{c(e^*)+\overline{U}_e}$ holds.

Next, we prove when condition (11) holds, there must exist contracts to implement the second-best outcome of $[P(LL)]$. The proof is simple as we have proved when condition (11) holds, SonBo contract exists and it implements the second-best outcome.

References

Bhattacharyya, Sugato and Francine Lafontaine. 1995. "Double-sided moral hazard and the nature of share contracts." *The RAND Journal of Economics* 26 (4):761–781.

Brickley, James A. 2002. "Royalty rates and upfront fees in share contracts: evidence from franchising." *Journal of Law, Economics, and Organization* 18 (2):511–535.

Griva, Krina and Nikolaos Vettas. 2015. "On two-part tariff competition in a homogeneous product duopoly." *International Journal of Industrial Organization* 41:30–41.

Hölmstrom, Bengt. 1982. "Moral hazard in teams." *The Bell Journal of Economics* 13 (2):324–340.

Innes, Robert D. 1990. "Limited liability and incentive contracting with ex-ante action choices." *Journal of Economic Theory* 52 (1):45–67.

Jewitt, Ian. 1988. "Justifying the first-order approach to principal-agent problems." *Econometrica* 56 (5):1177–1190.

Jewitt, Ian, Ohad Kadan, and Jeroen M Swinkels. 2008. "Moral hazard with bounded payments." *Journal of Economic Theory* 143 (1):59–82.

Kadan, Ohad and Jeroen M Swinkels. 2013. "Minimum payments and induced effort in moral hazard problems." *Games and Economic Behavior* 82:468–489.

Kim, Son Ku. 1997. "Limited liability and bonus contracts." *Journal of Economics & Management Strategy* 6 (4):899–913.

Kim, Son Ku and Susheng Wang. 1998. "Linear contracts and the double moral-hazard." *Journal of Economic Theory* 82 (2):342–378.

Milgrom, Paul R. 1981. "Good news and bad news: Representation theorems and applications." *The Bell Journal of Economics* 12 (2):380–391.

Park, Eun-Soo. 1995. "Incentive contracting under limited liability." *Journal of Economics & Management Strategy* 4 (3):477–490.

Pitegoff, Thomas M. and W. Michael Garner, editors. 2008. *Fundamentals of Franchising*. American Bar Association.

Poblete, Joaquín and Daniel Spulber. 2012. "The form of incentive contracts: agency with moral hazard, risk neutrality, and limited liability." *The RAND Journal of Economics* 43 (2):215–234.

Rogerson, William P. 1985. "The first-order approach to principal-agent problems." *Econometrica* 53 (6):1357–1367.

Romano, Richard E. 1994. "Double moral hazard and resale price maintenance." *The RAND Journal of Economics* 25 (2):455–466.

Suzuki, Yutaka. 2007. "Commitment problem, optimal incentive schemes, and relational contracts in agency with bilateral moral hazard." *Journal of International Economic Studies* 21:103–124.

Wang, Susheng and Tian Zhu. 2005. "Control allocation, revenue sharing, and joint ownership." *International Economic Review* 46 (3):895–915.

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