# Optimal Contract for Machine Repair and Maintenance

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A principal hires an agent to repair a machine when it is down and maintain it when it is up, and earns a flow revenue when the machine is up. Both the up and down times follow exponential distributions. If the agent exerts effort, the downtime is shortened, and uptime is prolonged. Effort, however, is costly to the agent and unobservable to the principal. We study optimal dynamic contracts that always induce the agent to exert effort while maximizing the principal's profits. We formulate the contract design problem as a stochastic optimal control model with incentive constraints in continuous time over an infinite horizon. Although we consider the contract space that allows payments and potential contract termination time to take general forms, the optimal contracts demonstrate simple and intuitive structures, making them easy to describe and implement in practice.

*Key words*: dynamic, moral hazard, optimal control, jump process, maintenance

## 1. Introduction

In this paper, we study a dynamic contract design problem over an infinite horizon, in which a principal hires an agent to more efficiently operate a production process ("machine"), which changes between two states: up and down. The state of the machine is public information. The "up" state yields a constant flow of revenue to the principal. The machine is subject to random shocks which causes it to go "down." When it is "down," the machine can be repaired to be "up" again. Without the agent, the machine stays in the up and down states for exponentially distributed random time periods with certain baseline rates. The agent has the expertise to improve maintenance and repair procedures by reducing the instantaneous rate for breaking down, and increasing the instantaneous rate to recover from the down state, if the agent exerts effort. Exerting effort is costly to the agent, and the effort cost may be different for repairing or maintaining the machine. Whether and when the agent puts in effort is the agent's private information. The principal would like to induce the agent's effort, and is able to commit to a long term contract, which involves payments and potential termination contingent on public information. We allow general forms of payments, including both instantaneous and flow payments. The principal is allowed to terminate the contract at any time, including terminating the contract with a probability less than one when the machine changes state. We also assume that the agent has limited liability. That is, the agent can decide to quit and never owes money to the principal.[1] Both players are risk neutral.

Although there is a wide literature on maintenance and repair, the majority of this literature is focused on optimal maintenance and repair conducted by a central decision maker, and has largely ignored the issues caused by agency. In many practical settings, however, maintenance and repair is conducted by an agent. Maintenance outsourcing is quite common in airline, aerospace, defense and mining industries, that often rely on complex, heavy and critical equipment (Tarakci et al. 2006). Instead of investing in the latest maintenance tools and facilities, and training in-house maintenance teams, firms outsource maintenance activities to specialized companies (McFadden and Worrells 2012). It may be hard for firms to observe whether maintenance companies put sufficient resources into providing best service, which gives rise to agency issues. Therefore, our paper makes a contribution to the maintenance/repair literature by tackling agency issues. In particular, we study a dynamic principal-agent framework, in which we obtain optimal contracts among history dependent ones. Despite the complexity of history dependent contracts, we demonstrate that the optimal contracts possess very simple structures that are easy to compute and implement. Further distinguished from the existing service/maintenance contract literature, we allow the agent to have limited liability and the ability to walk away at any point in time. Therefore, our contracts need to satisfy participation constraints, which guarantee that the agent stays before contract termination.

The paper also contributes to the dynamic contract design literature by considering an environment with *two* (machine) states. It is standard to formulate dynamic contract design problems as continuous time stochastic optimal control problems with incentive compatibility constraints, in which the agent's "promised utility" (see, for example, Spear and Srivastava 1987) constitutes the state space. Our state space, however, also needs to include the machine state, which yields dynamics that do not appear to arise in traditional settings without such a multi-state environment. The paper studies all of the following three possible cases, although the main body of the paper is focused on the third one: (1) the principal only needs the agent when the machine is down; (2) the principal only needs the agent when the machine is up; and (3) the principal needs the agent for both types of work.

The classical maintenance literature is focused on optimal scheduling in a centralized context (see, for example, Pierskalla and Voelker 1976, Paz and Leigh 1994, McCall 1965, Barlow and Proschan 1965, Gupta et al. 2001). Several papers consider maintenance outsourcing contracts involving a maintenance agent and a customer. In particular, Murthy and Asgharizadeh (1998) study a game-theoretic model in which an agent offers several options of contracts to a customer, including charging a fee for each repair during a given duration of time, or charging a lump sum fee for repairing the machine whenever it breaks down. The customer decides whether to hire the agent depending on the proposed contract. Murthy and Asgharizadeh (1999) extends the model to include multiple customers. Asgharizadeh and Murthy (2000) further allows the agent to choose the number of customers and the number of service channels besides a pricing strategy. Following this line of work, Tarakci et al. (2006) develop incentive contracts to achieve channel coordination. Kim et al. (2010) consider performance-based contracts for recovery services where the disruptions occur infrequently when the agent is risk-averse. They compare two types of widely used contracts, one based on sample-average downtime and the other based on cumulative downtime according to the supplier's ability to influence the frequency of disruptions. A clear distinction of our paper is that we consider time-dependent dynamic contracts while the aforementioned papers either consider static settings or repeated single-period settings. Other papers with static or repeated single-period settings include Tarakci et al. (2009), Wang (2010), Pakpahan and Iskandar (2015), Baker (2006), Cohen (1987), Tarakci et al. (2014). A common assumption in this literature is that the agent decides on the effort level (or, equivalently, capacity level) only once, then sticks to this level regardless of further outcomes. In many settings, an agent is often able to adjust effort choices over time. If a contract ignores such possibilities, the agent may lose the incentive to stick to the effort level as intended by the designer. Our model avoids this incentive issue because it is dynamic.

Plambeck and Zenios (2000) is the first paper to consider a dynamic principal-agent model of maintenance contract design in a discrete-time setting with a finite time horizon. In each period, if the machine is down, a risk-averse manager (agent) could choose between high and low effort levels, which further determine the probabilities that the machine comes back up in the following period. There is no moral hazard issue when the machine is up. More fundamentally, the agent in their model has access to borrowing and lending at the same rate, which means that they do not assume limited liability. Limited liability is a key assumption widely adopted in the dynamic contract literature (see, for example, Biais et al. 2010, Green and Taylor 2016, Sun and Tian 2017). Without it, the principal can essentially sell the entire enterprise to the agent, and therefore align incentives in a rather trivial fashion. The long-term optimal contract in Plambeck and Zenios (2000) is history-independent and renegotiation-proof. These nice properties rely critically on the borrowing and lending interest rates being exactly the same, and the agent's utility is additively separable and exponential. Following their optimal contract, in case the machine performance is bad for a period of time, the agent may have to borrow large amounts against future income, resulting in a negative total future utility. We, on the other hand, assume limited liability, which allows the agent to simply walk away (contract termination) instead of bearing a large debt.

The origin of the continuous time dynamic contract literature is often credited to the seminal paper Sannikov (2008), which considers a principal hiring an agent to control the drift of a Brownian motion. Several papers have applied similar techniques in different settings with applications mostly in corporate finance (see, for example, DeMarzo and Sannikov 2006, Biais et al. 2007, Fu 2015, to name a few).

Instead of controlling the drift of a Brownian motion, in our model, the agent exerts effort to change the arrival rates of Poisson processes. Previous literature has studied one-sided problems, i.e., either decreasing or increasing the arrival rate of a Poisson process. Biais et al. (2010), for example, considers a firm (principal) hiring a manager (agent) to exert private effort to decrease the arrival rate of large losses, modeled as a Poisson process, when the two players have different time discount rates. Myerson (2015) studies essentially the same model as in Biais et al. (2010), except that the two players share the same time discount rate. In contrast, Sun and Tian (2017) consider the case of increasing the arrival rate of a Poisson process by the agent's private effort. Varas (2017), Shan (2017), and Green and Taylor (2016) study similar models with a finite number of arrivals and additional features, such as adverse selection issues or multiple players.

Because of limited liability, the optimal contract structures are different for decreasing versus increasing arrival rates. The common theme between the one-sided cases is that the optimal dynamic contracts often involve letting the promised utility to take a constant jump upon an arrival, which is upward for the case of increasing the arrival rate, and downward for decreasing the arrival rate. Our paper generalizes the previous literature by studying contracts that induce the agent to alternatively increase and decrease two different arrival rates over time (increase the rate of repair, and decrease the rate of failure). The combined control problem is more complex, and the optimal solution more intricate. In particular, the dynamics of the promised utility following our optimal control policy is not a mere combination of one-sided control policies.

Specifically, whenever the machine is repaired from a down state, the agent needs to be at least rewarded with an amount (denoted as $\beta_{\mathbf{d}}$). This amount $\beta_{\mathbf{d}}$ is set to exactly compensate the agent's effort to repair when the machine is down, so that the agent is (barely) willing to exert effort. The reward could take the form of either an increase in promised utility or a direct payment. Similarly, whenever the machine breaks down from an upstate, the agent needs to be penalized with an amount (denoted as $\beta_{\mathbf{u}}$), set to be exactly enough to induce the agent's effort to maintain when the machine is up. However, due to limited liability, the principal cannot charge the agent money. Therefore the penalty $\beta_{\mathbf{u}}$ takes the form of a reduction of promised utility. When the agent's promised utility is already lower than $\beta_{\mathbf{u}}$, we cannot reduce the promised utility by $\beta_{\mathbf{u}}$ anymore since the agent is protected by limited liability. Instead, in the optimal contract, the principal applies random termination to incentivize the agent to exert effort when the promised utility is low. The exact optimal contract structure differ between the cases of $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$. In both cases, the optimal contracts possess interesting structures only if the revenue rate the principal accrues when the machine is up is high enough.

If $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, the structure of the optimal contract is not quite surprising. The aforementioned reward and penalties are always set at the minimum levels $\beta_{\mathbf{d}}$ and $\beta_{\mathbf{u}}$ respectively, and random termination never happens. If $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, however, the optimal contract has a much more complex and delicate structure, and it has the following two intricate features.

First, it is possible that the principal rewards the agent with an amount more than the minimum necessary to incentivize the effort, i.e, the incentive compatibility constraints are not always binding. This is in contrast to previous papers (see, for example, Sannikov 2008, Biais et al. 2010, Sun and Tian 2017), where the incentive compatibility constraints are always binding in the optimal contract.

Second, because of limited liability and incentive compatibility, the agent's continuation utility cannot be attained below a threshold when the machine is up. To mitigate this possibility, we need to introduce random termination, where we randomly decide whether to terminate the agent or let the continuation utility increase back up to the threshold for free. This random termination also appears in Myerson (2015), in which the threshold is fixed at $\beta_{\mathbf{u}}$ (using our paper's notation). In our paper, however, the threshold is endogenously determined and may be higher than $\beta_{\mathbf{u}}$. We use a "smooth-pasting" technique to derive this threshold. This technique is classical in optimal stopping problem (Dixit and Pindyck 1994) and has been used in optimal contract literature (see, for example, Zhu 2013, Chen et al. 2017).

Overall, we consider the aforementioned two features as the most interesting and intricate results of this paper. In particular, the first one appears new in the literature, while the second one constitutes a major technical challenge in the analysis. Therefore, Section 4.2 contains the most interesting results, while earlier sections allow readers to gradually ease into them.

Specifically, we introduce the model and derive the incentive compatibility constraints in Section 2. In Section 3, we derive simple incentive compatible contracts without termination. In Section 4.1 and 4.2, we characterize the optimal incentive compatible contract under the condition $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, respectively. Section 4.3 summarizes all the results thus far. In certain settings, it may be better for the principal not to always induce effort from the agent. Therefore, in Section 5, we numerically compare the optimal incentive compatible contracts and two other alternative contracts that only induce effort for one of the machine states. Formal derivation and analysis of these alternative contracts are in the e-companion.

## 2. Model

Consider a principal operating a process (e.g. a "machine") in a continuous time setting. At any time $t$, the state of the machine, $\theta_t$, is either up or down, denoted as $\mathbf{u}$ or $\mathbf{d}$, respectively. The principal receives a revenue at a positive rate $R$ per unit of time when the machine is up. When the machine is down, the revenue is zero.[2] The machine remains in the up state for an exponentially distributed random time with rate $\bar{\mu}_{\mathbf{u}} > 0$ before breaking down. Once down, it takes an exponentially distributed random time with rate $\underline{\mu}_{\mathbf{d}} > 0$ to repair the machine back to state $\mathbf{u}$. There are many settings in which the above described situation arises. For example, factories produce products to be sold for revenue when their equipment are working. Similarly, airlines only generate revenue when their planes are functioning. (Obviously, most airlines have more than one plane, and factories more than one machine. Our model can be considered as a building block for multi-machine settings.)

The principal hires an agent to improve the process. Whenever the agent exerts effort (for example, assigning sufficient personnel to this job), the instantaneous rate of breaking down from state $\mathbf{u}$ is reduced to $\mu_{\mathbf{u}} \in (0, \bar{\mu}_{\mathbf{u}})$.[3] Similarly, at state $\mathbf{d}$, the agent's effort increases the instantaneous rate of recovering to $\mu_{\mathbf{d}} > \underline{\mu}_{\mathbf{d}}$. Effort is costly to the agent, and not observable to the principal. Specifically, denote $c_{\mathbf{u}}$ and $c_{\mathbf{d}}$ to be the effort costs in states $\mathbf{u}$ and $\mathbf{d}$, respectively. The corresponding effort cost rate at time $t$ is

$$c(\theta_t) := c_{\mathbf{u}} \mathbb{1}_{\theta_t = \mathbf{u}} + c_{\mathbf{d}} \mathbb{1}_{\theta_t = \mathbf{d}}. \tag{1}$$

At any point in time $t$, the *public information* includes all the time epochs the machines changes state by time $t$. Formally, we denote a right-continuous counting process $N_t$ to represent the total number of public events, i.e., change of machine states, up to time $t$. Let $\mathcal{F}^N$ be the filtration generated by the initial state $\theta_0$ and the counting process $N = \{N_t\}$. Further denote an $\mathcal{F}^N$-predictable $\nu = \{\nu_t\}$ to represent the agent's effort process, such that $\nu_t \in \{0, 1\}$ for any time $t$. Specifically, $\nu_t = 1$ and $\nu_t = 0$ represent that the agent exerts effort and shirks at time $t$, respectively. Therefore, at any point in time $t$ with the state of the machine $\theta_t$ and the effort level $\nu_t$, the arrival rate of process $N$ is

$$\mu(\theta_t, \nu_t) := \left[\mu_{\mathbf{u}} \nu_t + \bar{\mu}_{\mathbf{u}}(1 - \nu_t)\right] \mathbb{1}_{\theta_t = \mathbf{u}} + \left[\mu_{\mathbf{d}} \nu_t + \underline{\mu}_{\mathbf{d}}(1 - \nu_t)\right] \mathbb{1}_{\theta_t = \mathbf{d}}. \tag{2}$$

We assume that the agent has limited liability, and we mainly focus our attention on contracts that always induce effort from the agent. (In Section 5 and the e-companion, we also consider contracts that induce effort only in one of the machine states.) Therefore, the principal needs to reimburse the aforementioned effort costs in real time as flow payments whenever effort is expected.

As a result, the effort cost $c(\theta_t)$ becomes shirking benefit if the agent shirks at time $t$. This is a standard treatment in the dynamic contracting literature (see, for example, Biais et al. 2010).

We further assume that the principal has the commitment power to a long-term contract based on public information. Overall, a dynamic contract $\Gamma = (L, \tau, q)$ includes a payment process $L$, a contract termination time $\tau$, and a stochastic termination process $q$.

Specifically, denote an $\mathcal{F}^N$-adapted process $L = \{L_t\}_{t\geq 0}$ to represent the cumulative payment from the principal to the agent up to time $t$. The payment includes an instantaneous one, $I_t$, and a flow with rate $\ell_t$ *beyond the background payment* $c(\theta_t)$ *that reimburses effort*, such that $dL_t = I_t + \ell_t dt$. Limited liability implies $I_t \geq 0$ and $\ell_t \geq 0$.

The contract not only includes payments, but also the possibility of terminating the agent at a random time $\tau$. We consider two ways of contract termination. First, at any point in time $t$ when the machine changes state (i.e., $dN_t = 1$), we allow the principal to terminate the contract randomly, with probability $q_t \in [0,1]$, where the probability $q_t$ depends on all of the information on machine state changes until time $t$, i.e., $\mathcal{F}^N_t$-measurable. Therefore, the contract contains an $\mathcal{F}^N$-adapted process $q = \{q_t\}_{t\geq 0}$ for random contract termination. Second, we also allow the principal to terminate the agent depending (deterministically) on history $\mathcal{F}^N_t$ without randomization. As will be clear later in the paper, allowing random termination is crucial to construct optimal contracts for certain model parameter settings. The principal and the agent are both risk-neutral and discount future cash flows at rate $r$.[4]

The principal's expected total discounted profit under a contract $\Gamma$ and effort process $\nu$ is defined as[5]

$$U(\Gamma, \nu, \theta_0) = \mathbb{E}\left[\int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t - c(\theta_t)dt) + e^{-r\tau}\underline{v}_\tau \middle| \theta_0\right], \tag{3}$$

where $\underline{v}_\tau$ is the principal's total discounted future profit after terminating the agent. This value clearly depends on the state of machine $\theta_\tau$ at the termination time $\tau$. It is easy to verify (see Lemma 4 in the Appendix) that $\underline{v}_\tau$ takes the following values for $\theta_\tau = \mathbf{u}$ and $\theta_\tau = \mathbf{d}$, respectively,

$$\underline{v}_\mathbf{u} := \frac{R}{r}\cdot\frac{r+\underline{\mu}_\mathbf{d}}{r+\bar{\mu}_\mathbf{u}+\underline{\mu}_\mathbf{d}}, \quad \text{and} \quad \underline{v}_\mathbf{d} := \frac{R}{r}\cdot\frac{\underline{\mu}_\mathbf{d}}{r+\bar{\mu}_\mathbf{u}+\underline{\mu}_\mathbf{d}}. \tag{4}$$

Given contract $\Gamma$ that always reimburses effort cost before termination, and an effort process $\nu$, the agent's expected total discounted utility is the cumulative payments minus the effort cost, expressed as the following

$$u(\Gamma, \nu, \theta_0) = \mathbb{E}\left\{\int_0^\tau e^{-rt}\left[dL_t + (1-\nu_t)c(\theta_t)dt\right] \middle| \theta_0\right\}. \tag{5}$$

Therefore, given either initial state $\theta_0 = \mathbf{u}$ or $\theta_0 = \mathbf{d}$, we can define a game between the two players, in which the principal designs an optimal contract $\Gamma$ that maximizes utility $U(\Gamma, \nu, \theta_0)$, anticipating the agent's effort process $\nu$ that maximizes $u(\Gamma,\nu,\theta_0)$. Throughout the paper, we focus on studying contracts that induce the agent to always exert effort (so called "incentive compatible" contracts).[6] In the e-companion Section EC.1 we provide a sufficient condition on model parameters such that it is indeed optimal for the principal to only focus on incentive compatible contracts.

**Incentive Compatibility** A contract $\Gamma$ is *incentive compatible* (IC) if in equilibrium, the agent has the incentive to always exert effort (to better maintain the machine so that the failure rate from state $\mathbf{u}$ drops to $\mu_{\mathbf{u}}$, and to faster repair the machine so that it comes back up at rate $\mu_{\mathbf{d}}$ from state $\mathbf{d}$), i.e. $\nu^* := \{\nu_t = 1\}_{\forall t\in[0,\tau]}$. That is, the contract is *incentive compatible* if[7]

$$u(\Gamma,\nu^*,\theta_0) \geq u(\Gamma,\nu,\theta_0)\ ,\quad \forall \mathcal{F}^N\text{-predictable effort process } \nu,\ \theta_0\in\{u,d\}. \tag{IC}$$

In this paper we focus on the class of incentive compatible contracts that always induce effort.

The contract design problem may be formulated as a stochastic optimal control problem, in which the state is the agent's promised utility at time $t$, defined as,

$$W_t(\Gamma,\nu) = \mathbb{E}\left\{\int_t^\tau e^{-r(s-t)}\left[dL_s + (1-\nu_s)c(\theta_s)ds\right]\middle|\mathcal{F}_t\right\}\mathbb{1}\{t\leq\tau\}. \tag{6}$$

It is clear that $W_0(\Gamma,\nu) = u(\Gamma,\nu,\theta_0)$ for $\theta_0$ consistent with $\mathcal{F}_0$. It is worth noting that the $\mathcal{F}^N$-adapted process $W_t$ is right-continuous, representing the agent's continuation utility *after* observing a potential arrival at time $t$ and *after* a potential instantaneous payment $I_t$. In comparison, the principal's control processes, $L_t$ and $q_t$, are $\mathcal{F}^N$-predictable and left-continuous. The principal schedules payments and stopping time through controlling the agent's promised utility. Therefore it is important to introduce process $W_{t-} = \lim_{s\uparrow t} W_s$, the left-hand limit of $W_t$, which is left-continuous and $\mathcal{F}^N$-predictable (assuming $W_{0-} = W_0$). At any time $t$, $W_{t-}$ captures the agent's continuation utility *before* knowing about the potential arrival and instantaneous payment at time $t$. Similarly, for the right-continuous state process $\theta_t$, we define left-continuous process $\theta_{t-} = \lim_{s\uparrow t}\theta_t$ to represent the state of the machine right before time $t$ for any $t>0$.

The contract also needs to ensure the agent's participation at any point in time. That is, the agent's promised utility needs to be non-negative (also called the *individual rationality* (IR) condition), i.e.,[8]

$$W_t \geq 0, \quad \forall t\geq 0. \tag{IR}$$

The following lemma provides the evolution of the agent's promised utility $W_t$ under any contract $\Gamma$, which is often called the *promise keeping* (PK) condition in the dynamic contract literature. It also provides an equivalent condition for (IC) in terms of the promised utility $W_t$.

LEMMA 1. *For any contract* $\Gamma$*, there exists an* $\mathcal{F}^N$*-predictable process* $H_t$ *such that for* $t\in[0,\tau]$*,*

$$dW_t = \Big\{ rW_{t-} - (1-\nu_t)c(\theta_t) - [(1-q_t)H_t - q_t W_{t-}]\,\mu(\theta_t,\nu_t) - \ell_t \Big\} dt \\ + [(1-X_t)H_t - X_t W_{t-}]\,dN_t - I_t, \qquad \text{(PK)}$$

*in which Bernoulli random variable* $X_t$ *takes value* 1 *with probability* $q_t$*.*

*Furthermore, contract* $\Gamma$ *satisfies (IC) if and only if*

$$\begin{aligned} (1-q_t)H_t - q_t W_{t-} &\le -\beta_{\mathbf{u}}, \quad \textit{for} \quad \theta_{t-} = \mathbf{u}, \ \textit{and} \\ (1-q_t)H_t - q_t W_{t-} &\ge \beta_{\mathbf{d}}, \quad \textit{for} \quad \theta_{t-} = \mathbf{d}, \end{aligned} \qquad (7)$$

*for all* $t\in[0,\tau]$*, where*

$$\beta_{\mathbf{u}} := \frac{c_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}} - \mu_{\mathbf{u}}} \quad \textit{and} \quad \beta_{\mathbf{d}} := \frac{c_{\mathbf{d}}}{\mu_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}}}. \qquad (8)$$

*Finally, we need* $-H_t \le W_{t-}$ *for all* $t\ge 0$ *in order to satisfy (IR).*

The (PK) condition is a standard result for the dynamics of the agent's promised utilities over time. The left-continuous, $\mathcal{F}^N$-predictable process $H_t$ captures the change of the continuation utility $W_{t-}$ before a potential instantaneous payment $I_t$ at time $t$.

To facilitate understanding, it is helpful to consider a heuristic derivation based on discrete time approximation. Consider a small time interval $[t,t+\delta)$. Assume that the agent's promised utility $W_{t-}$ evolves continuously to $W_{t+}$ over this interval, unless there is a change of machine state, with probability $\mu(\theta_t,\nu_t)\delta$. With a change of state, the agent's total future utility takes a jump to either $W_{t-}+H_t$ with probability $1-q_t$, or to 0 with probability $q_t$ (termination). Also taking into consideration the shirking benefit $(1-\nu_t)c(\theta_t)\delta$, flow payment $\ell_t\delta$, and time discount rate $r$ (for simplicity, ignore the instantaneous payment $I_t$), the above description of the discrete time approximation of the promised utility implies the following,

$$W_{t-} = (1-\nu_t)c(\theta_t)\delta + \ell_t\delta + \mu(\theta_t,\nu_t)\delta\,[q_t\cdot 0 + (1-q_t)(W_{t-}+H_t)] + [1-(\mu(\theta_t,\nu_t)+r)\delta]\,W_{t+}.$$

As $\delta$ approaches 0, replace it with $dt$, and rearrange terms, we observe that the smooth change $W_{t+}-W_{t-}$ equals

$$\Big\{ rW_{t-} - (1-\nu_t)c(\theta_t) - \ell_t - [(1-q_t)H_t - q_t W_{t-}]\,\mu(\theta_t,\nu_t) \Big\} dt,$$

which recovers the terms involving $dt$ in (PK). The change of machine state ($dN_t=1$) results in the agent's total future utility changing by either $H_t$ or $-W_{t-}$ (termination), depending on the outcome of the random variable $X_t$. Therefore, the change is $[(1-X_t)H_t - X_t W_{t-}]\,dN_t$. Finally, this total change can be delivered by a direct instantaneous payment $I_t$ in addition to the change in the promised utility $dW_t$. That is, $dW_t + I_t = [(1-X_t)H_t - X_t W_{t-}]\,dN_t$ when $dN_t = 1$.

The values $\beta_{\mathbf{d}}$ and $\beta_{\mathbf{u}}$ defined in equation (8) reflect the ratios between effort cost and improvement in the repair or failure rates, which reveal the intuition behind the (IC) condition. For an intuitive interpretation of these two important quantities, consider, for example, the up state. If the principal could charge the agent an amount $\beta_{\mathbf{u}}$ upon the machine breaking down, the agent is then indifferent between exerting effort or not. This is because over a small time period $\delta$, the shirking benefit, $c_{\mathbf{u}}\delta$, exactly compensates the additional expected charge, $\beta_{\mathbf{u}}(\bar{\mu}_{\mathbf{u}} - \mu_{\mathbf{u}})\delta$. Condition (7) states that instead of directly charging the agent, an incentive compatible contract needs to reduce the agent's promised utility by at least $\beta_{\mathbf{u}}$. The term $\beta_{\mathbf{d}}$ has a similar interpretation for the down state. Following standard IC conditions in Sannikov (2008) and Biais et al. (2010), one would only obtain the result that the magnitude of $H_t$ is larger than $\beta_{\mathbf{d}}$ or smaller than $-\beta_{\mathbf{u}}$. Our (IC) condition in Lemma 1, however, generalizes the standard form due to the consideration of contract termination. In Section 4.2, we show that the probability of random termination, $q_t$, could indeed be positive in the the optimal contract.

Later in the paper we show that the structure of the optimal contract, including whether and when the incentive compatibility constraints (7) are binding, depends on whether $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ or $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$. The intuitive interpretation of these conditions follows the definition of $\beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}}$. For example, if the costs of effort are the same in the two machine states (i.e., $c_{\mathbf{d}} = c_{\mathbf{u}}$), then $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ means that the agent is able to decrease the break down rate more than increase the recovery rate ($\bar{\mu}_{\mathbf{u}} - \mu_{\mathbf{u}} \geq \mu_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}}$).

Finally, (IR) requires that the agent's promised utility must be non-negative at all times, including right after a downward jump of the promised utility. As explained above, (IC) requires that in the up state a downward jump has to be at least $\beta_{\mathbf{u}}$. Therefore, when the state $\theta_t = \mathbf{u}$, we can only satisfy constraint (7) when $W_{t-} \geq \beta_{\mathbf{u}}$. When the machine is up and $W_{t-}$ becomes too low (say, lower than $\beta_{\mathbf{u}}$), however, the principal needs to randomize the promised utility to either 0 (termination), or back to a threshold. This is why we need the randomized termination process $q_t$ for the optimal contract. Interestingly, as we will show in Section 4, random termination only occurs if $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$. In fact randomization may even occur when $W_{t-} > \beta_{\mathbf{u}}$. If $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, on the other hand, the optimal contract always guarantees $W_{t-} \geq \beta_{\mathbf{u}}$ when the machine is up.

In the next section, we first introduce two simple and stationary incentive compatible contracts, which help us lay the foundation of the optimal incentive compatible contracts.

## 3. Benchmark contracts

Before introducing the optimal contract, it is worth studying simple incentive compatible contracts in this section. These contracts are stationary in nature – the contract terms only depend on the state of the machine and its transitions, and not on time otherwise. This implies that they never terminate the agent. In fact, if we do not allow contract termination, they are indeed optimal incentive compatible contracts. In the next section, however, we show that optimal contracts that allow termination are based upon, but outperform these simple ones. In particular, it is important to distinguish between the two cases $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, which are studied separately in the two subsections, respectively.

### 3.1. $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$

The contract is indeed very simple: the principal pays an instantaneous bonus $\beta_{\mathbf{d}} - \beta_{\mathbf{u}}$ when the machine recovers from state $\mathbf{d}$, followed by a flow payment with rate

$$\ell_1^* = \mu_{\mathbf{d}}\beta_{\mathbf{d}} + (r + \mu_{\mathbf{u}})\beta_{\mathbf{u}} \tag{9}$$

when the machine remains in state $\mathbf{u}$. We denote $\bar{\Gamma}$ to represent this contract.

In order to prove that $\bar{\Gamma}$ is incentive compatible, it is important to derive the agent's promised utility following this contract. In fact, we claim that the promised utility remains a constant for each machine state. Define $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$ as these two promised utilities when the machine's state is $\mathbf{d}$ and $\mathbf{u}$, respectively,

$$\bar{w}_{\mathbf{d}} = \frac{\mu_{\mathbf{d}}\beta_{\mathbf{d}}}{r}, \quad \text{and} \quad \bar{w}_{\mathbf{u}} = \bar{w}_{\mathbf{d}} + \beta_{\mathbf{u}}. \tag{10}$$

It is easy to verify that contract $\bar{\Gamma}$ is incentive compatible. In fact, whenever the machine breaks down, the promised utility changes from $\bar{w}_{\mathbf{u}}$ to $\bar{w}_{\mathbf{d}}$, with a downward jump of exactly $H_t = -\beta_{\mathbf{u}}$. Upon recovery from state $\mathbf{d}$, the promised utility first takes an upward jump of $\beta_{\mathbf{u}}$, and then the agent is given a direct payment of $\beta_{\mathbf{d}} - \beta_{\mathbf{u}}$ resulting in $H_t = \beta_{\mathbf{d}}$. Therefore, incentive compatibility constraints (7) are always binding, with aforementioned $H_t$ and $q_t = 0$. This further ensures that the agent always exerts effort. Regarding the promise keeping constraint, for state $\theta_t = \mathbf{u}$, if we set $W_t = \bar{w}_{\mathbf{u}}$ and $dL_t = \ell_1^* dt$, then (PK) becomes $dW_t = -\beta_{\mathbf{u}} dN_t$. Similarly, for state $\theta_t = \mathbf{d}$, setting $W_t = \bar{w}_{\mathbf{d}}$, and $dL_t = (\beta_{\mathbf{d}} - \beta_{\mathbf{u}})dN_t$, (PK) becomes $dW_t = \beta_{\mathbf{u}} dN_t$. Therefore, contract $\bar{\Gamma}$ and our claimed promised utilities (10) indeed satisfy both (PK) and (IC) constraints.

Besides the mathematical arguments above, it is in fact intuitive that contract $\bar{\Gamma}$ provides the incentive for the agent to exert effort. When the machine is down, the prospect of an instantaneous bonus followed by a flow payment provides the incentive for the agent to repair the machine faster. When the machine is up, the flow payment incentivizes the agent to better maintain and prolong the period of payment. In particular, the flow payment $\ell_1^*$ has two components. The first component is the interest payment $r\bar{w}_{\mathbf{u}}$, so that the agent's promised utility is kept at $\bar{w}_{\mathbf{u}}$. The second component is the information rent $\mu_{\mathbf{u}}\beta_{\mathbf{u}}$ whenever there is no arrival (machine breaking down).

Furthermore, because contract $\bar{\Gamma}$ never terminates the agent, we have the following expressions for the total discounted societal values (summation of the principal and the agent's utilities) at states $\mathbf{u}$ and $\mathbf{d}$, respectively (see Lemma 4 in the Appendix for the derivations).

$$\bar{v}_{\mathbf{d}} = \frac{\mu_{\mathbf{d}}(R - c_{\mathbf{u}}) - (r + \mu_{\mathbf{u}})c_{\mathbf{d}}}{r(r + \mu_{\mathbf{d}} + \mu_{\mathbf{u}})} \quad \text{and} \quad \bar{v}_{\mathbf{u}} = \frac{(r + \mu_{\mathbf{d}})(R - c_{\mathbf{u}}) - \mu_{\mathbf{u}} c_{\mathbf{d}}}{r(r + \mu_{\mathbf{d}} + \mu_{\mathbf{u}})}. \tag{11}$$

Consequently, the principal's utilities under contract $\bar{\Gamma}$ for state $\mathbf{u}$ and $\mathbf{d}$ are, respectively,

$$U(\bar{\Gamma}, \nu^*, \mathbf{u}) = \bar{U}_{\mathbf{u}} := \bar{v}_{\mathbf{u}} - \bar{w}_{\mathbf{u}} \quad \text{and} \quad U(\bar{\Gamma}, \nu^*, \mathbf{d}) = \bar{U}_{\mathbf{d}} := \bar{v}_{\mathbf{d}} - \bar{w}_{\mathbf{d}}. \tag{12}$$

Although this simple contract $\bar{\Gamma}$ is incentive compatible, it is actually not optimal, because it only uses the "carrot" of payments without the "stick" of termination. At the end of this section, Proposition 7 shows that this simple contract is actually the optimal incentive compatible contract if the principal is not allowed to terminate the agent. Besides introducing this simple contract to build intuition, we would like to clarify the simple contract's connection with the optimal contract. According to the optimal contract, it is possible that the promised utilities eventually become $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$ for states $\mathbf{u}$ and $\mathbf{d}$, respectively. From that point on, the optimal contract becomes identical to the simple contract $\bar{\Gamma}$, and the agent is never terminated. However, following the optimal contract, it is also possible that the promised utilities never reach $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$ before the agent is terminated.

Finally, it is clear that the society is better off with contract $\bar{\Gamma}$ compared with not hiring the agent at all if $\bar{v}_{\mathbf{u}}$ and $\bar{v}_{\mathbf{d}}$ are at least as high as $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$ defined in (4). In fact, when $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, one can verify that $\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}$ readily implies $\bar{v}_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$. Furthermore,

$$\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}} \tag{13}$$

is equivalent to

$$R \geq h_{\mathbf{d}} := \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right) \frac{\mu_{\mathbf{d}} c_{\mathbf{u}} + (r + \mu_{\mathbf{u}})\, c_{\mathbf{d}}}{\mu_{\mathbf{d}} \Delta\mu_{\mathbf{u}} + (r + \mu_{\mathbf{u}})\, \Delta\mu_{\mathbf{d}}}. \tag{14}$$

Intuitively, hiring the agent is beneficial only if the revenue rate $R$ is high enough. In Section 4.1, we demonstrate that the structure of the the optimal contracts depends critically on whether condition (13) holds.

### 3.2. $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$

The simple contract in this case, denoted as $\hat{\Gamma}$, can be described in one sentence: it pays the agent a flow payment with rate

$$\ell_2^* = (r + \mu_{\mathbf{u}} + \mu_{\mathbf{d}})\beta_{\mathbf{u}} \tag{15}$$

at state $\mathbf{u}$.

The promised utilities are the following two constants for the two machine states, respectively,

$$\hat{w}_{\mathbf{d}} = \frac{\mu_{\mathbf{d}}\beta_{\mathbf{u}}}{r}, \quad \text{and} \quad \hat{w}_{\mathbf{u}} = \hat{w}_{\mathbf{d}} + \beta_{\mathbf{u}}. \tag{16}$$

similar to $\bar{w}_{\mathbf{d}}$ and $\bar{w}_{\mathbf{u}}$ defined in (10). Similar to the analysis for $\bar{\Gamma}$, we can verify that contract $\hat{\Gamma}$ together with $\hat{w}_{\mathbf{d}}$ and $\hat{w}_{\mathbf{u}}$ satisfy (PK) and (IC). The expressions for the societal utility still follow (11). The principal's utilities under contract $\hat{\Gamma}$ are, therefore,

$$U(\hat{\Gamma}, \nu^*, \mathbf{u}) = \hat{U}_{\mathbf{u}} := \bar{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}} \text{ and } U(\hat{\Gamma}, \nu^*, \mathbf{d}) = \hat{U}_{\mathbf{d}} := \bar{v}_{\mathbf{d}} - \hat{w}_{\mathbf{d}}, \tag{17}$$

for machine states $\mathbf{u}$ and $\mathbf{d}$, respectively.

The overall feature of $\hat{\Gamma}$ for the case of $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$ is very similar to $\bar{\Gamma}$ for the case of $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$. Later in Section 4.2.1, we show that the agent's promised utility has a chance to eventually become $\hat{w}_{\mathbf{d}}$ and $\hat{w}_{\mathbf{u}}$ following the optimal contract. After reaching that point, the optimal contract becomes identical to $\hat{\Gamma}$, and the agent is never terminated. At the end of Section 4.2, we present Proposition 7, which shows that this simple contract is actually the optimal incentive compatible contract if terminating the agent is not allowed. Similar to $\ell_1^*$, the flow payment $\ell_2^*$ can also be decomposed into two components, the interest payment $r\hat{w}_{\mathbf{u}}$, and the information rent $\mu_{\mathbf{u}}\beta_{\mathbf{u}}$. Finally, parallel to the previous case, when $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, we have

$$\bar{v}_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}} \tag{18}$$

is equivalent to

$$R \geq h_{\mathbf{u}} := \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right) \frac{\mu_{\mathbf{u}} c_{\mathbf{d}} + (r + \mu_{\mathbf{d}}) c_{\mathbf{u}}}{\mu_{\mathbf{u}} \Delta\mu_{\mathbf{d}} + (r + \mu_{\mathbf{d}}) \Delta\mu_{\mathbf{u}}}, \tag{19}$$

and readily implies $\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}$.

Despite these similarities between $\bar{\Gamma}$ and $\hat{\Gamma}$, it is worth noting an important difference between them. Under contract $\hat{\Gamma}$ of the current case, the incentive compatibility constraint (7) is not binding. In fact, both the downward jump upon breaking down and the upward jump upon recovery are both $\beta_{\mathbf{u}}$ (i.e.,$H_t = \beta_{\mathbf{u}} \left(\mathbb{1}_{\theta_{t-} = \mathbf{d}} - \mathbb{1}_{\theta_{t-} = \mathbf{u}}\right)$). In particular, when the machine recovers, the upward jump is higher than what is required in constraint (7). Given our claim in the last paragraph, it means that the incentive compatibility constraint may not be binding in the optimal contract. This may appear surprising, given that we are not aware of other optimal dynamic contract with non-binding incentive compatibility constraint in the literature. We will explain why this phenomenon arises in our setting after introducing the optimal contract in Section 4.2.

## 4. Optimal Contract

In this section, we study and characterize in detail the optimal contracts that induce the agent to always exert effort before termination. Similar to the previous section, here we first study the case $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ before $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, in Sections 4.1 and 4.2, respectively. In the end we summarize main results for different cases in Section 4.3. It is worth noting that more interesting and intricate results of this paper, including non-binding incentive compatible constraint.

### 4.1. The Case $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$

The structure of the optimal contract in this case, although new, may not appear surprising to readers already familiar with the continuous time contracting literature (Biais et al. 2010, Sun and Tian 2017). However, this section provides a gentle preparation to the more complex and delicate structure in the optimal contract for the case $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$ next.

In Section 4.1.1, we first introduce the optimal contract under condition (13), which is equivalent to (14). Section 4.1.2 further provides the principal's value functions under the optimal contract and the proof of optimality. Finally, Section 4.1.3 studies what happens when the condition (13) does not hold.

**4.1.1. Optimal IC contract when $\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}$** In this subsection, we develop a contract $\Gamma_1^*$, and leave the proof of optimality to the next subsection. The contract keeps track of the agent's promised utility. Figure 1 depicts two sample trajectories of the agent's promised utility in the proposed contract where the machine starts at state $\theta_0 = \mathbf{d}$.

The promised utility starts from an initial promised utility $W_0 = w_{\mathbf{d}}^* \in (0, \bar{w}_{\mathbf{d}})$. While repairing the machine, this utility keeps decreasing (the exact form to be specified later) until either the machine is repaired or the utility reaches 0. If the machine has not recovered before the utility $W_t$ reaches 0, the principal terminates the agent. The dotted curve in Figure 1 represents this situation, where the promised utility decreases to zero at time $\tau$.

On the other hand, if the machine recovers at time $t$ with $W_{t-} > 0$, the utility $W_t$ takes an upward jump of level $\min\{\beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}} - W_{t-}\}$ and the agent is paid $(W_{t-} + \beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+$ instantaneously. See the solid curve in Figure 1, which represents another sample trajectory. In the time interval $[0, t_1)$, the promised utility is decreasing over time. At $t_1$, it jumps up by $\beta_{\mathbf{d}}$ because $W_{t_1-} < \bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}$. The corresponding instantaneous payment is 0. Then the contract continues with the agent maintaining the machine in the up state, while the promised utility keeps increasing until either it reaches $\bar{w}_{\mathbf{u}}$, or the machine breaks down. During $(t_1, t_2)$, the promised utility is increasing over time. At time $t_2$, the machine breaks down and the promised utility drops by $\beta_{\mathbf{u}}$. Again, in $(t_2, t_3)$, the agent is repairing the machine with the promised utility decreasing over time. After $t_3$, the machine

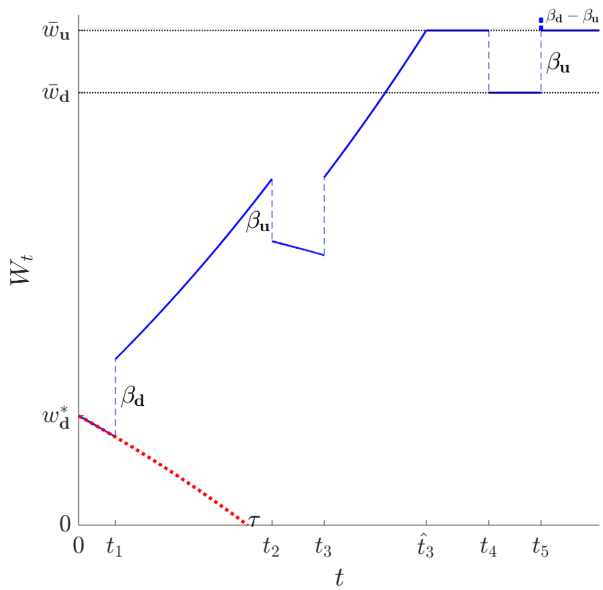



**Figure 1** Two sample trajectories of promised utility with $\mu_{\mathbf{u}}=6, \bar{\mu}_{\mathbf{u}}=9, \mu_{\mathbf{d}}=5, \underline{\mu}_{\mathbf{d}}=2, c_{\mathbf{u}}=0.8, c_{\mathbf{d}}=1, r=0.9, R=7.5$. In this case, $\bar{w}_{\mathbf{d}}=0.74$, $\bar{w}_{\mathbf{u}}=1.01$ and $\beta_{\mathbf{u}}=0.27<\beta_{\mathbf{d}}=0.33$. The policy starts from $W_0=w^*_{\mathbf{d}}=0.4685$. The two dashed horizontal lines represent the level of $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$, respectively. The upward jump level when the machine is repaired is $\beta_{\mathbf{d}}$ and the downward drop level when the machine breaks down is $\beta_{\mathbf{u}}$.

does not break down before the promised utility reaches $\bar{w}_{\mathbf{u}}$ at time $\hat{t}_3$, at which point the flow payment $\ell_1^*$ (defined in (9)) starts. After time $\hat{t}_3$, the agent's promised utility jumps back and forth between $\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$. The contract becomes exactly the same as the simple contract $\bar{\Gamma}$ introduced in the previous subsection. In the following, we provide a formal definition of the proposed optimal contract.

Definition 1. For a machine starting from state $\theta\in\{\mathbf{u},\mathbf{d}\}$, define contract $\Gamma_1^*(w)=(L^*,q^*,\tau^*)$ as the following, where $w\in[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}}]$ if the initial state is $\mathbf{u}$, and $w\in[0,\bar{w}_{\mathbf{d}}]$ if the initial state is $\mathbf{d}$.

i. The dynamics of the agent's promised utility $W_t$ follows

$$dW_t = \big[r(W_{t-}-\bar{w}_{\mathbf{d}})dt+\min\{\bar{w}_{\mathbf{u}}-W_{t-},\beta_{\mathbf{d}}\}dN_t\big]\mathbb{1}_{\theta_{t-}=\mathbf{d}} + \big[(rW_{t-}+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{u}}}dt-\beta_{\mathbf{u}}dN_t\big]\mathbb{1}_{\theta_{t-}=\mathbf{u}}, \qquad \text{(DW1)}$$

from the initial promised utility $W_0=w$.

ii. The payment to the agent follows $dL_t^*=\ell_1^*\mathbb{1}_{W_{t-}=\bar{w}_{\mathbf{u}}}\mathbb{1}_{\theta_{t-}=\mathbf{u}}dt+(W_{t-}+\beta_{\mathbf{d}}-\bar{w}_{\mathbf{u}})^+\mathbb{1}_{\theta_{t-}=\mathbf{d}}dN_t$.

iii. The random termination probability is $q_t^*=0$, (i.e. there is no random termination) and the termination time is $\tau^*=\min\{t:W_t=0\}$.

One can verify that the dynamics of $W_t$ in the proposed optimal contract follows (PK), with $H_t = \beta_{\mathbf{d}} \mathbb{1}_{\theta_{t-}=\mathbf{d}} - \beta_{\mathbf{u}} \mathbb{1}_{\theta_{t-}=\mathbf{u}}$, $dL_t = dL_t^*$ and $q_t = q_t^*$. Also, in the proposed optimal contract, the incentive compatible constraints (7) are binding, and the principal never randomly terminates the agent. It is only possible to terminate the agent when the machine is down (note that we do not terminate the agent exactly at the point when the machine goes down but when the promised utility reaches zero, e.g., after a long enough down period). On the other hand, when the machine is up, the agent's promised utility is always greater than $\beta_{\mathbf{u}}$. This is because if the initial state of the machine is up, the initial promised utility would be at least $\beta_{\mathbf{u}}$ and keeps going up until the first break down; after the agent has finished repair once, the promised utility would always jump to a level above $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$ to start the up state.

It is worth noting that payment in Definition 1 involves both instantaneous payment and flow payment. And payment only occur when the promised utility is high enough such that the optimal contract becomes the benchmark $\bar{\Gamma}$ defined in Section 3.1.

Remark 1 (Implementation). In practice, a principal can implement contract $\Gamma_1^*$ by stationing a meter that shows changing $W_t$ (promised utility to the agent) over time. At time $t$, if the machine is up, the meter keeps increasing at an ever-increasing speed $\mu_{\mathbf{u}}\beta_{\mathbf{u}} + rW_{t-}$ per period of time (where $\mu_{\mathbf{u}}\beta_{\mathbf{u}}$ is the information rent for keeping the machine running, and $rW_{t-}$ is the interest to the agent), and stops at $\bar{w}_{\mathbf{u}}$. When the machine is down, the meter keeps decreasing with a speed $-rW_{t-} + r\bar{w}_{\mathbf{d}}$ (where $r\bar{w}_{\mathbf{d}}$ is a constant punishment for not having finished repairing, and $rW_{t-}$ is again the interest to the agent). The agent is terminated when the meter reaches 0. When the machine breaks down, the meter jumps down $\beta_{\mathbf{u}}$. When the machine recovers, the meter jumps up by $\beta_{\mathbf{d}}$, unless the jump is clipped by $\bar{w}_{\mathbf{u}}$. The agent receives incentive payments of $r\bar{w}_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}$ per unit of time only when the meter reaches $\bar{w}_{\mathbf{u}}$. In addition, the agent is continuously reimbursed at rate $c_\theta$ for his effort cost when the machine's state is $\theta \in \{\mathbf{d}, \mathbf{u}\}$. This form of payment can be interpreted as a "base rate" of pay in addition to the aforementioned incentive pay, which is easy to explain in practice.

**4.1.2. Value functions and proof of optimality when $\bar{v}_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}$** In this section, we first heuristically derive the dynamics of the principal's utility, as a function of the agent's promised utility under the proposed optimal contract $\Gamma_1^*$ defined in Definition 1. Later, in Proposition 2, we prove that our derived value function is the actual optimal value function of the principal.

Specifically, let $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ represent the principal's utility at time $t$ when the agent's promised utility is $w$ if the machine's state is $\mathbf{d}$ and $\mathbf{u}$, respectively. Following a standard heuristic derivation (see Appendix D.1), we obtain the following system of differential equations. In particular, for state $\mathbf{d}$ and $w \in [0, \bar{w}_{\mathbf{d}}]$, the differential equation is

$$(\mu_{\mathbf{d}} + r)J_{\mathbf{d}}(w) = -c_{\mathbf{d}} + r(w - \bar{w}_{\mathbf{d}})J'_{\mathbf{d}}(w) + \mu_{\mathbf{d}} J_{\mathbf{u}}(\min\{w + \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) - \mu_{\mathbf{d}}(w + \beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+. \quad (20)$$

For state $\mathbf{u}$, the differential equation for $w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$ is

$$(\mu_{\mathbf{u}} + r)J_{\mathbf{u}}(w) = R - c_{\mathbf{u}} + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}})J'_{\mathbf{u}}(w) + \mu_{\mathbf{u}}J_{\mathbf{d}}(w - \beta_{\mathbf{u}}) \ , \tag{21}$$

with at $w = \bar{w}_{\mathbf{u}}$,

$$(\mu_{\mathbf{u}} + r)J_{\mathbf{u}}(\bar{w}_{\mathbf{u}}) = R - c_{\mathbf{u}} + \mu_{\mathbf{u}}J_{\mathbf{d}}(\bar{w}_{\mathbf{u}} - \beta_{\mathbf{u}}) - \ell_1^*. \tag{22}$$

The boundary conditions are

$$J_{\mathbf{u}}(0) = \underline{v}_{\mathbf{u}} \quad \text{and} \quad J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}, \tag{23}$$

reflecting that the principal receives baseline revenues $\underline{v}_{\mathbf{d}}$ and $\underline{v}_{\mathbf{u}}$ (defined in (4)), upon terminating the agent in states $\mathbf{d}$ and $\mathbf{u}$, respectively.

For the interval $[0, \beta_{\mathbf{u}}]$, we simply extend the function $J_{\mathbf{u}}(w)$ to be linear, that is,

$$J_{\mathbf{u}}(w) = J_{\mathbf{u}}(0) + \frac{J_{\mathbf{u}}(\beta_{\mathbf{u}}) - J_{\mathbf{u}}(0)}{\beta_{\mathbf{u}}} w, \ \text{for } w \in [0, \beta_{\mathbf{u}}]. \tag{24}$$

As we have demonstrated, the agent's promised utility never falls into the interior of this interval if we follow the optimal contract according to Definition 1. However, having an extended definition of $J_{\mathbf{u}}(w)$ for that interval is crucial for the the proof of optimality of the contract in Definition 1. This is because the optimality proof needs to argue that contract $\Gamma_1^*$ outperforms any other contract, and a generic contract may bring the promised utility to this interval at state $\mathbf{u}$.

Proposition 1. *The system of differential equations (20)-(22) with boundary conditions* (23) *and* (24) *has a unique solution: the pair of functions* $J_{\mathbf{u}}(w)$ *on* $[0, \bar{w}_{\mathbf{u}}]$ *and* $J_{\mathbf{d}}(w)$ *on* $[0, \bar{w}_{\mathbf{d}}]$*, both of which are strictly concave with* $J'_{\mathbf{u}}(w) \geq -1$ *and* $J'_{\mathbf{d}}(w) \geq -1$.

Following proposition 1, we can define $w_{\mathbf{d}}^*$ and $w_{\mathbf{u}}^*$ as unique maximizers of $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ on $[0, \bar{w}_{\mathbf{d}}]$ and $[0, \bar{w}_{\mathbf{u}}]$, respectively. Next, we show that functions $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are indeed the value functions of the principal under contract $\Gamma_1^*(w)$, starting from a promised utility $w$ at time 0 with the initial states $\theta_0 = \mathbf{d}$ and $\theta_0 = \mathbf{u}$, respectively.

Proposition 2. *For any state* $\theta \in \{u, d\}$ *and promised utility* $w \in [0, \bar{w}_\theta]$*, we have* $U(\Gamma_1^*(w), \nu^*, \theta) = J_\theta(w)$*. That is, functions* $J_{\mathbf{u}}(w)$ *and* $J_{\mathbf{d}}(w)$ *are equal to the principal's total discounted utilities of following contract* $\Gamma_1^*$ *when the initial state of the machine is* $\mathbf{u}$ *and* $\mathbf{d}$*, respectively.*

Figure 2 provides a numerical example of the principal's value functions $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$. To implement the contract, the principal needs to designate the initial promised utility $W_0$. The initial promised utility should be $w_{\mathbf{d}}^*$ if the machine starts at state $\theta_0 = \mathbf{d}$ and should be $w_{\mathbf{u}}^*$ if the machine starts at state $\theta_0 = \mathbf{u}$. Note that due to concavity, if $J_{\mathbf{u}}(\beta_{\mathbf{u}}) \geq J_{\mathbf{u}}(0)$, then $w_{\mathbf{u}}^* \geq \beta_{\mathbf{u}}$. Otherwise, the optimal initial promised utility $w^*_{\mathbf{u}} = 0$, and, in this case, it is better not to hire the agent if the initial state of the machine is $\mathbf{u}$.

Furthermore, it is worth noting that $J_{\mathbf{d}}(\bar{w}_{\mathbf{d}}) = \bar{U}_{\mathbf{d}}$ and $J_{\mathbf{u}}(\bar{w}_{\mathbf{u}}) = \bar{U}_{\mathbf{u}}$, where $\bar{U}_{\mathbf{d}}$ and $\bar{U}_{\mathbf{u}}$, defined in (12), are the principal's utilities under the simple contract $\bar{\Gamma}$ introduced in Section 3.1. This implies that $\Gamma^*_1$ always (weakly) outperforms $\bar{\Gamma}$. The suboptimality of the benchmark contract $\bar{\Gamma}$ is the difference between the peak of the value function $J_\theta$ and $\bar{U}_\theta$ if the system starts from state $\theta$.

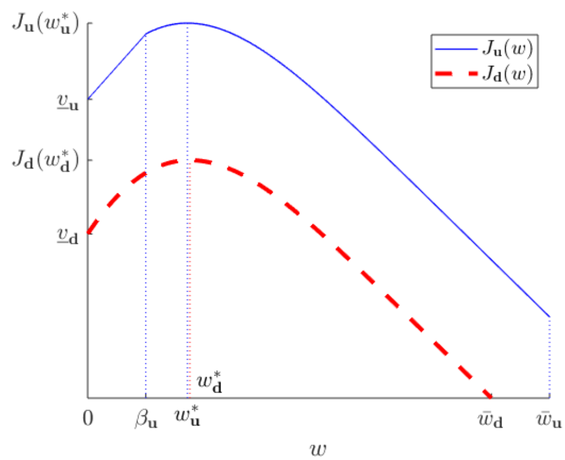



**Figure 2** Principal's Value functions with $\mu_{\mathbf{u}} = 6$, $\bar{\mu}_{\mathbf{u}} = 9$, $\mu_{\mathbf{d}} = 5$, $\underline{\mu}_{\mathbf{d}} = 2$, $c_{\mathbf{u}} = 0.8$, $c_{\mathbf{d}} = 1$, $r = 0.9$, and $R = 7.5$. In this case, $\bar{w}_{\mathbf{d}} = 0.74$, $\bar{w}_{\mathbf{u}} = 1.01$ and $\beta_{\mathbf{u}} = 0.27 < \beta_{\mathbf{d}} = 0.33$. $J_{\mathbf{u}}(w^*_{\mathbf{u}}) = 2.388$, $\underline{v}_{\mathbf{u}} = 2.031$ and $J_{\mathbf{u}}(\bar{w}_{\mathbf{u}}) = \bar{U}_{\mathbf{u}} = 1.012$. $J_{\mathbf{d}}(w^*_{\mathbf{d}}) = 1.746$, $\underline{v}_{\mathbf{d}} = 1.4$ and $J_{\mathbf{d}}(\bar{w}_{\mathbf{d}}) = \bar{U}_{\mathbf{d}} = 0.632$.

Finally, to show that the contract $\Gamma^*_1$ is indeed optimal, in the next proposition, we first demonstrate that functions $J_{\mathbf{u}}$ and $J_{\mathbf{d}}$ are upper bounds for the principal's utility under any incentive compatible contract $\Gamma$, if the machine starts at states $\mathbf{u}$ and $\mathbf{d}$, respectively.

Proposition 3. *For any incentive compatible contract* $\Gamma$ *and any initial state* $\theta \in \{\mathbf{u}, \mathbf{d}\}$*, we have* $J_\theta\big(u(\Gamma, \nu^*, \theta)\big) \geq U(\Gamma, \nu^*, \theta)$*, in which we extend the function* $J_\theta(w) = J_\theta(\bar{w}_\theta) - (w - \bar{w}_\theta)$ *for* $w > \bar{w}_\theta$*.*

Therefore, we know that for any incentive compatible contract $\Gamma$ and initial state $\theta$,

$$U(\Gamma, \nu^*, \theta) \leq J_\theta\left(u(\Gamma, \nu^*, \theta)\right) \leq J_\theta(w^*_\theta) = U\left(\Gamma^*_1(w^*_\theta), \nu^*, \theta\right),$$

where the first inequality follows from Proposition 3, the second inequality follows from the fact that $w^*_\theta$ is the maximizer of $J_\theta$, and the third equality follows from Proposition 2. This implies the following main result of this section.

Theorem 1. *The optimal incentive contract is* $\Gamma^*_1(w^*_\theta)$ *if* $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$*, condition* (14) *is satisfied and the machine starts from state* $\theta \in \{u, d\}$*. That is,* $U\big(\Gamma^*_1(w^*_\theta), \nu^*, \theta\big) \geq U(\Gamma, \nu^*, \theta)$ *for any incentive compatible contract* $\Gamma$ *and state* $\theta$*.*

**4.1.3.** $\bar{v}_d < \underline{v}_d$ In this section, we consider the case if (13), or equivalently, (14), is violated. That is, the revenue rate $R$ when the machine is up is not very high. Consider the following contract structure. If the machine starts at state $\mathbf{d}$, the principal does not hire the agent. If the machine starts at state $\mathbf{u}$, on the other hand, the principal hires the agent only to maintain the machine until it breaks down for the first time. During the maintenance period, the principal pays a constant flow payment with rate $(r+\mu_{\mathbf{u}})\beta_{\mathbf{u}}$. Furthermore, the agent's corresponding promised utility is maintained at $\beta_{\mathbf{u}}$, because

$$\mathbb{E}\left[\int_0^{\tau^*_{\mathbf{u}}} e^{-rt}(r+\mu_{\mathbf{u}})\beta_{\mathbf{u}}dt\right] = \beta_{\mathbf{u}},$$

where $\tau^*_{\mathbf{u}}$, the time in state $\mathbf{u}$, follows an exponential distribution with rate $\mu_{\mathbf{u}}$.

Here is a formal definition of the proposed contract.

Definition 2. Define contract $\Gamma^*_{\mathbf{u}}$ when the machine starts in state $\mathbf{u}$ as the following:

i. In state $\mathbf{u}$, the agent's promised utility $W_t$ is maintained at $\beta_{\mathbf{u}}$, which drops to 0 as soon as the state switches to $\mathbf{d}$. In state $\mathbf{d}$, $W_t$ remains to be 0.
ii. The payment to the agent follows $dL^*_t = (r+\mu_{\mathbf{u}})\beta_{\mathbf{u}}dt$ at state $\mathbf{u}$.
iii. Termination occurs when the state switches to $\mathbf{d}$, that is, $q^* = 1$ and $\tau^* = \min\{t : \theta_t = \mathbf{d}\}$.

It can be verified that the corresponding expected societal value starting from state $\mathbf{u}$ is

$$v_{\mathbf{u}} := \mathbb{E}\left[\int_0^{\tau^*_{\mathbf{u}}} e^{-rt}(R-c_{\mathbf{u}})dt + e^{-r\tau^*_{\mathbf{u}}}\underline{v}_{\mathbf{d}}\right] = \frac{R-c_{\mathbf{u}}+\mu_{\mathbf{u}}\underline{v}_{\mathbf{d}}}{r+\mu_{\mathbf{u}}}. \quad (25)$$

Intuitively, the aforementioned contract structure is desirable only if it out performs not hiring the agent at all starting from state $\mathbf{u}$. That is,

$$v_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}, \text{ or, equivalently, } R \geq g_{\mathbf{u}}, \quad (26)$$

in which we define

$$g_{\mathbf{u}} := \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right)\beta_{\mathbf{u}}. \quad (27)$$

For $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, it is easy to verify that $h_{\mathbf{d}} \geq g_{\mathbf{u}}$. (In particular, $h_{\mathbf{d}} = g_{\mathbf{u}}$ if $\beta_{\mathbf{d}} = \beta_{\mathbf{u}}$.)

The next result formally states that such a contract is indeed optimal when condition (14) is violated while (26) holds, that is,

$$g_{\mathbf{u}} \leq R < h_{\mathbf{d}}. \quad (28)$$

Theorem 2. *1. Contract $\Gamma^*_{\mathbf{u}}$ is incentive compatible.*

*2. The principal's utilities following Contract $\Gamma^*_{\mathbf{u}}$ are*

$$U(\Gamma^*_{\mathbf{u}}, \nu^*, \mathbf{d}) = \underline{v}_{\mathbf{d}} \quad \textit{and} \quad U(\Gamma^*_{\mathbf{u}}, \nu^*, \mathbf{u}) = v_{\mathbf{u}} - \beta_{\mathbf{u}}.$$

*3. Assume that condition (28) holds.*

*(i) For any incentive compatible contract* $\Gamma$*, we have*

$$\underline{v}_{\mathbf{d}} \geq U(\Gamma, \nu^*, \mathbf{d}),$$

*or, it is better not to hire the agent starting from state* $\mathbf{d}$*.*

*(ii) Furthermore, if* $v_{\mathbf{u}} - \beta_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$*, we have*

$$U(\Gamma^*_{\mathbf{u}}, \nu^*, \mathbf{u}) \geq U(\Gamma, \nu^*, \mathbf{u}),$$

*That is,* $\Gamma^*_{\mathbf{u}}$ *is the optimal incentive compatible contract.*

*(iii) Finally, if* $v_{\mathbf{u}} - \beta_{\mathbf{u}} < \underline{v}_{\mathbf{u}}$*, for any incentive compatible contract* $\Gamma$*, we have*

$$\underline{v}_{\mathbf{u}} \geq U(\Gamma, \nu^*, \mathbf{u}),$$

*or, it is better not to hire the agent starting from state* $\mathbf{u}$ *as well.*

Contract $\Gamma^*_{\mathbf{u}}$ suggests that the principal hire the agent only if the machine starts in the up state, and terminate the agent as soon as it breaks down. This is driven by the fact that we do not allow shirking so far in the paper. If we allow shirking instead, the principal may benefit from hiring the agent to exert effort only when the machine is up, while allowing the agent to shirk when the machine is down. In Section EC.2 of the e-companion, we provide the optimal contracts that motivate the agent to exert effort only when the machine is up (resp. down), and call it "maintenance contract" (resp, "repair contract"). It is clear that contract $\Gamma^*_{\mathbf{u}}$ is a particular "maintenance contract." Therefore, under condition (28), the optimal "maintenance contract" always outperforms the contract $\Gamma^*_{\mathbf{u}}$. Generally speaking, the principal may prefer the maintenance contract over a contract that always induces effort when, for example, when the agent's cost of effort to repair ($c_{\mathbf{d}}$) is so expensive that the principal is better off just hiring the agent to conduct maintenance and not repair.

The next result further states that if condition (26) is violated, it is also better for the principal to not hire the agent than motivating effort.

THEOREM 3. *If*

$$R < g_{\mathbf{u}}, \tag{29}$$

*we have* $\underline{v}_\theta \geq U(\Gamma, \nu^*, \theta)$ *for any incentive compatible contract* $\Gamma$ *and state* $\theta \in \{\mathbf{u}, \mathbf{d}\}$*, where* $g_{\mathbf{u}}$ *is defined in (27).*

Theorem 3 is intuitive in the sense that when revenue rate $R$ is not large enough compared to the cost, it is not worthwhile for the principal to pay the cost and payment to induce the agent to work.

### 4.2. The case $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$

If $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, the contract $\Gamma_1^*$ in Definition 1 is no longer incentive compatible. To see this, consider the situation where the promised utility $W_{t-} < \beta_{\mathbf{u}} - \beta_{\mathbf{d}}$ before the machine recovers. If the promised utility still jumps up by $\beta_{\mathbf{d}}$ upon the machine recovery at time $t$, then $W_t < \beta_{\mathbf{u}}$. At that point constraint (7) cannot be satisfied. That is, the principal cannot incentivize the agent to exert effort in maintaining the machine. As we will show in the following, the optimal contract needs to involve random termination when the agent's promised utility is low. Furthermore, when the promised utility is high, the optimal contract involves a region where one of the incentive compatible constraints in (7) is not binding. As we have alluded to in Section 3.2, this is quite peculiar, because, as far as we know, IC constraints are always binding in optimal contracts studied in the continuous time moral hazard literature (see, for example, Sannikov 2008, Biais et al. 2010, Shan 2017, Sun and Tian 2017).

The structure of this section mirrors Section 4.1. In Sections 4.2.1 and 4.2.2, we first study incentive compatible optimal contracts under condition (18). Finally, Section 4.2.3 studies what happens when condition (18) is violated.

**4.2.1. Optimal IC contract when $\bar{v}_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$** We first illustrate the structure of the optimal contract using Figure 3 before formally defining the optimal contract. Once again, the contract keeps track of the agent's promised utility $W_t$ over time. The dynamics of $W_t$, however, are more complicated than the optimal contract in Section 4.1.1. In particular, if $W_{t-} \in (0, \bar{w}_{\mathbf{d}})$ in state $\mathbf{d}$, the promised utility keeps decreasing until either the machine is repaired, or the promised utility reaches 0 and the agent is terminated. If $W_{t-} \in [\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}]$ in state $\mathbf{d}$, on the other hand, the promised utility remains a constant until the machine is repaired. If, upon recovery to state $\mathbf{u}$, the promised utility is below $\beta_{\mathbf{u}}$, however, the incentive compatibility constraint (7) implies that the machine cannot stay in state $\mathbf{u}$ at the current promised utility level. Instead, the principal randomly terminates the contract or resets the promised utility to be at or above $\beta_{\mathbf{u}}$.

Figure 3 depicts two sample trajectories following the proposed contract starting at state $\theta_0 = \mathbf{d}$ from an initial promised utility $W_0 = w_{\mathbf{d}}^* \in (0, \hat{w}_{\mathbf{d}})$. First, focus on the solid curve. The promised utility decreases over time while the agent exerts effort to repair the machine, until time $t_1$, when the machine recovers. At this point, the promised utility jumps up by $\beta_{\mathbf{d}}$ and the agent starts maintaining the machine at state $\mathbf{u}$. The promised utility keeps increasing until time $t_2$, when the machine breaks down. In the time interval $(t_2, t_4)$, the promised utility behaves the same way as it does in $(0, t_2)$, with the machine recovering at $t_3$. When the machine breaks down again at time $t_4$, however, the promised utility is already so high that it will still be above $\bar{w}_{\mathbf{d}}$ after a downward jump of $\beta_{\mathbf{u}}$. Because $W_{t-} \geq \bar{w}_{\mathbf{d}}$ at state $\mathbf{d}$, the promised utility is kept at this level as a constant, until the machine recovers at time $t_5$. At this point in time the promised utility takes an upward jump $rW_{t_5-}/\mu_{\mathbf{d}} > \beta_{\mathbf{d}}$, or, the IC constraint (7) at state $\mathbf{d}$ is not binding. After time $t_5$, the machine stays in state $\mathbf{u}$ while the promised utility increases to reach $\hat{w}_{\mathbf{u}}$ at time $\tilde{t}_5$, at which point the contract follows $\hat{\Gamma}$ as defined in Section 3.2. Note that following this sample trajectory, the structure of the optimal contract after time $t_4$ behaves differently from the optimal contract $\Gamma_1^*$ defined in Section 4.1 (because the promised utility remains constant even though the machine is down).

Now we focus on the other sample trajectory in Figure 3, the dotted curve. The machine is in state $\mathbf{d}$ during time intervals $[0, \hat{t}_1)$ and $[\hat{t}_2, \hat{t}_3)$, and in state $\mathbf{u}$ during $[\hat{t}_1, \hat{t}_2)$. The promised utility decreases in state $\mathbf{d}$ and increases in state $\mathbf{u}$. Right before the machine recovers for the second time, at $\hat{t}_3$, the promised utility is below $\beta_{\mathbf{u}} - \beta_{\mathbf{d}}$. Therefore, even an upward jump of $\beta_{\mathbf{d}}$ cannot raise the promised utility above $\beta_{\mathbf{u}}$. In light of the discussion in the beginning of this section, the agent is terminated with probability $q^*_{\hat{t}_3} = (\beta_{\mathbf{u}} - W_{\hat{t}_3})/\beta_{\mathbf{u}}$. On the other hand, with probability $1 - q^*_{\hat{t}_3}$, the agent's promised utility is reset to $\beta_{\mathbf{u}}$.

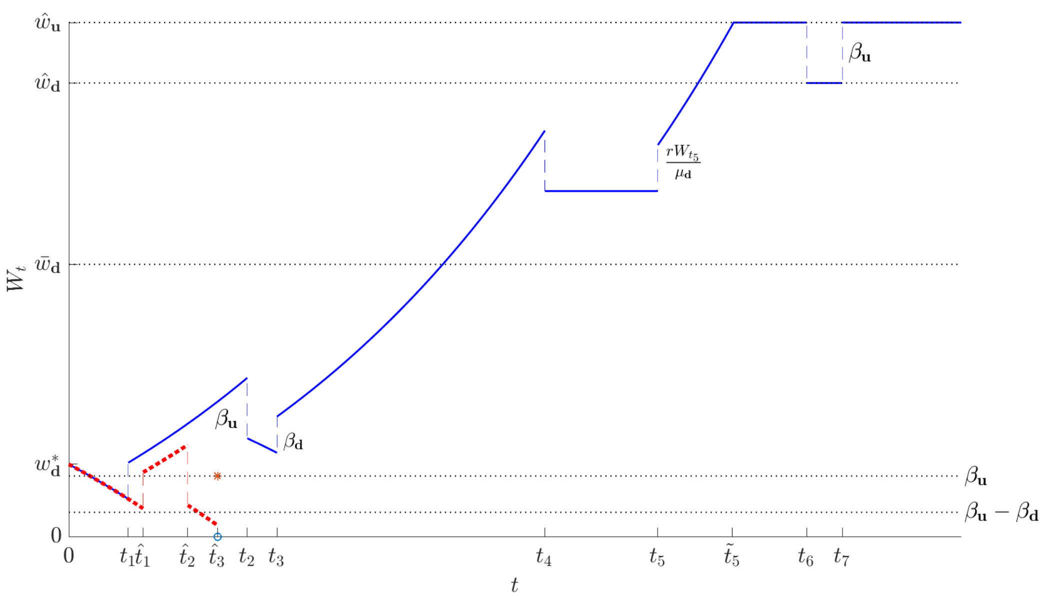



**Figure 3** Two sample trajectories of promised utility with model parameters $\mu_{\mathbf{u}} = 2, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 6, \Delta\mu_{\mathbf{d}} = 2, c_{\mathbf{u}} = 1, c_{\mathbf{d}} = 1.2, r = 0.8, R = 20$. In this case, $\bar{w}_{\mathbf{d}} = 3$, $\hat{w}_{\mathbf{d}} = 6$, $\hat{w}_{\mathbf{u}} = 7$ and $\beta_{\mathbf{u}} = 1 > \beta_{\mathbf{d}} = 0.6$. The policy starts from $w^*_{\mathbf{d}} = 1.194$. The solid curve represents a sample trajectories which brings the agent to the point of never terminated. The dotted curve represents another sample trajectory in which the agent is terminated due to a random draw at a point when the machine recovers.

It is clear that randomization needs to occur at state $\mathbf{u}$ if the promised utility is below the threshold $\beta_{\mathbf{u}}$. In fact, the threshold below which the random termination occurs does not have to be exactly $\beta_{\mathbf{u}}$. It can be at a more general level of $\hat{\beta} \geq \beta_{\mathbf{u}}$. In the contract depicted in Figure 3, we have $\hat{\beta} = \beta_{\mathbf{u}}$, but this equality does not necessarily always hold, and we may have $\hat{\beta} > \beta_{\mathbf{u}}$. That is, as long as the promised utility $W_t$ is below $\hat{\beta}$ in state $\mathbf{u}$, the agent is randomly terminated with probability $q^*_t = (\hat{\beta} - W_t)/\hat{\beta}$. If termination does not happen at the random draw, the promised utility is reset to $\hat{\beta}$.

Formally, we define the following contract, $\Gamma^*_{\hat{\beta}}$, and later show that the optimal contract follows this definition with an appropriately chosen value of $\hat{\beta} \geq \beta_{\mathbf{u}}$.

Definition 3. For any $\hat{\beta} \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$, define contract $\Gamma^*_{\hat{\beta}}(w) = (L^*, q^*, \tau^*)$ for $w \in [0, \hat{w}_\theta]$ if the initial state of the machine is $\theta \in \{\mathbf{u}, \mathbf{d}\}$.

i. The dynamics of the agent's promised utility $W_t$, follows

$$dW_t = \Bigg\{ r(W_{t-} - \bar{w}_{\mathbf{d}})\mathbb{1}_{W_{t-} < \bar{w}_{\mathbf{d}}} dt + \Bigg\{ \mathbb{1}_{W_{t-} \in (\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}]} \frac{rW_{t-}}{\mu_{\mathbf{d}}} + \mathbb{1}_{W_{t-} \in (\hat{\beta} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]} \beta_{\mathbf{d}} + \mathbb{1}_{W_{t-} < \hat{\beta} - \beta_{\mathbf{d}}} \Big[ (1 - X_t)(\hat{\beta} - W_{t-}) - X_t W_{t-} \Big] \Bigg\} dN_t \Bigg\} \mathbb{1}_{\theta_{t-} = \mathbf{d}} + [(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}) dt \mathbb{1}_{W_{t-} < \bar{w}_{\mathbf{u}}} - \beta_{\mathbf{u}} dN_t] \mathbb{1}_{\theta_{t-} = \mathbf{u}}, \tag{DW2}$$

from an initial promised utility $W_0 = w$.

ii. The payment to the agent follows $dL^*_t = \ell^*_2 \mathbb{1}_{\theta_{t-} = \mathbf{u}} \mathbb{1}_{W_{t-} = \hat{w}_{\mathbf{u}}} dt$.

iii. The random termination probability is $q^*_t = \hat{q}(W_{t-}) \mathbb{1}_{W_{t-} + \beta_{\mathbf{d}} < \hat{\beta}} \mathbb{1}_{\theta_{t-} = \mathbf{d}} dN_t$, in which

$$\hat{q}(w) = \frac{\hat{\beta} - (w + \beta_{\mathbf{d}})}{\hat{\beta}}, \tag{30}$$

and the termination time is $\tau^* = \min\{t : W_t = 0\}$.

It is worth noting that in contract $\Gamma^*_{\hat{\beta}}(w)$, constraint (7) is not always binding. Specifically, if $W_{t-} > \bar{w}_{\mathbf{d}}$, following the definition we have $q^*_t = 0$ and $H_t = rW_{t-}/\mu_{\mathbf{d}} > \beta_{\mathbf{d}}$. Before we rigorously prove the optimality of the contract, let us explain the intuition why constraint (7) is not always binding in the optimal contract, in two steps. First, we explain that social efficiency can be achieved in the optimal contract. Then we explain why achieving efficiency introduces slacks in the incentive compatible constraint (7) when $\beta_{\mathbf{d}} < \beta_{\mathbf{d}}$.

The principal and agent having the same time discount rate implies that they have the same total discounted valuation for any payments. Therefore, the societal value function is simply the principal's value function plus the agent's promised utility. Consequently, a contract that maximizes the principal's value function must also maximize the societal value function. Under condition (18), contract $\hat{\Gamma}$ introduced in Section 3.2 achieves social efficiency (maximizes the societal value functions at promised utility levels $\bar{w}_{\mathbf{u}}$ or $\bar{w}_{\mathbf{u}}$). Therefore, social efficiency must also be achievable at the same promised utility levels under the optimal contract.

If we had to force incentive compatible constraints to be always binding, the upward jump in the promised utility would be $\beta_{\mathbf{d}}$, smaller than the downward jump $\beta_{\mathbf{u}}$. Therefore, no matter where the promised utility starts from, a downward jump of $\beta_{\mathbf{u}}$ cannot be fully compensated by an upward jump of $\beta_{\mathbf{d}}$. As a result, starting from any finite promised utility value, a sample trajectory (however unlikely) with a sequence of very frequent state switches eventually drives the promised utility down to 0. The existence of such sample trajectories implies that the agent would be terminated with positive probability, and, hence, social efficiency would not be achievable. This contradicts the arguments in the last paragraph that the optimal contract should be able to achieve social efficiency. Therefore, in the optimal contract we cannot enforce IC constraints to be binding all the time.

**4.2.2. Value functions and proof of optimality when $\bar{v}_{\mathbf{u}} \geq \underline{v}_{\mathbf{u}}$** There are some important distinctions in the approach to determine the principal's value functions, in the case of $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, compared with the one in Section 4.1.2. This is because here we need to specify the threshold $\hat{\beta}$ that defines when/if the agent will be randomly terminated.

First, let $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ represent the principal's value functions for states $\mathbf{u}$ and $\mathbf{d}$, respectively. Following Definition 3 and similar heuristic derivation steps as in Appendix D.1, we obtain the following system of differential equations. In particular, for state $\mathbf{d}$, equation (20) in Section 4.1.2 becomes the following three equations

$$(\mu_{\mathbf{d}}+r)J_{\mathbf{d}}(w)=\mu_{\mathbf{d}}J_{\mathbf{u}}\left(\frac{r+\mu_{\mathbf{d}}}{r}w\right)-c_{\mathbf{d}},\ w\in[\bar{w}_{\mathbf{d}},\hat{w}_{\mathbf{d}}], \tag{31}$$

$$-c_{\mathbf{d}}+r(w-\bar{w}_{\mathbf{d}})J'_{\mathbf{d}}(w)=(\mu_{\mathbf{d}}+r)J_{\mathbf{d}}(w)-\mu_{\mathbf{d}}J_{\mathbf{u}}(w+\beta_{\mathbf{d}}),\quad w\in[\hat{\beta}-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}}],\ \text{and} \tag{32}$$

$$-c_{\mathbf{d}}+r(w-\bar{w}_{\mathbf{d}})J'_{\mathbf{d}}(w)=(\mu_{\mathbf{d}}+r)J_{\mathbf{d}}(w)-\mu_{\mathbf{d}}\left[\hat{q}(w)J_{\mathbf{u}}(0)+\big(1-\hat{q}(w)\big)J_{\mathbf{u}}(\hat{\beta})\right],\ w\in[0,\hat{\beta}-\beta_{\mathbf{d}}]. \tag{33}$$

For state $\mathbf{u}$, the differential equation is similar to (21) for $w\in[\hat{\beta},\hat{w}_{\mathbf{u}}]$. That is,

$$-c_{\mathbf{u}}+(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\hat{w}_{\mathbf{u}}}J'_{\mathbf{u}}(w)=(\mu_{\mathbf{u}}+r)J_{\mathbf{u}}(w)-R-\mu_{\mathbf{u}}J_{\mathbf{d}}(w-\beta_{\mathbf{u}})+\ell^*\mathbb{1}_{w=\hat{w}_{\mathbf{u}}},\ w\in[\hat{\beta},\hat{w}_{\mathbf{u}}] \tag{34}$$

Due to randomization, we may further extend function $J_{\mathbf{u}}(w)$ to the interval $[0,\hat{\beta}]$ as a linear function with a slope $a$, that is,

$$J_{\mathbf{u}}(w)=J_{\mathbf{u}}(0)+aw,\quad w\in[0,\hat{\beta}]. \tag{35}$$

The principal receives baseline revenues $\underline{v}_{\mathbf{d}}$ and $\underline{v}_{\mathbf{u}}$, as defined in (4), upon termination in states $\mathbf{d}$ and $\mathbf{u}$, respectively, which implies the following boundary conditions

$$J_{\mathbf{u}}(0)=\underline{v}_{\mathbf{u}}\quad\text{and}\quad J_{\mathbf{d}}(0)=\underline{v}_{\mathbf{d}}. \tag{36}$$

We first present the following result regarding general solutions to the aforementioned differential equations.

Lemma 2. *For any $a > -1$, there exists a unique pair of functions $J_{\mathbf{d}}^{a\hat{\beta}}$ and $J_{\mathbf{u}}^{a\hat{\beta}}$, in place of $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$, respectively, that satisfy (31)-(36), in which slope "a" appears in (35).*

*Furthermore, functions $J_{\mathbf{d}}^{a\hat{\beta}}(w)$ and $J_{\mathbf{u}}^{a\hat{\beta}}(w)$ are twice continuously differentiable, except for $J_{\mathbf{u}}^{a\hat{\beta}}(w)$ at $w = \hat{\beta}$.*

It is straightforward to show that it is sufficient to focus only on the case $a > -1$. Intuitively, this is because the slope $a$ represents how much the the principal's utility changes as the agent's promised utility increases. It can never be less than $-1$ because otherwise, decreasing the agent's promised utility by a direct monetary payment would generate a profit to the principal, which is impossible.

Next, we determine the threshold $\hat{\beta}$ for a given slope $a$. The idea is to set $\hat{\beta}$ such that function $J_{\mathbf{u}}^{a\hat{\beta}}(w)$ is differentiable at $\hat{\beta}$ if possible, so that we achieve "smooth pasting"[9] between (34) and (35). For this purpose we define the following function for $\hat{\beta} \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$,

$$f_a(\hat{\beta}) := \left( J_{\mathbf{u}}^{a\hat{\beta}\prime}(\hat{\beta}_-) - J_{\mathbf{u}}^{a\hat{\beta}\prime}(\hat{\beta}_+) \right) (r\hat{\beta} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}). \tag{37}$$

Function $f_a$ is a technical construction, and has good properties for us to study when the function $J_{\mathbf{u}}^{a\hat{\beta}}$'s left and right derivatives are the same at $\hat{\beta}$. Clearly, we can achieve "smooth pasting" if there exists a $\hat{\beta}$ such that $f_a(\hat{\beta}) = 0$. The following result guarantees that there exists at most one such $\hat{\beta}$.

Lemma 3. *For any $a > -1$, function $f_a(\hat{\beta})$ is increasing in $\hat{\beta}$ on $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$, and $\lim_{\beta \uparrow \bar{w}_{u-}} f(\hat{\beta}) > 0$. Therefore, the following quantity $\beta_a$ is well defined,*

$$\beta_a := \begin{cases} \beta_{\mathbf{u}}, & f_a(\beta_{\mathbf{u}}) \geq 0, \\ f_a^{-1}(0), & f_a(\beta_{\mathbf{u}}) < 0, \end{cases} \tag{38}$$

*in which $f_a^{-1}$ is the inverse function of $f_a$.*

Furthermore, as soon as the promised utility reaches $\hat{w}_{\mathbf{u}}$ in state $\mathbf{u}$, the contract $\Gamma^*_{\hat{\beta}}$ becomes identical to $\hat{\Gamma}$, and the agent will no longer be terminated. This implies the following boundary conditions

$$J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{d}}) = \bar{v}_{\mathbf{d}} - \hat{w}_{\mathbf{d}} \quad \text{and} \quad J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{u}}) = \bar{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}} \ , \tag{39}$$

in which $\bar{v}_{\mathbf{d}}$ and $\bar{v}_{\mathbf{u}}$ are the societal value function when the agent is never terminated, as defined in (11).

Now we are ready to uniquely determine the value $a$ in equation (35) for the value function.

Proposition 4. *There exists a unique $\bar{a} > 0$ such that*

$$\lim_{w \uparrow \hat{w}_{\mathbf{u}}} J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w) = J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{u}}) = \bar{v}_{\mathbf{u}} - \hat{w}_{\mathbf{u}} \quad \text{and} \quad \lim_{w \uparrow \hat{w}_{\mathbf{d}}} J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w) = J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(\hat{w}_{\mathbf{d}}) = \bar{v}_{\mathbf{d}} - \hat{w}_{\mathbf{d}}, \tag{40}$$

*where threshold $\beta_{\bar{a}}$ is defined according to (38). Furthermore, functions $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$ are both strictly concave, and,*

$$\lim_{w\uparrow\hat{w}_{\mathbf{u}}}\frac{d}{dw}J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)=\lim_{w\uparrow\hat{w}_{\mathbf{d}}}\frac{d}{dw}J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)=-1.$$

Similar to Proposition 2, the following result shows that $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$ specified in Proposition 4 are indeed the principal's total discounted utility under contract $\Gamma_{\beta_{\bar{a}}}(w)$, as stated in the next result.

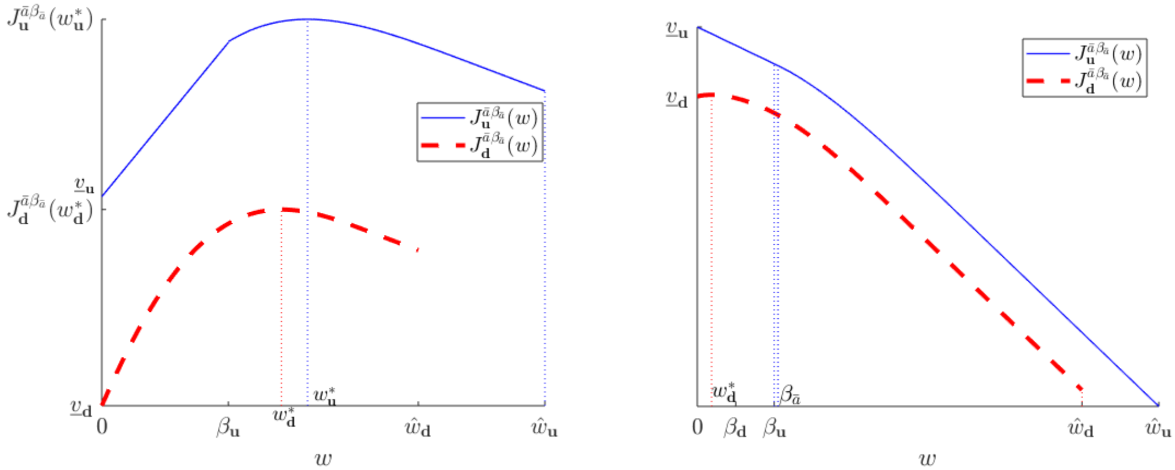
Proposition 5. *For any state $\theta\in\{\mathbf{u},\mathbf{d}\}$ and promised utility $w\in[0,\bar{w}_\theta]$, we have $U(\Gamma^*_{\beta_{\bar{a}}}(w),\nu^*,\theta)=J_\theta^{\bar{a}\beta_{\bar{a}}}(w)$. That is, values $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$ are equal to the principal's total discounted utilities of following contract $\Gamma^*_{\beta_{\bar{a}}}$ from the initial promised utility $w$ when the initial state of the machine is $\mathbf{u}$ and $\mathbf{d}$, respectively.*

Figures 4 depicts the principal's value functions $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$, similar to Figure 2 of Section 4.1. In particular, Figure 4(a) depicts a case where the threshold $\beta_{\bar{a}}=\beta_{\mathbf{u}}$, while Figure 4(b) depicts a case with $\beta_{\bar{a}}>\beta_{\mathbf{u}}$ with smooth pasting in play.

Intuitively, in optimal control problems with a finite number of actions, randomization between actions allows us to obtain a concave upper envelope of the value function. In our setting, randomization between contract termination (setting the promised utility $w$ to 0) and resetting the promised utility to $\beta_{\mathbf{u}}$ allows us to achieve a value function that is linear between 0 and $\beta_{\mathbf{u}}$, as we see in Figure 4(a). If the resulting value function is concave, then we can show that the control policy is indeed optimal. However, if the aforementioned randomization yields a value function such that the left derivative at $\beta_{\mathbf{u}}$ is smaller than the right derivative at this point, then the resulting value function is not concave. Whenever a value function is non-concave, it must be sub-optimal. This is because using randomization we should at least achieve its concave upper envelope. In our setting, this implies that we can increase the point where we reset the promised utility, from $\beta_{\mathbf{u}}$ to somewhere above it, until the value function becomes concave. Smooth pasting captures the intuition that the value function becomes "barely concave." As we see in Figure 4(b), in this parameter setting, randomization between 0 and $\beta_{\bar{a}}$ (instead of $\beta_{\mathbf{u}}$) yields a value function that is smooth at $\beta_{\bar{a}}$ for state $\mathbf{u}$.

Now we are ready to show that the contract $\Gamma^*_{\beta_{\bar{a}}}$ is indeed optimal. The following main result is parallel to a combination of Proposition 3 and Theorem 1 of the previous subsection.

Theorem 4. *For any incentive compatible contract $\Gamma$ and initial state $\theta\in\{\mathbf{u},\mathbf{d}\}$, we have $J_\theta^{\bar{a}\beta_{\bar{a}}}\left(u(\Gamma,\nu^*,\theta)\right)\geq U(\Gamma,\nu^*,\theta)$, in which we extend the function $J_\theta^{\bar{a}\beta_{\bar{a}}}(w)=J_\theta^{\bar{a}\beta_{\bar{a}}}(\bar{w}_\theta)-(w-\bar{w}_\theta)$ for $w>\bar{w}_\theta$.*




(a) Principal's Value functions (b) Principal's Value functions with smooth-pasting

**Figure 4** (a): $\mu_{\mathbf{u}} = 1.5, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 1.5, \Delta\mu_{\mathbf{d}} = 1, c_{\mathbf{u}} = 0.7, c_{\mathbf{d}} = 0.6, r = 0.6, R = 11$. In this case, $\bar w_{\mathbf{d}} = 1.5$, $\hat w_{\mathbf{d}} = 1.75$ and $\hat w_{\mathbf{u}} = 2.45$ and $\beta_{\mathbf{u}} = 0.7 > \beta_{\mathbf{d}} = 0.6$. $J_{\mathbf{u}}(w_{\mathbf{u}}^*) = 8.195$, $\underline{v}_{\mathbf{u}} = 5.602$ and $J_{\mathbf{u}}(\hat w_{\mathbf{u}}) = \bar U_{\mathbf{u}} = 7.147$. $J_{\mathbf{d}}(w_{\mathbf{d}}^*) = 5.414$, $\underline{v}_{\mathbf{d}} = 2.546$ and $J_{\mathbf{d}}(\hat w_{\mathbf{d}}) = \bar U_{\mathbf{d}} = 4.819$. (b): $\mu_{\mathbf{u}} = 8, \Delta\mu_{\mathbf{u}} = 4, \mu_{\mathbf{d}} = 6, \Delta\mu_{\mathbf{d}} = 5, c_{\mathbf{u}} = 4.8, c_{\mathbf{d}} = 3, r = 1.2, R = 16$. In this case, $\hat w_{\mathbf{d}} = 3.33$, $\hat w_{\mathbf{u}} = 4.33$ and $\beta_{\mathbf{u}} = 1.2 > \beta_{\mathbf{d}} = 0.6$. $\bar a = 0.501$ and $\beta_{\bar a} = 1.259$, $w_{\mathbf{d}}^* = 0.222$. $J_{\mathbf{u}}(w_{\mathbf{u}}^*) = 2.066$, $\underline{v}_{\mathbf{u}} = 2.066$ and $J_{\mathbf{u}}(\hat w_{\mathbf{u}}) = \bar U_{\mathbf{u}} = -4.095$. $J_{\mathbf{d}}(w_{\mathbf{d}}^*) = 0.964$, $\underline{v}_{\mathbf{d}} = 0.939$ and $J_{\mathbf{d}}(\hat w_{\mathbf{d}}) = \bar U_{\mathbf{d}} = -3.829$.

*Therefore, denoting $w_\theta^*$ to represent a maximizer of function $J_\theta^{\bar a\beta_{\bar a}}$, we have $U\left(\Gamma_{\beta_{\bar a}}^*(w_\theta^*), \nu^*, \theta\right) \geq U(\Gamma, \nu^*, \theta)$ for any incentive compatible contract $\Gamma$ and state $\theta$. That is, the optimal incentive compatible contract is $\Gamma_{\beta_{\bar a}}^*(w_\theta^*)$, if $\beta_{\mathbf{u}} > \beta_{\mathbf{d}}$, condition (19) holds, and the machine starts from state $\theta \in \{\mathbf{u}, \mathbf{d}\}$.*

It is worth noting that $J_{\mathbf{d}}^{\bar a\beta_{\bar a}}(\hat w_{\mathbf{d}}) = \hat U_{\mathbf{d}}$ and $J_{\mathbf{u}}^{\bar a\beta_{\bar a}}(\hat w_{\mathbf{u}}) = \hat U_{\mathbf{u}}$ where $\hat U_{\mathbf{d}}$ and $\hat U_{\mathbf{u}}$, defined in (17), are the principal's utility under the simple contract $\hat\Gamma$ of Section 3.2. This also implies that contract $\Gamma_{\hat\beta}^*$ always (weakly) outperforms $\hat\Gamma$. The difference between the peak of the value function $J_\theta^{\bar a\beta_{\bar a}}$ and $\hat U_\theta$ demonstrates the suboptimality of the benchmark contract $\hat\Gamma$ if the machine starts from state $\theta$. For example, in Figure 4(a), the difference between the optimal contract and the benchmark contract is captured in the difference between $J_{\mathbf{u}}(w_{\mathbf{u}}^*) = 8.195$ and $J_{\mathbf{u}}(\hat w_{\mathbf{u}}) = \bar U_{\mathbf{u}} = 7.147$, or $J_{\mathbf{d}}(w_{\mathbf{d}}^*) = 5.414$ and $J_{\mathbf{d}}(\hat w_{\mathbf{d}}) = \bar U_{\mathbf{d}} = 4.819$, when the machine starts from state **u** and **d**, respectively. In Figure 4(b), we have $J_{\mathbf{u}}(w_{\mathbf{u}}^*) = 2.066$ and $J_{\mathbf{u}}(\hat w_{\mathbf{u}}) = \bar U_{\mathbf{u}} = -4.095$; $J_{\mathbf{d}}(w_{\mathbf{d}}^*) = 0.964$ and $J_{\mathbf{d}}(\hat w_{\mathbf{d}}) = \bar U_{\mathbf{d}} = -3.829$. Therefore, in the case of Figure 4(b), the optimal contract is profitable, while the benchmark contract is not.

Furthermore, as we can see from Figure 4(b), where the threshold $\beta_{\bar a} > \beta_{\mathbf{u}}$, the function $J_{\mathbf{u}}^{\bar a\beta_{\bar a}}(w)$ is monotonically decreasing, or, the maximizer $w_{\mathbf{u}}^* = 0$. That is, if the initial state of the machine is **u**, it is better for the principal not to hire the agent than to motivate the agent's full effort. This is generally true, as confirmed in the following result.

Proposition 6. *If $\beta_{\bar{a}} > \beta_{\mathbf{u}}$, then we have the slope $\bar{a} < 0$.*

In other words, if it is optimal to hire the agent at the initial state $\mathbf{u}$, then the threshold $\beta_{\bar{a}}$ in contract $\Gamma^*_{\beta_{\bar{a}}}$ must be equal to $\beta_{\mathbf{u}}$. On the other hand, in Figure 4(b), we have $w^*_{\mathbf{d}} > 0$. Therefore, when smooth pasting is at work ($\beta_{\bar{a}} > \beta_{\mathbf{u}}$), it is better not to hire the agent if the initial state is $\mathbf{u}$. However, it may still be beneficial to hire the agent if the initial state is $\mathbf{d}$, although this benefit tends to be small.

**4.2.3.** $\bar{v}_{\mathbf{u}} < \underline{v}_{\mathbf{u}}$ Now we consider the case that (18), or, equivalently, (19), is violated. First, similar to (25) in Section 4.1.3, we define the following societal value for the case where the agent starts in state $\mathbf{d}$, exerts effort to repair the machine and is terminated once the machine is repaired,

$$v_{\mathbf{d}} := \mathbb{E}\left[-\int_0^{\tau^*_{\mathbf{d}}} e^{-rt} c_{\mathbf{d}} dt + e^{-r\tau^*} \underline{v}_{\mathbf{u}}\right] = \frac{\mu_{\mathbf{d}} \underline{v}_{\mathbf{u}} - c_{\mathbf{d}}}{r + \mu_{\mathbf{d}}}. \tag{41}$$

Here $\tau^*_{\mathbf{d}}$ represents the time that the machine is in state $\mathbf{d}$, which follows an exponential distribution with rate $\mu_{\mathbf{d}}$ when the agent exerts effort.

Similar to condition (26) in Section 4.1.3, we first consider the optimal contract under the following condition,

$$v_{\mathbf{d}} \geq \underline{v}_{\mathbf{d}}, \text{ or, equivalently, } R \geq g_{\mathbf{d}}, \tag{42}$$

in which we define

$$g_{\mathbf{d}} := \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right) \beta_{\mathbf{d}}. \tag{43}$$

And we have $g_{\mathbf{d}} < (=) h_{\mathbf{u}}$ for $\beta_{\mathbf{d}} < (=) \beta_{\mathbf{u}}$.

If the machine starts at state $\mathbf{u}$, the principal does not hire the agent. On the other hand, if the machine starts at state $\mathbf{d}$, then the promised utility starts from an initial value $W_0 \leq \bar{w}_{\mathbf{d}}$ and keeps decreasing according to $dW_t = r(W_{t-} - \bar{w}_{\mathbf{d}})dt$ until termination, when either $W_t$ reaches 0 or the machine recovers. If the machine recovers at a positive $W_{t-}$, then the agent is paid this promised utility $W_{t-}$ plus $\beta_{\mathbf{d}}$, which provides the incentive for the agent to exert effort to repair the machine. Formally, we have the following definition of a contract.

Definition 4. Define contract $\Gamma^*_{\mathbf{d}}(w)$ for $w \in [0, \bar{w}_{\mathbf{d}}]$ if the machine starts in state $\mathbf{d}$ as the following.

i. In state $\mathbf{d}$, the agent's promised utility $W_t$ follows

$$dW_t = r(W_{t-} - \bar{w}_{\mathbf{d}})dt - W_{t-} dN_t\ , \tag{DWd}$$

starting from $W_0 = w$. In state $\mathbf{u}$, $W_t$ remains 0.

ii. The payment to the agent follows $dL^*_t = (W_{t-} + \beta_{\mathbf{d}})dN_t$.

iii. The random termination probability is $q_t^* = \mathbb{1}_{\{\theta_t=\mathbf{u}\}}$, and the termination time is $\tau^* = \{t : W_t = 0\}$.

According to Definition 4, termination may occur when the machine is down for a long enough period, or at the time it recoves.

The next result formally establishes the optimality of the contract.

THEOREM 5. *1. Contract $\Gamma_{\mathbf{d}}^*(w)$ is incentive compatible.*

*2. The principal's value functions under contract $\Gamma_{\mathbf{d}}^*(w)$ are*

$$U\left(\Gamma_{\mathbf{d}}^*(w), \nu^*, \mathbf{u}\right) = \underline{v}_{\mathbf{u}} - w,$$
$$U\left(\Gamma_{\mathbf{d}}^*(w), \nu^*, \mathbf{d}\right) = (\underline{v}_{\mathbf{d}} - v_{\mathbf{d}})\left(1 - \frac{w}{\bar{w}_{\mathbf{d}}}\right)^{1+\frac{\mu_{\mathbf{d}}}{r}} - w + v_{\mathbf{d}},$$

*3. Assume that condition (19) is violated while (42) holds, that is*

$$g_{\mathbf{d}} \le R < h_{\mathbf{u}}. \tag{44}$$

*For any incentive compatible contract $\Gamma$, we have*

$$U\left(\Gamma_{\mathbf{d}}^*(w^*), \nu^*, \mathbf{d}\right) \ge U(\Gamma, \nu^*, \mathbf{d}) \quad \textit{and} \quad \underline{v}_{\mathbf{u}} \ge U(\Gamma, \nu^*, \mathbf{u}),$$

*where $w^*$ is a maximizer of $U\left(\Gamma_{\mathbf{d}}^*(w), \nu^*, \mathbf{d}\right)$ as a function of $w$.*

Contract $\Gamma_{\mathbf{d}}^*$ suggests that the principal hires the agent only if the machine starts in the down state, and terminates the agent as soon as the machine recovers. This is intuitive because $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$ implies that it is cheaper to motivate effort to repair than to maintain. The fact that the agent is terminated as soon as the machine is up is, again, due to the fact Theorem 5 is focused on incentive compatible contracts. If we allow shirking instead, the principal may benefit from hiring the agent to exert effort only when the machine is down, while allowing the agent to shirk when the machine is up. As mentioned in Section 4.1.3, we call this class of contract "repair contract," which also includes $\Gamma_{\mathbf{d}}^*$. Therefore, under condition (44), the optimal repair contract outperforms the contract $\Gamma_{\mathbf{d}}^*$ (the repair contract is analyzed in the e-companion).

Despite similarities, Theorem 5 is not quite the same as Theorem 2 for the previous case. Most prominently, the value function in (130) is non-linear, while in (95) it is piece-wise linear.

If the maximizer $w^* = 0$, Theorem 5 indicates that the principal should not hire the agent at all. Similar to Theorem 3 in Section 4.1.3, the following result indicates that the principal is also better off not hiring the agent if condition (42) is violated.

THEOREM 6. *If*

$$R \le g_{\mathbf{d}}, \tag{45}$$

*we have $\underline{v}_\theta \ge U(\Gamma, \nu^*, \theta)$ for any incentive compatible contract $\Gamma$ and state $\theta \in \{\mathbf{u}, \mathbf{d}\}$, where $g_{\mathbf{d}}$ is defined in (43).*

### 4.3. A summary

It is helpful to summarize the main results that we obtained throughout this section. For the case of $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, we have characterized model parameters into three regions that can be easily characterized by focusing on the revenue rate $R$, fixing other model parameters.

- $R > h_{\mathbf{d}}$: The incentive compatible constraints in equation (7) are always binding, and the dynamic contract $\Gamma_1^*$ demonstrates rich structures.
- $R \in [g_{\mathbf{u}}, h_{\mathbf{d}}]$: The principal may hire the agent and motivate effort only to maintain the machine.
- $R < g_{\mathbf{u}}$: No incentive compatible contract (including hiring an agent only to maintain or repair–analyzed in the e-companion) performs better for the principal than not hiring the agent at all. Furthermore, as we will demonstrate in the e-companion Section EC.1, for these model parameters, not hiring the agent is the best strategy for the principal, even among contracts that allow shirking.

Similarly, when $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, we also characterize model parameters into three regions of revenue $R$.

- $R > h_{\mathbf{u}}$: The optimal contract follows $\Gamma_{\hat{\beta}}^*(w)$, where the incentive compatible constraints in equation (7) may not be always binding and the agent may need to be terminated randomly.
- $R \in [g_{\mathbf{d}}, h_{\mathbf{u}}]$: The principal may hire the agent and motivate effort only to repair the machine.
- $R < g_{\mathbf{d}}$: Not hiring the agent is the best strategy for the principal.

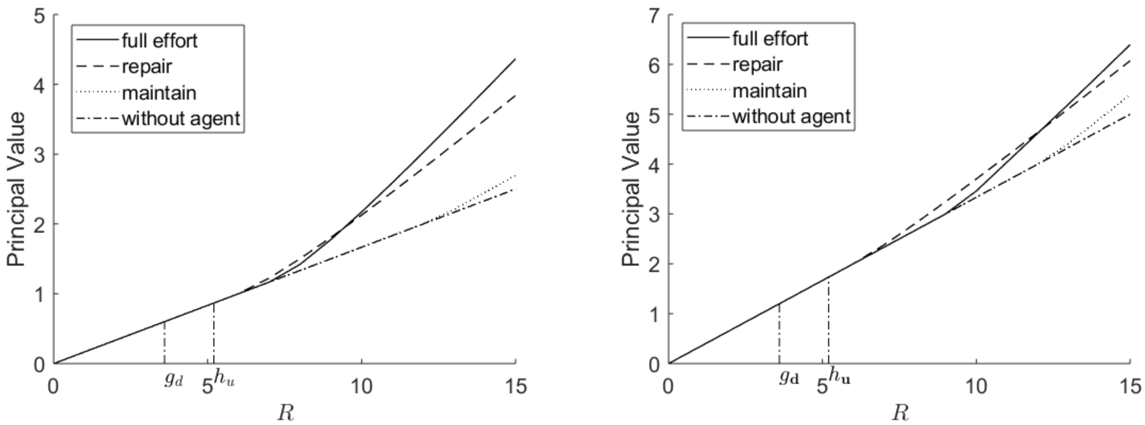
Finally, if we do not allow contract termination, the following Proposition shows that the simple contracts $\bar{\Gamma}$ and $\hat{\Gamma}$ introduced in Section 3 are optimal.

Proposition 7. *For any state $\theta \in \{\mathbf{u}, \mathbf{d}\}$ and incentive compatible contracts $\Gamma$ such that $\tau = \infty$, we have*

- $U(\bar{\Gamma}, \nu^*, \theta) \geq U(\Gamma, \nu^*, \theta)$ *if* $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$,
- $U(\hat{\Gamma}, \nu^*, \theta) \geq U(\Gamma, \nu^*, \theta)$ *if* $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$.

## 5. Numerical Comparison

So far, we have focused on analyzing optimal contracts that induce full effort from the agent before termination. However, these contracts are not necessarily optimal if the principal does not need to always induce full effort from the agent. In the e-companion, we provide sufficient conditions based on principal's value functions. One can use these conditions to verify if the optimal incentive compatible contracts that induce full effort are, in fact, optimal, even if we allow shirking. When the sufficient conditions are not satisfied, it may be preferable for the principal to hire the agent just to maintain or just to repair, and to allow the agent to shirk ("maintenance contract" and "repair contract" formally studied in the e-companion Section EC.2).



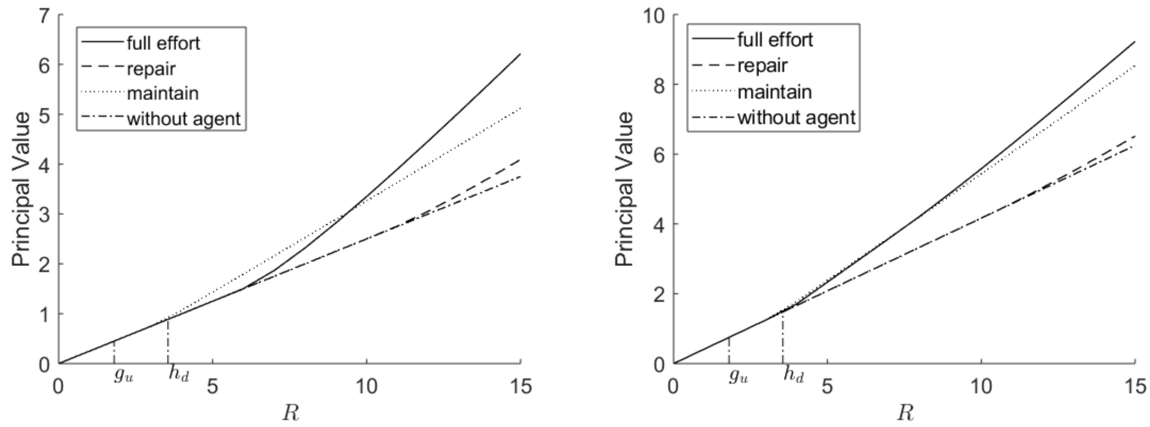

(a) Principal's value start at **d** (b) Principal's value start at **u**

**Figure 5** Principal's value under three contracts with $\mu_{\mathbf{u}} = 2, \bar{\mu}_{\mathbf{u}} = 4, \mu_{\mathbf{d}} = 3, \underline{\mu}_{\mathbf{d}} = 1, c_{\mathbf{u}} = 2, c_{\mathbf{d}} = 1.2, r = 1, R \in [0, 20]$ and $\beta_{\mathbf{u}} = 1 > \beta_{\mathbf{d}} = 0.6$. Here $g_{\mathbf{d}}$ and $h_{\mathbf{u}}$ are defined in (43) and (19), respectively.

In the following, we numerically compare the performance of the full effort incentive compatible contracts versus the repair only contract and maintenance only contract. In Figures 5 and 6, we vary revenue rate $R$, while keeping all other parameters the same. For each choice of model parameters, we calculate the principal's value for the three contracts and the value without any agents when the machine starts from state **d** and **u**, respectively. As $R$ increases, the principal's value under all the three contracts increase.

Figure 5 depicts the case of $\beta_{\mathbf{u}} > \beta_{\mathbf{d}}$. When $R \leq g_{\mathbf{d}}$, according to Theorem 6, it is not worthwhile to hire the agent when we only consider the full effort incentive compatible contracts. In the e-companion, Propositions EC.4 and EC.7 show that even under the maintenance only contract and repair only contract, the principal should not hire the agent when $R \leq g_{\mathbf{d}}$ (note that $g_{\mathbf{d}} < g_{\mathbf{u}}$ in this case). This is consistent with what we see in Figure 5, where the four curves coincide when $R < g_{\mathbf{d}}$. In fact, the region that they are all the same extends to $R > g_{\mathbf{d}}$, indicating that the optimal initial promised utility $w^*_{\mathbf{d}}$ or $w^*_{\mathbf{u}}$ is 0 in the optimal contracts, which is equivalent to not hiring the agent at all. Increasing $R$ further, hiring the agent starts making sense. First, the repair only contract outperforms the other two. In the e-companion, Theorem EC.2 implies that when $R \in [g_{\mathbf{d}}, h_{\mathbf{u}}]$, the repair only contract always outperforms the full effort contract $\Gamma^*_{\mathbf{d}}$. Intuitively, the repair only contract outperforms the maintenance only contract because $\beta_{\mathbf{u}} > \beta_{\mathbf{d}}$ implies that motivating effort to maintain is more costly than motivating effort to repair. When $R$ becomes large enough, on the other hand, full effort contract outperforms the other two one-sided contracts.

Similarly, Figure 6 depicts the case $\beta_{\mathbf{u}} < \beta_{\mathbf{d}}$. The observations and underlying reasons are parallel to Figure 5 and we do not repeat.




(a) Principal's value start at **d** (b) Principal's value start at **u**

**Figure 6** Principal's value under three contracts with $\mu_{\mathbf{u}} = 1.5, \bar{\mu}_{\mathbf{u}} = 3.5, \mu_{\mathbf{d}} = 3.5, \underline{\mu}_{\mathbf{d}} = 1.5, c_{\mathbf{u}} = 0.6, c_{\mathbf{d}} = 2, r = 1, R \in [0, 20]$ and $\beta_{\mathbf{u}} = 0.3 < \beta_{\mathbf{d}} = 1$. Here $g_{\mathbf{u}}$ and $h_{\mathbf{d}}$ are defined in (27) and (14), respectively.

It is clear that a very interesting extension of our paper would be one that studies optimal dynamic contracts that allow the agent to shirk. Unfortunately, this case seems to be very difficult to analyze, and the optimal contracts may involve complex structures that renders them impractical even for simple settings. Zhu (2013), for example, considers the optimal contract with shirking under Brownian motion. The evolution of the promised utility involves a sticky Brownian motion that is a mathematical construct with very little practical relevance. Therefore, we consider the pursuit of optimal contracts that allow shirking outside the scope of this paper, and leave it for future research.

## 6. Conclusion

We study an incentive design problem in continuous time over an infinite horizon. Specifically, a principal hires an agent to exert effort in order to repair a machine when the machine is down, and maintain the machine when it is up. The agent can adjust the effort level at any time, which is not observable to the principal. Our paper contributes to the service/maintenance literature by studying the optimal dynamic contract. Although we allow a general form of payment and random termination in the contract design, the structure of the optimal contract is overall simple and intuitive. In particular, payment over time and potential termination decisions are all based on the evolution of the agent's promised utility. Payment only occurs when the promised utility is high enough. Intuitively, the principal pays the agent a flow when the machine is up, which can be decomposed into an interest payment to maintain the promised utility, and an information rent. In the case that $\beta_{\mathbf{d}} > \beta_{\mathbf{u}}$, the principal also needs to use an instantaneous payment upon machine recovery to provide an appropriate incentive for the agent to repair the machine fast enough.

Our paper also contributes to the dynamic contract literature where an agent exerts effort to either increase or decrease the arrival rates of Poisson processes. We instead combine both directions (increase and decrease), which turns out to be a non-trivial generalization. In particular, we find two new features in the optimal contract, which are new in the dynamic contract literature: (1) The incentive compatibility constraints are not always binding. (2) When the agent's promised utility is low, the optimal contract needs to involve a random termination, if the agent is not terminated from the random draw, the promised utility is brought back up to a certain threshold. Different from Myerson (2015), which also involves random termination, our threshold is not fixed at one level, but endogenously determined depending on model parameters.

Our general approach applies to other operational settings beyond maintenance/repair. For example, consider a queuing control system where an agent needs to exert effort in order to increase either the service rate or the arrival rate (e.g., by marketing efforts). In this case and the number of customers in the queue could be considered as the state of the system, which is more than the two states studied in our model. We believe that the techniques and results derived in our paper serve as a necessary step for solving these more general problems.

## Endnotes

1. Limited liability is commonly assumed in contract theory, especially dynamic contract theory. Without it, the model and analysis becomes easy, or even trivial. For example, the principal could simply sell the entire enterprise to the agent up front, at a price that equals the efficient social surplus. This allows the principal to exact the entire surplus and leaves the agent with zero surplus.

2. It is without loss of generality to assume zero revenue rate when the machine is down. In fact, our results hold as long as the revenue rate is lower when the machine is down.

3. We assume two levels of effort for simplicity. The results do not change if the effort level is from an interval, and the effort cost is linear in effort level.

4. We assume equal discount rate between the two players, similar to Myerson (2015). This is mostly for simplicity, although one may also argue that having access to a complete financial market allows the two agents to hedge all risks and use the risk-free interest rate and risk-neutral probabilities. Interestingly, one of the main findings of Myerson (2015), the infinite back-loading issue when the two players share equal time discount, does not arise in our setting. We explain this phenomenon and the underlying reasons in more detail in Section 4. If the two players have different discount rates, the optimal contract structure appears to be much more intricate. We leave that for future research.

5. Note that the expectation here, as well as in (5), is taken with respective to the stochastic process generated from the effort process $\nu$. This explains that in (3) we need to specify $\nu$ as an

argument of the function. For ease of exposition, we omit the explicit dependence between the expectation and $\nu$ in the main text of the paper.

6. Allowing shirking complicates the analysis for dynamic contracts substantially. See, for example, Zhu (2013) for a reference of optimal contract design allowing shirking in a Brownian motion framework.

7. All the inequalities in this paper are to be understood as almost surely.

8. If one only considers (IR) for time 0, the contract design is trivial. The principal can extract the entire surplus of the first best outcome by offering zero utility to the agent.

9. The "smooth pasting" condition requires that the value function is differentiable at $\hat{\beta}$. This condition often arises in optimal stopping problems (Dixit and Pindyck 1994) and optimal contract design (Zhu 2013, Chen et al. 2017).

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## Appendix

### A. Summary of Notations

#### Model parameters

$R$: flow revenue rate to the principal when the machine is up.

$\bar{\mu}_{\mathbf{u}}$ and $\mu_{\mathbf{u}}$: base case and low break down rates of the machine, respectively.

$\mu_{\mathbf{d}}$ and $\underline{\mu}_{\mathbf{d}}$: base case and high recovery rates of the machine, respectively.

$c_{\mathbf{u}}$ and $c_{\mathbf{d}}$: cost of effort in maintaining and in repairing the machine, respectively, per unit of time.

$r$: principal and agent's discount rates.

#### Contracts and utilities

$\nu$ and $\nu^*$: generic and full effort process under the contracts.

$I$ and $\ell$: instantaneous and flow payments, respectively.

$L$: payment process $dL_t = I_t + \ell_t dt$.

$q$: a stochastic firing probability at time $t$.

$\tau$: termination time.

$\Gamma$: generic contract, $\Gamma = (L, \tau, q)$.

$\bar{\Gamma}$ and $\hat{\Gamma}$: simple contract introduced in Section 3.1 and 3.2, respectively.

$\Gamma_1^*$ and $\Gamma_{\beta_{\bar{a}}}^*$: optimal contracts for the case in Section 4.1.1 and 4.2.1, respectively.

$u$ and $U$: agent's and principal's utilities, respectively.

$W_t$: agent's promised utility.

#### Derived quantities

$\beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}}$ : defined in Lemma 1.

$\underline{v}_{\mathbf{d}}, \underline{v}_{\mathbf{u}}$: defined in (4).

$\bar{v}_{\mathbf{d}}, \bar{v}_{\mathbf{u}}$: defined in (11).

$\bar{w}_{\mathbf{u}}$ and $\bar{w}_{\mathbf{d}}$: defined in (10).

$\hat{w}_{\mathbf{u}}$ and $\hat{w}_{\mathbf{d}}$: defined in (16).

$w_\theta^*$, $\theta \in \{\mathbf{u}, \mathbf{d}\}$: maximizers of function $J_\theta(w)$.

#### Value functions

$J_{\mathbf{d}}$, $J_{\mathbf{u}}$: the principal's value function of the optimal contract under state $\mathbf{d}$ and $\mathbf{u}$, respectively.

$V_{\mathbf{d}}$, $V_{\mathbf{u}}$: the societal value function of the optimal contract under state $\mathbf{d}$ and $\mathbf{u}$, respectively.

## B. Proofs in Section 2

### B.1. Proof of Lemma 1

Since the proof would not depend on $\theta_0$, we omit the $\theta_0$ of the equations (3) and (5) throughout the proof. We first define a 2-variate counting process $\{N_t^n, N_t^f\}_{t\in[0,\tau]}$, in which $dN_t^f = X_t dN_t$, and $N_t^n = N_t - N_t^f$. If $\tau < \infty$, the principal terminates the collaboration with the agent, while the collaboration continues throughout the infinite time horizon if $\tau = \infty$. Also, $dN_t = dN_t^f + dN_t^n = X_t dN_t + (1-X_t)dN_t$.

For a generic contract $\Gamma$ and effort process $\nu$, we introduce the agent's total expected utility conditional on the information available at time $t$ as the following $\mathcal{F}_t^N$-adapted random variable,

$$u_t(\Gamma,\nu) = \mathbb{E}\left[\int_0^\tau e^{-rs}(dL_s + (1-\nu_s)c(\theta_s)ds)\middle|\mathcal{F}_t^N\right] = \int_0^{t\wedge\tau^-} e^{-rs}(dL_s + (1-\nu_s)c(\theta_s)ds) + e^{-rt}W_t(\Gamma,\nu). \tag{46}$$

Therefore, $u_0(\Gamma,\nu) = u(\Gamma,\nu)$.

Process $\{u_t\}_{t\geq 0}$ is an $\mathcal{F}^N$-martingale. Define processes

$$M_t^{n,\nu} = N_t^n - \int_0^t \mu(\theta_s,\nu_s)(1-q_s)ds, \text{and} \tag{47}$$

$$M_t^{f,\nu} = N_t^f - \int_0^t \mu(\theta_s,\nu_s)q_s ds, \tag{48}$$

which are $\mathcal{F}^N$-martingales. Following the Martingale Representation Theorem, (see Brémaud 1981), there exists a $\mathcal{F}^N$-predictable processes $H(\Gamma,\nu) = \{H_t(\Gamma,\nu)\}_{t\geq 0}$ such that

$$u_t(\Gamma,\nu) = u_0(\Gamma,\nu) + \int_0^{t\wedge\tau} e^{-rs}[H_s(\Gamma,\nu)dM_s^{n,\nu} - W_{s-}dM_s^{f,\nu}]\ , \quad \forall t\geq 0\ . \tag{49}$$

Differentiating (46) and (49) with respect to $t$ yields

$$du_t = e^{-rt}[H_t(\Gamma,\nu)dM_t^{n,\nu} - W_{t-}dM_t^{f,\nu}] = e^{-rt}(dL_t + (1-\nu_t)c(\theta_t)dt) - re^{-rt}W_t(\Gamma,\nu)dt + e^{-rt}dW_t(\Gamma,\nu),$$

which implies (PK).

Denote $\tilde{u}_t(\Gamma,\nu',\nu)$ to be a $\mathcal{F}_t^N$-measurable random variable, representing the agent's total payoff following an effort process $\nu'$ before time $t$ and $\nu$ after $t$, that is,

$$\tilde{u}_t(\Gamma,\nu',\nu) = \int_0^{t\wedge\tau} e^{-rs}(dL_s + (1-\nu_s)c(\theta_s))ds) + e^{-rt}W_t(\Gamma,\nu).$$

Therefore,

$$\tilde{u}_0(\Gamma,\nu',\nu) = u_0(\Gamma,\nu) = u(\Gamma,\nu)\ , \tag{50}$$

$$\mathbb{E}\left[\tilde{u}_\tau(\Gamma,\nu',\nu)|\,\mathcal{F}_0^N\right] = u(\Gamma,\nu')\ , \quad \text{and} \tag{51}$$

$$\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu)|\,\mathcal{F}_0^N\right] = u(\Gamma,\nu)\ , \quad \forall t\geq 0\ . \tag{52}$$

For any given sample trajectory $\{N_s\}_{0\leq s\leq t}$ and effort processes $\nu$ and $\nu^*$.

$$\begin{aligned}
\tilde{u}_t(\Gamma,\nu,\nu^*) =& u_t(\Gamma,\nu^*) + \int_0^{t\wedge\tau} e^{-rs}(1-\nu_s)c(\theta_s)ds \\
=& u_0(\Gamma,\nu^*) + \int_0^{t\wedge\tau} e^{-rs}[H_s(\Gamma,\nu^*)dM_s^{n,\nu^*} - W_{s-}dM_s^{f,\nu^*}] + \int_0^{t\wedge\tau} e^{-rs}(1-\nu_s)c(\theta_s)ds \\
=& u_0(\Gamma,\nu^*) + \int_0^{t\wedge\tau} e^{-rs}[H_s(\Gamma,\nu^*)dM_s^{n,\nu} - W_{s-}dM_s^{f,\nu}] + \int_0^{t\wedge\tau} e^{-rs}(1-\nu_s)c(\theta_s)ds \\
&+ \int_0^{t\wedge\tau} e^{-rs}[(1-q_s)H_s(\Gamma,\nu^*) - q_s W_{s-}](\mu(\theta_s,\nu_s) - \mu(\theta_s,1))ds,
\end{aligned}$$

where the first equality follows from (46), the second equality follows (49) and the third equality follows from (47) and (48). Consider any two times $t' < t$,

$$\begin{aligned}
\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu^*)\middle|\mathcal{F}_{t'}^N\right] =& u_0(\Gamma,\nu^*) + \int_0^{t'\wedge\tau} e^{-rs}[H_s(\Gamma,\nu^*)dM_s^{n,\nu} - W_{s-}dM_s^{f,\nu}] \\
&+ \int_0^{t'\wedge\tau} e^{-rs}\{(1-\nu_s)c(\theta_s) + [(1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-}](\mu(\theta_s,\nu_s) - \mu(\theta_s,1))\}ds \\
&+ \mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}\{(1-\nu_s)c(\theta_s) + [(1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-}](\mu(\theta_s,\nu_s) - \mu(\theta_s,1))\}\middle|\mathcal{F}_{t'}^N\right] \\
=& \tilde{u}_{t'}(\Gamma,\nu,\nu^*) + \mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}(\bar{\mu}_{\mathbf{u}} - \mu_{\mathbf{u}})(\nu_s-1)[-\beta_{\mathbf{u}} - (1-q_s)H_s(\Gamma,\nu^*) + q_sW_{s-}]\mathbb{1}_{\theta_s=\mathbf{u}}ds\middle|\mathcal{F}_{t'}^N\right] \\
&+ \mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}(\mu_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}})(\nu_s-1)[-\beta_{\mathbf{d}} + (1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-}]\mathbb{1}_{\theta_s=\mathbf{d}}ds\middle|\mathcal{F}_{t'}^N\right],
\end{aligned} \tag{53}$$

where the second equality follows from equation (8).

If condition (7) holds for all $s \geq 0$, then (53) implies that $\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu^*)\middle|\mathcal{F}_{t'}^N\right] \leq \tilde{u}_{t'}(\Gamma,\nu,\nu^*)$. Therefore, $\{\tilde{u}_t\}_{t\geq 0}$ is a super-martingale. Take $t' = 0$, we have

$$u(\Gamma,\nu^*) = \tilde{u}_0(\Gamma,\nu,\nu^*) \geq \mathbb{E}\left[\tilde{u}_\tau(\Gamma,\nu,\nu^*)\middle|\mathcal{F}_0^N\right] = u(\Gamma,\nu),$$

in which the first equality follows from (50) and the last equality from (51), while the inequality follows from Doob's Optional Stopping Theorem. Therefore, the agent prefers the effort process $\nu^*$ to any other effort process $\nu$, which implies that $\Gamma$ satisfies (IC) if condition (7) holds for all $s \geq 0$.

If, on the other hand, $(1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-} > -\beta_{\mathbf{u}}$ for $s \in \Omega_{\mathbf{u}} \subset [0,t]$ with $\theta_{s-} = \mathbf{u}$, where $\Omega_{\mathbf{u}}$ is a positive measure set, define effort process $\nu$ to be such that

$$\nu_s = \begin{cases} 1\ ,\ (1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-} \leq -\beta_{\mathbf{u}} \\ 0\ ,\ (1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-} > -\beta_{\mathbf{u}} \end{cases} \quad \text{for } s \in [0,t] \text{ where } \theta_{s-} = \mathbf{u},$$

and $\nu_s = 1$ for $s > t$ where $\theta_{s-} = \mathbf{u}$ and $\nu_s = 1$ for $\theta_{s-} = \mathbf{d}$ $\forall s$. Therefore, $\tilde{u}_t(\Gamma,\nu,\nu^*) = \tilde{u}_t(\Gamma,\nu,\nu)$, and

$$\mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}(\bar{\mu}_{\mathbf{u}} - \mu_{\mathbf{u}})(\nu_s-1)[-\beta_{\mathbf{u}} - (1-q_s)H_s(\Gamma,\nu^*) + q_sW_{s-}]\mathbb{1}_{\theta_s=\mathbf{u}}ds\middle|\mathcal{F}_{t'}^N\right] > 0,$$

while

$$\mathbb{E}\left[\int_{t'\wedge\tau}^{t\wedge\tau} e^{-rs}(\mu_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}})(\nu_s-1)[-\beta_{\mathbf{d}} + (1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-}]\mathbb{1}_{\theta_s=\mathbf{u}}ds\middle|\mathcal{F}_{t'}^N\right] = 0.$$

Equation (53) then implies that $\mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu^*)\middle|\mathcal{F}_0^N\right] > \tilde{u}_0(\Gamma,\nu,\nu^*)$, and, therefore,

$$u(\Gamma,\nu^*) = \tilde{u}_0(\Gamma,\nu,\nu^*) < \mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu^*)\middle|\mathcal{F}_0^N\right] = \mathbb{E}\left[\tilde{u}_t(\Gamma,\nu,\nu)\middle|\mathcal{F}_0^N\right] = u(\Gamma,\nu),$$

in which the last equality follows from (52). The same logic applies if we can consider the situation when $(1-q_s)H_s(\Gamma,\nu^*) - q_sW_{s-} < \beta_{\mathbf{d}}$ for $s \in \Omega_{\mathbf{d}} \subset [0,t]$ with $\theta_{s-} = \mathbf{d}$ and a positive measure set $\Omega_{\mathbf{d}}$. Therefore, the agent prefers effort process $\nu'$ over $\nu^*$, which implies that $\Gamma$ does not satisfy (IC) if condition (7) does not hold. Q.E.D.

### B.2. Lemma 4 and its proof

LEMMA 4. *Define* $\underline{\nu} := \{\nu_t = 0\}_{\forall t}$. *For* $\theta_0 \in \{u,d\}$, *we have*

$$\mathbb{E}\left[\int_0^\infty e^{-rt}R\mathbb{1}_{\theta_t=\mathbf{u}}dt\middle|\theta_0,\underline{\nu}\right] = \underline{v}_{\theta_0}\ . \tag{54}$$

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}} - c(\theta_t))dt\middle|\theta_0,\nu^*\right] = \bar{v}_{\theta_0}\ . \tag{55}$$

*where* $\underline{v}_{\theta_0}$ *and* $\bar{v}_{\theta_0}$ *are defined in equation (4) and (11), respectively.*

**Proof.** We first calculate (55) with $\theta_0 = \mathbf{d}$ which is the societal value when the machine starts with state $\mathbf{d}$ and the agent always exerts effort. We define $t_k$ as the time of occurrence of the $k$th transition of the states, and $t_0 = 0$. Further define $\tau_k := t_k - t_{k-1}$. Therefore $\tau_{2k+1}$ follows an exponential distribution with rate $\mu_{\mathbf{u}}$, and $\tau_{2k+2}$ follows an exponential distribution with rate $\mu_{\mathbf{d}}$ where $k \in \mathbb{N}$. Then

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}} - c(\theta_t,\nu_t^*))dt \middle| d,\nu^*\right] &= \sum_{k=0}^\infty \left\{\mathbb{E}\left[\int_{t_{2k}}^{t_{2k+1}} e^{-rt}(R-c_{\mathbf{u}})dt\right] + \mathbb{E}\left[\int_{t_{2k+1}}^{t_{2k+2}} e^{-rt} - c_{\mathbf{d}}dt\right]\right\} \\ &= \sum_{k=0}^\infty \left\{\mathbb{E}\left[\int_{t_{2k}}^{t_{2k+1}} e^{-rt}dt\right](R-c_{\mathbf{u}}) + \mathbb{E}\left[\int_{t_{2k+1}}^{t_{2k+2}} e^{-rt}dt\right]\cdot(-c_{\mathbf{d}})\right\},\end{aligned} \tag{56}$$

where

$$\begin{aligned}\mathbb{E}\left[\int_{t_{2k}}^{t_{2k+1}} e^{-rt}dt\right] &= \frac{\mathbb{E}\left[e^{-rt_{2k}}\right](1-\mathbb{E}\left[e^{-r\tau_{2k+1}}\right])}{r} \\ &= \frac{\mathbb{E}\left[e^{-r\sum_{i=1}^{2k}\tau_{2k}}\right](1-\mathbb{E}\left[e^{-r\tau_{2k+1}}\right])}{r} \\ &= \frac{\alpha^k\beta^k(1-\alpha)}{r},\end{aligned}$$

where $\alpha = \mathbb{E}\left[e^{\tau_1}\right] = \dfrac{\mu_{\mathbf{d}}}{r+\mu_{\mathbf{d}}}$ and $\beta = \mathbb{E}\left[e^{\tau_2}\right] = \dfrac{\mu_{\mathbf{u}}}{r+\mu_{\mathbf{u}}}$. In the same way, $\mathbb{E}\left[\displaystyle\int_{t_{2k+1}}^{t_{2k+2}} e^{-rt}dt\right] = \dfrac{\alpha^{k+1}\beta^k(1-\beta)}{r}$. Furthermore, the expression of $\alpha$ and $\beta$ yields $\dfrac{1-\alpha}{r} = \dfrac{1}{r+\mu_{\mathbf{d}}}$ and $\dfrac{1-\beta}{r} = \dfrac{1}{r+\mu_{\mathbf{u}}}$. Following equation (56),

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}} - c(\theta_t))dt \middle| d,\nu^*\right] \\ =&\sum_{k=0}^\infty \left\{\frac{\alpha^{k+1}\beta^k}{r+\nu_{\mathbf{u}}}(R-c_{\mathbf{u}}) + \frac{\alpha^k\beta^k}{r+\nu_{\mathbf{d}}}(-c_{\mathbf{d}})\right\} \\ =&\frac{\alpha}{1-\alpha\beta}\frac{1}{r+\nu_{\mathbf{u}}}(R-c_{\mathbf{u}}) + \frac{1}{1-\alpha\beta}\frac{1}{r+\nu_{\mathbf{d}}}(-c_{\mathbf{d}}) \\ =&\frac{\mu_{\mathbf{d}}(R-c_{\mathbf{u}}) - (r+\mu_{\mathbf{u}})c_{\mathbf{d}}}{r(r+\mu_{\mathbf{u}}+\mu_{\mathbf{d}})}.\end{aligned}$$

The same logical steps yields (55) for the case of $\theta_0 = \mathbf{d}$, and also (54) and (55) for the case of $\theta_0 = \mathbf{u}$. Q.E.D.

## C. Optimality Condition

The following lemma states conditions for functions $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$ such that they are upper bounds of the principal's utility $U(\Gamma)$ under any contract $\Gamma$. This verification result serves as an optimality condition for later sections.

LEMMA 5. *Suppose $J_{\mathbf{d}}(w) : [0,\infty) \to \mathbb{R}$ and $J_{\mathbf{u}}(w) : [\beta_{\mathbf{u}},\infty) \to \mathbb{R}$ are differentiable, concave, upper-bounded functions, with $J'_{\mathbf{d}}(w) \geq -1$, $J'_{\mathbf{u}}(w) \geq -1$, and $J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$. Consider any incentive compatible contract $\Gamma$, which yields the agent's expected utility $u(\Gamma,\nu^*) = W_0$, followed by the promised utility process $\{W_t\}_{t\geq 0}$ according to (PK) and satisfy (IC). Define a stochastic process $\{\Phi_t\}_{t\geq 0}$ as*

$$\begin{aligned}\Phi_t :=& R\mathbb{1}_{\theta_t=\mathbf{u}} + J'_{\theta_t}(W_{t-})(rW_{t-} - [-q_tW_{t-} + (1-q_t)H_t]\mu(\theta_t,\nu_t)) - rJ_{\theta_t}(W_{t-}) \\ &+ \mu(\theta_t,\nu_t)q_t[J_{\hat{\theta}_t}(0) - J_{\theta_t}(W_{t-})] + \mu(\theta_t,\nu_t)(1-q_t)[J_{\hat{\theta}_t}(W_{t-}+H_t) - J_{\theta_t}(W_{t-})] - c(\theta_t)\ .\end{aligned} \tag{57}$$

*where $\theta_t \in \{u,d\}$ and $\hat{\theta}_t = \mathbb{1}_{\theta_t=\mathbf{d}}\cdot u + \mathbb{1}_{\theta_t=\mathbf{u}}\cdot d$, and $J_{\mathbf{u}}$ is extended such that $J_{\mathbf{u}}(0) = \underline{v}_{\mathbf{u}}$. If the process $\{\Phi_t\}_{t\geq 0}$ is non-positive almost surely, then we have $J_\theta(u(\Gamma,\nu^*,\theta)) \geq U(\Gamma,\nu^*,\theta)$.*

**Proof.** We define the following function to represent the value function as a function of time $t$,

$$J(t)=\begin{cases}J_{\mathbf{d}}(W_{t-}) & \text{if} \quad \theta_{t-}=\mathbf{d}\ ,\\ J_{\mathbf{u}}(W_{t-}) & \text{if} \quad \theta_{t-}=\mathbf{u}\ .\end{cases} \tag{58}$$

Following the Itô's Formula for jump processes (see, for example, Bass 2011, Theorem 17.5) and (PK), we obtain

$$e^{-r\tau}J(\tau)=e^{-r0}J(0)+\int_0^\tau[e^{-rt}dJ(t)-re^{-rt}J(t)dt]= \quad J(0)+\int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t=\mathbf{u}}dt+c(\theta_t)dt+dL_t)+\int_0^\tau e^{-rt}\mathcal{A}_t, \tag{59}$$

where

$$\begin{aligned}\mathcal{A}_t=&dJ(t)-rJ(t)dt+R\mathbb{1}_{\theta_t=\mathbf{u}}dt-c(\theta_t)dt-dL_t\\ =&J'(t)[rW_{t-}-\mu(\theta_t,\nu_t)(-q_tW_{t-}+(1-q_t)H_t)-\ell_t]dt-rJ(t)dt+J(t+)-J(t)+R\mathbb{1}_{\theta_t=\mathbf{u}}dt-c(\theta_t)dt-dL_t\\ =&J'(t)[rW_{t-}-\mu(\theta_t,\nu_t)(-q_tW_{t-}+(1-q_t)H_t)-\ell_t]dt-rJ(t)dt+R\mathbb{1}_{\theta_t=\mathbf{u}}dt-c(\theta_t)dt-dL_t\\ &+[J_{\theta_t}(W_{t-}-I_t)-J_{\theta_t}(W_{t-})](1-dN_t)+[J_{\hat{\theta}_t}(W_{t-}+[(1-X_t)H_t-X_tW_{t-}]-I_t)-J_{\theta_t}(W_{t-})]dN_t\\ =&J'(t)[rW_{t-}-\mu(\theta_t,\nu_t)(-q_tW_{t-}+(1-q_t)H_t)-\ell_t]dt-rJ(t)dt+R\mathbb{1}_{\theta_t=\mathbf{u}}dt-c(\theta_t)dt-dL_t\\ &+[J_{\theta_t}(W_{t-}-I_t)-J_{\theta_t}(W_{t-})](1-dN_t)+\big[J_{\hat{\theta}_t}(W_{t-}+[(1-X_t)H_t-X_tW_{t-}]-I_t)-J_{\hat{\theta}_t}(W_{t-}+[(1-X_t)H_t-X_tW_{t-}])\big]dN_t\\ &+\big[J_{\hat{\theta}_t}(W_{t-}+[(1-X_t)H_t-X_tW_{t-}])-J_{\theta_t}(W_{t-})\big]dN_t.\end{aligned}$$

Further define

$$\mathcal{B}_t:=[J_{\hat{\theta}_t}(W_{t-}+H_t)-J_{\theta_t}(W_{t-})](dN_t^n-\mu(\theta_t,\nu_t)(1-q_t)dt)+[J_{\hat{\theta}_t}(0)-J_{\theta_t}(W_{t-})](dN_t^f-\mu(\theta_t,\nu_t)q_tdt).$$

Because function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are concave, $J'_{\mathbf{d}}(w)\geq -1$ and $J'_{\mathbf{u}}(w)\geq -1$, we have

$$\begin{aligned}\mathcal{A}_t=&\ dJ(t)-rJ(t)dt+R\mathbb{1}_{\theta_t=\mathbf{u}}dt-dL_t\\ \leq&\ J'(t)[rW_{t-}-\mu(\theta_t,\nu_t)(-q_tW_{t-}+(1-q_t)H_t)]dt-rJ(t)dt+R\mathbb{1}_{\theta_t=\mathbf{u}}dt-dL_t-J'(t)\ell_tdt-J'_{\theta_t}(W_{t-})I_t(1-dN_t)\\ -&\ J'_{\hat{\theta}_t}(W_{t-}+[(1-X_t)H_t-X_tW_{t-}])I_tdN_t+\big[J_{\hat{\theta}_t}(W_{t-}+[(1-X_t)H_t-X_tW_{t-}])-J_{\theta_t}(W_{t-})\big]dN_t-c(\theta_t)dt\\ \leq&\ J'(t)[rW_{t-}-\mu(\theta_t,\nu_t)(-q_tW_{t-}+(1-q_t)H_t)]dt-rJ(t)dt+R\mathbb{1}_{\theta_t=\mathbf{u}}dt\\ +&\ \big[J_{\hat{\theta}_t}(W_{t-}+[(1-X_t)H_t-X_tW_{t-}])-J_{\theta_t}(W_{t-})\big]dN_t-c(\theta_t)dt\\ =&\ R\mathbb{1}_{\theta_t=\mathbf{u}}dt+J'_{\theta_t}(t)[rW_{t-}-\mu(\theta_t,\nu_t)(-q_tW_{t-}+(1-q_t)H_t)]dt-rJ_{\theta_t}(t)dt\\ +&\ \big[J_{\hat{\theta}_t}(W_{t-}+H_t)-J_{\theta_t}(W_{t-})\big]dN_t^n+\big[J_{\hat{\theta}_t}(0)-J_{\theta_t}(W_{t-})\big]dN_t^f-c(\theta_t)dt\\ =&\mathcal{B}_t+\Phi_tdt\ .\end{aligned}$$

Therefore, if $\Phi_t\leq 0$, we must have $\mathcal{A}_t\leq\mathcal{B}_t$ almost surely. Taking the expectation on both sides of (59), we immediately have

$$J_{\theta_0}(u(\Gamma,\nu,\theta_0))=J(0)\geq\mathbb{E}\left[e^{-r\tau}J(\tau)+\int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt-c(\theta_t)dt-dL_t)\middle|\theta_0\right]=u(\Gamma,\nu,\theta_0),$$

where we use the fact that $\int_0^\tau e^{-rt}\mathcal{B}_tdt$ is a martingale and $J(\tau)=J_{\theta_\tau}(0)=v_\tau$. Q.E.D.

To prove that a contract is optimal among all incentive compatible contracts, we only need to verify if $\Phi_t$ defined in (57) is non-positive.

## D. Proofs and derivations in Section 4.1

### D.1. Heuristic derivation of equations (20)-(22)

If the machine's current state is $\mathbf{d}$, consider a small time interval $[t,t+\delta]$, during which the principal reimburses the agent's effort cost $c_{\mathbf{d}}\delta$. With probability $\mu_{\mathbf{d}}\delta$, the machine recovers after this interval and changes to state $\mathbf{u}$, the principal pays the agent $(w+\beta_{\mathbf{d}}-\bar{w}_{\mathbf{u}})^+$, and, correspondingly, the promised utility jumps up to $\min\{w+\beta_{\mathbf{d}},\bar{w}_{\mathbf{u}}\}$. With probability $1-\mu_{\mathbf{d}}\delta$, on the other hand, the machine stays in $\mathbf{d}$, and the promised utility evolves to $w+r(w-\bar{w}_{\mathbf{d}})\delta$. Therefore, we have

$$J_{\mathbf{d}}(w)=-c_{\mathbf{d}}\delta+e^{-r\delta}\Big\{\mu_{\mathbf{d}}\delta\big[-(w+\beta_{\mathbf{d}}-\bar{w}_{\mathbf{u}})^++J_{\mathbf{u}}(\min\{w+\beta_{\mathbf{d}},\bar{w}_{\mathbf{u}}\})\big]$$

$$+(1-\mu_{\mathbf{d}}\delta)J_{\mathbf{d}}(w+r(w-\bar{w}_{\mathbf{d}})\delta)\Big\}+o(\delta).$$

Subtracting $J_{\mathbf{d}}(w)$ and dividing $\delta$ on both sides, then letting $\delta$ approach 0, we obtain equation (20).

Similarly, consider the machine's current state at $\mathbf{u}$. and a small time interval $[t, t+\delta]$, when the principal collects revenue $R\delta$ and the agent's promised utility $w \geq \beta_{\mathbf{u}}$. With probability $\mu_{\mathbf{u}}\delta$, the machine breaks down and changes to state $\mathbf{d}$, and the promised utility drops to $w-\beta_{\mathbf{u}}$. With probability $1-\mu_{\mathbf{u}}\delta$, on the other hand, the machine stays in $\mathbf{u}$, the promised utility evolves to $w+(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\delta$ if $w<\bar{w}_{\mathbf{u}}$, and the principal pays the agent $\ell^*\delta$ if $w=\bar{w}_{\mathbf{u}}$ while the promised utility stays at $\bar{w}_{\mathbf{u}}$. Therefore,

$$J_{\mathbf{u}}(w)=(R-c_{\mathbf{u}})\delta+e^{-r\delta}\Big\{\mu_{\mathbf{u}}\delta J_{\mathbf{d}}(w-\beta_{\mathbf{u}})+(1-\mu_{\mathbf{u}}\delta)\big[J_{\mathbf{u}}(w+(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\delta\mathbb{1}_{w<\bar{w}_{\mathbf{u}}})-\ell^*\mathbb{1}_{w=\bar{w}_{\mathbf{u}}}\big]\Big\}+o(\delta).$$

Following similar steps as before, we obtain equations (21) and (22).

## D.2. Proof of Proposition 1

It is helpful to consider the societal value functions, defined below as the summation of the principal and the agent's utilities,

$$V_{\mathbf{d}}(w)=J_{\mathbf{d}}(w)+w \quad \text{and} \quad V_{\mathbf{u}}(w)=J_{\mathbf{u}}(w)+w. \tag{60}$$

Following (20)-(24), we obtain the following system of differential equations for $V_{\mathbf{d}}$ and $V_{\mathbf{u}}$,

$$(\mu_{\mathbf{d}}+r)V_{\mathbf{d}}(w)=\mu_{\mathbf{d}}V_{\mathbf{u}}(\min\{w+\beta_{\mathbf{d}},\bar{w}_{\mathbf{u}}\})-c_{\mathbf{d}}-r(\bar{w}_{\mathbf{d}}-w)V_{\mathbf{d}}'(w)\ ,\ w\in[0,\bar{w}_{\mathbf{d}}], \tag{61}$$

$$(\mu_{\mathbf{u}}+r)V_{\mathbf{u}}(w)=-c_{\mathbf{u}}+R+\mu_{\mathbf{u}}V_{\mathbf{d}}(w-\beta_{\mathbf{u}})+(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_{\mathbf{u}}}V_{\mathbf{u}}'(w)\ ,\ w\in[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}}], \tag{62}$$

$$V_{\mathbf{u}}(w)=V_{\mathbf{u}}(0)+\frac{V_{\mathbf{u}}(\beta_{\mathbf{u}})-V_{\mathbf{u}}(0)}{\beta_{\mathbf{u}}}w, \tag{63}$$

$$V_{\mathbf{u}}(0)=\underline{v}_{\mathbf{u}} \quad \text{and} \quad V_{\mathbf{d}}(0)=\underline{v}_{\mathbf{d}}. \tag{64}$$

Furthermore, as soon as the promised utility reaches $\bar{w}_{\mathbf{u}}$ at state $\mathbf{u}$, contract $\Gamma_1^*$ becomes identical to the simple contract studied in Section 3.1. This implies the following boundary conditions

$$V_{\mathbf{d}}(\bar{w}_{\mathbf{d}})=\bar{v}_{\mathbf{d}} \quad \text{and} \quad V_{\mathbf{u}}(\bar{w}_{\mathbf{u}})=\bar{v}_{\mathbf{u}}, \tag{65}$$

in which $\bar{v}_{\mathbf{d}}$ and $\bar{v}_{\mathbf{u}}$ are defined in (11). Equivalently, we prove that the system of differential equations (61) and (62) with boundary conditions (63), (64) and (65) has a unique solution: the pair of functions $V_{\mathbf{u}}(w)$ on $[0,\bar{w}_{\mathbf{u}}]$ and $V_{\mathbf{d}}(w)$ on $[0,\bar{w}_{\mathbf{d}}]$, both of which are increasing and strictly concave.

First, we prove that (61) and (62) with boundary conditions (64) and (65) has a unique solution: the pair of functions $V_{\mathbf{u}}(w)$ on $[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}}]$ and $V_{\mathbf{d}}(w)$ on $[0,\bar{w}_{\mathbf{d}}]$. Next we write the proof for the two cases $\beta_{\mathbf{d}}>\beta_{\mathbf{u}}$ and $\beta_{\mathbf{d}}=\beta_{\mathbf{u}}$ separately.

**D.2.1.** $\beta_{\mathbf{d}}>\beta_{\mathbf{u}}$ Recall that function $V_{\mathbf{d}}$ and $V_{\mathbf{u}}$ satisfies the system of differential equations (61) and (62).

**Case 1.** $\bar{w}_{\mathbf{u}}\leq\beta_{\mathbf{d}}$. Since for $w\in[0,\bar{w}_{\mathbf{d}}]$, $V_{\mathbf{u}}(\min\{w+\beta_{\mathbf{d}},\bar{w}_{\mathbf{u}}\})=V_{\mathbf{u}}(\bar{w}_{\mathbf{u}})=\bar{v}_{\mathbf{u}}$, we could rearrange equation (61) as

$$(\mu_{\mathbf{d}}+r)V_{\mathbf{d}}(w)=\mu_{\mathbf{d}}\bar{v}_{\mathbf{u}}-c_{\mathbf{d}}-r(\bar{w}_{\mathbf{d}}-w)V_{\mathbf{d}}'(w).$$

The above equation in $[0,\bar{w}_{\mathbf{d}})$ is a linear differential equation with boundary condition. The solution is

$$V_{\mathbf{d}}(w)=\bar{v}_{\mathbf{d}}+b_1(\bar{w}_{\mathbf{d}}-w)^{\frac{r+\mu_{\mathbf{d}}}{r}} \quad \text{for } w\in[0,\bar{w}_{\mathbf{d}}], \tag{66}$$

with $b_1=(\underline{v}_{\mathbf{d}}-\bar{v}_{\mathbf{d}})\bar{w}_{\mathbf{d}}^{(r+\mu_{\mathbf{d}})/r}<0$. (Followed by the condition (13). )

Then, (66) implies that $V_{\mathbf{d}}'(w)=-b_1(r+\mu_{\mathbf{d}})(\bar{w}_{\mathbf{d}}-w)^{\mu_{\mathbf{d}}/r}/r>0$, $V_{\mathbf{d}}''(w)=b_1(r+\mu_{\mathbf{d}})\mu_{\mathbf{d}}(\bar{w}_{\mathbf{d}}-w)^{\mu_{\mathbf{d}}-r/r}/r^2<0$ for $w\in[0,\bar{w}_{\mathbf{d}}]$. Hence, $V_{\mathbf{d}}$ is increasing and strictly concave in $[0,\bar{w}_{\mathbf{d}}]$. Furthermore, it can be verified that $V_{\mathbf{d}}'(\bar{w}_{\mathbf{d}}-)=0$. Next, we show that $V_{\mathbf{u}}$ is also increasing and strictly concave in $[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}}]$. Rearranging equation (62) in $[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}}]$ as

$$(\mu_{\mathbf{u}}+r)V_{\mathbf{u}}(w)=-c_{\mathbf{u}}+R+\mu_{\mathbf{u}}\left(\bar{V}_{\mathbf{d}}+b_1(\bar{w}_{\mathbf{d}}-w+\beta_{\mathbf{u}})^{\frac{r+\mu_{\mathbf{d}}}{r}}\right)+(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_{\mathbf{u}}}V_{\mathbf{u}}'(w). \tag{67}$$

The above equation in $[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}})$ is a linear differential equation with boundary condition. It is easy to verify that $\lim_{w\to\bar{w}_{\mathbf{u}}-}V_{\mathbf{u}}'(w)=0$ with $V_{\mathbf{u}}(\bar{w}_{\mathbf{u}})=\bar{v}_{\mathbf{u}}$. Equation (67) implies that

$$V_{\mathbf{u}}''(w)=\frac{\mu_{\mathbf{u}}(V_{\mathbf{u}}'(w)-V_{\mathbf{d}}'(w-\beta_{\mathbf{u}}))}{rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}}} \quad \text{for } w\in[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}}), \tag{68}$$

and

$$V_{\mathbf{u}}'''(w) = \frac{\mu_{\mathbf{u}}(V_{\mathbf{u}}''(w) - V_{\mathbf{d}}''(w-\beta_{\mathbf{u}})) - rV_{\mathbf{u}}''(w)}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \quad \text{for } w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}). \tag{69}$$

Since $V_{\mathbf{d}}'(\bar{w}_{\mathbf{d}}) = 0$, then equation (68) implies that $\lim_{w\to w_{\mathbf{u}}-} V_{\mathbf{u}}''(w) = 0$. Furthermore, with $V_{\mathbf{d}}'(w-\beta_{\mathbf{u}}) < 0$ for $w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$, we can show that there exists $\epsilon > 0$ such that $V_{\mathbf{u}}''(w) < 0$ and $V_{\mathbf{u}}'(w) > 0$ for $w \in [\bar{w}_{\mathbf{u}} - \epsilon, \bar{w}_{\mathbf{u}})$. Hence, $V_{\mathbf{u}}(w)$ is increasing and strictly concave in $[\bar{w}_{\mathbf{u}} - \epsilon, \bar{w}_{\mathbf{u}})$. Assume there exists $\tilde{w} \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}} - \epsilon)$ such that $V_{\mathbf{u}}''(\tilde{w}) \geq 0$. There must be $\hat{w} = \max\{w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}} - \epsilon) | V_{\mathbf{u}}''(w) = 0\}$, and $V_{\mathbf{u}}''(w) < 0$, $\forall w > \hat{w}$. However, this contradicts $V_{\mathbf{u}}'''(\hat{w}) = \dfrac{-\mu_{\mathbf{u}} V_{\mathbf{d}}''(\hat{w} - \beta_{\mathbf{u}})}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} > 0$ which is implied by equation (69). Therefore, we must have $V_{\mathbf{u}}$ to be increasing and strictly concave in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$. Furthermore, it can be verified that $V_{\mathbf{u}}(w) = \bar{v}_{\mathbf{u}}$ for $w \in [\bar{w}_{\mathbf{u}}, \infty)$ and $V_{\mathbf{d}}(w) = \bar{v}_{\mathbf{d}}$ for $w \in [\bar{w}_{\mathbf{d}}, \infty)$ solves (61) and (62).

**Case 2.** $\bar{w}_{\mathbf{u}} > \beta_{\mathbf{d}}$. Rearranging (61) as

$$(\mu_{\mathbf{d}} + r)V_{\mathbf{d}}(w) = \mu_{\mathbf{d}}\bar{v}_{\mathbf{u}} - c_{\mathbf{d}} - r(\bar{w}_{\mathbf{d}} - w)V_{\mathbf{d}}'(w), \text{ for } w \in [\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \infty) \text{ , and} \tag{70}$$

$$(\mu_{\mathbf{d}} + r)V_{\mathbf{d}}(w) = \mu_{\mathbf{d}}V_{\mathbf{u}}(w + \beta_{\mathbf{d}}) - c_{\mathbf{d}} - r(\bar{w}_{\mathbf{d}} - w)V_{\mathbf{d}}'(w), \text{ for } w \in [0, \bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}). \tag{71}$$

We then show the result according to the following steps.

1. Demonstrate the solution of (70) as a parametric function $V_{\mathbf{d}}^b$, with parameter $b$.
2. Show that the solution of (71) and (62) are a pair of unique and twice continuously differentiable equations for any $b$, called as $V_{\mathbf{d}}^b$ and $V_{\mathbf{u}}^b$.
3. Show that for $b < 0$, $V_{\mathbf{d}}^b$ and $V_{\mathbf{u}}^b$ are concave and increasing.
4. Show that $V_{\mathbf{d}}^b(0)$ is increasing in $b$, which implies that the boundary condition $V_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$ uniquely determines $b$, and therefore the solution of the original system of differential equations.

**Step 1.** The solution to the linear ordinary differential equation (70) on $[\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$ must have the following form, for any scalar $b$.

$$V_{\mathbf{d}}^b(w) = \bar{v}_{\mathbf{d}} + b(w_{\mathbf{d}} - w)^{\frac{r+\mu_{\mathbf{d}}}{r}} \quad \text{for } w \in [\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}], \tag{72}$$

Also define $V_{\mathbf{d}}^b(w) = \bar{v}_{\mathbf{d}}$ for $w \in [\bar{w}_{\mathbf{d}}, \infty]$, which satisfies (70), so that $V_{\mathbf{d}}^b$ is continuously differentiable on $[\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \infty)$.

**Step 2.** Using (72) as the boundary condition, we show that the system of differential equations (71) and (62) has a unique pair of solutions (called $V_{\mathbf{d}}^b$ and $V_{\mathbf{u}}^b$, on $(0, \bar{w}_{\mathbf{d}})$, $(\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$), which are continuously differentiable. In fact, the system of differential equations (71) and (62) are equivalent to a sequence of initial value problems over the intervals $[\bar{w}_{\mathbf{d}} - (k+1)(\beta_{\mathbf{d}} - \beta_{\mathbf{u}}), \bar{w}_{\mathbf{d}} - k(\beta_{\mathbf{d}} - \beta_{\mathbf{u}})]$ for $V_{\mathbf{d}}$ and $[\bar{w}_{\mathbf{u}} - k(\beta_{\mathbf{d}} - \beta_{\mathbf{u}}), \bar{w}_{\mathbf{u}} - (k-1)(\beta_{\mathbf{d}} - \beta_{\mathbf{u}}))$ for $V_{\mathbf{u}}$, $k = 1, 2, \ldots$. This sequence of initial value problems satisfy the Cauchy-Lipschitz Theorem and, therefore, bear unique solutions. Also define $V_{\mathbf{u}}^b(w) = \bar{v}_{\mathbf{u}}$ for $w \in [\bar{w}_{\mathbf{u}}, \infty)$, which satisfies (62), so that $V_{\mathbf{u}}^b$ is continuously differentiable on $[\bar{w}_{\mathbf{u}}, \infty)$. Also, computing $V_b'(\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}})$ from (72), and comparing it with (71), we see that $V_{\mathbf{d}}^b$ is continuously differentiable at $\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}$, and therefore $V_{\mathbf{d}}^b$ and $V_{\mathbf{u}}^b$ are continuously differentiable $[0, \infty)$ and $[\beta_{\mathbf{u}}, \infty)$, respectively. Furthermore, we could derive the expressions for $V_{\mathbf{d}}^{b''}$ and $V_{\mathbf{u}}^{b''}$ following (71) and (62), respectively,

$$V_{\mathbf{u}}^{b''}(w) = \frac{\mu_{\mathbf{u}}(V_{\mathbf{u}}^{b'}(w) - V_{\mathbf{d}}^{b'}(w - \beta_{\mathbf{u}}))}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \text{ , and} \tag{73}$$

$$V_{\mathbf{d}}^{b''}(w) = \frac{\mu_{\mathbf{d}}(V_{\mathbf{u}}^{b'}(w + \beta_{\mathbf{d}}) - V_{\mathbf{d}}^{b'}(w))}{r(w_{\mathbf{d}} - w)}. \tag{74}$$

**Step 3.** Next, we argue that for $b < 0$, $V_{\mathbf{d}}^b$ and $V_{\mathbf{u}}^b$ are concave and increasing. Equation (72) implies that $V_{\mathbf{d}}^b$ is increasing and strictly concave on $[\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$, and therefore $V_b^{d''}(w) < 0$ in this interval. We could firstly prove that $V_{\mathbf{u}}^b$ is strictly concave and increasing in $[\bar{w}_{\mathbf{u}} + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}})$ in the same way in Case 1. Next, we want to show that $V_{\mathbf{d}}^b$ is strictly concave in $[\bar{w}_{\mathbf{u}} + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}})$.

In the following, we prove two lemmas to establish the result.

Lemma 6. *For any $w \leq \bar{w}_{\mathbf{u}}$, if $V_{\mathbf{u}}^b$ is strictly concave in $[w + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}})$ and $V_{\mathbf{d}}^b$ is strictly concave in $[w - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}})$, then $V_{\mathbf{d}}^b$ is strictly concave in $[w + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}})$.*

**Proof.** $V_{\mathbf{d}}^b$ is strictly concave in $[w-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}})$ implies that $V_{\mathbf{d}}^{b''}(w)<0$ in this interval. Assume that there exists $\tilde{w}_b\in[w+\beta_{\mathbf{u}}-2\beta_{\mathbf{d}},w-\beta_{\mathbf{d}})$ such that $V_{\mathbf{d}}^{b''}(\tilde{w}_b)\geq 0$, then following step 2, $V_{\mathbf{d}}^b$ twice continuously differentiable implies that there must exist $\hat{w}_b=\max\{w\in[w+\beta_{\mathbf{u}}-2\beta_{\mathbf{d}},w-\beta_{\mathbf{d}})|V_{\mathbf{d}}^{b''}(w)=0\}$, and $V_{\mathbf{d}}^{b''}(w)<0$, $\forall w>\hat{w}_b$. Equation (74) implies that

$$V_{\mathbf{u}}^{b'}(\hat{w}_b+\beta_{\mathbf{d}})=V_{\mathbf{d}}^{b'}(\hat{w}_b)\ . \tag{75}$$

Furthermore, since $V_{\mathbf{u}}^b$ is strictly concave in $[w+\beta_{\mathbf{u}}-\beta_{\mathbf{d}},\bar{w}_{\mathbf{u}})$ and $\hat{w}_b+\beta_{\mathbf{d}}\geq w+\beta_{\mathbf{u}}$, we have $V_u^{b''}(\hat{w}_b+\beta_{\mathbf{d}})<0$. Then equation (73) implies that $V_{\mathbf{u}}^{b'}(\hat{w}_b+\beta_{\mathbf{d}})-V_{\mathbf{d}}^{b'}(\hat{w}_b+\beta_{\mathbf{d}}-\beta_{\mathbf{u}})<0$. With equation (75), we have $V_{\mathbf{d}}^{b'}(\hat{w}_b)<V_{\mathbf{d}}^{b'}(\hat{w}_b+\beta_{\mathbf{d}}-\beta_{\mathbf{u}})$ which contradicts with $V_{\mathbf{d}}^{b''}(w)<0$, $\forall w>\hat{w}_b$. Q.E.D.

Lemma 7. *For $w\leq\bar{w}_{\mathbf{u}}+\beta_{\mathbf{u}}-\beta_{\mathbf{d}}$, if $V_{\mathbf{d}}^b$ is strictly concave in $[w-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}}]$ and $V_{\mathbf{u}}^b$ is strictly concave in $[w,\bar{w}_{\mathbf{u}}]$, then $V_{\mathbf{u}}^b$ is strictly concave in $[w+\beta_{\mathbf{u}}-\beta_{\mathbf{d}},\bar{w}_{\mathbf{u}})$.*

The proof of Lemma 7 follows the same steps as the proof of Lemma 6.

With Lemmas 6 and 7, we prove that if $V_{\mathbf{u}}^b$ is strictly concave in $[w+\beta_{\mathbf{u}}-\beta_{\mathbf{d}},\bar{w}_{\mathbf{u}})$ and $V_{\mathbf{d}}^b$ is strictly concave in $[w-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}})$, then $V_{\mathbf{u}}^b$ is strictly concave in $[w+2\beta_{\mathbf{u}}-2\beta_{\mathbf{d}},\bar{w}_{\mathbf{u}})$ and $V_{\mathbf{d}}^b$ is strictly concave in $[w+\beta_{\mathbf{u}}-2\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}})$. Hence, by induction, we can prove that $V_{\mathbf{d}}^b$ is strictly concave and increasing in $[0,\bar{w}_{\mathbf{d}})$ and $V_{\mathbf{u}}^b$ is strictly concave and increasing in $[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}})$.

**Step 4.** Finally, we show that $V_{\mathbf{d}}^b(0)$ is strictly increasing in $b$ for $b<0$, which allows us to uniquely determine $b$ that satisfies $V_{\mathbf{d}}^b(0)=\underline{v}_{\mathbf{d}}$. For given $b_1<b_2<0$, define $X_{\mathbf{d}}(w):=V_{\mathbf{d}}^{b1}(w)-V_{\mathbf{d}}^{b2}(w)$ and $X_{\mathbf{u}}(w):=V_{\mathbf{u}}^{b1}(w)-V_{\mathbf{u}}^{b2}(w)$. Equations (61) and (62) imply that

$$(\mu_{\mathbf{d}}+r)X_{\mathbf{d}}(w)=\mu_{\mathbf{d}}X_{\mathbf{u}}(w+\beta_{\mathbf{d}})-r(\bar{w}_{\mathbf{d}}-w)X_{\mathbf{d}}'(w)\ \text{, and}$$

$$(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_{\mathbf{u}}}X_{\mathbf{u}}'(w)=-\mu_{\mathbf{u}}X_{\mathbf{d}}(w-\beta_{\mathbf{u}})+(\mu_{\mathbf{u}}+r)X_{\mathbf{u}}(w).$$

Equation (72) implies that $X_{\mathbf{d}}(w)=(b_1-b_2)(\bar{w}_{\mathbf{d}}-w)^{\frac{r+\mu_{\mathbf{d}}}{r}}$ for $[\bar{w}_{\mathbf{u}}+\beta_{\mathbf{u}}-\beta_{\mathbf{d}},\bar{w}_{\mathbf{u}})$, which is strictly concave and increasing. Following the same logic as in step 3, we can prove that $X_{\mathbf{d}}$ is strictly concave and increasing on $[0,\bar{w}_{\mathbf{d}}]$ and $X_{\mathbf{u}}$ is strictly concave and increasing on $[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}}]$. Hence,

$$V_{\mathbf{d}}^{b1}(0)-V_{\mathbf{d}}^{b2}(0)=X_{\mathbf{d}}(0)<X_{\mathbf{d}}(\bar{w}_{\mathbf{d}})=0.$$

Because $V_{\mathbf{d}}^0(0)=\bar{v}_{\mathbf{d}}>\underline{v}_{\mathbf{d}}$, and $\lim_{b\to-\infty}V_{\mathbf{d}}^b(0)<V_{\mathbf{d}}^b(\bar{w}_{\mathbf{u}}-\beta_{\mathbf{d}})=-\infty$, there must exist a unique $b^*<0$ such that $V_{\mathbf{d}}^{b*}(0)=\bar{v}_{\mathbf{d}}$, and $V_{\mathbf{d}}^{b*}(w)$ and $V_{\mathbf{u}}^{b*}(w)$ are strictly concave and increasing on $[0,\bar{w}_{\mathbf{d}}]$ and $[\beta_{\mathbf{u}},\bar{w}_{\mathbf{u}}]$, respectively.

**D.2.2.** $\beta_{\mathbf{d}}=\beta_{\mathbf{u}}$ Let $\beta_{\mathbf{d}}=\beta_{\mathbf{u}}=\beta$, then equations (61) and (62) become

$$(\mu_{\mathbf{d}}+r)V_{\mathbf{d}}(w)=\mu_{\mathbf{d}}V_{\mathbf{u}}(w+\beta)-c_{\mathbf{d}}-r(\bar{w}_{\mathbf{d}}-w)V_{\mathbf{d}}'(w),\ \text{for } w\in[0,\bar{w}_{\mathbf{d}}),\ \text{and} \tag{76}$$

$$(\mu_{\mathbf{u}}+r)V_{\mathbf{u}}(w)=-c_{\mathbf{u}}+R+\mu_{\mathbf{u}}V_{\mathbf{d}}(w-\beta)+(rw+\mu_{\mathbf{u}}\beta)\mathbb{1}_{w<\bar{w}_{\mathbf{u}}}V_{\mathbf{u}}'(w),\ \text{for } w\in[\beta,\bar{w}_{\mathbf{u}}), \tag{77}$$

since $w+\beta\leq w_{\mathbf{u}}$ for $w\in[0,\bar{w}_{\mathbf{d}})$. Let $\breve{w}=w-\beta$ in equation (77), we have

$$(\mu_{\mathbf{u}}+r)V_{\mathbf{u}}(\breve{w}+\beta)=-c_{\mathbf{u}}+R+\mu_{\mathbf{u}}V_{\mathbf{d}}(\breve{w})+(rw+(r+\mu_{\mathbf{u}})\beta)V_{\mathbf{u}}'(\breve{w}+\beta),\ \text{for } \breve{w}\in[0,\bar{w}_{\mathbf{d}}). \tag{78}$$

Differentiate (78) with respect to $\breve{w}$ on both sides, we obtain

$$\mu_{\mathbf{u}}V_{\mathbf{u}}'(\breve{w}+\beta)=\mu_{\mathbf{u}}V_{\mathbf{d}}'(\breve{w})+(rw+(r+\mu_{\mathbf{u}})\beta)V_{\mathbf{u}}''(\breve{w}+\beta),\ \text{for } \breve{w}\in[0,\bar{w}_{\mathbf{d}}). \tag{79}$$

Equations (76), (78) and (79) together imply that

$$\begin{aligned}&(\mu_{\mathbf{u}}+r)[r(\bar{w}_{\mathbf{d}}-w)V_{\mathbf{d}}'(w)+c_{\mathbf{d}}+(\mu_{\mathbf{d}}+r)V_{\mathbf{d}}(w)]\\&=\mu_{\mathbf{d}}(-c_{\mathbf{u}}+R)+\mu_{\mathbf{d}}\mu_{\mathbf{u}}V_{\mathbf{d}}(w)+(rw+(r+\mu_{\mathbf{u}})\beta)[r(\bar{w}_{\mathbf{d}}-w)V_{\mathbf{d}}''(w)+\mu_{\mathbf{d}}V_{\mathbf{d}}'(w)],\ \text{for } w\in[0,\bar{w}_{\mathbf{d}})\ ,\end{aligned} \tag{80}$$

Differentiate (80) with respect to $w$ on both sides, we obtain

$$\begin{aligned}&[\mu_{\mathbf{u}}r(\bar{w}_{\mathbf{d}}-w)-(rw+(r+\mu_{\mathbf{u}})\beta)(\mu_{\mathbf{d}}-r)]V_{\mathbf{d}}''(w)\\&=(rw+(r+\mu_{\mathbf{u}})\beta_{\mathbf{u}})r(\bar{w}_{\mathbf{d}}-w)V_{\mathbf{d}}'''(w),\ \text{for } w\in[0,\bar{w}_{\mathbf{d}}).\end{aligned} \tag{81}$$

Further, we define:

$$z(w):=\frac{[\mu_{\mathbf{u}}r(\bar{w}_{\mathbf{d}}-w)-(rw+(r+\mu_{\mathbf{u}})\beta)(\mu_{\mathbf{d}}-r)]}{(rw+(r+\mu_{\mathbf{u}})\beta_{\mathbf{u}}+c_{\mathbf{u}})r(\bar{w}_{\mathbf{d}}-w)},\ \text{for } w\in[0,\bar{w}_{\mathbf{d}})\ .$$

Then equation (81) is equivalent to

$$\frac{V_{\mathbf{d}}'''(w)}{V_{\mathbf{d}}''(w)} = z(w) \ ,$$

Solving the differential equation, we obtain $V_{\mathbf{d}}''(w) = C_0 e^{\int z(w)}$. With the boundary condition $V_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}} < \bar{v}_{\mathbf{d}}$, we could calculate $C_0$ with $C_0 < 0$. Hence, $V_{\mathbf{d}}$ is strictly concave and increasing in $[0, \bar{w}_{\mathbf{d}})$. In the same way we used in the step 4 of the case $\beta_{\mathbf{d}} > \beta_{\mathbf{u}}$, we could establish that $V_{\mathbf{u}}$ is also strictly concave and increasing in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}})$.

Second, combining with boundary condition (63), we further prove that $V_{\mathbf{u}}$ is increasing and concave in $[0, \bar{w}_{\mathbf{u}}]$. Following condition (13), (18) and $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, we have

$$R \geq (r + \bar{\mu}_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}})\beta_{\mathbf{u}}. \tag{82}$$

Following (62), we have

$$V_{\mathbf{u}}'(\beta_{\mathbf{u}+}) = \frac{(\mu_{\mathbf{u}} + r)V_{\mathbf{u}}(\beta_{\mathbf{u}}) + c_{\mathbf{u}} - R - \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}}}{r\beta_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} \geq 0,$$

which implies that

$$V_{\mathbf{u}}(\beta_{\mathbf{u}}) \geq \frac{R - c_{\mathbf{u}} + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}}}{\mu_{\mathbf{u}} + r} = \left[(r + \mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \frac{\Delta\mu_{\mathbf{u}} R}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}} - c_{\mathbf{u}}\right] / (r + \mu_{\mathbf{u}}) \geq \underline{v}_{\mathbf{u}}, \tag{83}$$

where the second inequality follows from (82). Also, this implies that $V_{\mathbf{u}}'(\beta_{\mathbf{u}-}) = \frac{V_{\mathbf{u}}(\beta_{\mathbf{u}}) - \underline{v}_{\mathbf{u}}}{\beta_{\mathbf{u}}} \geq 0$ and,

$$\begin{aligned}
(r + \bar{\mu}_{\mathbf{u}})\beta_{\mathbf{u}}(V_u'(\beta_{\mathbf{u}-}) - V_{\mathbf{u}}'(\beta_{\mathbf{u}+})) &= (r + \bar{\mu}_{\mathbf{u}})(V_{\mathbf{u}}(\beta_{\mathbf{u}}) - \underline{v}_{\mathbf{u}}) - (\mu_{\mathbf{u}} + r)V_{\mathbf{u}}(\beta_{\mathbf{u}}) - c_{\mathbf{u}} + R + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}} \\
&\geq \Delta\mu_{\mathbf{u}}\underline{v}_{\mathbf{u}} - (r + \bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}} - c_{\mathbf{u}} + R + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}} \\
&\geq R + \mu_{\mathbf{u}}\underline{v}_{\mathbf{d}} - (r + \mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} - c_{\mathbf{u}} \\
&= R + \frac{[\mu_{\mathbf{u}}\underline{\mu}_{\mathbf{d}} - (r + \mu_{\mathbf{u}})(r + \underline{\mu}_{\mathbf{d}})]R}{r(r + \bar{\mu}_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}})} - c_{\mathbf{u}} \\
&= \frac{\Delta\mu_{\mathbf{u}} R}{(r + \bar{\mu}_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}})} - c_{\mathbf{u}} \geq 0,
\end{aligned} \tag{84}$$

where the first inequality follows from (83) and the last inequality follows from (82). Finally, (84) implies that $V_{\mathbf{u}}'(\beta_{\mathbf{u}-}) \geq V_u'(\beta_{\mathbf{u}+})$. Q.E.D.

Furthermore, equations (61) and (62) imply that

$$\begin{aligned}
V_{\mathbf{u}}''(w) &= \frac{\mu_{\mathbf{u}}(V_{\mathbf{u}}'(w) - V_{\mathbf{d}}'(w - \beta_{\mathbf{u}}))}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}}, \text{ for } w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}), \\
V_{\mathbf{d}}''(w) &= \frac{\mu_{\mathbf{d}}(V_{\mathbf{u}}'(w + \beta_{\mathbf{d}}) - V_{\mathbf{d}}'(w))}{r(w_{\mathbf{d}} - w)}, \text{ for } w \in [0, \bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}), \text{ and} \\
V_{\mathbf{d}}''(w) &= \frac{-V_{\mathbf{d}}'(w)}{r(w_{\mathbf{d}} - w)}, \text{ for } w \in [\bar{w}_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}).
\end{aligned}$$

Then the concavity of $V_{\mathbf{d}}$ and $V_{\mathbf{u}}$ implies that

$$V_{\mathbf{u}}'(w) < V_{\mathbf{d}}'(w - \beta_{\mathbf{u}}), \text{ for } w \in [\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}), \text{ and} \tag{85}$$

$$V_{\mathbf{u}}'(w + \beta_{\mathbf{d}}) < V_{\mathbf{d}}'(w), \text{ for } w \in [0, \bar{w}_{\mathbf{d}}). \tag{86}$$

## D.3. Proof of Proposition 2

Following (58) and (59), we obtain that under contract $\Gamma_1^*$ in Definition 1,

$$e^{-r\tau} J(\tau) = J(0) + \int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t = \mathbf{u}}dt + c(\theta_t)dt + dL_t) + \int_0^\tau e^{-rt}\mathcal{A}_t^*, \tag{87}$$

where

$$\mathcal{A}_t^* = dJ(t) - rJ(t)dt + R\mathbb{1}_{\theta_t = \mathbf{u}}dt - dL_t - c(\theta_t)dt$$

$$
\begin{aligned}
&= J'(t)[rW_{t-} - \mu(\theta_t,1)H_t^* - \ell_t^*]dt - rJ(t)dt + J(t+) - J(t) + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - c(\theta_t)dt - dL_t^* \\
&= \left\{J_{\mathbf{u}}'(t)(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{u}}} - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}}\right\}dt\mathbb{1}_{\theta_t=\mathbf{u}} + \left\{J_{\mathbf{d}}'(W_{t-})r(W_{t-} - \bar{w}_{\mathbf{d}})dt - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}}\right\}dt\mathbb{1}_{\theta_t=\mathbf{d}} \\
&+ [J_{\mathbf{u}}(\min\{W_{t-}+\beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) - J_{\mathbf{d}}(W_{t-})]dN_t\mathbb{1}_{\theta_t=\mathbf{d}} + [J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})]dN_t\mathbb{1}_{\theta_t=\mathbf{u}} + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t^* \\
&= \{R - c_{\mathbf{u}} + J_{\mathbf{u}}'(W_{t-})(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{u}}} - rJ_{\mathbf{u}}(W_{t-})dt + \mu_{\mathbf{u}}(J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})) \\
&+ (r\bar{w}_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}=\bar{w}_{\mathbf{u}}}\}\mathbb{1}_{\theta_t=\mathbf{u}}dt \\
&+ \{J_{\mathbf{d}}'(W_{t-})r(W_{t-} - \bar{w}_{\mathbf{d}}) - rJ_{\mathbf{d}}(W_{t-})dt + \mu_{\mathbf{d}}(J_{\mathbf{u}}(\min\{W_{t-}+\beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) - J_{\mathbf{d}}(W_{t-})) - \mu_{\mathbf{d}}(W_{t-}+\beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+ - c_{\mathbf{d}}\}\mathbb{1}_{\theta_t=\mathbf{d}}dt + \mathcal{B}_t^* \\
&= \mathcal{B}_t^*,
\end{aligned}
$$

in which the last equality follows from (20) and (21), and

$$\mathcal{B}_t^* = [J_{\mathbf{u}}(\min\{W_{t-}+\beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}) - J_{\mathbf{d}}(W_{t-}) - (W_{t-}+\beta_{\mathbf{d}} - \bar{w}_{\mathbf{u}})^+](dN_t - \mu_{\mathbf{d}}dt)\mathbb{1}_{\theta_t=\mathbf{d}} + [J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})](dN_t - \mu_{\mathbf{u}}dt)\mathbb{1}_{\theta_t=\mathbf{u}}.$$

Taking the expectation on both sides of (87), we immediately have

$$J_{\theta_0}(w) = J(0) = \mathbb{E}\left[e^{-r\tau}J(\tau) + \int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t^*)\middle|\theta_0\right] = u(\Gamma_1^*(w), \nu^*, \theta_0),$$

where $u(\Gamma_1^*, \nu^*, \theta_0) = w$ and we apply the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t^* dt$ is a martingale and $J(\tau) = J_{\theta_\tau}(0) = v_\tau$. Q.E.D.

## D.4. Proof of Proposition 3

From Proposition 1, we know that $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are concave, $J_{\mathbf{d}}'(w) \geq -1$, and $J_{\mathbf{u}}'(w) \geq -1$. Recall Lemma 5, to show that $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are upper bounds of principal's utility under any incentive compatible contract, we only need to show that $\Phi_t \leq 0$ holds almost surely if $\nu_t = 1$. From (57), we have

$$\Phi_t = \Phi_t^u \mathbb{1}_{\theta_t=\mathbf{u}} + \Phi_t^d \mathbb{1}_{\theta_t=\mathbf{d}},$$

where

$$
\begin{aligned}
\Phi_t^u :=& R + J_{\mathbf{u}}'(W_{t-})rW_{t-} + \mu_{\mathbf{u}}q_t[W_{t-}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] \\
&+ \mu_{\mathbf{u}}(1-q_t)[-H_tJ_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})] - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}},
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_t^d :=& J_{\mathbf{d}}'(W_{t-})rW_{t-} + \mu_{\mathbf{d}}q_t[W_{t-}J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] \\
&+ \mu_{\mathbf{d}}(1-q_t)[-H_tJ_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(W_{t-}+H_t) - J_{\mathbf{d}}(W_{t-})] - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}}.
\end{aligned}
$$

We have $\Phi_t \leq 0$ if $\Phi_t^d \leq 0$ and $\Phi_t^u \leq 0$. First, we prove that $\Phi_t^u \leq 0$ by considering the following optimization problem,

$$
\begin{aligned}
\max_{q_t, H_t} \quad & q_t[W_{t-}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_tJ_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})], \\
s.t. \quad & 0 \leq q_t \leq 1, \quad -q_tW_{t-} + (1-q_t)H_t \leq -\beta_{\mathbf{u}}.
\end{aligned}
$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = -\beta_{\mathbf{u}}. \tag{88}$$

by the KKT conditions. Define the following dual variables for the binding constraints

$$x_{\mathbf{u}} = -(J_{\mathbf{u}}'(W_{t-}) - J_{\mathbf{d}}'(W_{t-}-\beta_{\mathbf{u}})) \geq 0,$$

in which the inequality follows from (85), and

$$y_{\mathbf{u}} = (W_{t-} - \beta_{\mathbf{u}})\left(\frac{J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{d}}(0)}{W_{t-}-\beta_{\mathbf{u}}} - J_{\mathbf{d}}'(W_{t-}-\beta_{\mathbf{u}})\right) \geq 0,$$

where the inequality follows from the concavity of $J_{\mathbf{d}}$ and the fact that $W_{t-} \geq \beta_{\mathbf{u}}$ for any incentive compatible contract. One can verify that

$$[W_{t-}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] - [-H_t^*J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t^*) - J_{\mathbf{u}}(W_{t-})] = -y_{\mathbf{u}} - (H_t^* + W_{t-})x_{\mathbf{u}}, \tag{89}$$

$$(1-q_t^*)(J'_{\mathbf{u}}(W_{t-})-J'_{\mathbf{d}}(W_{t-}+H_t^*))=(q_t^*-1)x_{\mathbf{u}}. \tag{90}$$

Therefore, (88) implies that

$$\Phi_t^u \le R+J'_{\mathbf{u}}(W_{t-})rW_{t-}+\mu_{\mathbf{u}}[\beta_{\mathbf{u}}J'_{\mathbf{u}}(W_{t-})+J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}})-J_{\mathbf{u}}(W_{t-})]-rJ_{\mathbf{u}}(W_{t-})-c_{\mathbf{u}}=0,$$

where the equality follows from (21).

Following similar logic, we prove that $\Phi_t^u \le 0$ by considering the following optimization problem,

$$\begin{aligned}\max_{q_t,H_t}\quad & q_t[W_{t-}J'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(0)-J_{\mathbf{d}}(W_{t-})]+(1-q_t)[-H_tJ'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(W_{t-}+H_t)-J_{\mathbf{d}}(W_{t-})],\\ s.t.\quad & 0\le q_t\le 1,\quad -q_tW_{t-}+(1-q_t)H_t\ge\beta_{\mathbf{d}}, W_{t-}+H_t\ge\beta_{\mathbf{u}},\end{aligned}$$

and verify that the optimal solution is

$$q_t^*=0 \quad\text{and}\quad H_t^*=\beta_{\mathbf{d}}. \tag{91}$$

by the KKT conditions. Again define the following dual variables for the binding constraints

$$\begin{aligned}x_{\mathbf{d}}&=J'_{\mathbf{d}}(W_{t-})-J'_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}})\ge 0,\quad\text{and}\\ y_{\mathbf{d}}&=(W_{t-}+\beta_{\mathbf{d}})\Big(\frac{J_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}})-J_{\mathbf{u}}(0)}{W_{t-}-\beta_{\mathbf{d}}}-J'_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}})\Big)\ge 0.\end{aligned}$$

We can verify that

$$[W_{t-}J'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(0)-J_{\mathbf{d}}(W_{t-})]-[-H_tJ'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(W_{t-}+H_t^*)-J_{\mathbf{d}}(W_{t-})]=-y_{\mathbf{d}}+(H_t^*+W_{t-})x_{\mathbf{d}}, \tag{92}$$

$$(1-q_t^*)(J'_{\mathbf{d}}(W_{t-})-J'_{\mathbf{u}}(W_{t-}+H_t^*))=(1-q_t^*)x_{\mathbf{d}}. \tag{93}$$

Therefore, (91) implies that

$$\Phi_t^u \le J'_{\mathbf{d}}(W_{t-})rW_{t-}+\mu_{\mathbf{d}}[-\beta_{\mathbf{d}}J'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}})-J_{\mathbf{d}}(W_{t-})]-rJ_{\mathbf{d}}(W_{t-})-c_{\mathbf{d}}=0,$$

where the equality follows from (20). Q.E.D.

### D.5. Proof of Theorem 2

First, it is easy to verify that contract $\Gamma_{\mathbf{u}}^*$ is incentive compatible. Next, we define two functions $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ as

$$J_{\mathbf{d}}(w)=\underline{v}_{\mathbf{d}}-w. \tag{94}$$

and

$$J_{\mathbf{u}}(w)=\begin{cases} v_{\mathbf{u}}-w, & \text{for } w\in[\beta_{\mathbf{u}},\infty),\\ \underline{v}_{\mathbf{u}}+(v_{\mathbf{u}}-\underline{v}_{\mathbf{u}}-\beta_{\mathbf{u}})w/\beta_{\mathbf{u}}, & \text{for } w\in[0,\beta_{\mathbf{u}}).\end{cases} \tag{95}$$

Under condition (28), $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$ are concave, $J'_{\mathbf{d}}(w)\ge -1$, and $J'_{\mathbf{u}}\ge -1$. Hence, following Lemma 5, we have $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are upper bounds of the principal's utility under state $\mathbf{d}$ and $\mathbf{u}$, respectively if $\Phi_t\le 0$, where $\Phi_t$ is defined by (57). Furthermore,

$$\Phi_t=\Phi_t^{\mathbf{u}}\mathbb{1}_{\theta_t=\mathbf{u}}+\Phi_t^{\mathbf{d}}\mathbb{1}_{\theta_t=\mathbf{d}},$$

where

$$\begin{aligned}\Phi_t^{\mathbf{u}}&=R-rW_{t-}+\mu_{\mathbf{u}}[-q_tW_{t-}+(1-q_t)H_t]-r(v_{\mathbf{u}}-W_{t-})+\mu_{\mathbf{u}}q_t\underline{v}_{\mathbf{d}}+\mu_{\mathbf{u}}(1-q_t)(\underline{v}_{\mathbf{d}}-W_{t-}-H_t)-\mu_{\mathbf{u}}(v_{\mathbf{u}}-W_{t-})-c_{\mathbf{u}}\\ &=R-c_{\mathbf{u}}-rv_{\mathbf{u}}+\mu_{\mathbf{u}}\underline{v}_{\mathbf{d}}-\mu_{\mathbf{u}}v_{\mathbf{u}}=0,\end{aligned}$$

where the first equality follows from $J'_{\mathbf{u}}(W_{t-})=-1$ for $W_{t-}\ge\beta_{\mathbf{u}}$,and the third equality follows from (25). Therefore,

$$\begin{aligned}\Phi_t^{\mathbf{d}}&=-rW_{t-}+\mu_{\mathbf{d}}[-q_tW_{t-}+(1-q_t)H_t]-r(\underline{v}_{\mathbf{d}}-W_{t-})+\mu_{\mathbf{d}}q_t\underline{v}_{\mathbf{u}}+\mu_{\mathbf{d}}(1-q_t)J_{\mathbf{u}}(W_{t-}+H_t)-\mu_{\mathbf{d}}(\underline{v}_{\mathbf{d}}-W_{t-})-c_{\mathbf{d}}\\ &=-c_{\mathbf{d}}-(r+\mu_{\mathbf{d}})\underline{v}_{\mathbf{d}}+\mu_{\mathbf{d}}q_t\underline{v}_{\mathbf{u}}+\mu_{\mathbf{d}}(1-q_t)V_{\mathbf{u}}(W_{t-}+H_t)\\ &\le -c_{\mathbf{d}}-(r+\mu_{\mathbf{d}})\underline{v}_{\mathbf{d}}+\mu_{\mathbf{d}}v_{\mathbf{u}}\le 0,\end{aligned}$$

where the first inequality follows by taking $q_t=0$ and $H_t=\beta_{\mathbf{d}}$, and the second inequality follows from (28).

Next, we can easily verify that the performance of $\Gamma_{\mathbf{u}}^*$ is

$$U(\Gamma_{\mathbf{u}}^*,\nu^*,\mathbf{d})=J_{\mathbf{d}}(0)=\underline{v}_{\mathbf{d}}$$

and

$$U(\Gamma_{\mathbf{u}}^*,\nu^*,\mathbf{u})=J_{\mathbf{u}}(\beta_{\mathbf{u}})=v_{\mathbf{u}}-\beta_{\mathbf{u}}$$

Starting from state $\mathbf{d}$, it is optimal to let $W_0=0$, hence $\underline{v}_d\ge U(\Gamma,\nu^*,\mathbf{d})$. Starting from state $\mathbf{u}$, if $v_{\mathbf{u}}-\beta_{\mathbf{u}}\ge\underline{v}_{\mathbf{u}}$, it is optimal to let $W_0=\beta_{\mathbf{u}}$ and if $v_{\mathbf{u}}-\beta_{\mathbf{u}}<\underline{v}_{\mathbf{u}}$, it is optimal to let $W_0=0$. Hence, $U(\Gamma^*(\beta_{\mathbf{u}}).\nu^*,\mathbf{u})\ge U(\Gamma,\nu^*,\mathbf{u})$ if $v_{\mathbf{u}}-\beta_{\mathbf{u}}\ge\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{u}}\ge U(\Gamma,\nu^*,\mathbf{u})$ if $v_{\mathbf{u}}-\beta_{\mathbf{u}}<\underline{v}_{\mathbf{u}}$. Q.E.D.

### D.6. Proof of Theorem 3

It suffices to show that if (29) is satisfied, then the principal's value functions $J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w$ and $J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w$ satisfy the optimality condition $\Phi_t \le 0$ where $\Phi_t$ is defined by (57). In fact,

$$\Phi_t = \Phi_t^{\mathbf{u}} \mathbb{1}_{\theta_t = \mathbf{u}} + \Phi_t^{\mathbf{d}} \mathbb{1}_{\theta_t = \mathbf{d}},$$

where

$$\begin{aligned}
\Phi_t^{\mathbf{u}} &= R - rW_{t-} + \mu_{\mathbf{u}}[-q_t W_{t-} + (1-q_t)H_t] - r(\underline{v}_{\mathbf{u}} - W_{t-}) + \mu_{\mathbf{u}} q_t \underline{v}_{\mathbf{d}} + \mu_{\mathbf{u}}(1-q_t)(\underline{v}_{\mathbf{d}} - W_{t-} - H_t) - \mu_{\mathbf{u}}(\underline{v}_{\mathbf{u}} - W_{t-}) - c_{\mathbf{u}} \\
&= R - c_{\mathbf{u}} - r\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}} \underline{v}_{\mathbf{d}} - \mu_{\mathbf{u}} \underline{v}_{\mathbf{u}} = R - c_{\mathbf{u}} - (r + \mu_{\mathbf{u}}) \frac{(r + \underline{\mu}_{\mathbf{d}})R}{r(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})} + \mu_{\mathbf{u}} \frac{\underline{\mu}_{\mathbf{d}} R}{r(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})} \\
&= \frac{\Delta \mu_{\mathbf{u}} R}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}} - c_{\mathbf{u}} = \frac{\Delta \mu_{\mathbf{u}}}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}} (R - (r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}) \beta_{\mathbf{u}}) < 0,
\end{aligned}$$

and

$$\begin{aligned}
\Phi_t^{\mathbf{d}} &= -rW_{t-} + \mu_{\mathbf{d}}[-q_t W_{t-} + (1-q_t)H_t] - r(\underline{v}_{\mathbf{d}} - W_{t-}) + \mu_{\mathbf{d}} q_t \underline{v}_{\mathbf{u}} + \mu_{\mathbf{d}}(1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t) - \mu_{\mathbf{d}}(\underline{v}_{\mathbf{d}} - W_{t-}) - c_{\mathbf{d}} \\
&= -c_{\mathbf{d}} - r\underline{v}_{\mathbf{d}} + \mu_{\mathbf{d}} \underline{v}_{\mathbf{u}} - \mu_{\mathbf{d}} \underline{v}_{\mathbf{d}} = -c_{\mathbf{d}} - (r + \mu_{\mathbf{d}}) \frac{\underline{\mu}_{\mathbf{d}} R}{r(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})} + \mu_{\mathbf{d}} \frac{(r + \underline{\mu}_{\mathbf{d}})R}{r(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})} \\
&= \frac{\Delta \mu_{\mathbf{d}} R}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}} - c_{\mathbf{d}} = \frac{\Delta \mu_{\mathbf{d}}}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}} (R - (r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}) \beta_{\mathbf{d}}) < 0,
\end{aligned}$$

where the inequalities follow from (29). Q.E.D.

## E. Results and Proofs in Section 4.2

### E.1. Proof of Lemma 2

Using (35) and (36) as boundary conditions, (33) is a linear differential equation with boundary condition. The solution is

$$J_{\mathbf{d}}^{a\hat{\beta}}(w) = aw + \frac{\mu_{\mathbf{d}} \underline{v}_{\mathbf{u}} - c_{\mathbf{d}}}{\mu_{\mathbf{d}} + r} + C_1 (\bar{w}_{\mathbf{d}} - w)^{\frac{r + \mu_{\mathbf{d}}}{r}}, \text{ for } w \in [0, \min\{\hat{\beta} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\}], \tag{96}$$

with

$$C_1 = -\frac{\left[\frac{\Delta \mu_{\mathbf{d}} R}{r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}} - c_{\mathbf{d}}\right] \bar{w}_{\mathbf{d}}^{-\frac{r + \mu_{\mathbf{d}}}{r}}}{r + \mu_{\mathbf{d}}} < 0, \tag{97}$$

in which the inequality follows from (18). Therefore, we can solve $J_{\mathbf{u}}^{a\hat{\beta}}$ for $\left[\hat{\beta}, \min\{\hat{\beta} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\} + \beta_{\mathbf{u}}\right]$ using (34), (35) and (96). By induction, we can solve $J_{\mathbf{d}}^{a\hat{\beta}}$ in $[0, \bar{w}_{\mathbf{d}}]$ and $J_{\mathbf{u}}^{a\hat{\beta}}$ in $[0, \bar{w}_{\mathbf{u}}]$. These are a sequence of initial value problems satisfying the Cauchy-Lipschitz Theorem, and, therefore, bear unique solutions. Furthermore, $J_{\mathbf{u}}^{a\hat{\beta}}$ is $C^2\left([0, \bar{w}_{\mathbf{u}}] \setminus \{\hat{\beta}\}\right)$ and $J_{\mathbf{d}}^{a\hat{\beta}}$ is $C^3\left([0, \bar{w}_{\mathbf{d}}] \setminus \{\hat{\beta} - \beta_{\mathbf{d}}\}\right)$. For $w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}}]$, (31) and (34) together imply that

$$(rw + \mu_{\mathbf{u}} \beta_{\mathbf{u}}) \mathbb{1}_{w < \hat{w}_{\mathbf{u}}} J_{\mathbf{u}}'(w) = (\mu_{\mathbf{u}} + r) J_{\mathbf{u}}(w) - R - \frac{\mu_{\mathbf{u}} \mu_{\mathbf{d}}}{\mu_{\mathbf{d}} + r} J_{\mathbf{u}} \left( \frac{r + \mu_{\mathbf{d}}}{\mu_{\mathbf{d}}} (w - \beta_{\mathbf{u}}) \right) + \frac{\mu_{\mathbf{u}} c_{\mathbf{d}}}{\mu_{\mathbf{d}} + r} + \ell^* \mathbb{1}_{w = \hat{w}_{\mathbf{u}}}. \tag{98}$$

If we define $w_0 := \bar{w}_{\mathbf{u}}$ and $w_n := (\mu_{\mathbf{d}} w_{n-1})/(r + \mu_{\mathbf{d}}) + \beta_{\mathbf{u}}$ for $n = 1, 2, 3 \ldots$, then $\hat{w}_{\mathbf{u}} = \lim_{n \to \infty} w_n$. Furthermore, (98) is equivalent to a sequence of initial value problems over the intervals $[w_n, w_{n+1}]$, $n = 1, 2, \ldots$. This sequence of initial value problem again satisfy the Cauchy-Lipschitz Theorem and bear unique solutions. Furthermore, if $\hat{\beta} < \bar{w}_{\mathbf{d}} + \beta_{\mathbf{d}}$, then $J_{\mathbf{u}}^{a\hat{\beta}}$ is $C^2([0, \hat{w}_{\mathbf{u}}) \setminus \{\hat{\beta}\})$, $J_{\mathbf{d}}^{a\hat{\beta}}$ is $C^3([0, \hat{w}_{\mathbf{d}}) \setminus \{\hat{\beta} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\})$ and if $\hat{\beta} \ge \bar{w}_{\mathbf{d}} + \beta_{\mathbf{d}}$, then $J_{\mathbf{u}}^{a\hat{\beta}}$ is $C^2([0, \hat{w}_{\mathbf{u}}) \setminus \{\hat{\beta}, (\mu_{\mathbf{d}} \hat{\beta})/(r + \mu_{\mathbf{d}}) + \beta_{\mathbf{u}}\})$, $J_{\mathbf{d}}^{a\hat{\beta}}$ is $C^3([0, \hat{w}_{\mathbf{d}}) \setminus \{\hat{\beta} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}, \hat{\beta} + (\mu_{\mathbf{d}} \beta_{\mathbf{u}})/(r + \mu_{\mathbf{d}})\})$. Then, we could derive the expressions for $J_{\mathbf{u}}^{a\hat{\beta}''}$, $J_{\mathbf{d}}^{a\hat{\beta}''}$ and $J_{\mathbf{d}}^{a\hat{\beta}'''}$ following (31), (33) and (34), respectively,

$$J_{\mathbf{u}}^{a\hat{\beta}''}(w) = \frac{\mu_{\mathbf{u}} \left( J_{\mathbf{u}}^{a\hat{\beta}'}(w) - J_{\mathbf{d}}^{a\hat{\beta}'}(w - \beta_{\mathbf{u}}) \right)}{rw + \mu_{\mathbf{u}} \beta_{\mathbf{u}}}, \text{ for } w \in (\hat{\beta}, \hat{w}_{\mathbf{u}}), \tag{99}$$

$$J_{\mathbf{u}}^{a\hat{\beta}''}(w) = \frac{\mu_{\mathbf{u}}\left(J_{\mathbf{u}}^{a\hat{\beta}'}(w) - J_{\mathbf{u}}^{a\hat{\beta}'}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(w-\beta_{\mathbf{u}})\right)\right)}{rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}}}, \text{ for } w \in [\bar{w}_u, \hat{w}_{\mathbf{u}}), \tag{100}$$

$$J_{\mathbf{d}}^{a\hat{\beta}''}(w) = \frac{\mu_{\mathbf{d}}\left(J_{\mathbf{u}}^{a\hat{\beta}'}(w+\beta_{\mathbf{d}}) - J_{\mathbf{d}}^{a\hat{\beta}'}(w)\right)}{r(\bar{w}_{\mathbf{d}}-w)}, \text{ for } w \in [0,\bar{w}_{\mathbf{d}}) \setminus \{\hat{\beta}-\beta_{\mathbf{d}}\}, \tag{101}$$

$$J_{\mathbf{d}}^{a\hat{\beta}''}(w) = J_{\mathbf{u}}^{a\hat{\beta}''}\left(w + \frac{rw}{\mu_{\mathbf{d}}}\right), \text{ for } w \in (\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}}), \text{ and} \tag{102}$$

$$J_{\mathbf{d}}^{a\hat{\beta}'''}(w) = \frac{\mu_{\mathbf{d}}\left(J_{\mathbf{u}}^{a\hat{\beta}''}(w+\beta_{\mathbf{d}}) - J_{\mathbf{d}}^{a\hat{\beta}''}(w)\right) + rJ_{\mathbf{d}}^{a\hat{\beta}''}(w)}{r(\bar{w}_{\mathbf{d}}-w)}, \text{ for } w \in [0,\bar{w}_{\mathbf{d}}) \setminus \{\hat{\beta}-\beta_{\mathbf{d}}\}. \tag{103}$$

Q.E.D.

## E.2. Proof of Lemma 3

Following (34), we can calculate for $\hat{\beta} \in [\beta_u, \bar{w}_u)$:

$$\begin{aligned}
J_{\mathbf{u}}^{a\hat{\beta}'}(\hat{\beta}_+) &= \frac{(r+\mu_{\mathbf{u}})J_{\mathbf{u}}(\hat{\beta}) - \mu_{\mathbf{u}}J_{\mathbf{d}}(\hat{\beta}-\beta_{\mathbf{u}}) - R + c_{\mathbf{u}}}{r\hat{\beta}+\mu_{\mathbf{u}}\beta_{\mathbf{u}}} \\
&= \frac{(r+\mu_{\mathbf{u}})(\underline{v}_{\mathbf{u}}+a\hat{\beta}) - \mu_{\mathbf{u}}\left[a(\hat{\beta}-\beta_{\mathbf{u}}) + \frac{\mu_{\mathbf{d}}\underline{v}_{\mathbf{u}}-c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} + C_1(\bar{w}_{\mathbf{d}}-\hat{\beta}+\beta_{\mathbf{u}})^{\frac{r+\mu_{\mathbf{d}}}{r}}\right] - R + c_{\mathbf{u}}}{r\hat{\beta}+\mu_{\mathbf{u}}\beta_{\mathbf{u}}} \\
&= a + \frac{(r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} - \mu_{\mathbf{u}}\left[\frac{\mu_{\mathbf{d}}\underline{v}_{\mathbf{u}}-c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} + C_1(\bar{w}_{\mathbf{d}}-\hat{\beta}+\beta_{\mathbf{u}})^{\frac{r+\mu_{\mathbf{d}}}{r}}\right] - R + c_{\mathbf{u}}}{r\hat{\beta}+\mu_{\mathbf{u}}\beta_{\mathbf{u}}},
\end{aligned} \tag{104}$$

where $C_1$ follows (97). Furthermore, following equation (37) and (104), we have for $\hat{\beta} \in [\beta_u, \bar{w}_u)$,

$$f_a(\hat{\beta}) = -(r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}}\left[\frac{\mu_{\mathbf{d}}\underline{v}_{\mathbf{u}}-c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} + C_1(\bar{w}_{\mathbf{d}}-\beta+\beta_{\mathbf{u}})^{\frac{r+\mu_{\mathbf{d}}}{r}}\right] + R - c_{\mathbf{u}}, \tag{105}$$

and $f_a(\hat{\beta})$ is increasing in $[\beta_{\mathbf{u}}, \bar{w}_{\mathbf{u}}]$ because $C_1 < 0$. Therefore,

$$\begin{aligned}
\lim_{\hat{\beta}\uparrow\bar{w}_{u-}} f_a(\hat{\beta}) &= -(r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}}\left[\frac{\mu_{\mathbf{d}}\underline{v}_{\mathbf{u}}-c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r}\right] + R - c_{\mathbf{u}} \\
&= \frac{r\Delta\mu_{\mathbf{u}} + \mu_{\mathbf{u}}\Delta\mu_{\mathbf{d}} + \mu_{\mathbf{d}}\Delta\mu_{\mathbf{u}}}{(\mu_{\mathbf{d}}+r)(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})}R - \frac{\mu_{\mathbf{u}}c_{\mathbf{d}}}{\mu_{\mathbf{d}}+r} - c_{\mathbf{u}} \geq 0,
\end{aligned}$$

where the last inequality follows from the condition (19). Q.E.D.

## E.3. Proof of Proposition 4

We show the result following three steps.

1. Show that $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, \hat{w}_{\mathbf{d}})$, and $J_{\mathbf{u}}^{a\beta_a}$ is concave in $[0, \hat{w}_{\mathbf{u}})$ and strictly concave in $[\beta_a, \hat{w}_{\mathbf{u}})$.
2. Show that for any $w \geq 0$, derivatives $\frac{d}{dw}J_{\mathbf{u}}^{a\beta_a}(w)$ and $\frac{d}{dw}J_{\mathbf{d}}^{a\beta_a}(w)$ are increasing in $a$.
3. There exists unique $\bar{a} > -1$ such that (40) is satisfied, and the corresponding functions $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}(w)$ and $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}(w)$ are both concave with derivatives greater than or equal to $-1$.

**Step 1.** For any $a > -1$, if $\beta_a = \beta_{\mathbf{u}}$, then $J_{\mathbf{u}}^{a\beta_a}$ is $C^2([0,\bar{w}_{\mathbf{u}}) \setminus \{\beta_{\mathbf{u}}\})$ and $J_{\mathbf{d}}^{a\beta_a}$ is $C^3([0,\bar{w}_{\mathbf{d}}) \setminus \{\beta_{\mathbf{u}}-\beta_{\mathbf{d}}\})$. Otherwise, if $\beta_a > \beta_{\mathbf{u}}$, then $J_{\mathbf{u}}^{a\beta_a}$ is $C^2([0,\hat{w}_{\mathbf{u}}))$ and $J_{\mathbf{d}}^{a\beta_a}$ is $C^3([0,\bar{w}_{\mathbf{d}})\setminus\{\bar{w}_{\mathbf{d}}\})$. Following (96) and (97), we have $J_{\mathbf{d}}^{a\beta_a}(w)$ is strictly concave with $J_{\mathbf{d}}^{a\beta_a'}(w) > a$ in the interval $[0, \beta_a - \beta_{\mathbf{d}})$. We claim that $J_{\mathbf{d}}^{a\beta_a''}((\beta_a-\beta_{\mathbf{d}})_+) < 0$. If $\beta_a > \beta_{\mathbf{u}}$, then this result directly follows by smooth pasting. Otherwise, if $\beta_a = \beta_{\mathbf{u}}$, equation (101) implies that

$$J_{\mathbf{d}}^{a\beta_a''}((\beta_{\mathbf{u}}-\beta_{\mathbf{d}})_+) = \frac{\mu_{\mathbf{d}}\left(J_{\mathbf{u}}^{a\beta_a'}(\beta_{\mathbf{u}+}) - J_{\mathbf{d}}^{a\beta_a'}(\beta_{\mathbf{u}}-\beta_{\mathbf{d}})\right)}{r(\bar{w}_{\mathbf{d}}-w)} < 0,$$

where the inequality follows from $a - J_{\mathbf{u}}^{a\beta'_a}(\beta_{\mathbf{u}+}) \geq 0$ which is implied by the definition of $\beta_a$ and $J_{\mathbf{d}}^{a\beta'_a}(\beta_{\mathbf{u}} - \beta_{\mathbf{d}}) > a$. Next, we prove that $J_{\mathbf{u}}^{a\beta_a}(w)$ is strictly concave in $[\beta_a, \min\{\beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}]$. First, following (99), we have

$$J_{\mathbf{u}}^{a\beta''_a}(\beta_{a+}) = \frac{\mu_{\mathbf{u}}\left(J_{\mathbf{u}}^{a\beta'_a}(\beta_{a+}) - J_{\mathbf{d}}^{a\beta'_a}(\beta_a - \beta_{\mathbf{u}})\right)}{r\beta_a + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} < 0,$$

where the inequality follows from $J_{\mathbf{u}}^{a\beta'_a}(\beta_{a+}) \leq a$ and $J_{\mathbf{d}}^{a\beta'_a}(\beta_a - \beta_{\mathbf{u}}) > a$. Assume that there exists $w \in (\beta_a, \min\{\beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}]$ such that $J_{\mathbf{u}}^{a\beta''_a}(w) \geq 0$, then $J_{\mathbf{u}}^{a\beta_a}$ being twice continuously differentiable implies that there must exist $\hat{w} = \min\{w \in (\beta_a, \min\{\beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}] | J_{\mathbf{u}}^{a\beta''_a}(w) = 0\}$, such that $J_{\mathbf{u}}^{a\beta''_a}(w) < 0$ for $w < \hat{w}$. Equation (99) implies that

$$J_{\mathbf{u}}^{a\beta'_a}(\hat{w}) = J_{\mathbf{d}}^{a\beta'_a}(\hat{w} - \beta_{\mathbf{u}}).$$

Since $J_{\mathbf{d}}^{a\beta_a}$ is concave in the interval $[0, \min\{\beta_a - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\}]$, equation (101) implies that $J_{\mathbf{u}}^{a\beta'_a}(\hat{w} + \beta_{\mathbf{d}} - \beta_{\mathbf{u}}) < J_{\mathbf{d}}^{a\beta'_a}(\hat{w} - \beta_{\mathbf{u}})$, which further implies that

$$J_{\mathbf{u}}^{a\beta'_a}(\hat{w} + \beta_{\mathbf{d}} - \beta_{\mathbf{u}}) < J_{\mathbf{u}}^{a\beta'_a}(\hat{w}),$$

which contradicts with $J_{\mathbf{u}}^{a\beta''_a}(w) < 0$ for $w < \hat{w}$. Hence, $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[\beta_a, \min\{\beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{u}}\}]$.

Next we prove two lemmas.

LEMMA 8. *For any $w \geq 0$, if $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, w + \beta_a - \beta_{\mathbf{d}}]$ and $J_{\mathbf{u}}^{a\beta_a}$ is concave in $[\beta_a, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$ for any $w \geq 0$, then $J_{\mathbf{d}}^{a\beta_a}$ is also strictly concave in $[0, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$.*

**Proof.** Assume that there exists $w \in [w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$ such that $J_{\mathbf{d}}^{a\beta''_a}(w) \geq 0$, then the fact that $J_{\mathbf{d}}^{a\beta_a}$ is twice continuously differentiable implies that there must exist $\tilde{w} = \min\{w \in (w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}] | J_{\mathbf{d}}^{a\beta''_a}(\tilde{w}) = 0\}$, such that $J_{\mathbf{d}}^{a\beta''_a}(w) < 0$ for $w < \tilde{w}$. Equation (103) implies that

$$J_{\mathbf{d}}^{a\beta'''_a}(\hat{w}) = \frac{\mu_{\mathbf{d}} J_{\mathbf{u}}^{a\beta''_a}(w + \beta_{\mathbf{d}})}{r(\bar{w}_{\mathbf{d}} - \hat{w})} < 0,$$

where the inequality follows from $J_{\mathbf{u}}^{a\beta_a}$ being concave in $[0, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$. This contradicts with $J_{\mathbf{d}}^{a\beta''_a}(\tilde{w}) = 0$ and $J_{\mathbf{d}}^{a\beta''_a}(w) < 0$ for $w < \tilde{w}$. Q.E.D.

LEMMA 9. *For any $w \geq 0$, if $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[0, w]$ and $J_{\mathbf{d}}^{a\beta_a}$ is concave in $[0, w - \beta_{\mathbf{d}}]$ for any $w \geq 0$, then $J_{\mathbf{u}}^{a\beta_a}$ is also strictly concave in $[w, w + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$.*

The proof for Lemma 9 follows the same logic as Lemma 8, and is omitted here. Equipped with Lemmas 8 and 9, we prove that if $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[\beta_a, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$ and $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, w + \beta_a - \beta_{\mathbf{d}}]$, then $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[\beta_a, w + \beta_a + 2\beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$ and $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$. Hence, by induction, $J_{\mathbf{u}}^{a\beta_a}$ is strictly concave in $[\beta_a, \bar{w}_{\mathbf{u}})$ and $J_{\mathbf{d}}^{a\beta_a}$ is strictly concave in $[0, \bar{w}_{\mathbf{d}})$.

We have $J_{\mathbf{d}}^{a\beta'_a}(\bar{w}_{\mathbf{d}}-) > J_{\mathbf{u}}^{a\beta'_a}(\bar{w}_{\mathbf{u}})$ from (99) and $J_{\mathbf{d}}^{a\beta'_a}(\bar{w}_{\mathbf{d}}+) = J_{\mathbf{u}}^{a\beta'_a}(\bar{w}_{\mathbf{u}})$ from (31). Hence, $J_{\mathbf{d}}^{a\beta'_a}(\bar{w}_{\mathbf{d}}-) > J_{\mathbf{d}}^{a\beta'_a}(\bar{w}_{\mathbf{d}}+)$. Finally, we prove that $J_{\mathbf{u}}^{a\beta''_a}(w+) < 0$ for $w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}})$. If there exists $w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}})$ such that $J_{\mathbf{u}}^{a\beta''_a}(w+) \geq 0$, then there must exist $\check{w} = \min\{w \in [\bar{w}_{\mathbf{u}}, \hat{w}_{\mathbf{u}}) | J_{\mathbf{u}}^{a\beta''_a}(w+) = 0\}$, such that $J_{\mathbf{u}}^{a\beta''_a}(w+) < 0$ for $w < \check{w}$. Finally, (100) implies that

$$J_{\mathbf{u}}^{a\hat{\beta}'}(\check{w}) - J_{\mathbf{u}}^{a\hat{\beta}'}\left(\frac{r + \mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\check{w} - \beta_{\mathbf{u}})_+\right) = 0,$$

which contradicts with

$$J_{\mathbf{u}}^{a\hat{\beta}'}(\check{w}) = J_{\mathbf{u}}^{a\hat{\beta}'}\left(\frac{r + \mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\check{w} - \beta_{\mathbf{u}})_+\right) + \int_{\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\check{w} - \beta_{\mathbf{u}})}^{\check{w}} J_{\mathbf{u}}^{a\hat{\beta}''}(x)dx < J_{\mathbf{u}}^{a\hat{\beta}'}\left(\frac{r + \mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\check{w} - \beta_{\mathbf{u}})\right),$$

where the inequality follows from $\check{w} > \frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\check{w} - \beta_{\mathbf{u}})$ and $J_{\mathbf{u}}^{a\beta''_a}(w_+) < 0$ for $w < \check{w}$. Following (102), $J_{\mathbf{d}}^{a\beta''_a}$ is also strictly concave in $[\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}})$.

**Step 2.** We show that for any $w \geq 0$, $dJ_{\mathbf{u}}^{a\beta_a}/dw$ and $dJ_{\mathbf{d}}^{a\beta_a}/dw$ are increasing in $a$. To do so, we define

$$g_{\mathbf{d}}(w) := \frac{dJ_{\mathbf{d}}^{a\beta_a}}{da}(w_+) \quad \text{and} \quad g_{\mathbf{u}}(w) := \frac{dJ_{\mathbf{u}}^{a\beta_a}}{da}(w_+).$$

It suffices to prove that $g_{\mathbf{d}}(w)$ and $g_{\mathbf{u}}(w)$ are well-defined and strictly increasing in $w$.

- For $w \in [0, \beta_a)$, we have $g_{\mathbf{u}}(w) = w$, which is strictly increasing in $w$. For $w \in [0, \beta_a - \beta_{\mathbf{d}}]$, $g_{\mathbf{d}}(w) = w$ which is also strictly increasing in $w$.
- For $w = \beta_a$, we have

$$g_{\mathbf{u}}(\beta_{a+}) = \lim_{\epsilon \downarrow 0} \frac{J_{\mathbf{u}}^{a+\epsilon\beta_a}(\beta_a) - J_{\mathbf{u}}^{a\beta_a}(\beta_a)}{\epsilon} + \frac{J_{\mathbf{u}}^{a\beta_{a+\epsilon}}(\beta_a) - J_{\mathbf{u}}^{a\beta_a}(\beta_a)}{\epsilon} \cdot \frac{d\beta_a}{da}$$
$$= \lim_{\epsilon \downarrow 0} \frac{J_{\mathbf{u}}^{a+\epsilon\beta_a}(\beta_a) - J_{\mathbf{u}}^{a\beta_a}(\beta_a)}{\epsilon} = \beta_a = g_{\mathbf{u}}(\beta_{a-}),$$

where the second equality follows from $J_{\mathbf{u}}^{a\beta_{a+\epsilon}}(\beta_a) = J_{\mathbf{u}}^{a\beta_a}(\beta_a)$ because $\beta_{a+\epsilon} \geq \beta_a$ for any $\epsilon \geq 0$.
- For $J_{\mathbf{u}}^{a\beta_a}(w)$ on $[\beta_a, \bar{w}_{\mathbf{u}}]$ and $J_{\mathbf{d}}^{a\beta_a}(w)$ on $[\beta_a - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$, taking derivatives with respect to $a$ on both sides of (32) and (34), we know that $g_{\mathbf{d}}(w)$ and $g_{\mathbf{u}}(w)$ satisfies the following system of equations:

$$(\mu_{\mathbf{d}} + r) g_{\mathbf{d}}(w) = \mu_{\mathbf{d}} g_{\mathbf{u}}(w + \beta_{\mathbf{d}}) - r(\bar{w}_{\mathbf{d}} - w) g'_{\mathbf{d}}(w) \ , \quad w \in [0, \bar{w}_{\mathbf{d}}], \text{ and} \tag{106}$$
$$(\mu_{\mathbf{u}} + r) g_{\mathbf{u}}(w) = \mu_{\mathbf{u}} g_{\mathbf{d}}(w - \beta_{\mathbf{u}}) + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}) g'_{\mathbf{u}}(w). \quad w \in [\beta_a, \bar{w}_{\mathbf{u}}] \tag{107}$$

In the following, we prove that $g_{\mathbf{d}}(w)$ and $g_{\mathbf{u}}(w)$ are also strictly increasing on $[\beta_a - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}]$ and $[\beta_a, \bar{w}_{\mathbf{u}}]$, respectively. Following equation (107), we have

$$g'_{\mathbf{u}}(\beta_{a_+}) = \frac{(\mu_{\mathbf{u}} + r) g_{\mathbf{u}}(\beta_a) - \mu_{\mathbf{u}} g_{\mathbf{d}}(\beta_a - \beta_{\mathbf{u}})}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}}$$
$$= \frac{(\mu_{\mathbf{u}} + r)\beta_a - \mu_{\mathbf{u}} [\beta_a - \beta_{\mathbf{u}}]}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}}$$
$$\geq \frac{(\mu_{\mathbf{u}} + r)\beta_{\mathbf{u}}}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}} > 0,$$

where the second inequality follows from $\beta_a \geq \beta_{\mathbf{u}}$. Then we claim that $g_{\mathbf{u}}(w)$ is strictly increasing in $[\beta_a, \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$. If not, then there exists $w \in (\beta_a, \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$ such that $g'_{\mathbf{u}}(w) \geq 0$. Therefore, we must have $\hat{w} = \min\{w \in (\beta_a, \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}] | g'_{\mathbf{u}}(w) = 0\}$ and $g'_{\mathbf{u}}(w) > 0$ for $w < \hat{w}$. Equation (107) implies that

$$(r + \mu_{\mathbf{u}}) g_{\mathbf{u}}(\hat{w}) = \mu_{\mathbf{u}} g_{\mathbf{d}}(\hat{w} - \beta_{\mathbf{u}}).$$

The fact that $g_{\mathbf{d}}(w)$ is increasing in $[0, \beta_a - \beta_{\mathbf{d}}]$ implies that $(\mu_{\mathbf{d}} + r) g_{\mathbf{d}}(w - \beta_{\mathbf{u}}) < \mu_{\mathbf{d}} g_{\mathbf{u}}(\hat{w} - \beta_{\mathbf{u}} + \beta_{\mathbf{d}})$, which further implies that

$$(r + \mu_{\mathbf{u}}) g_{\mathbf{u}}(\hat{w}) = \mu_{\mathbf{u}} g_{\mathbf{d}}(\hat{w} - \beta_{\mathbf{u}}) < \mu_{\mathbf{u}} \frac{\mu_{\mathbf{d}}}{\mu_{\mathbf{d}} + r} g_{\mathbf{u}}(\hat{w} - \beta_{\mathbf{u}} + \beta_{\mathbf{d}}),$$

which contradicts $g'_{\mathbf{u}}(w) > 0$ for $w < \hat{w}$. We establish the final results by proving the next two claims.

Lemma 10. *If $g_{\mathbf{d}}$ is strictly increasing in $[0, w + \beta_a - \beta_{\mathbf{d}}]$ and $g_{\mathbf{u}}$ is strictly increasing in $[0, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$ for any $w \geq 0$, then $g_{\mathbf{d}}$ is also increasing in $[w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$.*

**Proof.** If there exists $w \in (w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}]$ such that $g'_{\mathbf{d}}(w) \leq 0$, then we must have $\tilde{w} = \min\{w \in (w + \beta_a - \beta_{\mathbf{d}}, w + \beta_a + \beta_{\mathbf{u}} - 2\beta_{\mathbf{d}}] | g'_{\mathbf{d}}(w) = 0\}$ such that $g'_{\mathbf{d}}(w) > 0$ for $w < \tilde{w}$. Differentiating (106), we obtain that

$$g''_{\mathbf{d}}(\hat{w}) = \frac{\mu_{\mathbf{d}}(g'_{\mathbf{u}}(\hat{w} + \beta_{\mathbf{d}}) - g'_{\mathbf{d}}(\hat{w}))}{r(\bar{w}_{\mathbf{d}} - w)} > 0,$$

where the inequality holds because $g_{\mathbf{u}}$ is increasing on $[0, w + \beta_a + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$. However, this contradicts $g'_{\mathbf{d}}(\hat{w}) = 0$ and $g'_{\mathbf{d}}(w) > 0$ for $w < \tilde{w}$. Q.E.D.

Lemma 11. *If $g_{\mathbf{u}}$ is strictly concave in $[0, w]$ and $g_{\mathbf{d}}$ is concave in $[0, w - \beta_{\mathbf{d}}]$ for any $w \geq 0$, then $g_{\mathbf{u}}$ is also strictly concave in $[w, w + \beta_{\mathbf{u}} - \beta_{\mathbf{d}}]$.*

The logic of the proof of Lemma 11 is similar to that of Lemma 10, and is therefore omitted here. Following Lemmas 10 and 11, we can prove by induction that $g_{\mathbf{u}}$ is strictly concave in $[\beta_a, \bar{w}_{\mathbf{u}})$ and $g_{\mathbf{d}}$ is strictly concave in $[0, \bar{w}_{\mathbf{d}})$.

- For $J_{\mathbf{u}}^{a\beta_a}(w)$ on $[\bar{w}_{\mathbf{u}},\hat{w}_{\mathbf{u}})$ and $J_{\mathbf{d}}^{a\beta_a}(w)$ on $[\bar{w}_{\mathbf{d}},\hat{w}_{\mathbf{d}}]$, taking derivatives with respect to $a$ on both sides of (31) and (98), we know that $g_{\mathbf{d}}(w)$ and $g_{\mathbf{u}}(w)$ satisfies the following system of equations,

$$(\mu_{\mathbf{d}}+r)g_{\mathbf{d}}(w)=\mu_{\mathbf{d}}g_{\mathbf{u}}\left(w+\frac{rw}{\mu_{\mathbf{d}}}\right) \text{ for } w\in[\bar{w}_{\mathbf{d}},\hat{w}_{\mathbf{d}}], \text{ and} \tag{108}$$

$$(\mu_{\mathbf{u}}+r)g_{\mathbf{u}}(w)=\frac{\mu_{\mathbf{u}}\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}+r}g_{\mathbf{u}}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(w-\beta_{\mathbf{u}})\right)+(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\hat{w}_{\mathbf{u}}}g_{\mathbf{u}}'(w) \text{ for } w\in[\bar{w}_{\mathbf{u}},\hat{w}_{\mathbf{u}}]. \tag{109}$$

Since (108) implies that $g_{\mathbf{d}}'(w)=g_{\mathbf{u}}'\left(w+\frac{rw}{\mu_{\mathbf{d}}}\right)$ for $w\in[\bar{w}_{\mathbf{d}},\hat{w}_{\mathbf{d}}]$, we just need to show that $g_{\mathbf{u}}'(w)>0$ for $w\in[\bar{w}_{\mathbf{u}},\hat{w}_{\mathbf{u}})$. We have proved that $g_u'(w)>0$ for $w\in[0,\bar{w}_{\mathbf{u}}]$. If there exists $w\in(\bar{w}_{\mathbf{u}},\hat{w}_{\mathbf{u}})$ such that $g_{\mathbf{u}}'(w)\le 0$, then there must be $\breve{w}=\min\{w\in(\bar{w}_{\mathbf{u}},\hat{w}_{\mathbf{u}})|g_{\mathbf{u}}'(w)=0\}$, such that $g_u'(w)>0$ for $w>\breve{w}$. Then, (109) implies that

$$(\mu_{\mathbf{u}}+r)g_{\mathbf{u}}(\breve{w})=\frac{\mu_{\mathbf{u}}\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}+r}g_{\mathbf{u}}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})\right).$$

However, this contradicts with

$$g_{\mathbf{u}}(\breve{w})=g_{\mathbf{u}}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})\right)+\int_{\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})}^{w}g_{\mathbf{u}}'(x)dx>g_{\mathbf{u}}\left(\frac{r+\mu_{\mathbf{d}}}{\mu_{\mathbf{d}}}(\breve{w}-\beta_{\mathbf{u}})\right).$$

**Step 3.** Since for any $w\ge 0$, derivatives $\dfrac{d}{dw}J_{\mathbf{u}}^{a\beta_a}(w)$ and $\dfrac{d}{dw}J_{\mathbf{d}}^{a\beta_a}(w)$ are increasing in $a$, with boundary condition (36), $J_{\mathbf{u}}^{a\beta_a}(w)$ and $J_{\mathbf{d}}^{a\beta_a}(w)$ are also increasing in $a$. For $a$ approaching $-1$, we have $\lim_{w\uparrow\bar{w}_{\mathbf{u}}}J_{\mathbf{u}}^{a\beta_a}(w)<\underline{v}_{\mathbf{u}}-\hat{w}_{\mathbf{u}}\le\bar{v}_{\mathbf{u}}-\hat{w}_{\mathbf{u}}$. For $a$ approaching $\infty$, we have $\lim_{w\uparrow\bar{w}_{\mathbf{u}}}J_{\mathbf{u}}^{a\beta_a}(w)\to\infty$. Hence, there exists a unique $a>0$, denoted as $\bar{a}$, such that $\lim_{w\uparrow\bar{w}_{\mathbf{u}}}J_{\mathbf{u}}^{a\beta_a}(w)=\bar{v}_{\mathbf{u}}-\hat{w}_{\mathbf{u}}$. Following (31), we have $\lim_{w\uparrow\hat{w}_{\mathbf{d}}}J_{\mathbf{d}}^{a\beta_a}(w)=\bar{v}_{\mathbf{d}}-\hat{w}_{\mathbf{d}}$.

Then, (31) and (98) imply that $J_{\mathbf{d}}^{a\beta_a}(\hat{w}_{\mathbf{d}})=\bar{v}_{\mathbf{d}}-\hat{w}_{\mathbf{d}}$, $J_{\mathbf{u}}^{a\beta_a}(\hat{w}_{\mathbf{u}})=\bar{v}_{\mathbf{u}}-\hat{w}_{\mathbf{u}}$, $\lim_{w\uparrow\hat{w}_{\mathbf{u}}}J_{\mathbf{u}}^{a\beta_a'}(w)=-1$, and $\lim_{w\uparrow\hat{w}_{\mathbf{d}}}J_{\mathbf{d}}^{a\beta_a'}(w)=-1$. Hence, (40) is satisfied and the corresponding functions $J_{\mathbf{u}}^{\bar{a}\beta_{\bar{a}}}$ on $[0,\hat{w}_{\mathbf{u}}]$ and $J_{\mathbf{d}}^{\bar{a}\beta_{\bar{a}}}$ on $[0,\hat{w}_{\mathbf{d}}]$ are strictly concave. Further, the derivatives of $J_{\mathbf{u}}^{a\beta_a}$ and $J_{\mathbf{d}}^{a\beta_a}$ are greater than or equal to $-1$.

Finally, following (99), (101), (102) and the concavity of $J_{\mathbf{d}}^{a\beta_a}$ and $J_{\mathbf{u}}^{a\beta_a}$, we have

$$J_{\mathbf{u}}^{a\beta_a'}(w)<J_{\mathbf{d}}^{a\beta_a'}(w-\beta_{\mathbf{u}}), \text{ for } w\in(\beta,\hat{w}_{\mathbf{u}}), \tag{110}$$

$$J_{\mathbf{u}}^{a\beta_a'}(w+\beta_{\mathbf{d}})<J_{\mathbf{d}}^{a\beta_a'}(w), \text{ for } w\in[0,\bar{w}_{\mathbf{d}})\setminus\{\beta_{\bar{a}}-\beta_{\mathbf{d}}\},$$

$$J_{\mathbf{u}}^{a\beta_a'}(\beta_{\bar{a}+}),J_{\mathbf{u}}^{a\beta_a'}(\beta_{\bar{a}-})<J_{\mathbf{d}}^{a\beta_a'}(\beta_{\bar{a}}-\beta_{\mathbf{d}}), \text{ and} \tag{111}$$

$$J_{\mathbf{d}}^{a\beta_a'}(w)=J_{\mathbf{u}}^{a\beta_a'}\left(w+\frac{rw}{\mu_{\mathbf{d}}}\right) \text{ for } w\in(\bar{w}_{\mathbf{d}},\hat{w}_{\mathbf{d}}]. \tag{112}$$

Q.E.D.

## E.4. Proof of Proposition 5

Following definition (58) and equation (59), we obtain that under contract $\Gamma_{\beta_a}^*$ in Definition 3,

$$e^{-r\tau}J(\tau)=J(0)+\int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t=\mathbf{u}}dt+c(\theta_t)dt+dL_t)+\int_0^\tau e^{-rt}\mathcal{A}_t^*, \tag{113}$$

where

$$\begin{aligned}
\mathcal{A}_t^*&=dJ(t)-rJ(t)dt+R\mathbb{1}_{\theta_t=\mathbf{u}}dt-c(\theta_t)dt-dL_t\\
&=J'(t)[rW_{t-}-\mu(\theta_t,1)H_t^*-\ell_t^*]dt-rJ(t)dt+J(t+)-J(t)+R\mathbb{1}_{\theta_t=\mathbf{u}}dt-dL_t^*-c(\theta_t)dt\\
&=\left\{J_{\mathbf{u}}'(t)(rW_{t-}+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\hat{w}_{\mathbf{u}}}-rJ_{\mathbf{u}}(W_{t-})-c_{\mathbf{u}}\right\}dt\mathbb{1}_{\theta_t=\mathbf{u}}+\left\{J_{\mathbf{d}}'(W_{t-})r(W_{t-}-\bar{w}_{\mathbf{d}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{d}}}dt-rJ_{\mathbf{d}}(W_{t-})-c_{\mathbf{d}}\right\}dt\mathbb{1}_{\theta_t=\mathbf{d}}\\
&+\left\{\left[J_{\mathbf{u}}\left(W_{t-}+\frac{rW_{t-}}{\mu_{\mathbf{d}}}\right)-J_{\mathbf{d}}(W_{t-})\right]\mathbb{1}_{W_{t-}\ge\bar{w}_{\mathbf{d}}}+[J_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}})-J_{\mathbf{d}}(W_{t-})]\mathbb{1}_{W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}})}\right.\\
&\left.+[(J_{\mathbf{u}}(\beta_a)-J_{\mathbf{d}}(W_{t-}))(1-X_t)+(J_{\mathbf{u}}(0)-J_{\mathbf{d}}(W_{t-}))X_t]\mathbb{1}_{W_{t-}<\beta_a-\beta_{\mathbf{d}}}\right\}dN_t\mathbb{1}_{\theta_t=\mathbf{d}}
\end{aligned}$$

$$
\begin{aligned}
&+ [J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})]dN_t \mathbb{1}_{\theta_t=\mathbf{u}} + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t^* \\
&= \{R + J_{\mathbf{u}}'(W_{t-})(rW_t + c_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{u}}} - rJ_{\mathbf{u}}(W_{t-})dt + \mu_{\mathbf{u}}(J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})) \\
&+ (r\bar{w}_{\mathbf{u}} + \mu_{\mathbf{u}}\beta_{\mathbf{u}} + c_{\mathbf{u}})\mathbb{1}_{W_{t-}=\bar{w}_{\mathbf{u}}}\}\mathbb{1}_{\theta_t=\mathbf{u}} - c_{\mathbf{u}}dt \\
&+ \big\{J_{\mathbf{d}}'(W_{t-})r(W_{t-}-\bar{w}_{\mathbf{d}})\mathbb{1}_{W_{t-}<\bar{w}_{\mathbf{d}}} - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}} + [\mu_{\mathbf{d}}q_t^*(J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})) + \mu_{\mathbf{d}}(1-q_t^*)(J_{\mathbf{u}}(\beta_a) - J_{\mathbf{d}}(W_{t-}))]\mathbb{1}_{W_{t-}<\beta_a-\beta_{\mathbf{d}}} \\
&+ \mu_{\mathbf{d}}[J_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}}) - J_{\mathbf{d}}(W_{t-})]\mathbb{1}_{W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}})} + \mu_{\mathbf{d}}\left[J_{\mathbf{u}}\left(W_{t-} + \frac{rW_{t-}}{\mu_{\mathbf{d}}}\right) - J_{\mathbf{d}}(W_{t-})\right]\Bigg\}\mathbb{1}_{\theta_t=\mathbf{d}}dt + \mathcal{B}_t^* \\
&= \mathcal{B}_t^*,
\end{aligned}
$$

in which the last equality follows from (32), (33), (34) and

$$
\begin{aligned}
\mathcal{B}_t^* = \{&\left[J_{\mathbf{u}}\left(W_{t-} + \frac{rW_{t-}}{\mu_{\mathbf{d}}}\right) - J_{\mathbf{d}}(W_{t-})\right]\mathbb{1}_{W_{t-}\geq\bar{w}_{\mathbf{d}}}(dN_t - \mu_{\mathbf{d}}dt) + [J_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}}) - J_{\mathbf{d}}(W_{t-})]\mathbb{1}_{W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}})}(dN_t - \mu_{\mathbf{d}}dt) \\
&+ [(J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-}))(X_t dN_t - \mu_{\mathbf{d}}q_t^* dt) + (J_{\mathbf{u}}(\beta_a) - J_{\mathbf{d}}(W_{t-}))((1-X_t)dN_t - \mu_{\mathbf{d}}(1-q_t^*)dt)]\mathbb{1}_{W_{t-}<\beta_a-\beta_{\mathbf{d}}}\}\mathbb{1}_{\theta_t=\mathbf{d}} \\
&+ [J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})](dN_t - \mu_{\mathbf{u}}dt)\mathbb{1}_{\theta_t=\mathbf{u}}.
\end{aligned}
$$

Taking the expectation on both sides of (113), we obtain

$$
J_{\theta_0}(w) = J(0) = \mathbb{E}\left[e^{-r\tau}J(\tau) + \int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c(\theta_t)dt - dL_t^*)\right] = u(\Gamma_{\beta_{\bar{a}}}^*(w), \nu^*, \theta_0),
$$

where $u(\Gamma_{\beta_{\bar{a}}}^*, \nu^*, \theta_0) = w$, and we apply the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t^* dt$ is a martingale and $J(\tau) = J_{\theta_\tau}(0) = v_\tau$. Q.E.D.

## E.5. Proof of Theorem 4

From Proposition 4, we know that $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are concave, $J_{\mathbf{d}}'(w) \geq -1$ and $J_{\mathbf{u}}'(w) \geq -1$. Given Lemma 5, we only need to show $\Phi_t \leq 0$ holds almost surely if $\nu_t = 1$. From (57), we have

$$
\Phi_t = \Phi_t^u \mathbb{1}_{\theta_t=\mathbf{u}} + \Phi_t^d \mathbb{1}_{\theta_t=\mathbf{d}},
$$

where

$$
\begin{aligned}
\Phi_t^u := &R + J_{\mathbf{u}}'(W_{t-})rW_{t-} + \mu_{\mathbf{u}}q_t[W_{t-}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] \\
&+ \mu_{\mathbf{u}}(1-q_t)[-H_t J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})] - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}},
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_t^d := &J_{\mathbf{d}}'(W_t)rW_{t-} + \mu_{\mathbf{d}}q_t[W_{t-}J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] \\
&+ \mu_{\mathbf{d}}(1-q_t)[-H_t J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(W_{t-}+H_t) - J_{\mathbf{d}}(W_{t-})] - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}}.
\end{aligned}
$$

We have $\Phi_t \leq 0$ if $\Phi_t^d \leq 0$ and $\Phi_t^u \leq 0$. First, we prove that $\Phi_t^u \leq 0$ by considering the following optimization problem,

$$
\begin{aligned}
\max_{q_t, H_t} \quad & q_t[W_{t-}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_t J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})] \\
s.t. \quad & 0 \leq q_t \leq 1, \; -q_t W_{t-} + (1-q_t)H_t \leq -\beta_{\mathbf{u}}.
\end{aligned}
$$

In the following, we verify that the optimal solution is

$$
q_t^* = 0 \quad \text{and} \quad H_t^* = -\beta_{\mathbf{u}}. \tag{114}
$$

using the KKT conditions. Define the following dual variables for the binding constraints

$$
x_{\mathbf{u}} = -(J_{\mathbf{u}}'(W_{t-}) - J_{\mathbf{d}}'(W_{t-}-\beta_{\mathbf{u}})) \geq 0,
$$

in which the inequality follows from (116), and

$$
\begin{aligned}
y_u &= (W_{t-}-\beta_{\mathbf{u}})(J_{\mathbf{u}}'(W_{t-}) - J_{\mathbf{d}}'(W_{t-}-\beta_{\mathbf{u}})) - W_{t-}J_{\mathbf{u}}'(W_{t-}) - J_{\mathbf{d}}(0) + \beta_{\mathbf{u}}J_{\mathbf{u}}'(W_{t-}) + J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) \\
&= (W_{t-}-\beta_{\mathbf{u}})\left(\frac{J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{d}}(0)}{W_{t-}-\beta_{\mathbf{u}}} - J_{\mathbf{d}}'(W_{t-}-\beta_{\mathbf{u}})\right) \geq 0,
\end{aligned}
$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-}J'_{\mathbf{u}}(W_{t-})+J_{\mathbf{d}}(0)-J_{\mathbf{u}}(W_{t-})]-[-H_t^*J'_{\mathbf{u}}(W_{t-})+J_{\mathbf{d}}(W_{t-}+H_t^*)-J_{\mathbf{u}}(W_{t-})]=-y_{\mathbf{u}}-(H_t^*+W_{t-})x_{\mathbf{u}}, \tag{115}$$

$$(1-q_t^*)(J'_{\mathbf{u}}(W_{t-})-J'_{\mathbf{d}}(W_{t-}-H_t^*))=(q_t^*-1)x_{\mathbf{u}}. \tag{116}$$

Therefore, (114) implies that

$$\Phi_t^u \le R+J'_{\mathbf{u}}(W_{t-})rW_{t-}+\mu_{\mathbf{u}}[\beta_{\mathbf{u}}J'_{\mathbf{u}}(W_{t-})+J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}})-J_{\mathbf{u}}(W_{t-})]-rJ_{\mathbf{u}}(W_{t-})-c_{\mathbf{u}}=0,$$

where the equality follows from (34).

Following similar logic, we prove that $\Phi_t^d \le 0$ by considering the following optimization problem,

$$\begin{aligned}\max_{q_t,H_t} \quad & q_t[W_{t-}J'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(0)-J_{\mathbf{d}}(W_{t-})]+(1-q_t)[-H_tJ'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(W_{t-}+H_t)-J_{\mathbf{d}}(W_{t-})],\\ s.t. \quad & 0\le q_t\le 1,\ -q_tW_{t-}+(1-q_t)H_t\ge\beta_{\mathbf{d}},\ W_t+H_t\ge\beta_{\mathbf{u}}.\end{aligned}$$

In the following, we verify that the optimal solution is

$$q_t^*=0 \text{ and } H_t^*=\frac{rW_{t-}}{\mu_{\mathbf{d}}} \quad \text{if} \quad W_{t-}\ge\bar{w}_{\mathbf{d}}, \tag{117}$$

$$q_t^*=0 \text{ and } H_t^*=\beta_{\mathbf{d}} \quad \text{if} \quad W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}}], \text{ and} \tag{118}$$

$$q_t^*=\frac{\beta_a-\beta_{\mathbf{d}}-W_{t-}}{\beta_a} \text{ and } H_t^*=-W_{t-}+\beta_a \quad \text{if} \quad W_{t-}<\beta_a-\beta_{\mathbf{d}}, \tag{119}$$

using the KKT conditions.

- If $W_{t-}\ge\bar{w}_{\mathbf{d}}$, define the following dual variable for the binding constraint

$$\begin{aligned}y_{\mathbf{d}}&=J_{\mathbf{u}}(W_{t-}+H_t^*)-J_{\mathbf{u}}(0)-(W_{t-}+H_t^*)J'_{\mathbf{d}}(W_{t-})\\&=\frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}}\left[\frac{J_{\mathbf{u}}\left(\frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}}\right)-J_{\mathbf{u}}(0)}{\frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}}}-J'_{\mathbf{u}}\left(\frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}}\right)\right]\ge 0,\end{aligned}$$

  where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(0)-J_{\mathbf{d}}(W_{t-})]-[-H_t^*J'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(W_{t-}+H_t^*)-J_{\mathbf{d}}(W_{t-})]=-y_{\mathbf{d}}, \tag{120}$$

$$(1-q_t^*)(J'_{\mathbf{d}}(W_{t-})-J'_{\mathbf{u}}(W_{t-}+H_t^*))=J'_{\mathbf{d}}(W_{t-})-J'_{\mathbf{u}}\left(\frac{(r+\mu_{\mathbf{d}})W_{t-}}{\mu_{\mathbf{d}}}\right)=0, \tag{121}$$

  where (121) follows from (112).

- If $W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}}]$, define the following dual variables for the binding constraints,

$$x_{\mathbf{d}}=J'_{\mathbf{d}}(W_{t-})-J'_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}})\ge 0,$$

  in which the inequality follows from (111), and

$$\begin{aligned}y_{\mathbf{d}}&=(W_{t-}+\beta_{\mathbf{d}})(J'_{\mathbf{d}}(W_{t-})-J'_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}}))-W_{t-}J'_{\mathbf{d}}(W_{t-})-J_{\mathbf{u}}(0)-\beta_{\mathbf{d}}J'_{\mathbf{u}}(W_{t-})+J_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}})\\&=(W_{t-}+\beta_{\mathbf{d}})\left(\frac{J_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}})-J_{\mathbf{u}}(0)}{W_{t-}+\beta_{\mathbf{d}}}-J'_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}})\right)\ge 0,\end{aligned}$$

  where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(0)-J_{\mathbf{d}}(W_{t-})]-[-H_t^*J'_{\mathbf{d}}(W_{t-})+J_{\mathbf{u}}(W_{t-}+H_t^*)-J_{\mathbf{d}}(W_{t-})]=-y_{\mathbf{d}}+(W_{t-}+H_t^*)x_{\mathbf{d}}, \tag{122}$$

$$(1-q_t^*)(J'_{\mathbf{d}}(W_{t-})-J'_{\mathbf{u}}(W_{t-}+H_t^*))=(1-q_t^*)x_{\mathbf{d}}. \tag{123}$$

- If $W_{t-}<\beta_a-\beta_{\mathbf{d}}$ and $\beta_a=\beta_{\mathbf{u}}$, define the following dual variables for the binding constraints

$$x_{\mathbf{d}}=J'_{\mathbf{d}}(W_{t-})-\frac{J_{\mathbf{u}}(\beta_{\mathbf{u}})-J_{\mathbf{u}}(0)}{\beta_{\mathbf{u}}}=J'_{\mathbf{d}}(W_{t-})-a>0$$

in which the inequality follows from (96), and

$$\alpha = (1-q_t^*)(a - J'_{\mathbf{u}}(\beta_{\mathbf{u}})) \geq 0,$$

in which the inequality follows from the definition of $\beta_a$. One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t^* J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-}+H_t^*) - J_{\mathbf{d}}(W_{t-})] = (W_{t-}+H_t^*)x_{\mathbf{d}}, \tag{124}$$

$$(1-q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-}+H_t^*)) = (1-q_t^*)x_{\mathbf{d}} + \alpha. \tag{125}$$

- If $W_{t-} < \beta_a - \beta_{\mathbf{d}}$ and $\beta_a > \beta_{\mathbf{u}}$, define the following dual variable for the binding constraint

$$x_{\mathbf{d}} = J'_{\mathbf{d}}(W_{t-}) - a > 0$$

in which the inequality follows from (96). One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t^* J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-}+H_t^*) - J_{\mathbf{d}}(W_{t-})] = (W_{t-}+H_t^*)x_{\mathbf{d}}, \tag{126}$$

$$(1-q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-}+H_t^*)) = (1-q_t^*)x_{\mathbf{d}}. \tag{127}$$

where (127) follows from $J'_{\mathbf{u}}(\beta_a) = a$.

Therefore, (117), (118) and (119) together imply that

$$\begin{aligned}
\Phi_t^d \leq & -rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}} + \mu_{\mathbf{d}}\left[J_{\mathbf{u}}\left(W_{t-} + \frac{rW_{t-}}{\mu_{\mathbf{d}}}\right) - J_{\mathbf{u}}(W_{t-})\right]\mathbb{1}_{W_{t-}\geq \bar{w}_{\mathbf{d}}} \\
& + [J'_{\mathbf{d}}(W_{t-})rW_{t-} + \mu_{\mathbf{d}}[-\beta_{\mathbf{d}}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-}+\beta_{\mathbf{d}}) - J_{\mathbf{d}}(W_{t-})]]\mathbb{1}_{W_{t-}\in[\beta_a-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}}]} \\
& + [J'_{\mathbf{d}}(W_{t-})rW_{t-} + \mu_{\mathbf{d}}q_t^*[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] \\
& + \mu_{\mathbf{d}}(1-q_t^*)[(\beta_a - W_{t-})J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(\beta_a) - J_{\mathbf{d}}(W_{t-})]\mathbb{1}_{W_{t-}<\beta_a-\beta_{\mathbf{d}}} = 0,
\end{aligned}$$

where the equality follows from (31), (32) and (33). Q.E.D.

### E.6. Proof of Proposition 6

For any $\bar{a} \geq 0$, (105) implies that $f_a(\beta_{\mathbf{u}}) \geq 0$. Therefore, the definition of $\beta_a$ implies that $\beta_{\bar{a}} = \beta_{\mathbf{u}}$. Hence, if $\beta_{\bar{a}} > \beta_{\mathbf{u}}$, then $\bar{a} < 0$ . Q.E.D.

### E.7. Proof of Theorem 5

First, it is easy to verify that $\Gamma^*_{\mathbf{d}}(w)$ is incentive compatible. Following definition 4, we obtain the following equation for the principal's value function at state $\mathbf{d}$,

$$(\mu_{\mathbf{d}} + r)J_{\mathbf{d}}(w) = r(w - \bar{w}_{\mathbf{d}})J'_{\mathbf{d}}(w) + \mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} - \mu_{\mathbf{d}}(w+\beta_{\mathbf{d}}) - c_{\mathbf{d}}, w\in[0,\bar{w}_{\mathbf{d}}] \ , \tag{128}$$

with boundary condition $J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$. By solving this differential equation, we obtain that under state $\mathbf{d}$,

$$J_{\mathbf{d}}(w) = (\underline{v}_{\mathbf{d}} - v_{\mathbf{d}})\left(1 - \frac{w}{\bar{w}_{\mathbf{d}}}\right)^{1+\frac{\mu_{\mathbf{d}}}{r}} - w + v_{\mathbf{d}} \ . \tag{129}$$

For state $\mathbf{u}$, the societal value function is a constant,

$$J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w, \tag{130}$$

Following similar logic to the one we use in the proof of proposition 2, we can show that the principal's utilities following contract $\Gamma^*_{\mathbf{d}}(w)$ are $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ in states $\mathbf{d}$ and $\mathbf{u}$, respectively. Under condition (44), $J_{\mathbf{d}}$ and $J_{\mathbf{u}}$ are concave, $J'_{\mathbf{d}}(w) \geq -1$, and $J'_{\mathbf{u}} \geq -1$. Hence, it suffices to prove that $\Phi_t \leq 0$ where $\Phi_t$ is defined in (57). To this end, we let

$$\Phi_t = \Phi_t^{\mathbf{u}}\mathbb{1}_{\theta_t=\mathbf{u}} + \Phi_t^{\mathbf{d}}\mathbb{1}_{\theta_t=\mathbf{d}} \ ,$$

where

$$\begin{aligned}
\Phi_t^{\mathbf{u}} &= R - rW_{t-} + \mu_{\mathbf{u}}[-q_tW_{t-} + (1-q_t)H_t] - r(\underline{v}_{\mathbf{u}} - W_{t-}) + \mu_{\mathbf{u}}q_t\underline{v}_{\mathbf{d}} + \mu_{\mathbf{u}}(1-q_t)J_{\mathbf{d}}(W_{t-}+H_t) - \mu_{\mathbf{u}}(\underline{v}_{\mathbf{u}} - W_{t-}) - c_{\mathbf{u}} \\
&= R - c_{\mathbf{u}} - (r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}}q_t\underline{v}_{\mathbf{d}} + \mu_{\mathbf{u}}(1-q_t)V_{\mathbf{d}}(W_{t-}+H_t)
\end{aligned}$$

$$\leq R - c_{\mathbf{u}} - (r+\mu_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \mu_{\mathbf{u}} v_d \leq 0,$$

where the first equality follows from taking $q_t = 0$ and $\underline{v}_{\mathbf{d}} \leq V_{\mathbf{d}}(W_{t-}+H_t) \leq v_{\mathbf{d}}$, and the second inequality from the opposition of (18). Therefore,

$$\Phi_t^{\mathbf{d}} = J'_{\mathbf{d}}(W_{t-})(rW_{t-} - \mu_{\mathbf{d}}[-q_t W_{t-} + (1-q_t)H_t]) - rJ_{\mathbf{d}}(W_{t-}) + \mu_{\mathbf{d}} q_t \underline{v}_{\mathbf{u}} + \mu_{\mathbf{d}}(1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t) - \mu_{\mathbf{d}} J_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}}.$$

We prove that $\Phi_t^{\mathbf{d}} \leq 0$ by considering the following optimization problem,

$$\begin{aligned} \max_{q_t, H_t} \quad & J'_{\mathbf{d}}(W_{t-})[q_t W_{t-} - (1-q_t)H_t] + q_t \underline{v}_{\mathbf{u}} + (1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t), \\ s.t. \quad & 0 \leq q_t \leq 1, \; q_t W_{t-} + (1-q_t)H_t \geq \beta_d, \end{aligned}$$

and verify that its optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = \beta_{\mathbf{d}} \;, \tag{131}$$

following the KKT conditions. Define the following dual variable for the binding constraint

$$\alpha = J'_{\mathbf{d}}(W_{t-}) + 1 \geq 0 \;,$$

in which the inequality follows from $J'_{\mathbf{d}}(W_{t-}) \geq -1$. One can verify

$$J'_{\mathbf{d}}(W_{t-})(W_{t-} + H_t^*) + W_{t-} + H_t^* = (W_{t-} + H_t^*)\alpha, \text{ and} \tag{132}$$

$$(1-q_t^*)(J'_{\mathbf{d}}(W_{t-}) + 1) = (1-q_t^*)\alpha. \tag{133}$$

Therefore, (131) implies that

$$\begin{aligned} \Phi_t^{\mathbf{d}} &\leq J'_{\mathbf{d}}(W_{t-})(rW_{t-} - \mu_{\mathbf{d}}\beta_{\mathbf{d}}) - rJ_{\mathbf{d}}(W_{t-}) + \mu_{\mathbf{d}}(\underline{v}_{\mathbf{u}} - W_{t-} - \beta_{\mathbf{d}}) - \mu_{\mathbf{d}} J_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}} \\ &= J'_{\mathbf{d}}(W_{t-}) r(W_{t-} - \bar{w}_{\mathbf{d}}) - (r+\mu_{\mathbf{d}}) J_{\mathbf{d}}(W_{t-}) + \mu_{\mathbf{d}}(\underline{v}_{\mathbf{u}} - W_{t-} - \beta_{\mathbf{d}}) - c_{\mathbf{d}} = 0, \end{aligned}$$

where the second equality follows from (128). In summary, we have $U(\Gamma^*_{\mathbf{d}}(w), \nu^*, \mathbf{d}) \geq U(\Gamma, \nu^*, \mathbf{d})$ and $\underline{v}_{\mathbf{u}} \geq U(\Gamma, \nu^*, \mathbf{u})$. Q.E.D.

### E.8. Proof of Theorem 6

The proof of this theorem follows the same logic as the proof of Theorem 3, and is omitted here.

## F. Proofs in Section 4.3

### F.1. Proof of Proposition 7

**F.1.1.** $\beta_{\mathbf{d}} \geq \beta_u$ According to Lemma 1, under any incentive compatible contract without termination, the agent's promised utility satisfies equation (PK) with $q_t = 0$, $H_t \geq \beta_{\mathbf{d}}$ if $\theta_t = \mathbf{d}$ and $H_t \leq -\beta_{\mathbf{u}}$ if $\theta_t = \mathbf{u}$. Rearranging equation (PK) and replacing $\nu$ with $\nu^*$, $q_t = 0$ and $X_t = 0$, we obtain that

$$dW_t = \{(rW_{t-} - \mu_{\mathbf{d}} H_t)dt + H_t dN_t\}\mathbb{1}_{\theta_t = \mathbf{d}} + \{(rW_{t-} - \mu_{\mathbf{u}} H_t)dt + H_t dN_t\}\mathbb{1}_{\theta_t = \mathbf{u}} - dL_t.$$

For any contract that starts at state $\mathbf{d}$ and agent's utility $W_{t-} < \bar{w}_{\mathbf{d}}$, we have $rW_{t-} - \mu_{\mathbf{d}} H_t \leq rW_{t-} - \mu_{\mathbf{d}}\beta_{\mathbf{d}} = r(W_{t-} - \bar{w}_{\mathbf{d}}) < 0$. This implies that before the machine recovers, the utility $W_t$ keeps decreasing. Therefore, starting from any promised utility below $\bar{w}_{\mathbf{d}}$ when the machine's state is $\mathbf{d}$, there is a positive probability that the promised utility decreases to 0 before the machine is repaired, which contradicts the requirement of $\tau = \infty$.

Similarly, for any contract that starts at state $\mathbf{u}$ and agent's utility $W_{t-} < \bar{w}_{\mathbf{u}}$, there is a positive probability that the agent is terminated. This is because at state $\mathbf{u}$, in order to incentivize the agent, the utility needs to drop by at least $\beta_{\mathbf{u}}$ when the machine breaks down, which implies that it is possible that the utility at state $\mathbf{d}$ is smaller than $\bar{w}_{\mathbf{u}} - \beta_{\mathbf{u}} = \bar{w}_{\mathbf{d}}$.

Furthermore, Propositions 2 and 3 imply that $J_{\mathbf{d}}(w)$ is decreasing for $w > \bar{w}_{\mathbf{d}}$ and $J_{\mathbf{u}}(w)$ is decreasing for $w > \bar{w}_{\mathbf{u}}$, and are optimal value functions starting from the agent's initial utility $w$ and with initial state $\mathbf{d}$ and $\mathbf{u}$, respectively. Therefore, the initial $w$ for the required optimal contract should be $\bar{w}_{\mathbf{d}}$ and $\bar{w}_{\mathbf{u}}$ with the initial state $\mathbf{d}$ and $\mathbf{u}$, respectively. The corresponding optimal contract is the simple contract $\bar{\Gamma}$.

**F.1.2.** $\beta_{\mathbf{d}} < \beta_u$ At state **d**, the machine should start the promised utility with $W_{t-} \geq \bar{w}_{\mathbf{d}}$, and, at state **u**, the machine should start the promised utility with $W_{t-} \geq \bar{w}_{\mathbf{u}}$.

Furthermore, at state **d**, the promised utility starts with $W_{t-} \in [\bar{w}_{\mathbf{d}}, \hat{w}_{\mathbf{d}})$. If the upward jump $-H_t > rW_{t-}/\mu_{\mathbf{d}}$, then (PK) implies that $rW_{t-} - \mu_{\mathbf{d}} H_t < 0$, and the agent is terminated with positive probability. On the other hand, if $H_t \geq -rW_{t-}/\mu_{\mathbf{d}}$, since $W_t < \hat{w}_{\mathbf{d}}$, we have $rW_{t-}/\mu_{\mathbf{d}} < \beta_{\mathbf{u}}$. If the machine recovers and then breaks down soon afterwards, then the upward jump of the promised utility is $rW_{t-}/\mu_{\mathbf{d}}$, while the downward jump is at least $\beta_{\mathbf{u}}$. Hence, in a cycle of up and down, the continuation utility can decrease by at least $\beta_{\mathbf{u}} - rW_{t-}/\mu_{\mathbf{d}}$. Therefore, after a finite number of such cycles, the promised utility at state **d** will drop below $\bar{w}_{\mathbf{d}}$. Again, the agent is then terminated, with a positive probability.

Hence, in order to ensure $\tau = \infty$, the starting promised utility at state **d** needs to be greater than $\hat{w}_{\mathbf{d}}$, and at state **u** greater than $\hat{w}_{\mathbf{u}}$. Furthermore, Propositions 4 and 5 imply that $J_{\mathbf{d}}(w)$ is decreasing for $w > \hat{w}_{\mathbf{d}}$ and $J_{\mathbf{u}}(w)$ is decreasing for $w > \hat{w}_{\mathbf{u}}$. Therefore, the initial promised utility $w$ for the required optimal contract should be $\hat{w}_{\mathbf{d}}$ and $\hat{w}_{\mathbf{u}}$ for initial states **d** and **u**, respectively. The corresponding optimal contract is the simple contract $\hat{\Gamma}$. Q.E.D.

# E-companion: Optimal One Sided Contracts

The main body of the paper studies the optimal contract when the agent is responsible for both maintaining and repairing the machine (call it "combined contract") and these contracts induce full effort from the agent before termination. Results in Section 4 indicate that for a set of given model parameters, it is fairly easy to obtain optimal incentive compatible contracts and the corresponding value functions. In this e-companion, we first provide sufficient conditions based on computed corresponding value functions, which can be used to verify if the optimal incentive compatible contracts that obtain full effort from the agent are, in fact, optimal, even if we allow shirking.

When the sufficient conditions are not satisfied, it may be preferable for the principal to hire the agent just to maintain or just to repair, and to allow the agent to shirk. In Section EC.2 and EC.3 of this e-companion, we consider two one sided contracts where the agent is only responsible for one of the two duties. A "maintenance contract" only induces the agent to exert effort when the machine is up in order to decrease the arrival rate of failures. Similarly, a "repair contract" only induces the agent to exert effort when the machine is down to increase the rate of recovery. Studying these two types of contracts is relevant because as we showed in Section 5, one of these two contracts may outperform the optimal combined contract.

As it turns out, these two contract design problems are not special cases of the model studied in the main body of the paper. To see this, consider the example of maintenance contracts. In this setting, the machine recovers with a rate of $\underline{\mu}_{\mathbf{d}}$ without the agent's effort. In the optimal combined contract, the agent's promised utility is increased by at least $\beta_{\mathbf{d}}$ when the state changes from down to up, in which $\beta_{\mathbf{d}} = c_{\mathbf{d}} / \left(\mu_{\mathbf{d}} - \underline{\mu}_{\mathbf{d}}\right)$. In the maintenance contract setting, we cannot simply set $c_{\mathbf{d}} = 0$ and $\mu_{\mathbf{d}} = \underline{\mu}_{\mathbf{d}}$, because the corresponding $\beta_{\mathbf{d}}$ would not be well defined. In fact, the principal does not need to reward the agent when the state changes from down to up. Consequently, how the promised utility should change in this case is not immediately clear.

## EC.1. Incentive Compatibility where agents are responsible for both maintenance and repair

Following the optimality condition presented in Lemma EC.4, we first obtain the following sufficient condition for optimality of maintaining incentive compatibility in the problem where agents are responsible for both maintenance and repair. Since the sufficient condition is based on the principal's value functions, it is convenient to summarize the definition of value functions under different parameter regions:

- $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, $R \geq h_d$: Principal's value function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are defined by (20)-(24) in Section 4.1.2. ($h_{\mathbf{d}}$ is defined in (14))

- $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, $R \in [g_{\mathbf{u}}, h_d)$: Principal's value function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are defined by (94)-(95) in the proof of theorem 2. ($g_{\mathbf{u}}$ is defined in (27))
- $\beta_{\mathbf{d}} \geq \beta_{\mathbf{u}}$, $R < g_{\mathbf{u}}$: Principal's value function $J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w$ and $J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w$.
- $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, $R \geq h_u$: Principal's value function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are defined by (31)-(36) in Section 4.2.2 with $\bar{a}$ defined in proposition 4. ($h_{\mathbf{u}}$ is defined in (19))
- $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, $R \in [g_{\mathbf{d}}, h_u)$: Principal's value function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ are defined by (129)-(130) in the proof of theorem 5. ($g_{\mathbf{d}}$ is defined in (43))
- $\beta_{\mathbf{d}} < \beta_{\mathbf{u}}$, $R < g_{\mathbf{d}}$: Principal's value function $J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w$ and $J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w$.

Proposition EC.1. *It is optimal to always induce full effort from the agent before contract termination if function $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ summarized above satisfy the following two conditions,*

$$\varphi_{\mathbf{d}}(w) := rJ_{\mathbf{d}}(w) + \underline{\mu}_{\mathbf{d}} J_{\mathbf{d}}(w) - rwJ'_{\mathbf{d}}(w) - \underline{\mu}_{\mathbf{d}} \max_{-h \leq w} \{-hJ'_{\mathbf{d}}(w) + J_{\mathbf{u}}(w+h)\} \geq 0, \ \textit{for } w \geq 0, \quad \text{(EC.1)}$$

*and*

$$\varphi_{\mathbf{u}}(w) := rJ_{\mathbf{u}}(w) + \bar{\mu}_{\mathbf{u}} J_{\mathbf{u}}(w) - R - rwJ'_{\mathbf{u}}(w) - \bar{\mu}_{\mathbf{u}} \max_{-h \leq w} \{-hJ'_{\mathbf{u}}(w) + J_{\mathbf{d}}(w+h)\} \geq 0, \ \textit{for } w \geq 0. \quad \text{(EC.2)}$$

It is worth noting that Proposition EC.1 is a parallel result to condition (54) in Biais et al. (2010), Proposition 8 in DeMarzo and Sannikov (2006) and Proposition 6 in Varas (2017). However, our conditions are more complex than the corresponding conditions in the literature, involving solving a single dimensional maximization problem in both (EC.1) and (EC.2). This complexity is due to the key difference between our paper and the aforementioned continuous time dynamic contracting papers: in all the other papers, the agent is only responsible for one task whereas in ours, the agent is responsible for two tasks. This induces complexity because the principal's value function will further depend on the machine's states **u** and **d**.

Specifically, imagine, for the moment, that we replace the term $J_{\mathbf{u}}(w-h)$ in (EC.1) by $J_{\mathbf{d}}(w-h)$, so that there would be only one state. (the down state) It is easy to verify that in this case, concavity of the value function $J_{\mathbf{d}}(w-h)$ implies that the optimal $h$ in this maximization problem should be 0. (The intuitive interpretation is that there is no change in the agent's promised utility associated with arrivals during the period when the agent is allowed to shirk.) Consequently, the expression $\varphi_{\mathbf{d}}(w)$ would be greatly simplified to be a monotone function, which yields a sufficient condition only involving evaluating the value function at its boundaries. In our case, however, concavity of functions $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ do not guarantee that the optimal $h$ takes value 0. (That is, in general contracts allowing shirking, the agent's promised utility still needs to include jumps as the machine changes states when the agent shirks.) This exactly explains the reason why our verification conditions are more complex than those in the aforementioned literature, and highlights the distinct feature of our set-up with two machine states.

Fortunately, the principal's value functions $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ defined in the previous sections are, in fact, quite easy to compute. Therefore, conditions (EC.1) and (EC.2) can be easily verified numerically for any model parameter settings. From Sections 4.1 and 4.2, we learn that the optimal incentive compatible contracts take three forms depending on model parameters. Specifically, the three regions can be characterized by dividing the value of revenue rate $R$ into three intervals, fixing all other model parameters. The following result indicates that if the value of $R$ belongs to the lowest interval, sufficient conditions (EC.1) and (EC.2) are guaranteed to hold. If $R$ is moderate, on the other hand, sufficient conditions (EC.1) and (EC.2) do not hold. Therefore, we only need to check conditions (EC.1) and (EC.2) if revenue $R$ is high enough.

COROLLARY EC.1. *(i) If $\beta_d \geq \beta_u$ and condition (29) holds, or, if $\beta_d < \beta_u$ and condition (45) holds, then conditions (EC.1) and (EC.2) hold.*

*(ii) If $\beta_d \geq \beta_u$ and condition (28) holds, or, if $\beta_d < \beta_u$ and condition (44) holds, then conditions (EC.1) and (EC.2) do not hold.*

Corollary EC.1(i) implies that if $R$ is in the lowest interval, then not hiring the agent is not only the optimal incentive compatible contract, but also the best strategy among all contracts. In this case the principal's value function is a linear function with slope $-1$, which allows us to easily verify conditions (EC.1) and (EC.2). In comparison, Corollary EC.1(ii) implies that if $R$ takes moderate values (in the middle interval defined in (28) or (44)), the principal may be better off allowing shirking at some point in time before terminating the contract.

Note that the intervals defined in (28) and (44) are empty when $\beta_{\mathbf{u}} = \beta_{\mathbf{d}}$. That is, the middle interval only occurs if the ratios between effort cost and repair rate and maintenance rate improvement are not balanced, or, between the two types of efforts (repairing and maintaining) one of them is more favored than the other. In this case, the optimal incentive compatible contract dictates the principal to hire the agent only if the machine starts in the favored state, and to terminate the agent as soon as the state changes. If we allow shirking instead, the principal may benefit from hiring the agent to exert effort when the machine is in the favored state, while allowing the agent to shirk when the machine is in the other state and wait for the favored state to come back. This, again, provides us the motivation to study the optimal one-sided contracts.

## EC.2. Optimal Maintenance Contract

In this section, we consider the contract design problem where the agent only has the expertise of maintenance work which means when the machine is up, he could decrease the rate that machine breaks down from $\bar{\mu}_{\mathbf{u}}$ to $\mu_{\mathbf{u}}$ and when the machine is down, agent does not work and the machine recovers with rate $\underline{\mu}_{\mathbf{d}}$. Correspondingly, we need to change the arrival rate of process $N$ in (2) as

$$\mu_m(\theta_t, \nu_t) = [\mu_{\mathbf{u}}\nu_t + \bar{\mu}_{\mathbf{u}}(1-\nu_t)]\mathbb{1}_{\theta_t=\mathbf{u}} + \underline{\mu}_{\mathbf{d}}\mathbb{1}_{\theta_t=\mathbf{d}},$$

and the effort cost rate (1) at $t$ as

$$c_m(\theta_t) = c_{\mathbf{u}}\mathbb{1}_{\theta_t=\mathbf{u}}.$$

With these new definitions, we need to change the agent's expected total utility (5) by substituting $c(\theta_t)$ with $c_m(\theta_t)$. Without the agent, the principal's total discounted future profit for states $\mathbf{u}$ and $\mathbf{d}$ are $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$, respectively where $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$ are defined in equation (4). The principal's expected total discounted profit under a contract $\Gamma_m$ and effort process $\nu = \{\nu_t\}_{\forall t\in[0,\tau]}$ such that $\nu_t = 0$ when $\theta_t = \mathbf{d}$ is still defined as (3). Denote the full effort process as $\nu_m := \{(\nu_m)_t = \mathbb{1}_{\theta_t=\mathbf{u}}\}_{\forall t\in[0,\tau]}$. A maintenance contract $\Gamma_m$ is incentive compatible if $u(\Gamma_m, \nu_m, \theta_0) \geq u(\Gamma_m, \nu, \theta_0)$ for any effort process $\nu = \{\nu_t\}_{\forall t\in[0,\tau]}$ such that $\nu_t = 0$ when $\theta_t = \mathbf{d}$. Furthermore, the following result is parallel to Lemma 1.

Lemma EC.1. *In the maintenance setting, for any contract $\Gamma_m$, there exists $\mathcal{F}_t$-predictable processes $H_t$ such that for $t \in [0, \tau)$,*

$$dW_t = \{rW_{t-} - (1-\nu_t)c_m(\theta_t) - [(1-q_t)H_t - q_tW_{t-}]\mu_m(\nu_t, \theta_t)\}dt - dL_t + [(1-X_t)H_t - X_tW_{t-}]dN_t, \tag{PKm}$$

*in which Bernoulli random variable $X_t$ takes value 1 with probability $q_t$. Furthermore, contract $\Gamma_m$ is incentive compatible if and only if*

$$-q_tW_{t-} + (1-q_t)H_t \leq -\beta_{\mathbf{u}} \quad \text{for } \theta_{t-} = \mathbf{u},\ \forall t \in [0, \tau]. \tag{EC.3}$$

*Finally, we need $-H_t \leq W_{t-}$ for all $t \geq 0$ in order to satisfy (IR).*

Similar to the combined contract, constraint (EC.3) implies that any incentive compatible maintenance contract must satisfy the condition $W_{t-} \geq \beta_{\mathbf{u}}$ when $\theta_{t-} = \mathbf{u}$.

Next, we propose a maintenance contract and prove its optimality following similar approaches in Sections 4.1 and 4.2. The general idea is that the promised utility increases at rate $rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}}$ in state $\mathbf{u}$, and drops $\beta_{\mathbf{u}}$ whenever the machine breaks down. In state $\mathbf{d}$, the promised utility stays at a constant, and takes an upward jump of $rW_{t-}/\underline{\mu}_{\mathbf{d}}$ when the machine recovers, which collects the expected interest accrued during state $\mathbf{d}$. At the end of an up state, if a downward jump brings the promised utility to below the following threshold,

$$\underline{w}_m := \frac{\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}} + r}\beta_{\mathbf{d}},$$

the upward jump at the end of the down state cannot bring it back to $\beta_{\mathbf{u}}$ anymore. Because the promised utility has to be higher than $\beta_{\mathbf{u}}$ in state $\mathbf{u}$ in order to induce full effort, if the promised utility jumps down to below $\underline{w}_m$, then the principal should randomly terminate the agent, or reset it back to $\underline{w}_m$. Similar to before, payment starts when the promised utility reaches the upper threshold

$$\bar{w}_m := \frac{\underline{\mu}_{\mathbf{d}} + r}{r}\beta_{\mathbf{d}}.$$

The exact dynamics is represented in the following definition.

Definition EC.1. The contract $\Gamma_m^*(w) = (L^*, q^*, \tau^*)$ is defined as the following.

i. The dynamics of the agent's promised utility $W_t$ follows

$$dW_t = \begin{cases} (rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})dt - \beta_{\mathbf{u}} dN_t, & \theta_t = \mathbf{u},\ \beta_{\mathbf{u}} \le W_{t-} \le \bar{w}_m \\ -X_t W_{t-} + (1 - X_t)\left(\underline{w}_m - W_{t-}\right), & \theta_t = \mathbf{d},\ W_{t-} < \underline{w}_m \\ \left(rW_{t-}/\underline{\mu}_{\mathbf{d}}\right) dN_t, & \theta_t = \mathbf{d},\ W_{t-} \ge \underline{w}_m \end{cases}, \tag{DWm}$$

from an initial promised utility $W_0 = w$.

ii. The payment process follow $dL_t^* = \left(2\underline{\mu}_{\mathbf{d}} + r\right)\beta_{\mathbf{u}} \mathbb{1}_{W_{t-} = \bar{w}_m} \mathbb{1}_{\theta_t = \mathbf{u}} dt$.

iii. The random termination probability process for $W_{t-} < \underline{w}_m$ is $q_t^* = \hat{q}(W_{t-})$, in which

$$\hat{q}(w) := 1 - w/\underline{w}_m,$$

and the termination time is $\tau^* = \min\{t : W_t = 0\}$.

Furthermore, the following set of differential equations define the principal's value functions $J_{\mathbf{d}}^m$ and $J_{\mathbf{u}}^m$.

$$(\underline{\mu}_{\mathbf{d}} + r) J_{\mathbf{d}}^m(w) = \underline{\mu}_{\mathbf{d}} J_{\mathbf{u}}^m\left(\frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} w\right), \quad w \ge \underline{w}_m, \tag{EC.4}$$

$$J_{\mathbf{d}}^m(w) = \hat{q}(w) J_{\mathbf{d}}^m(0) + (1 - \hat{q}(w)) J_{\mathbf{d}}^m\left(\underline{w}_m\right), \quad w < \underline{w}_m, \text{ and} \tag{EC.5}$$

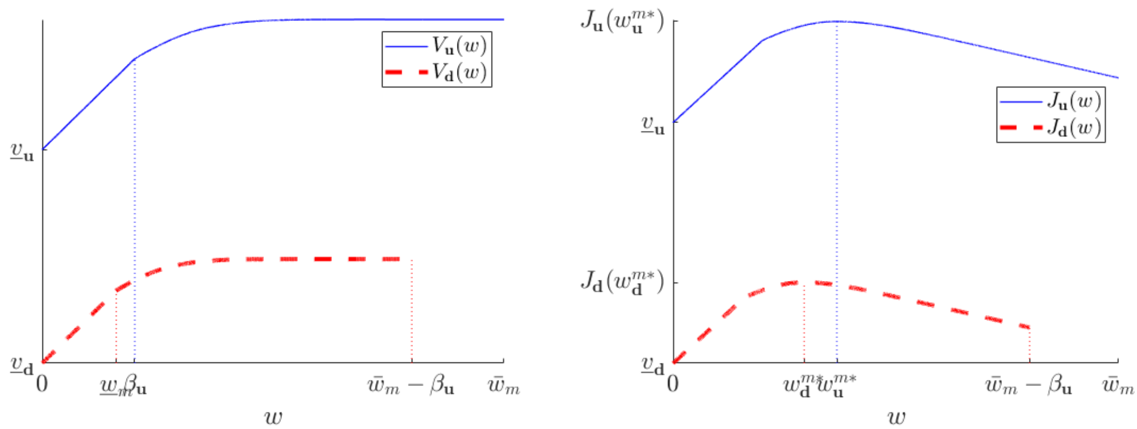
$$\begin{aligned} -c_{\mathbf{u}} + (rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}) \mathbb{1}_{w < \bar{w}_m} J_{\mathbf{u}}^{m\prime}(w) = (\mu_{\mathbf{u}} + r) J_{\mathbf{u}}^m(w) + \left(2\underline{\mu}_{\mathbf{d}} + r\right)\beta_{\mathbf{u}} \mathbb{1}_{w = \bar{w}_m} \\ - R - \mu_{\mathbf{u}} J_{\mathbf{d}}^m(w - \beta_{\mathbf{u}}),\ w \in [\beta_{\mathbf{u}}, \bar{w}_m]\ . \end{aligned} \tag{EC.6}$$

with boundary conditions

$$J_{\mathbf{d}}^m(0) = \underline{v}_{\mathbf{d}}, \qquad J_{\mathbf{d}}^m\left(\bar{w}_m - \beta_{\mathbf{u}}\right) = \frac{\underline{\mu}_{\mathbf{d}}(R - c_{\mathbf{u}})}{r\left(r + \mu_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}}\right)} - (\bar{w}_m - \beta_{\mathbf{u}}), \tag{EC.7}$$

$$J_{\mathbf{u}}^m(0) = \underline{v}_{\mathbf{u}}, \qquad J_{\mathbf{u}}^m\left(\bar{w}_m\right) = \frac{(r + \underline{\mu}_{\mathbf{d}})(R - c_{\mathbf{u}})}{r\left(r + \mu_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}}\right)} - \bar{w}_m. \tag{EC.8}$$

Similar to Proposition 1, the next proposition establishes the concavity of the principal's value functions.




(a) Societal's Value functions (b) Principal's Value functions

**Figure EC.1** Value functions with $\mu_{\mathbf{u}} = 5, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 2, \Delta\mu_{\mathbf{d}} = 1, c_{\mathbf{u}} = 0.1, c_{\mathbf{d}} = 1.3, r = 0.5, R = 10$.

Proposition EC.2. *The system of differential equations* (EC.4)-(EC.6) *with boundary conditions* (EC.7) *and* (EC.8) *has a unique solution: the pair of functions, $J_{\mathbf{u}}^m(w)$ on $[0, \bar{w}_m]$, and $J_{\mathbf{d}}^m(w)$ on $[0, \bar{w}_m - \beta_{\mathbf{u}}]$, both of which are strictly concave and $J_{\mathbf{u}}^{m'}(w) \geq -1$, $J_{\mathbf{d}}^{m'}(w) \geq -1$.*

The next result shows that $J_{\mathbf{d}}^m$ and $J_{\mathbf{u}}^m$ are indeed the principal's value function if the initial promised utility is $w$ starting from states $\mathbf{d}$ and $\mathbf{u}$, respectively.

Proposition EC.3. *For promised utility $w \in [0, \bar{w}_m - \beta_{\mathbf{u}}]$, we have $U(\Gamma_m^*(w), \nu_m, \mathbf{d}) = J_{\mathbf{d}}^m(w)$. For promised utility $w \in [0, \bar{w}_m]$, we have $U(\Gamma_m^*(w), \nu_m, \mathbf{u}) = J_{\mathbf{u}}^m(w)$.*

Furthermore, we can find $w_{\mathbf{d}}^{m*}$ and $w_{\mathbf{u}}^{m*}$ as the maximizers of $J_{\mathbf{u}}^m$ and $J_{\mathbf{d}}^m$ respectively, and start the promised utility from them.

Similar to Section 4.1 and 4.2, we define the societal value functions as the summation of the principal and the agent's utilities, $V_{\mathbf{d}}^m(w) = J_{\mathbf{d}}^m(w) + w$ and $V_{\mathbf{u}}^m(w) = J_{\mathbf{u}}^m(w) + w$. Figure EC.1 provides a numerical example of societal value functions $V_{\mathbf{d}}^m$ and $V_{\mathbf{u}}^m$ and the principal's value functions $J_{\mathbf{d}}^m$ and $J_{\mathbf{u}}^m$.

From Section EC.1, we know that for the combined contract, the sufficient conditions that guarantee the optimality of the full effort contract are relatively complicated. For the maintenance contract setting, we can show that, the following simple condition is necessary and sufficient for the principal to want to hire and induce full effort from the agent,

$$R \geq \left(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\right)\beta_{\mathbf{u}} = g_{\mathbf{u}}. \tag{EC.9}$$

The next theorem shows that under condition (EC.9), functions $J_{\mathbf{d}}^m$ and $J_{\mathbf{u}}^m$ are upper bounds for the principal's utility under any maintenance contract $\Gamma_m$.

Theorem EC.1. *Under condition (EC.9), for any contract $\Gamma_m$ and any initial state $\theta \in \{\mathbf{u}, \mathbf{d}\}$ that satisfies (EC.3), we have $J_\theta\big(u(\Gamma_m^*, \nu, \theta)\big) \geq U(\Gamma_m, \nu, \theta)$, in which we extend the function $J_{\mathbf{d}}^m(w) = J_{\mathbf{d}}^m(\bar{w}_m - \beta_{\mathbf{u}}) - (w - \bar{w}_m + \beta_{\mathbf{u}})$ for $w > \bar{w}_m - \beta_{\mathbf{u}}$ and $J_{\mathbf{u}}^m(w) = J_{\mathbf{u}}^m(\bar{w}_m) - (w - \bar{w}_m)$ for $w > \bar{w}_m$.*

*We have $U\big(\Gamma_m^*(w_\theta^{m*}), \nu_m, \theta\big) \geq U(\Gamma_m, \nu, \theta)$ for any contract $\Gamma_m$ and state $\theta$. That is, the optimal contract is $\Gamma_m^*(w_\theta^{m*})$ and the machine starts from state $\theta \in \{\mathbf{u}, \mathbf{d}\}$.*

It is worth noting that Theorem EC.1 shows that contract $\Gamma_m^*$ defined in Definition EC.1 is optimal among any maintenance contract $\Gamma_m$. This result is stronger than Theorem 1 and 4, which only show that contracts $\Gamma_1^*$ and $\Gamma_{\hat{\beta}}^*$ in Sections 4.1.1 and 4.2.1 are optimal among incentive compatible contracts.

The next proposition shows that if condition (EC.9) is violated, then the principal is better off not hiring the agent, even if we take contracts that allow shirking into consideration.

Proposition EC.4. *Assuming condition (EC.9) does not hold, that is,*

$$R < \Big(r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}\Big)\,\beta_{\mathbf{u}} = g_{\mathbf{u}}. \tag{EC.10}$$

*We have $\underline{v}_\theta \geq U(\Gamma_m, \nu_m, \theta)$ for any maintenance contract $\Gamma_m$ and state $\theta \in \{\mathbf{d}, \mathbf{u}\}$.*

Figure EC.2 depicts two sample trajectories of the agent's promised utility according to $\Gamma_m^*(w_{\mathbf{u}}^{m*})$ where the machine starts at state $\theta_0 = \mathbf{u}$. In state u, the promised utility increases over time until the machine breaks down or the promised utility reaches $\bar{w}_m$. According to the solid curve in Figure EC.2, the machine changes states at times $t_1$, $t_2$, $t_3$, $t_4$, and $t_5$. Between $[0, t_1]$, the promised utility increases over time while the agent is maintaining the machine. At time $t_1$, the machine breaks down and the promised utility drops by $\beta_{\mathbf{u}}$. Once the machine is in state $\mathbf{d}$, the agent does not need to work, and the promised utility remains a constant, until the machine recovers at time $t_2$. Whenever the machine recovers at time $t$, the utility $W_{t-}$ takes an upward jump of $\dfrac{rW_{t-}}{\underline{\mu}_{\mathbf{d}}}$. This upward jump happens at time $t_2$ following the solid curve. After $t_2$, the promised utility increases again while the agent maintains the machine, until time $\hat{t}_3$ when the promised utility reaches $\bar{w}_m$. At this point, the flow payment starts. After time $t_3$, the agent's promised utility is jumping back and forth between $\bar{w}_m$ when the machine is up and $\bar{w}_m - \beta_{\mathbf{u}}$ when the machine is down.

Now we focus on the other sample trajectory in Figure EC.2, the dotted curve. The machine is in state $\mathbf{u}$ during time intervals $[0, \tilde{t}_1]$, $[\tilde{t}_2, \tau]$ and in state $\mathbf{d}$ during $[\tilde{t}_1, \tilde{t}_2]$. The promised utility increases in state $\mathbf{u}$ and stays at a constant in state $\mathbf{d}$. Right after the machine breaks down at time $\tau$, the promised utility jumps to below $\underline{w}_m$. Consequently, even an upward jump of $\dfrac{rW_{t-}}{\underline{\mu}_{\mathbf{d}}}$ cannot raise the promised utility to above $\beta_{\mathbf{u}}$. Therefore, at time $\tilde{t}_3$ the agent is terminated with probability $\hat{q}(W_{\tilde{t}_3-})$. On the other hand, with probability $1 - \hat{q}(W_{\tilde{t}_3-})$, the agent's promised utility is reset to $\beta_{\mathbf{u}}$ (the "*" in the figure) and continues increasing.

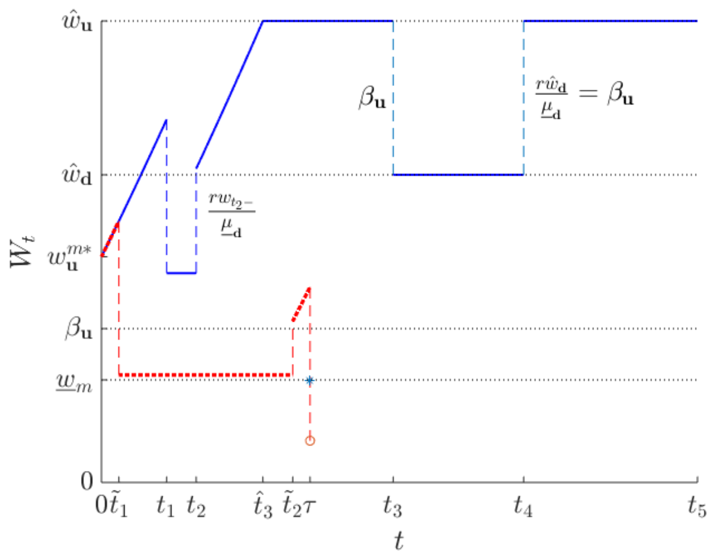



**Figure EC.2** Two sample trajectories of promised utility with model parameters $\mu_{\mathbf{u}} = 5, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 2, \Delta\mu_{\mathbf{d}} = 1, c_{\mathbf{u}} = 0.1, c_{\mathbf{d}} = 1.3, r = 0.5, R = 10$. The policy starts from $w_{\mathbf{u}}^{m*} = 0.146$. The solid curve represents a sample trajectory which brings the agent to the point of never to be terminated. The dotted curve represents another sample trajectory in which the agent is terminated due to a random draw at a point when the machine breaks down.

## EC.3. Optimal Repair Contract

In this section, we consider the contract design problem where the agent only has the expertise to repair. That is, when the machine is down, the agent is able to decrease the recovery rate from $\underline{\mu}_{\mathbf{d}}$ to $\mu_{\mathbf{d}}$ with effort. When the machine is up, on the other hand, the agent does not work and the machine breaks down with rate $\bar{\mu}_{\mathbf{u}}$. Correspondingly, we need to change the the arrival rate of process $N$ in (2) as

$$\mu_r(\theta_t, \nu_t) = \bar{\mu}_{\mathbf{u}} \mathbb{1}_{\theta_t = \mathbf{u}} + [\mu_{\mathbf{d}} \nu_t + \underline{\mu}_{\mathbf{d}} (1 - \nu_t)] \mathbb{1}_{\theta_t = \mathbf{d}},$$

and the effort cost rate (1) at $t$ as

$$c_r(\theta_t) = c_{\mathbf{d}} \mathbb{1}_{\theta_t = \mathbf{d}}.$$

With these new definitions, we need to change the agent's expected total utility (5) by substituting $c(\theta_t)$ with $c_r(\theta_t)$. Without the agent, the principal's total discounted future profit for states $\mathbf{u}$ and $\mathbf{d}$ are $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$, respectively, where $\underline{v}_{\mathbf{u}}$ and $\underline{v}_{\mathbf{d}}$ are defined in (4). The principal's expected total discounted profit under a contract $\Gamma_r$ and effort process $\nu = \{\nu_t\}_{\forall t \in [0,\tau]}$ such that $\nu_t = 0$ when $\theta_t = \mathbf{u}$ is still defined as (3). Denote the full effort process as $\nu_r := \{(\nu_r)_t = \mathbb{1}_{\theta_t = \mathbf{d}}\}_{\forall t \in [0,\tau]}$. A contract $\Gamma_r$ is incentive compatible if $u(\Gamma_r, \nu_r, \theta_0) \geq u(\Gamma_r, \nu, \theta)$ for any effort process $\nu = \{\nu_t\}_{\forall t \in [0,\tau]}$ such that $\nu_t = 0$ when $\theta_t = \mathbf{u}$. Again, the following result is parallel to Lemma 1.

LEMMA EC.2. *In a repair setting, for any contract $\Gamma_r$, there exists $\mathcal{F}_t$-predictable processes $H_t$ such that*

$$dW_t = \{rW_{t-} - (1-\nu_t)c_r(\theta_t) - [(1-q_t)H_t - q_tW_{t-}]\mu_r(\nu_t,\theta_t)\}dt - dL_t + [(1-X_t)H_t - X_tW_{t-}]dN_t, t\in[0,\tau) \tag{PKr}$$

*in which Bernoulli random variable $X_t$ takes value 1 with probability $q_t$. Furthermore, contract $\Gamma_r$ is incentive compatible if and only if*

$$-q_tW_{t-} + (1-q_t)H_t \geq \beta_{\mathbf{d}} \quad \text{for } \theta_t = \mathbf{d},\ \forall t\in[0,\tau]. \tag{EC.11}$$

*Finally, we need $-H_t \leq W_{t-}$ for all $t \geq 0$ in order to satisfy (IR).*

In the following, we directly propose a repair contract and prove the optimality following the similar approach in Section 4.1 and 4.2.

DEFINITION EC.2. The contract $\Gamma_r^*(w) = (L^*, q^*, \tau^*)$ is defined as

i. The dynamics of the agent's promised utility $W_t$, follows

$$dW_t = \left[r(W_{t-} - \bar{w}_{\mathbf{d}})dt + \min\left\{\frac{\bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}}+r}\bar{w}_{\mathbf{d}} - W_{t-}, \beta_{\mathbf{d}}\right\}dN_t\right]\mathbb{1}_{\theta_t=\mathbf{d}} + \frac{rW_{t-}}{\bar{\mu}_{\mathbf{u}}}dN_t\mathbb{1}_{\theta_t=\mathbf{u}} \tag{DWr}$$

from an initial promised utility $W_0 = w$.

ii. The payment to the agent follows $dL_t^* = (W_{t-} + \beta_{\mathbf{d}} - \bar{\mu}_{\mathbf{u}}/(\bar{\mu}_{\mathbf{u}}+r)\bar{w}_{\mathbf{d}})^+ dN_t\mathbb{1}_{\theta_t=\mathbf{d}}$.

iii. The random termination probability $q_t^* = 0$ and the termination time $\tau^* = \min\{t : W_t = 0\}$.

Furthermore, the principal's value functions are determined by the following set of differential equations

$$(\mu_{\mathbf{d}}+r)J_{\mathbf{d}}^r(w) = -c_{\mathbf{d}} + r(w-\bar{w}_{\mathbf{d}})J_{\mathbf{d}}^{r'}(w) + \mu_{\mathbf{d}}J_{\mathbf{u}}^r\left(\min\left\{w+\beta_{\mathbf{d}}, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right\}\right) - \mu_{\mathbf{d}}\left(w+\beta_{\mathbf{d}} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right)^+, \tag{EC.12}$$

$$(\bar{\mu}_{\mathbf{u}}+r)J_{\mathbf{u}}^r(w) = R + \bar{\mu}_{\mathbf{u}}J_{\mathbf{d}}^r\left(\frac{r+\bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}}}w\right), \tag{EC.13}$$

with boundary conditions

$$J_{\mathbf{d}}^r(0) = \underline{v}_d, \qquad J_{\mathbf{d}}^r(\bar{w}_{\mathbf{d}}) = \frac{\mu_{\mathbf{d}}R - (r+\bar{\mu}_{\mathbf{u}})c_{\mathbf{d}}}{r(r+\bar{\mu}_{\mathbf{u}}+\mu_{\mathbf{d}})} - \bar{w}_{\mathbf{d}}, \tag{EC.14}$$

$$J_{\mathbf{u}}^r(0) = \underline{v}_u, \qquad J_{\mathbf{u}}^r\left(\frac{\bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}}+r}\bar{w}_{\mathbf{d}}\right) = \frac{(r+\mu_{\mathbf{d}})R - \bar{\mu}_{\mathbf{u}}c_{\mathbf{d}}}{r(r+\bar{\mu}_{\mathbf{u}}+\mu_{\mathbf{d}})} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}. \tag{EC.15}$$

Similar to Proposition 1, the next Proposition establishes the concavity of the principal's value functions.

PROPOSITION EC.5. *The system of differential equations (EC.12) and (EC.13) with boundary conditions (EC.14) and (EC.15) has a unique solution: the pair of functions, $J_{\mathbf{u}}^r(w)$ on $\left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right]$, and $J_{\mathbf{d}}^r(w)$ on $[0, \bar{w}_{\mathbf{d}}]$, both of which are strictly concave and $J_{\mathbf{u}}^{r'}(w) \geq -1$, $J_{\mathbf{d}}^{r'}(w) \geq -1$.*

The next result shows that $J^r_{\mathbf{d}}(w)$ and $J^r_{\mathbf{u}}(w)$ are indeed the principal's value function if the initial promised utility is $w$ starting from states $\mathbf{d}$ and $\mathbf{u}$, respectively.

Proposition EC.6. *For promised utility* $w \in [0, \bar{w}_{\mathbf{d}}]$, *we have* $U(\Gamma^*_r, \nu_r, \theta) = J^r_{\mathbf{d}}(w)$. *For promised utility* $w \in \left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right]$, *we have* $U(\Gamma^*_r(w), \nu_r, \theta) = J^r_{\mathbf{u}}(w)$.

Furthermore, we can find $w^{r*}_{\mathbf{d}}$ and $w^{r*}_{\mathbf{u}}$ as the maximizers of $J^r_{\mathbf{d}}(w)$ and $J^r_{\mathbf{u}}(w)$ respectively, and start the promised utility from them.

Similar to Section 4.1 and 4.2, we define the societal value functions as the summation of the principal and the agent's utilities, $V^r_{\mathbf{d}}(w) = J^r_{\mathbf{d}}(w) + w$ and $V^r_{\mathbf{u}}(w) = J^r_{\mathbf{u}}(w) + w$. Figure EC.3 provides a numerical example of societal value functions $V^r_{\mathbf{d}}$ and $V^r_{\mathbf{u}}$ and the principal's value functions $J^r_{\mathbf{d}}$ and $J^r_{\mathbf{u}}$.

From Section EC.1, we know that for the combined contract, the sufficient conditions that guarantee the optimality of the full effort contract are relatively complicated. For the repair contract setting, we can show that, the following simple condition is necessary and sufficient for the principal to want to hire and induce full effort from the agent,

$$R \geq (r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})\beta_{\mathbf{d}} = g_{\mathbf{d}} \tag{EC.16}$$

The next theorem shows that under condition (EC.16), functions $J^r_{\mathbf{d}}$ and $J^r_{\mathbf{u}}$ are upper bounds for the principal's utility under any (not necessarily incentive compatible) contract $\Gamma_r$.

Theorem EC.2. *Under condition* (EC.16), *for any repair contract* $\Gamma_r$ *and any initial state* $\theta \in \{\mathbf{u}, \mathbf{d}\}$ *that satisfies (EC.11), we have* $J_\theta\big(u(\Gamma^*_r, \nu, \theta)\big) \geq U(\Gamma_r, \nu, \theta)$, *in which we extend the function* $J_{\mathbf{d}}(w) = J_{\mathbf{d}}(\bar{w}_{\mathbf{d}}) - (w - \bar{w}_{\mathbf{d}})$ *for* $w > \bar{w}_{\mathbf{d}}$ *and* $J_{\mathbf{u}}(w) = J_{\mathbf{u}}\left(\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right) - \left(w - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right)$ *for* $w > \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}$.

*We have* $U\big(\Gamma^*_r(w^{r*}_\theta), \nu_r, \theta\big) \geq U(\Gamma_r, \nu, \theta)$ *for any repair contract* $\Gamma_r$ *and state* $\theta$. *That is, the optimal contract is* $\Gamma^*_r(w^{r*}_\theta)$ *when the machine starts from state* $\theta \in \{\mathbf{u}, \mathbf{d}\}$.

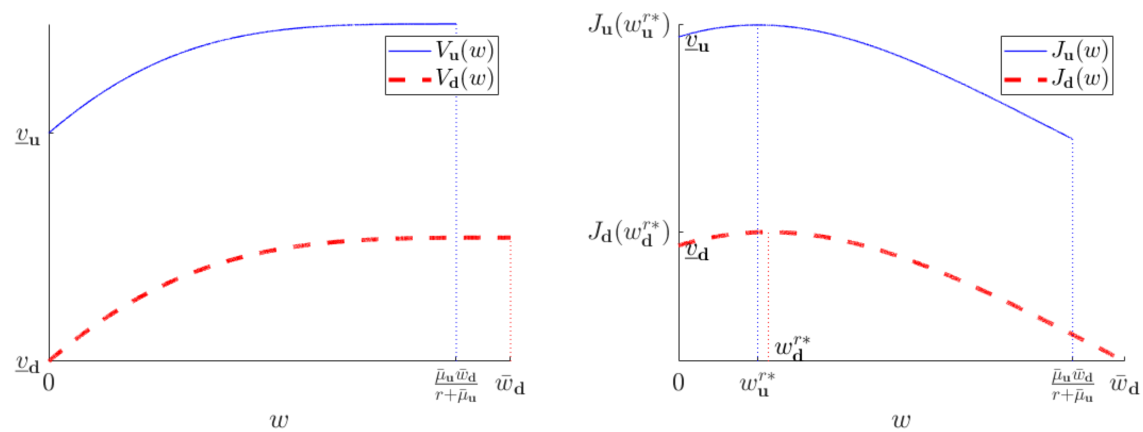
Therefore, $\Gamma^*_r$ is, in fact, the optimal contract among any repair contract $\Gamma_r$. Similar to Proposition EC.4, the next proposition shows that if condition (EC.16) is violated, then the principal is better off not hiring the agent.

Proposition EC.7. *Assuming condition* (EC.16) *does not hold, that is,*

$$R < (r + \underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}})\beta_{\mathbf{d}} = g_{\mathbf{d}}. \tag{EC.17}$$

*We have* $\underline{v}_\theta \geq U(\Gamma_r, \nu_r, \theta)$ *for any repair contract* $\Gamma_r$ *and state* $\theta \in \{\mathbf{d}, \mathbf{u}\}$.

Figure EC.4 depicts two sample trajectories of the agent's promised utility according to contract $\Gamma^*_r(w^{r*}_{\mathbf{u}})$, where the machine starts at state $\theta_0 = \mathbf{u}$. In state u, the agent does not need to work,




(a) Societal's Value functions (b) Principal's Value functions

**Figure EC.3** Value functions with $\mu_{\mathbf{u}}=5, \Delta\mu_{\mathbf{u}}=1, \mu_{\mathbf{d}}=2, \Delta\mu_{\mathbf{d}}=1, c_{\mathbf{u}}=1.3, c_{\mathbf{d}}=0.9, r=0.8, R=16$.

and the promised utility always remains a constant, until the machine breaks down, at time $t_1$ on the solid curve and at $\hat{t}_1$ on the dotted curve. Whenever the machine breaks down, the utility $W_{t-}$ takes an upward jump of level $\frac{r}{\bar{\mu}_{\mathbf{u}}}W_{t-}$, after which the machine is in state $\mathbf{d}$ and the agent starts to exert effort. In this state, the promised utility keeps decreasing until either the machine recovers, as depicted by the solid curve between time $t_1$ and $t_2$, or the promised utility decreases to zero and the contract terminates, as depicted by time $\tau$ on the dotted curve. If the machine recovers at time $t$ with $W_{t-}>0$, the utility takes an upward jump of level $\min\left\{\frac{\bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}}+r}\bar{w}_{\mathbf{d}}-W_{t-}, \beta_{\mathbf{d}}\right\}$, and the agent is paid $\left(W_{t-}+\beta_{\mathbf{d}}-\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{(\bar{\mu}_{\mathbf{u}}+r)}\right)^+$ instantaneously, as what happens at time $t_4$ or $t_6$ following the solid curve. After the first payment, the promised utility remains constant $\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}/(\bar{\mu}_{\mathbf{u}}+r)$ at state $\mathbf{u}$ and $\bar{w}_{\mathbf{d}}$ at state $\mathbf{d}$.

## EC.4. Proofs

This section collects all the proofs in this e-companion.

### EC.4.1. Proofs in Section EC.1

To discuss the optimality of the full effort incentive compatible contract under any contracts that even allow shirking, we need to consider a larger contract space in which the principal does not need to induce full effort from the agent. First, the principal's utility is revised to be

$$U(\Gamma,\nu,\theta_0)=\mathbb{E}\left[\int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt-dL_t)+e^{-r\tau}\underline{v}_\tau\middle|\theta_0\right], \tag{EC.18}$$

and the agent's utility is changed to be

$$u(\Gamma,\nu,\theta_0)=\mathbb{E}\left[\int_0^\tau e^{-rt}\left[dL_t-\nu_t c(\theta_t)dt\right]\middle|\theta_0\right]. \tag{EC.19}$$

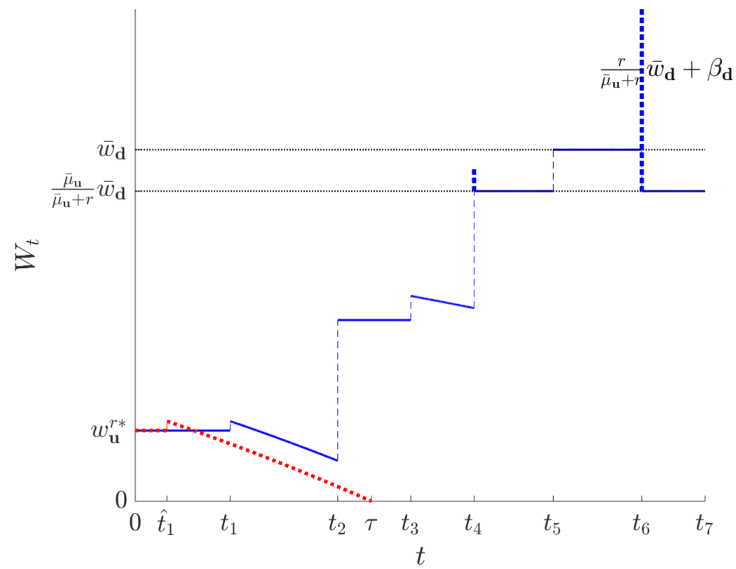



**Figure EC.4** Two sample trajectories of promised utility with model parameters $\mu_{\mathbf{u}} = 5, \Delta\mu_{\mathbf{u}} = 1, \mu_{\mathbf{d}} = 2, \Delta\mu_{\mathbf{d}} = 1, c_{\mathbf{u}} = 1.3, c_{\mathbf{d}} = 0.9, r = 0.8, R = 16$. The policy starts from $w_{\mathbf{u}}^{r*} = 0.4525$. The solid curve represents a sample trajectory which brings the agent to the point of never terminated. The dotted curve represents another sample trajectory in which the agent is terminated.

A more general version of Lemma 1 is presented in the following:

LEMMA EC.3. *For any contract $\Gamma$, there exists an $\mathcal{F}^N$-predictable process $H_t$ such that for $t \in [0, \tau]$,*

$$dW_t = \{rW_{t-} + \nu_t c(\theta_t) - [(1-q_t)H_t - q_t W_{t-}]\mu(\theta_t, \nu_t) - \ell_t\}dt + [(1-X_t)H_t - X_t W_{t-}]dN_t - I_t, \tag{EC.20}$$

*in which Bernoulli random variable $X_t$ takes value 1 with probability $q_t$. Furthermore, the necessary and sufficient condition for the effort process $\nu$ to maximize agent's utility (EC.19) given $\Gamma$ is that*

$$\begin{aligned} \nu_t = 1 \text{ if and only if } & -q_t W_{t-} + (1-q_t)H_t \le -\beta_{\mathbf{u}}, \ell_t \ge c_{\mathbf{u}}, \text{ for } \theta_t = \mathbf{u}, \text{ and} \\ & -q_t W_{t-} + (1-q_t)H_t \ge \beta_{\mathbf{d}}, \ell_t \ge c_{\mathbf{d}}, \text{ for } \theta_t = \mathbf{d} \end{aligned} \tag{EC.21}$$

*for all $t \in [0, \tau]$.*

Correspondingly, a more general optimality condition (compared to Lemma 5) is presented in the following,

LEMMA EC.4. *Suppose $J_{\mathbf{d}}(w) : [0, \infty) \to \mathbb{R}$ and $J_{\mathbf{u}}(w) : [0, \infty) \to \mathbb{R}$ are differentiable, concave, upper-bounded functions, with $J'_{\mathbf{d}}(w) \ge -1$, $J'_{\mathbf{u}}(w) \ge -1$, and $J_{\mathbf{d}}(0) = \underline{v}_{\mathbf{d}}$. Consider any contract $\Gamma$, which yields the agent's expected utility $u(\Gamma, \nu) = W_0$, followed by the continuation utility process $\{W_t\}_{t \ge 0}$ according to (PK). Define a stochastic process $\{\Phi_t\}_{t \ge 0}$ as*

$$\Phi_t := R\mathbb{1}_{\theta_t = \mathbf{u}} + J'_{\theta_t}(W_{t-})(rW_{t-} - [-q_t W_{t-} + (1-q_t)H_t]\mu(\theta_t, \nu_t)) - rJ_{\theta_t}(W_{t-})$$

$$+\,\mu(\theta_t,\nu_t)q_t[J_{\hat{\theta}_t}(0)-J_{\theta_t}(W_{t-})]+\mu(\theta_t,\nu_t)(1-q_t)[J_{\hat{\theta}_t}(W_{t-}+H_t)-J_{\theta_t}(W_{t-})]-\nu_t c(\theta_t)\ . \tag{EC.22}$$

*where* $\theta_t\in\{u,d\}$ *and* $\hat{\theta}_t=\mathbb{1}_{\theta_t=\mathbf{d}}\cdot u+\mathbb{1}_{\theta_t=\mathbf{u}}\cdot d$*. Also,* $\nu_t=0$ *if constraints (EC.21) are not satisfied at time* $t$ *and* $\nu_t=1$ *if constraints (EC.21) are satisfied at time* $t$*. If the process* $\{\Phi_t\}_{t\geq 0}$ *is non-positive almost surely, then we have* $J_\theta(u(\Gamma,\nu,\theta))\geq U(\Gamma,\nu,\theta)$*.*

**EC.4.1.1. Proof of Proposition EC.1** We have shown that $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ summarized at the beginning of Section EC.1 are upper bounds of the societal utility of any incentive compatible contracts starting from states **d** and **u**, respectively, under different conditions. Or, equivalently, they satisfy that $J'_{\mathbf{d}}(w)\geq -1$, $J'_{\mathbf{u}}(w)\geq -1$, and boundary conditions $J_{\mathbf{d}}(0)=\underline{v}_{\mathbf{d}}$ and $J_{\mathbf{u}}(0)=\underline{v}_{\mathbf{u}}$, and that $\Phi_t$ defined in (57) (or equivalently (EC.22) with $\nu_t=1$) is non-positive almost surely. Hence, to prove that they are upper bounds of any contracts, we need to further verify that if $\Phi_t$ defined in (EC.22) is non-positive almost surely when $\nu_t=0$. Hence, following (EC.22), the following conditions

$$rJ_{\mathbf{d}}(W_{t-})\geq -\underline{\mu}_{\mathbf{d}}J_{\mathbf{d}}(W_{t-})+rW_{t-}J'_{\mathbf{d}}(W_{t-})+\underline{\mu}_{\mathbf{d}}[q_tW_{t-}-(1-q_t)H_t]J'_{\mathbf{d}}(W_{t-})+\underline{\mu}_{\mathbf{d}}(q_tJ_{\mathbf{u}}(0)+(1-q_t)J_{\mathbf{u}}(W_{t-}+H_t)),\quad W_{t-}\geq 0,$$

and

$$rJ_{\mathbf{u}}(W_{t-})\geq -\bar{\mu}_{\mathbf{u}}J_{\mathbf{u}}(W_{t-})+rW_{t-}J'_{\mathbf{u}}(W_{t-})+\bar{\mu}_{\mathbf{u}}[q_tW_{t-}+(1-q_t)H_t]J'_{\mathbf{u}}(W_{t-})+\bar{\mu}_{\mathbf{u}}(q_tJ_{\mathbf{d}}(0)+(1-q_t)J_{\mathbf{d}}(W_{t-}-H_t)),\quad W_t\geq 0,$$

for any $-H_t\leq W_{t-}$ and $q_t\in[0,1]$ imply that it is optimal to induce effort from the agent before contract termination. They are further equivalent to

$$rJ_{\mathbf{d}}(w)\geq -\underline{\mu}_{\mathbf{d}}J_{\mathbf{d}}(w)+rwJ'_{\mathbf{d}}(w)+\underline{\mu}_{\mathbf{d}}\max_{q\in[0,1],-h\leq w}\{[qw-(1-q)h]J'_{\mathbf{d}}(w)+(qJ_{\mathbf{u}}(0)+(1-q)J_{\mathbf{u}}(w+h))\}\,,w\geq 0, \tag{EC.23}$$

and

$$rJ_{\mathbf{u}}(w)\geq -\bar{\mu}_{\mathbf{u}}J_{\mathbf{u}}(w)+rwJ'_{\mathbf{u}}(w)+\bar{\mu}_{\mathbf{u}}\max_{q\in[0,1],-h\leq w}\{[qw-(1-q)h]J'_{\mathbf{u}}(w)+(qJ_{\mathbf{d}}(0)+(1-q)J_{\mathbf{d}}(w+h))\}\,,w\geq 0, \tag{EC.24}$$

respectively. In the following, we first consider the optimization problem in (EC.23),

$$\begin{aligned}
&\max_{q\in[0,1],-h\leq w}\{[qw-(1-q)h]J'_{\mathbf{d}}(w)+(qJ_{\mathbf{u}}(0)+(1-q)J_{\mathbf{u}}(w+h))\}\\
=&\max_{q\in[0,1],-h\leq w}\{q[wJ'_{\mathbf{d}}(w)+J_{\mathbf{u}}(0)]+(1-q)[-hJ'_{\mathbf{d}}(w)+J_{\mathbf{u}}(w+h)]\}\\
=&\max_{q\in[0,1]}\left\{q[wJ'_{\mathbf{d}}(w)+J_{\mathbf{u}}(0)]+(1-q)\max_{-h\leq w}[-hJ'_{\mathbf{d}}(w)+J_{\mathbf{u}}(w+h)]\right\}.
\end{aligned}$$

Because $\max_{-h\le w}[-hJ'_{\mathbf{d}}(w)+J_{\mathbf{u}}(w+h)]\ge[wJ'_{\mathbf{d}}(w)+J_{\mathbf{u}}(0)]$, we know that the optimal solution to the above optimization problem should be $q^*=0$. Similarly, the optimal solution in the optimization problem in (EC.24) should also be $q^*=0$. Plugging $q^*=0$ into (EC.23) and (EC.24) yields (EC.1) and (EC.2), respectively. Q.E.D.

**EC.4.1.2. Proof of Corollary EC.1** If $\beta_{\mathbf{d}}\ge\beta_{\mathbf{u}}$ and condition (29) holds, or, if $\beta_{\mathbf{d}}<\beta_{\mathbf{u}}$ and condition (45) holds, then $J_{\mathbf{d}}(w)=\underline{v}_{\mathbf{d}}-w$ and $J_{\mathbf{u}}(w)=\underline{v}_{\mathbf{u}}-w$. In these cases, we have

$$\begin{aligned}\varphi_{\mathbf{d}}(w)&=(r+\underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}}-w)+rw-\underline{\mu}_{\mathbf{d}}[h+\underline{v}_{\mathbf{u}}-w-h]\\&=(r+\underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}}-w)+rw-\underline{\mu}_{\mathbf{d}}(-w+\underline{v}_{\mathbf{u}})\\&=(r+\underline{\mu}_{\mathbf{d}})\underline{v}_{\mathbf{d}}-\underline{\mu}_{\mathbf{d}}\underline{v}_{\mathbf{u}}=0,\end{aligned}$$

and

$$\begin{aligned}\varphi_{\mathbf{u}}(w)&=(r+\bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}}-w)-R+rw-\bar{\mu}_{\mathbf{u}}[h+\underline{v}_{\mathbf{d}}-w-h]\\&=(r+\bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}}-w)-R+rw-\bar{\mu}_{\mathbf{u}}(\underline{v}_{\mathbf{d}}-w)\\&=(r+\bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}}-R-\bar{\mu}_{\mathbf{u}}\underline{v}_{\mathbf{d}}=0.\end{aligned}$$

Hence, conditions (EC.1) and (EC.2) hold in these situations.

If $\beta_{\mathbf{d}}\ge\beta_{\mathbf{u}}$ and condition (28) holds, then $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ follow (94) and (95). Then we have

$$\begin{aligned}\varphi_{\mathbf{d}}(w)&=(r+\underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}}-w)+rw-\underline{\mu}_{\mathbf{d}}\max_{-h\le w}\{-hJ'_{\mathbf{d}}(w)+J_{\mathbf{u}}(w+h)\}\\&=(r+\underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}}-w)+rw-\underline{\mu}_{\mathbf{d}}(\beta_{\mathbf{u}}-w+v_{\mathbf{u}}-\beta_{\mathbf{u}})\\&=(r+\underline{\mu}_{\mathbf{d}})\underline{v}_{\mathbf{d}}-\underline{\mu}_{\mathbf{d}}v_{\mathbf{u}}\\&=\underline{\mu}_{\mathbf{d}}(\underline{v}_{\mathbf{u}}-v_{\mathbf{u}})<0\ ,\end{aligned}$$

where the second equality follows from $h^*=\beta_{\mathbf{u}}-w$ and the inequality follows from (28).

If $\beta_{\mathbf{d}}<\beta_{\mathbf{u}}$ and condition (44) holds, then $J_{\mathbf{d}}(w)$ and $J_{\mathbf{u}}(w)$ follow (129) and (130). Then we have

$$\begin{aligned}\varphi_{\mathbf{u}}(w)&=(r+\bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}}-w)-R+rw-\bar{\mu}_{\mathbf{u}}\max_{h\le w}\{-hJ'_{\mathbf{u}}(w)+J_{\mathbf{d}}(w+h)\}\\&=(r+\bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}}-w)-R+rw-\bar{\mu}_{\mathbf{u}}(\bar{w}_{\mathbf{d}}-w+v_{\mathbf{d}}-\bar{w}_{\mathbf{d}})\\&=(r+\bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}}-R-\bar{\mu}_{\mathbf{u}}v_{\mathbf{d}}\\&=\bar{\mu}_{\mathbf{u}}(\underline{v}_{\mathbf{d}}-v_{\mathbf{d}})<0\ ,\end{aligned}$$

where the second equality follows from $h^*=\bar{w}_{\mathbf{d}}-w$ and the inequality follows from (44). Hence, in the above two scenarios, the sufficient conditions (EC.1) and (EC.2) do not hold. Q.E.D.

### EC.4.2. Proofs in Section EC.2

**EC.4.2.1. Proof of Proposition EC.2** First, we could rearrange (EC.4)-(EC.6) as the following

$$(\underline{\mu}_{\mathbf{d}}+r)J_{\mathbf{d}}^m(w)=\underline{\mu}_{\mathbf{d}}J_{\mathbf{u}}^m\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}w\right)\ ,\qquad w\in[0,\bar{w}_m-\beta_{\mathbf{u}}]\ . \tag{EC.25}$$

and

$$-c_{\mathbf{u}}+(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_m}J_{\mathbf{u}}^{m'}(w)=(\mu_{\mathbf{u}}+r)J_{\mathbf{u}}^m(w)-R-\mu_{\mathbf{u}}J_{\mathbf{d}}^m(w-\beta_{\mathbf{u}})+\left[\left(2\underline{\mu}_{\mathbf{d}}+r\right)\beta_{\mathbf{u}}\right]\mathbb{1}_{w=\bar{w}_m}, \tag{EC.26}$$

$$w\in[\beta_{\mathbf{u}},\bar{w}_m]\ ,$$

$$J_{\mathbf{u}}^m(w)=J_{\mathbf{u}}^m(0)+\frac{J_{\mathbf{u}}^m(\beta_{\mathbf{u}})-J_{\mathbf{u}}^m(0)}{\beta_{\mathbf{u}}}w\ ,\quad w\in[0,\beta_{\mathbf{u}}]\ .$$

Then, the corresponding differential equations for $V_{\mathbf{d}}^m(w)$ and $V_{\mathbf{u}}^m(w)$ are

$$(\underline{\mu}_{\mathbf{d}}+r)V_{\mathbf{d}}^m(w)=\underline{\mu}_{\mathbf{d}}V_{\mathbf{u}}^m\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}w\right),\qquad w\in[0,\bar{w}_m-\beta_{\mathbf{u}}], \tag{EC.27}$$

$$(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_m}V_{\mathbf{u}}^{m'}(w)=(\mu_{\mathbf{u}}+r)V_{\mathbf{u}}^m(w)+c_{\mathbf{u}}-R-\mu_{\mathbf{u}}V_{\mathbf{d}}^m(w-\beta_{\mathbf{u}}),\quad w\in[\beta_{\mathbf{u}},\bar{w}_m], \tag{EC.28}$$

$$V_{\mathbf{u}}^m(w)=aw+\underline{v}_{\mathbf{u}},\quad w\in[0,\beta_{\mathbf{u}}]. \tag{EC.29}$$

From equation (EC.27), we observe that $V_{\mathbf{d}}^{m'}(w)=V_{\mathbf{u}}^{m'}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}w\right)$ and $V_{\mathbf{d}}^{m''}(w)=\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}V_{\mathbf{u}}^{m''}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}w\right)$. Hence, $V_{\mathbf{d}}^m$ is increasing and strictly concave if and only if so is $V_{\mathbf{u}}^m$. Combining (EC.27) and (EC.28), we obtain

$$(rw+\mu_{\mathbf{u}}\beta_{\mathbf{u}}+c_{\mathbf{u}})\mathbb{1}_{w<\bar{w}_m}V_{\mathbf{u}}^{m'}(w)=(\mu_{\mathbf{u}}+r)V_{\mathbf{u}}^m(w)-\frac{\mu_{\mathbf{u}}\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}+r}V_{\mathbf{u}}^m\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}(w-\beta_{\mathbf{u}})\right)-(R-c_{\mathbf{u}})\quad , \tag{EC.30}$$

$$w\in[\beta_{\mathbf{u}},\bar{w}_m].$$

We then show the result according to the following steps.

1. Show that the solution to (EC.30) is unique and twice continuously differentiable except at $w=\beta$ for any $a>0$. Call it $V_a$.
2. Argue that $V_{\mathbf{u}}^m$ is left-continuous at $\bar{w}_m$, which is $\lim_{w\to\bar{w}_m-}V_{\mathbf{u}}(w)=V_{\mathbf{u}}(\bar{w}_m)$.
3. For any $a>0$, show that $V_a$ is concave.
4. Show that $\lim_{w\to\bar{w}_m-}V_a(w)$ is increasing in $a$ for $a>0$, which implies that the boundary condition $V_a(\bar{w}_m)=\frac{(r+\underline{\mu}_{\mathbf{d}})(R-c_{\mathbf{u}})}{r(r+\mu_{\mathbf{u}}+\underline{\mu}_{\mathbf{d}})}$ uniquely determines $a$, and therefore the solution to the original differential equation. Furthermore, $\lim_{w\to\bar{w}_m-}V_{\mathbf{u}}(w)=V_{\mathbf{u}}(\bar{w}_m)$ implies that $\lim_{w\to\bar{w}_m-}V_{\mathbf{u}}'(w)=0$. Hence, the solution $V_{\mathbf{u}}$ is increasing and concave.

**Step 1.** Define $w_0 := 0$ and $w_n := \frac{\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}+r} w_{n-1} + \beta_{\mathbf{u}}$ for $n = 1,2,3....$ Then, we can verify that $\lim_{n\to\infty} w_n = \bar{w}_m$. Applying (EC.29) as the boundary condition, we show that differential equation (EC.30) has a unique solution (call it $V_a(w)$, on the interval $(\beta_{\mathbf{u}}, \bar{w}_m)$), which is continuously differentiable. In fact, differential equation (EC.30) is equivalent to a sequence of initial value problems over the intervals $[w_n, w_{n+1}), n = 1,2,....$ This sequence of initial value problems satisfy the Cauchy-Lipschitz Theorem and, therefore, bear unique solutions.

Furthermore, we could derive the expression of $V_a''(w)$ following (EC.30), as

$$V_a''(w) = \frac{\mu_{\mathbf{u}}\left[V_a'(w) - V_a'\left(\frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}}(w-\beta_{\mathbf{u}})\right)\right]}{rw + \mu_{\mathbf{u}}\beta_{\mathbf{u}}}, \text{ for } w \in (\beta_{\mathbf{u}}, \bar{w}_m). \tag{EC.31}$$

**Step 2.** The sequence of initial value problems in step 1 do not attain $\bar{w}_m$, so we first argue that $V_{\mathbf{u}}$ is left-continuous at $\bar{w}_m$. According to the contract $\Gamma_r^*$, if the contract starts with $W_0 = \bar{w}_m - \epsilon$ with sufficiently small $\epsilon > 0$, the probability that $W_t$ eventually reaches $\bar{w}_m$ approaches 1 as $\epsilon$ approaches 0. Therefore, we have $\lim_{\epsilon\to 0+} V_a(\bar{w}_m - \epsilon) = V_a(\bar{w}_m)$.

**Step 3.** Next, we show that if $a > 0$, $V_a$ is increasing and concave on $[0, \bar{w}_m)$. Equation (EC.30) implies that

$$V_{a+}'(\beta_{\mathbf{u}}) = a + \frac{c_{\mathbf{u}} - \frac{\Delta\mu_{\mathbf{u}}R}{r+\underline{\mu}_{\mathbf{d}}+\mu_{\mathbf{u}}}}{(r+\bar{\mu}_{\mathbf{u}})\beta_{\mathbf{u}}} < a,$$

where the inequality follows from (EC.9). Also, equation (EC.31) implies that $V_{a+}''(\beta_{\mathbf{u}}) < 0$. Then, we claim that $V_a''(w) < 0$ for $w \in (\beta_{\mathbf{u}}, \bar{w}_m)$. We proceed the proof by contradiction. Assuming that there exists $\breve{w} \in (\beta_{\mathbf{u}}, \bar{w}_m)$ such that $V_a''(\breve{w}) \geq 0$, because $V_a$ is twice continuously differentiable on $(\beta_{\mathbf{u}}, \bar{w}_m)$, there must exist $\tilde{w} = \max\{w \in (\beta_{\mathbf{u}}, \bar{w}_m) | V_a''(w) = 0\}$, and $V_a''(w) < 0, \forall w < \tilde{w}$. Equation (EC.31) implies that $V_a'(\tilde{w}) = V_a'\left(\frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}}(\tilde{w}-\beta_{\mathbf{u}})\right)$. However, this contradicts

$$V_a'(\tilde{w}) = V_a'\left(\frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}}(\tilde{w}-\beta_{\mathbf{u}})\right) + \int_{\frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}}(\tilde{w}-\beta_{\mathbf{u}})}^{\tilde{w}} V_a''(x)dx < V_a'\left(\frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}}(\tilde{w}-\beta_{\mathbf{u}})\right),$$

in which the inequality follows from the fact that for any $w \in (\beta_{\mathbf{u}}, \bar{w}_m)$, we must have $w > \frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}}(w-\beta_{\mathbf{u}})$. Hence, $V_a$ should be concave on the interval $[0, \bar{w}_m)$.

**Step 4.** Finally, we show that $\lim_{w\uparrow\bar{w}_m} V_a(w)$ is strictly increasing in $a$ for $a > 0$, which allows us to uniquely determine $a$ that satisfies $V_a(\bar{w}_m) = \frac{(r+\underline{\mu}_{\mathbf{d}})(R-c_{\mathbf{u}})}{r(r+\mu_{\mathbf{u}}+\underline{\mu}_{\mathbf{d}})}$. For any $0 < a_1 < a_2$, it can be seen that $V_{a_1}(w) < V_{a_2}(w)$, $V_{a_1}'(w) < V_{a_2}'(w)$, for $w \in [0, \beta_{\mathbf{u}})$ from (EC.29). We claim that $V_{a_1}' < V_{a_2}'$ for $w \in (\beta_{\mathbf{u}}, \bar{w}_m)$. Otherwise, because $V_{a_1} - V_{a_2}$ is continuously differentiable, there must exist $w' = \max\left\{w \left| V_{a_1}'(w) = V_{a_2}'(w), w \in (\beta_{\mathbf{u}}, \bar{w}_m)\right.\right\}$ and $V_{a_1}'(w) < V_{a_2}'(w)$ for $w < w'$. Equation (EC.30) implies that

$$(r+\mu_{\mathbf{u}})(V_{a_1}(w') - V_{a_2}(w')) = \frac{\underline{\mu}_{\mathbf{d}}\mu_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}}+r}\left[V_{a_1}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}(w'-\beta_{\mathbf{u}})\right) - V_{a_2}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}(w'-\beta_{\mathbf{u}})\right)\right].$$

However, this contradicts

$$0 > V_{a_1}(w') - V_{a_2}(w') - \left[ V_{a_1}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}(w'-\beta_{\mathbf{u}})\right) - V_{a_2}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}(w'-\beta_{\mathbf{u}})\right)\right] = \int_{\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}(w'-\beta_{\mathbf{u}})}^{w'} V'_{a_1}(x) - V'_{a_2}(x)dx\,.$$

Therefore, we must have $V'_{a_1}(w) - V'_{a_2}(w) < 0$ for $w \in (\beta_{\mathbf{u}}, \bar{w}_m)$, which implies that $V_{a_1}(w) - V_{a_2}(w) < 0$ for $w \in (0, \bar{w}_m)$. This implies that $\lim_{w \uparrow \bar{w}_m} V_a(w)$ is strictly increasing in $a$ for $a > 0$. Because $\lim_{a \downarrow 0} \lim_{w \uparrow \bar{w}_m} V_a(w) \le \underline{v}_{\mathbf{u}}$ and $\lim_{a \uparrow \infty} \lim_{w \uparrow \bar{w}_m} V_a(w) > \lim_{a \uparrow \infty} V_a(\beta_{\mathbf{u}}) = \infty$, there must exist a unique $\bar{a} > 0$ such that $\lim_{w \uparrow \bar{w}_m} V_{\bar{a}}(w) = \bar{V}_{\mathbf{u}}$. Further, with equation (EC.30), we are able to verify that $\lim_{w \uparrow \bar{w}_m} V'_{\mathbf{u}}(w) = 0$. Hence, the solution $V_1$ is concave and increasing on $[0, \bar{w}_m]$ and strictly concave on $(\beta_{\mathbf{u}}, \bar{w}_m)$. Q.E.D.

### EC.4.3. Proof of Proposition EC.3

Following Itô's Formula for jump processes (see, for example, Bass 2011, Theorem 17.5) and (DWm), we obtain

$$e^{-r\tau}J(\tau) = e^{-r0}J(0) + \int_0^\tau [e^{-rt}dJ(t) - re^{-rt}J(t)dt] = \quad J(0) + \int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t=\mathbf{u}}dt + c_m(\theta_t)dt + dL_t) + \int_0^\tau e^{-rt}\mathcal{A}_t. \tag{EC.32}$$

Following definition (EC.3) and equation (EC.32), we obtain, under contract $\Gamma_r^*$,

$$e^{-r\tau}J(\tau) = J(0) + \int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t=\mathbf{u}}dt + c_m(\theta_t)dt + dL_t) + \int_0^\tau e^{-rt}\mathcal{A}_t^*, \tag{EC.33}$$

where

$$\begin{aligned}
\mathcal{A}_t^* =& dJ(t) - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t - c_m(\theta_t)dt \\
=& \left\{ J'_{\mathbf{u}}(W_{t-})(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_m} - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}} \right\} dt\mathbb{1}_{\theta_t=\mathbf{u}} - rJ_{\mathbf{d}}(W_{t-})dt\mathbb{1}_{\theta_t=\mathbf{d}} \\
&+ \left[ J_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}W_{t-}\right) - J_{\mathbf{d}}(W_{t-})\right]\mathbb{1}_{W_{t-}\ge \underline{w}_m} dN_t \mathbb{1}_{\theta_t=\mathbf{d}} \\
&+ \left[ J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})\right]\mathbb{1}_{W_{t-}-\beta_{\mathbf{u}}\ge \underline{w}_m} dN_t \mathbb{1}_{\theta_t=\mathbf{u}} + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t^* \\
&+ \left[ (J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-}))(1-X_t) + (J_{\mathbf{d}}(\underline{w}_m) - J_{\mathbf{u}}(W_{t-}))X_t \right] \mathbb{1}_{W_{t-}-\beta_{\mathbf{u}}<\underline{w}_m} \\
=& \left\{ R - c_{\mathbf{u}} + J'_{\mathbf{u}}(W_{t-})(rW_{t-} + \mu_{\mathbf{u}}\beta_{\mathbf{u}})\mathbb{1}_{W_{t-}<\bar{w}_m} - rJ_{\mathbf{u}}(W_{t-}) + \mu_{\mathbf{u}}(J_{\mathbf{d}}(W_{t-} - \beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})) \right. \\
&\left. - \left[ \left(2\underline{\mu}_{\mathbf{d}} + r\right)\beta_{\mathbf{u}} \right] \mathbb{1}_{W_{t-}=\bar{w}_m} \right\} \mathbb{1}_{\theta_t=\mathbf{u}}dt \\
&+ \left\{ -rJ_{\mathbf{d}}(W_{t-})dt + \underline{\mu}_{\mathbf{d}}\left[ J_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}W_{t-}\right) - J_{\mathbf{d}}(W_{t-})\right]\right\} \mathbb{1}_{W_{t-}\ge \underline{w}_m}\mathbb{1}_{\theta_t=\mathbf{d}}dt + \mathcal{B}_t^* \\
=& \mathcal{B}_t^*,
\end{aligned}$$

in which the last equality follows from (EC.25) and (EC.26), and

$$\mathcal{B}_t^* = \left[ J_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}}W_{t-}\right) - J_{\mathbf{d}}(W_{t-})\right]\mathbb{1}_{W_{t-}\ge \underline{w}_m}(dN_t - \underline{\mu}_{\mathbf{d}}dt)\mathbb{1}_{\theta_t=\mathbf{d}}$$

$$+\Big\{\Big[(J_{\mathbf{d}}(0)-J_{\mathbf{u}}(W_{t-}))(X_t dN_t-\mu_{\mathbf{u}}\hat{q}_t(W_{t-}-\beta_{\mathbf{u}})dt)$$
$$+(J_{\mathbf{d}}(\underline{w}_m)-J_{\mathbf{u}}(W_{t-}))((1-X_t)dN_t-\mu_{\mathbf{u}}(1-\hat{q}(W_{t-}-\beta_{\mathbf{u}}))dt)\Big]\mathbb{1}_{W_{t-}-\beta_{\mathbf{u}}<\underline{w}_m}$$
$$+\big[J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}})-J_{\mathbf{u}}(W_{t-})\big](dN_t-\mu_{\mathbf{u}}dt)\mathbb{1}_{W_{t-}-\beta_{\mathbf{u}}\geq \underline{w}_m}\Big\}\mathbb{1}_{\theta_t=\mathbf{u}}.$$

Taking the expectation on both sides of (EC.33), we obtain

$$J_{\mathbf{d}}(w)=J(0)=\mathbb{E}^{\Gamma(w),\nu^*}\left[e^{-r\tau}J(\tau)+\int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt-c_m(\theta_t)dt-dL_t^*)\right],$$

where we apply the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t^* dt$ is a martingale. Q.E.D.

### EC.4.4. Proof of Theorem EC.1

First, we provide a parallel result of Lemma EC.3,

LEMMA EC.5. *For any contract $\Gamma_m$, there exists an $\mathcal{F}^N$-predictable process $H_t$ such that for $t\in[0,\tau]$,*

$$dW_t=\{rW_{t-}+\nu_t c_m(\theta_t)-[(1-q_t)H_t-q_tW_{t-}]\mu(\theta_t,\nu_t)-\ell_t\}dt+[(1-X_t)H_t-X_tW_{t-}]dN_t-I_t, \tag{EC.34}$$

*in which Bernoulli random variable $X_t$ takes value 1 with probability $q_t$. Furthermore, the necessary and sufficient condition for the effort process $\nu$ to maximize agent's utility given $\Gamma_m$ is that*

$$\nu_t=1 \text{ if and only if } -q_tW_{t-}+(1-q_t)H_t\leq -\beta_{\mathbf{u}}, \ell_t\geq c_{\mathbf{u}}, \text{ for } \theta_t=\mathbf{u} \tag{EC.35}$$

*for all $t\in[0,\tau]$.*

Correspondingly, a general optimality condition (parallel to Lemma EC.4) is presented in the following,

LEMMA EC.6. *Suppose $J_{\mathbf{d}}(w):[0,\infty)\to\mathbb{R}$ and $J_{\mathbf{u}}(w):[0,\infty)\to\mathbb{R}$ are differentiable, concave, upper-bounded functions, with $J_{\mathbf{d}}'(w)\geq -1$, $J_{\mathbf{u}}'(w)\geq -1$, and $J_{\mathbf{d}}(0)=\underline{v}_{\mathbf{d}}$. Consider any contract $\Gamma$, which yields the agent's expected utility $u(\Gamma,\nu)=W_0$, followed by the continuation utility process $\{W_t\}_{t\geq 0}$ according to (EC.34). Define a stochastic process $\{\Phi_t\}_{t\geq 0}$ as*

$$\Phi_t:=R\mathbb{1}_{\theta_t=\mathbf{u}}+J_{\theta_t}'(W_{t-})(rW_{t-}-[-q_tW_{t-}+(1-q_t)H_t]\mu(\theta_t,\nu_t))-rJ_{\theta_t}(W_{t-})$$
$$+\mu(\theta_t,\nu_t)q_t[J_{\hat{\theta}_t}(0)-J_{\theta_t}(W_{t-})]+\mu(\theta_t,\nu_t)(1-q_t)[J_{\hat{\theta}_t}(W_{t-}+H_t)-J_{\theta_t}(W_{t-})]-\nu_tc_m(\theta_t)\ . \tag{EC.36}$$

*where $\theta_t\in\{u,d\}$ and $\hat{\theta}_t=\mathbb{1}_{\theta_t=\mathbf{d}}\cdot u+\mathbb{1}_{\theta_t=\mathbf{u}}\cdot d$. Also, $\nu_t=0$ if constraints (EC.35) are not satisfied at time $t$ and $\nu_t=1$ if constraints (EC.35) are satisfied at time $t$. If the process $\{\Phi_t\}_{t\geq 0}$ is non-positive almost surely, then we have $J_\theta(u(\Gamma,\nu,\theta))\geq U(\Gamma,\nu,\theta)$.*

From Proposition EC.2, we know that $J_{\mathbf{d}}^m(w)$ and $J_{\mathbf{u}}^m(w)$ are concave and $J_{\mathbf{d}}^{m'}(w) \geq -1$, $J_{\mathbf{u}}^{m'}(w) \geq -1$. Then to prove Theorem EC.1, we only need to show that $\Phi_t \leq 0$ holds almost surely. First, if $\theta_t = \mathbf{d}$, then $\nu_t = 0$. Following (EC.36), we have

$$\begin{aligned}\Phi_t :=& J_{\mathbf{d}}'(W_{t-})(rW_{t-} + [q_t W_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{d}}(W_{t-}) \\ &+ \underline{\mu}_{\mathbf{d}} q_t [J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + \underline{\mu}_{\mathbf{d}}(1-q_t)[J_{\mathbf{u}}(W_{t-}+H_t) - J_{\mathbf{d}}(W_{t-})] \ . \end{aligned} \tag{EC.37}$$

We need to consider the following optimization problem,

$$\begin{aligned}\max_{q_t, H_t} \quad & q_t[W_{t-}J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + (1-q_t)[-H_t J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(W_{t-}+H_t) - J_{\mathbf{d}}(W_{t-})], \\ s.t. \quad & 0 \leq q_t \leq 1 (y_{\mathbf{d}}, z_{\mathbf{d}}), W_{t-} + H_t \geq \beta_{\mathbf{u}}, -H_t \leq W_{t-}.\end{aligned}$$

In the following, we verify that its optimal solution is

$$q_t^* = 0,\ H_t^* = \frac{rW_{t-}}{\underline{\mu}_{\mathbf{d}}} \quad \text{if} \quad W_{t-} \geq \underline{w}_m, \text{ and} \tag{EC.38}$$

$$q_t^* = 1 - \frac{W_{t-}(\underline{\mu}_{\mathbf{d}} + r)}{\beta_{\mathbf{u}} \underline{\mu}_{\mathbf{d}}},\ H_t^* = \beta_{\mathbf{u}} - W_{t-} \quad \text{if} \quad W_{t-} < \underline{w}_m. \tag{EC.39}$$

by the KKT conditions.

- If $W_{t-} \geq \dfrac{\underline{\mu}_{\mathbf{d}} \beta_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r}$, define the following dual variable for the binding constraint

$$\begin{aligned} y_{\mathbf{d}} &= W_{t-}J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(0) + H_t^* J_{\mathbf{d}}'(W_{t-}) - J_{\mathbf{u}}(W_{t-} + H_t^*) \\ &= \frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} W_{t-} \left[ J_{\mathbf{u}}' \left( \frac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} W_{t-} \right) - \frac{J_{\mathbf{u}}\left(\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}} W_{t-}\right) - J_{\mathbf{u}}(0)}{\frac{\underline{\mu}_{\mathbf{d}}+r}{\underline{\mu}_{\mathbf{d}}} W_{t-}} \right] \geq 0, \end{aligned}$$

  where the inequality follows from concavity. One can verify that

$$[W_{t-}J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t^* J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t^*) - J_{\mathbf{d}}(W_{t-})] = -y_{\mathbf{d}}, \tag{EC.40}$$

$$(1 - q_t^*)(J_{\mathbf{d}}'(W_{t-}) - J_{\mathbf{u}}'(W_{t-} + H_t^*)) = 0, \tag{EC.41}$$

  where (EC.41) follows from $J_{\mathbf{d}}'(W_{t-}) - J_{\mathbf{u}}'\left(\dfrac{\underline{\mu}_{\mathbf{d}} + r}{\underline{\mu}_{\mathbf{d}}} W_{t-}\right) = 0$.

- If $W_{t-} < \dfrac{\underline{\mu}_{\mathbf{d}} \beta_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r}$, one can verify that

$$[W_{t-}J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [H_t^* J_{\mathbf{d}}'(W_{t-}) + J_{\mathbf{u}}(W_{t-} - H_t^*) - J_{\mathbf{d}}(W_{t-})] = 0, \tag{EC.42}$$

$$(1 - q_t^*)(J_{\mathbf{d}}'(W_{t-}) - J_{\mathbf{u}}'(W_{t-} - H_t^*)) = 0, \tag{EC.43}$$

  where (EC.42) follows from

$$J_{\mathbf{u}}(0) - J_{\mathbf{u}}(\beta_{\mathbf{u}}) + \beta_{\mathbf{u}} J_{\mathbf{d}}'(W_{t-}) = \beta_{\mathbf{u}} \left[ J_{\mathbf{d}}'(W_{t-}) - \frac{J_{\mathbf{u}}(\beta_{\mathbf{u}}) - J_{\mathbf{u}}(0)}{\beta_{\mathbf{u}}} \right] = 0,$$

  and (EC.43) follows from $J_{\mathbf{d}}'(W_{t-}) - J_{\mathbf{u}}'(\beta_{\mathbf{u}}) = 0$.

Therefore, (EC.38) and (EC.39) implies that in (EC.37),

$$\Phi_t \le -rJ_{\mathbf{d}}(W_{t-}) + \left\{ J'_{\mathbf{d}}(W_{t-})rW_{t-} + \underline{\mu}_{\mathbf{d}} \left[ -\frac{rW_{t-}}{\underline{\mu}_{\mathbf{d}}} J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}\left( \frac{r+\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}}} W_{t-} \right) - J_{\mathbf{d}}(W_{t-}) \right] \right\} \mathbb{1}_{W_{t-}\ge \underline{w}_m}$$
$$+ \Big\{ J'_{\mathbf{d}}(W_{t-})rW_{t-} + \underline{\mu}_{\mathbf{d}} q_t^* [W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})]$$
$$+ \underline{\mu}_{\mathbf{d}}(1-q_t^*)[(W_{t-}-\beta_{\mathbf{u}})J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(\beta_{\mathbf{u}}) - J_{\mathbf{d}}(W_{t-})] \Big\} \mathbb{1}_{W_{t-}<\underline{w}_m} = 0.$$

where the last equality follows from equation (EC.25),(EC.26) and $J'_{\mathbf{u}}(W_{t-}) = -1$ for $W_{t-} \ge \bar{w}_m$ and $J'_{\mathbf{d}}(W_{t-}) = -1$ for $W_{t-} \ge \bar{w}_m - \beta_{\mathbf{u}}$.

If $\theta_t = \mathbf{u}$ and $\nu_t = 1$, then following (EC.36), we have

$$\Phi_t := R + J'_{\mathbf{u}}(W_{t-})(rW_{t-} + [q_t W_{t-} - (1-q_t)H_t]\mu_{\mathbf{u}}) - rJ_{\mathbf{u}}(W_{t-})$$
$$+ \mu_{\mathbf{u}} q_t [J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + \mu_{\mathbf{u}}(1-q_t)[J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})] \ . \tag{EC.44}$$

We need to consider the following optimization problem,

$$\max_{q_t, H_t} \quad q_t[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_t J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})],$$
$$s.t. \quad 0 \le q_t \le 1, \ -q_t W_{t-} + (1-q_t)H_t \le -\beta_{\mathbf{u}}.$$

In the following we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = -\beta_{\mathbf{u}} \tag{EC.45}$$

by the KKT conditions. Define the following dual variables for the binding constraints

$$x_{\mathbf{u}} = -(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}))$$
$$= -\left( J'_{\mathbf{d}}\left( \frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r} \right) - J'_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) \right) \ge 0,$$

where the inequality follows from the concavity of $J_{\mathbf{d}}$, and

$$y_u = (W_{t-}-\beta_{\mathbf{u}})(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}})) - W_{t-}J'_{\mathbf{u}}(W_{t-}) - J_{\mathbf{d}}(0) + \beta_{\mathbf{u}} J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}})$$
$$= (W_{t-}-\beta_{\mathbf{u}})\left(\frac{J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{d}}(0)}{W_{t-}-\beta_{\mathbf{u}}} - J'_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}})\right) \ge 0.$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] - [-H_t^* J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t^*) - J_{\mathbf{u}}(W_{t-})] = -y_{\mathbf{u}} - (H_t^* + W_{t-})x_{\mathbf{u}}, \tag{EC.46}$$
$$(1-q_t^*)(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-}+H_t^*)) = (q_t^*-1)x_{\mathbf{u}}. \tag{EC.47}$$

Therefore, (EC.45) implies that in (EC.44),

$$\Phi_t \le R + J'_{\mathbf{u}}(W_{t-})rW_{t-} + \mu_{\mathbf{u}}[\beta_{\mathbf{u}} J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}-\beta_{\mathbf{u}}) - J_{\mathbf{u}}(W_{t-})] - rJ_{\mathbf{u}}(W_{t-}) - c_{\mathbf{u}} = 0.$$

where the equality follows from (EC.6).

If $\theta_t = \mathbf{u}$ and $\nu_t = 0$, then following (EC.36), we have

$$\begin{aligned}\Phi_t :=& R + J'_{\mathbf{u}}(W_{t-})(rW_{t-} + \bar{\mu}_{\mathbf{u}}[q_t W_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{u}}(W_{t-}) \\ &+ \bar{\mu}_{\mathbf{u}} q_t [J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + \bar{\mu}_{\mathbf{u}}(1-q_t)[J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})] \; . \end{aligned} \tag{EC.48}$$

We need to consider the following optimization problem,

$$\begin{aligned}\max_{q_t, H_t} \quad & q_t[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_t J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})], \\ s.t. \quad & 0 \le q_t \le 1 (y,z), \; -H_t \le W_{t-} (x). \end{aligned}$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = -\frac{rW_{t-}}{\underline{\mu}_{\mathbf{d}} + r} \tag{EC.49}$$

following the KKT conditions. Define the following dual variable for the binding constraint

$$\begin{aligned} y &= -H_t^* J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-} + H_t^*) - W_{t-}J'_{\mathbf{u}}(W_{t-}) - J_{\mathbf{d}}(0) \\ &= J_{\mathbf{d}}\Big(\frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r}\Big) - J_{\mathbf{d}}(0) - \frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r} J'_{\mathbf{d}}\Big(\frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r}\Big) \ge 0, \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{d}}$. One can verify

$$[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] - [-H_t^* J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t^*) - J_{\mathbf{u}}(W_{t-})] = -y, \tag{EC.50}$$

$$(1-q_t^*)(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-}+H_t^*)) = 0, \tag{EC.51}$$

Therefore, (EC.49) implies that in (EC.48),

$$\begin{aligned}\Phi_t \le & R + J'_{\mathbf{u}}(W_{t-})\left(rW_{t-} + \frac{rW_{t-}}{\underline{\mu}_{\mathbf{d}} + r}\right) - rJ_{\mathbf{u}}(W_{t-}) + \bar{\mu}_{\mathbf{u}}\left[J_{\mathbf{d}}\Big(\frac{\underline{\mu}_{\mathbf{d}} W_{t-}}{\underline{\mu}_{\mathbf{d}} + r}\Big) - J_{\mathbf{u}}(W_{t-})\right] \\ =& R + \left(r + \frac{r\bar{\mu}_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r}\right) W_{t-}J'_{\mathbf{u}}(W_{t-}) - \Big(r + \bar{\mu}_{\mathbf{u}} - \frac{\bar{\mu}_{\mathbf{u}}\underline{\mu}_{\mathbf{d}}}{\underline{\mu}_{\mathbf{d}} + r}\Big) J_{\mathbf{u}}(W_{t-}) \\ \le & R + \left(r + \frac{r\bar{\mu}_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r}\right)\left(W_{t-}\frac{J_{\mathbf{u}}(W_{t-}) - J_{\mathbf{u}}(0)}{W_{t-}} - J_{\mathbf{u}}(W_{t-})\right) \\ =& R - \left(r + \frac{r\bar{\mu}_{\mathbf{u}}}{\underline{\mu}_{\mathbf{d}} + r}\right)\underline{v}_{\mathbf{u}} = 0, \end{aligned}$$

where the second inequality follows from the concavity of $J_{\mathbf{u}}$ and the last equality follows from (4).
Q.E.D.

### EC.4.5. Proof of Proposition EC.4

It suffices to show that if (EC.9) is not satisfied, then the following principal's value functions

$$J_{\mathbf{u}}(w)=\underline{v}_{\mathbf{u}}-w,\ \text{and}\ J_{\mathbf{d}}(w)=\underline{v}_{\mathbf{d}}-w.$$

satisfies the optimality condition $\Phi_t\le 0$, where $\Phi_t$ is defined in (EC.36). If $\theta_t=\mathbf{d}$, then $\nu_t=0$ and

$$\begin{aligned}\Phi_t&=-rW_{t-}-\underline{\mu}_{\mathbf{d}}[q_tW_{t-}-(1-q_t)H_t]-(r+\underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}}-W_{t-})+\underline{\mu}_{\mathbf{d}}q_t\underline{v}_{\mathbf{u}}+\underline{\mu}_{\mathbf{d}}(1-q_t)(\underline{v}_{\mathbf{u}}-W_{t-}-H_t)\\&=-(r+\underline{\mu}_{\mathbf{d}})\underline{v}_{\mathbf{d}}+\underline{\mu}_{\mathbf{d}}\underline{v}_{\mathbf{u}}=0,\end{aligned}$$

and when $\theta_t=\mathbf{u}$ and $\nu_t=1$, then

$$\begin{aligned}\Phi_t&=R-rW_{t-}-c_{\mathbf{u}}-\mu_{\mathbf{u}}[q_tW_{t-}-(1-q_t)H_t]-r(\underline{v}_{\mathbf{u}}-W_{t-})+\mu_{\mathbf{u}}q_t\underline{v}_{\mathbf{d}}+\mu_{\mathbf{u}}(1-q_t)(\underline{v}_{\mathbf{d}}-W_{t-}-H_t)-\mu_{\mathbf{u}}(\underline{v}_{\mathbf{u}}-W_{t-})\\&=R-c_{\mathbf{u}}-r\underline{v}_{\mathbf{u}}+\mu_{\mathbf{u}}\underline{v}_{\mathbf{d}}-\mu_{\mathbf{u}}\underline{v}_{\mathbf{u}}=R-c_{\mathbf{u}}-(r+\mu_{\mathbf{u}})\frac{(r+\underline{\mu}_{\mathbf{d}})R}{r(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})}+\mu_{\mathbf{u}}\frac{\underline{\mu}_{\mathbf{d}}R}{r(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})}\\&=\frac{\Delta\mu_{\mathbf{u}}R}{r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}}-c_{\mathbf{u}}=\frac{\Delta\mu_{\mathbf{u}}}{r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}}(R-(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})\beta_{\mathbf{u}})<0,\end{aligned}$$

where the inequalities follow from the fact that (EC.9) is not satisfied. If $\theta_t=\mathbf{u}$ and $\nu_t=0$, then

$$\begin{aligned}\Phi_t&=R-rW_{t-}-\bar{\mu}_{\mathbf{u}}[q_tW_{t-}-(1-q_t)H_t]-(r+\bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}}-W_{t-})+\bar{\mu}_{\mathbf{u}}q_t\underline{v}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}(1-q_t)(\underline{v}_{\mathbf{d}}-W_{t-}-H_t)\\&=R-(r+\bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}}+\bar{\mu}_{\mathbf{u}}\underline{v}_{\mathbf{d}}=0.\end{aligned}$$

Q.E.D.

## EC.5. Proofs in Section EC.3

### EC.5.1. Proof of Proposition EC.5

First, based on (EC.12) and (EC.13), we write the differential equations for $V^r_{\mathbf{d}}$ and $V^r_{\mathbf{u}}$ as the following,

$$(\mu_{\mathbf{d}}+r)V_{\mathbf{d}}(w)=-c_{\mathbf{d}}-r(\bar{w}_{\mathbf{d}}-w)V'_{\mathbf{d}}(w)+\mu_{\mathbf{d}}V_{\mathbf{u}}\left(\min\left\{w+\beta_{\mathbf{d}},\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}\right\}\right),\ \text{for}\ w\in[0,\bar{w}_{\mathbf{d}}]\ , \tag{EC.52}$$

$$(\bar{\mu}_{\mathbf{u}}+r)V_{\mathbf{u}}(w)=R+\bar{\mu}_{\mathbf{u}}V_{\mathbf{d}}\left(\frac{r+\bar{\mu}_{\mathbf{u}}}{\bar{\mu}_{\mathbf{u}}}w\right),\ \text{for}\ w\in\left[0,\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}\right]. \tag{EC.53}$$

Combining equations (EC.52) and (EC.53) yields

$$r(\bar{w}_{\mathbf{d}}-w)V'_{\mathbf{d}}(w)+(r+\mu_{\mathbf{d}})V_{\mathbf{d}}(w)=-c_{\mathbf{d}}+\mu_{\mathbf{d}}\left[\frac{R+\bar{\mu}_{\mathbf{u}}V_{\mathbf{d}}\left(\min\left\{\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w+\beta_{\mathbf{d}}),\bar{w}_{\mathbf{d}}\right\}\right)}{r+\bar{\mu}_{\mathbf{u}}}\right]. \tag{EC.54}$$

Rearrange equation (EC.54) as

$$(r+\mu_{\mathbf{d}})V_{\mathbf{d}}(w)-rV_{\mathbf{d}}'(w)(w-\bar{w}_{\mathbf{d}})+c_{\mathbf{d}}-\frac{\mu_{\mathbf{d}}R}{r+\bar{\mu}_{\mathbf{u}}}=\frac{\mu_{\mathbf{d}}\bar{\mu}_{\mathbf{u}}}{r+\bar{\mu}_{\mathbf{u}}}V_{\mathbf{d}}(\bar{w}_{\mathbf{d}}),\ \text{for } w\in\left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}-\beta_{\mathbf{d}},\infty\right),\ \text{and} \tag{EC.55}$$

$$(r+\mu_{\mathbf{d}})V_{\mathbf{d}}(w)-rV_{\mathbf{d}}'(w)(w-\bar{w}_{\mathbf{d}})+c_{\mathbf{d}}-\frac{\mu_{\mathbf{d}}R}{r+\bar{\mu}_{\mathbf{u}}}=\frac{\mu_{\mathbf{d}}\bar{\mu}_{\mathbf{u}}}{r+\bar{\mu}_{\mathbf{u}}}V_{\mathbf{d}}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w+\beta_{\mathbf{d}})\right),\ \text{for } w\in\left(0,\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}-\beta_{\mathbf{d}}\right). \tag{EC.56}$$

We then show the result according to the following steps.

1. Demonstrate the solution of (EC.55) as a parametric function $V_b$, with parameter $b$.
2. Show that the solution of (EC.56) (which we call $V_b$) is unique and twice continuously differentiable for any $b$.
3. Show that $V_b$ is convex and decreasing for $b>0$ and concave and increasing for $b<0$.
4. Show that $V_b(0)$ is increasing in $b$ for $b<0$, which implies that the boundary condition $V_b(0)=\underline{v}_{\mathbf{d}}$ uniquely determines $b$, and therefore the solution of the original differential equation.

**Step 1.** The solution to the linear ordinary differential equation (EC.55) on $\left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}},\bar{w}_{\mathbf{d}}\right)$ must have the following form, for any scalar $b$,

$$V_b(w)=\frac{\mu_{\mathbf{d}}R-(r+\bar{\mu}_{\mathbf{u}})c_{\mathbf{d}}}{r\,(r+\bar{\mu}_{\mathbf{u}}+\mu_{\mathbf{d}})}+b(\bar{w}_{\mathbf{d}}-w)^{\frac{r+\mu_{\mathbf{d}}}{r}},\quad \text{for } w\in\left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}}\right). \tag{EC.57}$$

Also define $V_b(w)=\frac{\mu_{\mathbf{d}}R-(r+\bar{\mu}_{\mathbf{u}})c_{\mathbf{d}}}{r\,(r+\bar{\mu}_{\mathbf{u}}+\mu_{\mathbf{d}})}$ for $w\in[\bar{w}_{\mathbf{d}},\infty)$, which satisfies (EC.55), so that $V_b$ is continuously differentiable on $\left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}-\beta_{\mathbf{d}},\infty\right)$.

**Step 2.** Using (EC.57) as the boundary condition, we show that differential equation (EC.56) has a unique solution, (which we call $V_b(w)$) on $\left(0,\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}-\beta_{\mathbf{d}}\right)$, which is continuously differentiable. In fact, differential equation (EC.56) is equivalent to a sequence of initial value problems. This sequence of initial value problems satisfy the Cauchy-Lipschitz Theorem and, therefore, bear unique solutions. Also, computing $V_b'\left(\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}-\beta_{\mathbf{d}}\right)$ from (EC.57), and comparing it with (EC.56), we see that $V_b$ is continuously differentiable at $\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}-\beta_{\mathbf{d}}$, and therefore on $[0,\infty)$.

Furthermore, deriving expressions for $V_b''(w)$ following (EC.56) and (EC.57), respectively, confirms that $V_b$ is twice continuously differentiable on $[0,\infty)$. In particular, (EC.56) implies that

$$V_b''(w)=\frac{\mu_{\mathbf{d}}\left[V_b'\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w+\beta_{\mathbf{d}})\right)-V_b'(w)\right]}{r(\bar{w}_{\mathbf{d}}-w)}\ . \tag{EC.58}$$

**Step 3.** Next, we argue that in order to satisfy the boundary condition $V_b(0)=\underline{v}_{\mathbf{d}}$, we must have $b<0$. Equivalently, we show that if $b>0$, $V_b$ must be convex and decreasing, which violates $V_b(0)=\underline{v}_{\mathbf{d}}<V_b(\bar{w}_{\mathbf{d}})$. In fact, if $b>0$, (EC.57) implies that $V_b$ is decreasing and convex on $\left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}-\beta_{\mathbf{d}},\bar{w}_{\mathbf{d}}\right)$, and therefore $V_b''(w)>0$ in this interval. Then, we show that $V_b''(w)>0$ for $w\in[0,\bar{w}_{\mathbf{d}})$. We prove this by contradiction. If there exists $\breve{w} \in [0, \bar{w} - \beta_{\mathbf{d}})$, such that $V_b''(\breve{w}) \le 0$, then $V_b$ being twice continuously differentiable implies that there must exist $\tilde{w} = \max\left\{ w \in \left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}\right) \middle| V_b''(w) = 0 \right\}$ such that $V_b''(w) > 0$ for all $w > \tilde{w}$. Equation (EC.58) implies that $V_b'\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(\tilde{w}+\beta_{\mathbf{d}})\right) = V_b'(\tilde{w})$. However, we this contradicts with

$$V_b'\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(\tilde{w}+\beta_{\mathbf{d}})\right) = V_b'(w) + \int_{\tilde{w}}^{\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(\tilde{w}+\beta_{\mathbf{d}})} V_b''(x)dx > V_b'(\tilde{w}) \ .$$

Therefore, we must have $V_b''(w) > 0$ and $V_b$ is decreasing on $[0, \bar{w}_{\mathbf{d}})$ if $b > 0$. If $b = 0$, $V_b(w)$ is a constant following (EC.56) and (EC.57), which also contradicts the boundary condition. Therefore we must have $b < 0$.

The same logic implies that for $b < 0$, $V_b$ must best be increasing and strictly concave on $[0, \bar{w}_{\mathbf{d}})$.

**Step 4.** Finally, we show that $V_b(0)$ is strictly increasing in $b$ for $b < 0$, which allows us to uniquely determine $b$ that satisfies $V_b(0) = \underline{v}_{\mathbf{d}}$. For any $b_1 < b_2 < 0$, it can be verified that $V_{b_1}(w) < V_{b_2}(w)$, $V_{b_1}'(w) > V_{b_2}'(w)$, for $w \in \left[\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}, \bar{w}_{\mathbf{d}}\right)$ from (EC.57). We claim that $V_{b_1}' > V_{b_2}'$ for $w \in [0, \bar{w}]$. Otherwise, because $V_{b_1} - V_{b_2}$ is continuously differentiable, there must exist $w' = \max\left\{ w \middle| V_{b_1}'(w) = V_{b_2}'(w), w \in \left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}} - \beta_{\mathbf{d}}\right) \right\}$ such that $V_{b_1}'(w) > V_{b_2}'(w)$ for $w > w'$. Equation (EC.56) implies that

$$\frac{\mu_{\mathbf{d}}\bar{\mu}_{\mathbf{u}}}{r+\bar{\mu}_{\mathbf{u}}}\left[V_{b_1}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})\right) - V_{b_2}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})\right)\right] = (r+\mu_{\mathbf{d}})(V_{b_1}(w') - V_{b_2}(w')).$$

However, this contradicts

$$\begin{aligned} 0 &> V_{b_1}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})\right) - V_{b_2}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})\right) \\ &= V_{b_1}(w') - V_{b_2}(w') + \int_0^{\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}(w'+\beta_{\mathbf{d}})-w'} V_{b_1}'(w'+x) - V_{b_2}'(w'+x)dx \ . \end{aligned}$$

Therefore, we must have $V_{b_1}'(w) - V_{b_2}'(w) > 0$ for $w \in [0, \bar{w}_{\mathbf{d}})$, which further implies that $V_{b_1}(w) - V_{b_2}(w) < 0$ for $w \in [0, \bar{w}_{\mathbf{d}})$. As a result, $V_b(0)$ is strictly increasing in $b$ for $b < 0$. Because $\lim_{b\uparrow 0} V_b(0) \le \underline{v}_{\mathbf{d}}$ and $\lim_{b\downarrow -\infty} V_b(0) > V_b(\bar{w}_1 - \beta_{\mathbf{d}}) = -\infty$ , there must exist a unique $b^* < 0$ such that $V_{b^*}'(0) = \underline{v}_{\mathbf{d}}$. Hence, the solution $V_{b^*}$ is strictly concave and increasing in $\left[0, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{r+\bar{\mu}_{\mathbf{u}}}\right]$. Q.E.D.

### EC.5.2. Proof of Proposition EC.6

Following definition (EC.2) and equation (EC.32), we obtain, under contract $\Gamma_r^*$,

$$e^{-r\tau}J(\tau) = J(0) + \int_0^\tau e^{-rt}(-R\mathbb{1}_{\theta_t=\mathbf{u}}dt + c_r(\theta_t)dt + dL_t) + \int_0^\tau e^{-rt}\mathcal{A}_t^*, \tag{EC.59}$$

where

$$\mathcal{A}_t^* = dJ(t) - rJ(t)dt + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c_r(\theta_t)dt - dL_t$$

$$
\begin{aligned}
=& -rJ_{\mathbf{u}}(W_{t-})dt\mathbb{1}_{\theta_t=\mathbf{u}} + \{J'_{\mathbf{d}}(W_{t-})r(W_{t-}-\bar{w}_{\mathbf{d}})dt - rJ_0(W_{t-})dt\}\,\mathbb{1}_{\theta_t=\mathbf{d}} + R\mathbb{1}_{\theta_t=\mathbf{u}}dt - dL_t^*\\
&+\left[J_{\mathbf{u}}\left(\min\left\{\frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}, W_{t-}+\beta_{\mathbf{d}}\right\}\right) - J_{\mathbf{d}}(W_{t-})\right]dN_t\mathbb{1}_{\theta_t=\mathbf{u}} + \left[J_{\mathbf{d}}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}\right) - J_{\mathbf{u}}(W_{t-})\right]dN_t\mathbb{1}_{\theta_t=\mathbf{u}}\\
=&\left\{R - rJ_{\mathbf{u}}(W_t) + \bar{\mu}_{\mathbf{u}}\left(J_{\mathbf{d}}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}W_{t-}\right) - J_{\mathbf{u}}(W_{t-})\right)\right\}\mathbb{1}_{\theta_t=\mathbf{u}}dt\\
&+\left\{J'_{\mathbf{d}}(W_{t-})r(W_{t-}-\bar{w}_{\mathbf{d}}) - rJ_{\mathbf{d}}(W_{t-})dt + \mu_{\mathbf{d}}\left(J_{\mathbf{u}}\left(\min\left\{W_{t-}+\beta_{\mathbf{d}}, \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right\}\right) - J_{\mathbf{d}}(W_{t-})\right)\right.\\
&\left.-\mu_{\mathbf{d}}\left(W_{t-}+\beta_{\mathbf{d}} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right)^+ - c_{\mathbf{d}}\right\}\mathbb{1}_{\theta_t=\mathbf{d}}dt + \mathcal{B}_t^*\\
=&\mathcal{B}_t^*,
\end{aligned}
$$

in which the last equality follows from (EC.12), (EC.13) and

$$
\begin{aligned}
\mathcal{B}_t^* =& \left[J_{\mathbf{u}}\left(W_{t-}+\beta_{\mathbf{d}} - \left(W_{t-}+\beta_{\mathbf{d}} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right)^+\right) - J_{\mathbf{d}}(W_{t-}) - \left(W_{t-}+\beta_{\mathbf{d}} - \frac{\bar{\mu}_{\mathbf{u}}\bar{w}_{\mathbf{d}}}{\bar{\mu}_{\mathbf{u}}+r}\right)^+\right](dN_t - \mu_{\mathbf{d}}dt)\mathbb{1}_{\theta_t=\mathbf{d}}\\
&+\left[J_{\mathbf{d}}\left(\frac{\bar{\mu}_{\mathbf{u}}+r}{\bar{\mu}_{\mathbf{u}}}W_{t-}\right) - J_{\mathbf{u}}(W_{t-})\right](dN_t - \bar{\mu}_{\mathbf{u}}dt)\mathbb{1}_{\theta_t=\mathbf{u}}.
\end{aligned}
$$

Taking the expectation on both sides of (EC.59), we immediately have

$$
J_{\mathbf{d}}(w) = J(0) = \mathbb{E}^{\Gamma(w),\nu^*}\left[e^{-r\tau}J(\tau) + \int_0^\tau e^{-rt}(R\mathbb{1}_{\theta_t=\mathbf{u}}dt - c_r(\theta_t)dt - dL_t^*)\right],
$$

where we apply the fact that $\int_0^\tau e^{-rt}\mathcal{B}_t^*dt$ is a martingale. Q.E.D.

### EC.5.3. Proof of Theorem EC.2

Again, we provide a parallel result of Lemma EC.3,

Lemma EC.7. *For any contract $\Gamma_r$, there exists an $\mathcal{F}^N$-predictable process $H_t$ such that for $t \in [0,\tau]$,*

$$
dW_t = \{rW_{t-} + \nu_t c_r(\theta_t) - [(1-q_t)H_t - q_tW_{t-}]\mu(\theta_t,\nu_t) - \ell_t\}dt + [(1-X_t)H_t - X_tW_{t-}]dN_t - I_t, \tag{EC.60}
$$

*in which Bernoulli random variable $X_t$ takes value 1 with probability $q_t$. Furthermore, the necessary and sufficient condition for the effort process $\nu$ to maximize agent's utility given $\Gamma_m$ is that*

$$
\nu_t = 1 \text{ if and only if } -q_tW_{t-} + (1-q_t)H_t \geq \beta_{\mathbf{d}}, \ell_t \geq c_{\mathbf{d}}, \text{ for } \theta_t = \mathbf{d} \tag{EC.61}
$$

*for all $t \in [0,\tau]$.*

Correspondingly, a more general optimality condition (parallel to Lemma EC.4) is presented in the following, Lemma EC.8. *Suppose $J_{\mathbf{d}}(w):[0,\infty)\to\mathbb{R}$ and $J_{\mathbf{u}}(w):[0,\infty)\to\mathbb{R}$ are differentiable, concave, upper-bounded functions, with $J'_{\mathbf{d}}(w)\geq -1$, $J'_{\mathbf{u}}(w)\geq -1$, and $J_{\mathbf{d}}(0)=\underline{v}_{\mathbf{d}}$. Consider any contract $\Gamma$, which yields the agent's expected utility $u(\Gamma,\nu)=W_0$, followed by the continuation utility process $\{W_t\}_{t\geq 0}$ according to (EC.60). Define a stochastic process $\{\Phi_t\}_{t\geq 0}$ as*

$$\begin{aligned}\Phi_t := & R\mathbb{1}_{\theta_t=\mathbf{u}} + J'_{\theta_t}(W_{t-})(rW_{t-} - [-q_tW_{t-} + (1-q_t)H_t]\mu(\theta_t,\nu_t)) - rJ_{\theta_t}(W_{t-}) \\ & + \mu(\theta_t,\nu_t)q_t[J_{\hat{\theta}_t}(0) - J_{\theta_t}(W_{t-})] + \mu(\theta_t,\nu_t)(1-q_t)[J_{\hat{\theta}_t}(W_{t-}+H_t) - J_{\theta_t}(W_{t-})] - \nu_t c_r(\theta_t)\ .\end{aligned} \tag{EC.62}$$

*where $\theta_t\in\{u,d\}$ and $\hat{\theta}_t = \mathbb{1}_{\theta_t=\mathbf{d}}\cdot u + \mathbb{1}_{\theta_t=\mathbf{u}}\cdot d$. Also, $\nu_t=0$ if constraints (EC.61) are not satisfied at time $t$ and $\nu_t=1$ if constraints (EC.61) are satisfied at time $t$. If the process $\{\Phi_t\}_{t\geq 0}$ is non-positive almost surely, then we have $J_\theta(u(\Gamma,\nu,\theta))\geq U(\Gamma,\nu,\theta)$.*

From Proposition EC.5, we know that $J^r_{\mathbf{d}}(w)$ and $J^r_{\mathbf{u}}(w)$ are concave and $J^{r'}_{\mathbf{d}}(w)\geq -1$, $J^{r'}_{\mathbf{u}}(w)\geq -1$. In order to show Theorem EC.2, we only need to show that $\Phi_t\leq 0$ holds almost surely. First, if $\theta_t=\mathbf{u}$, then $\nu_t=0$ and following (EC.62), we have

$$\begin{aligned}\Phi_t := & R + J'_{\mathbf{u}}(W_{t-})(rW_{t-} + \bar{\mu}_{\mathbf{u}}[q_tW_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{u}}(W_{t-}) \\ & + \bar{\mu}_{\mathbf{u}}q_t[J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + \bar{\mu}_{\mathbf{u}}(1-q_t)[J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_t)]\ .\end{aligned} \tag{EC.63}$$

We need to consider the following optimization problem,

$$\begin{aligned}\max_{q_t,H_t}\quad & q_t[W_tJ'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] + (1-q_t)[-H_tJ'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H_t) - J_{\mathbf{u}}(W_{t-})], \\ s.t.\quad & 0\leq q_t\leq 1, -H_t\leq W_{t-}.\end{aligned}$$

In the following, we verify that the optimal solution is

$$q^*_t = 0 \quad \text{and} \quad H^*_t = \frac{rW_{t-}}{\bar{\mu}_{\mathbf{u}}}, \tag{EC.64}$$

using the KKT conditions. Define the following dual variable for the binding constraint

$$\begin{aligned}y &= -H^*_tJ'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H^*_t) - W_{t-}J'_{\mathbf{u}}(W_{t-}) - J_{\mathbf{d}}(0) \\ &= J_{\mathbf{d}}\left(\frac{(r+\bar{\mu}_{\mathbf{u}})W_{t-}}{\bar{\mu}_{\mathbf{u}}}\right) - J_{\mathbf{d}}(0) - \frac{(r+\bar{\mu}_{\mathbf{u}})W_{t-}}{\bar{\mu}_{\mathbf{u}}}J'_{\mathbf{d}}\left(\frac{(r+\bar{\mu}_{\mathbf{u}})W_{t-}}{\bar{\mu}_{\mathbf{u}}}\right)\geq 0,\end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{d}}$. One can verify

$$[W_{t-}J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(0) - J_{\mathbf{u}}(W_{t-})] - [-H^*_tJ'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{d}}(W_{t-}+H^*_t) - J_{\mathbf{u}}(W_{t-})] = -y, \tag{EC.65}$$

$$(1-q^*_t)(J'_{\mathbf{u}}(W_{t-}) - J'_{\mathbf{d}}(W_{t-}+H^*_t)) = 0. \tag{EC.66}$$

Therefore, (EC.64) implies that in (EC.63),

$$\Phi_t := R - rJ_{\mathbf{u}}(W_t) + \bar{\mu}_{\mathbf{u}}\left[J_{\mathbf{d}}\left(W_t + \frac{rW_t}{\bar{\mu}_{\mathbf{u}}}\right) - J_{\mathbf{u}}(W_t)\right] = 0\ .$$

where the equality follows from (EC.13).

If $\theta_t = \mathbf{d}$ and $\nu_t = 1$, then following (EC.62), we have

$$\Phi_t := J'_{\mathbf{d}}(W_{t-})(rW_{t-} + \mu_{\mathbf{d}}[q_t W_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{d}}(W_{t-}) \\ + \mu_{\mathbf{d}} q_t [J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + \mu_{\mathbf{d}}(1-q_t)[J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})] \ . \tag{EC.67}$$

We need to consider the following optimization problem,

$$\begin{aligned} \max_{q_t, H_t} \quad & q_t[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + (1-q_t)[-H_t J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})], \\ s.t. \quad & 0 \le q_t \le 1, -q_t W_{t-} + (1-q_t)H_t \ge \beta_{\mathbf{d}}. \end{aligned}$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = \beta_{\mathbf{d}}. \tag{EC.68}$$

using the KKT conditions. Define the following dual variables for the binding constraints,

$$x_{\mathbf{d}} = J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) = J'_{\mathbf{u}}\left(\frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{r + \bar{\mu}_{\mathbf{u}}}\right) - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) \ge 0.$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$, and

$$\begin{aligned} y_{\mathbf{d}} &= (W_{t-} + \beta_{\mathbf{d}})(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}})) - W_{t-}J'_{\mathbf{d}}(W_{t-}) - J_{\mathbf{u}}(0) - \beta_{\mathbf{d}} J'_{\mathbf{u}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) \\ &= (W_{t-} + \beta_{\mathbf{d}})\left(\frac{J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{u}}(0)}{W_{t-} - \beta_{\mathbf{d}}} - J'_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}})\right) \ge 0, \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t^*) - J_{\mathbf{d}}(W_{t-})] = -y_{\mathbf{d}} - (H_t^* + W_{t-})x_{\mathbf{d}}, \tag{EC.69}$$

$$(1-q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H_t^*)) = (1-q_t^*)x_{\mathbf{d}}. \tag{EC.70}$$

Therefore, (EC.68) implies that in (EC.63),

$$\Phi_t \le J'_{\mathbf{d}}(W_{t-})rW_{t-} + \mu_{\mathbf{d}}[-\beta_{\mathbf{d}} J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + \beta_{\mathbf{d}}) - J_{\mathbf{d}}(W_{t-})] - rJ_{\mathbf{d}}(W_{t-}) - c_{\mathbf{d}} = 0,$$

where the equality follows from (EC.12).

If $\theta_t = \mathbf{d}$ and $\nu_t = 0$, then following (EC.63), we have

$$\Phi_t := J'_{\mathbf{d}}(W_{t-})(rW_{t-} + \underline{\mu}_{\mathbf{d}}[q_t W_{t-} - (1-q_t)H_t]) - rJ_{\mathbf{d}}(W_{t-}) \\ + \underline{\mu}_{\mathbf{d}} q_t [J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + \underline{\mu}_{\mathbf{d}}(1-q_t)[J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})] \ . \tag{EC.71}$$

We need to consider the following optimization problem,

$$\max_{q_t, H_t} \quad q_t[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] + (1-q_t)[-H_t J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t) - J_{\mathbf{d}}(W_{t-})],$$

$$s.t. \qquad 0 \le q_t \le 1, -H_t \le W_{t-}.$$

In the following, we verify that the optimal solution is

$$q_t^* = 0 \quad \text{and} \quad H_t^* = -\frac{rW_{t-}}{\bar{\mu}_{\mathbf{u}} + r}. \tag{EC.72}$$

using the KKT conditions. Define the following dual variable for the binding constraint

$$\begin{aligned} y_{\mathbf{d}} &= W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) + H_t^* J'_{\mathbf{d}}(W_{t-}) - J_{\mathbf{u}}(W_{t-} + H_t^*) \\ &= \frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r} \left[ J'_{\mathbf{u}} \left( \frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r} \right) - \frac{J_{\mathbf{u}} \left( \frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r} \right) - J_{\mathbf{u}}(0)}{\frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r}} \right] \ge 0 \ . \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{u}}$. One can verify

$$[W_{t-}J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(0) - J_{\mathbf{d}}(W_{t-})] - [-H_t^* J'_{\mathbf{d}}(W_{t-}) + J_{\mathbf{u}}(W_{t-} + H_t^*) - J_{\mathbf{d}}(W_{t-})] = -y_{\mathbf{d}}, \tag{EC.73}$$

$$(1 - q_t^*)(J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}}(W_{t-} + H_t^*)) = 0, \tag{EC.74}$$

where (EC.74) follows from $J'_{\mathbf{d}}(W_{t-}) - J'_{\mathbf{u}} \left( \frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r} \right) = 0$. Therefore, (EC.72) implies that in (EC.71),

$$\begin{aligned} \Phi_t :=& J'_{\mathbf{d}}(W_{t-}) \left( rW_{t-} + \frac{r\underline{\mu}_{\mathbf{d}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r} \right) - rJ_{\mathbf{d}}(W_{t-}) + \underline{\mu}_{\mathbf{d}} \left[ J_{\mathbf{u}} \left( \frac{\bar{\mu}_{\mathbf{u}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r} \right) - J_{\mathbf{d}}(W_{t-}) \right] \\ =& J'_{\mathbf{d}}(W_{t-}) \left( rW_{t-} + \frac{r\underline{\mu}_{\mathbf{d}} W_{t-}}{\bar{\mu}_{\mathbf{u}} + r} \right) - rJ_{\mathbf{d}}(W_{t-}) + \underline{\mu}_{\mathbf{d}} \left[ \frac{R}{\bar{\mu}_{\mathbf{u}} + r} - \frac{r}{r + \bar{\mu}_{\mathbf{u}}} J_{\mathbf{d}}(W_{t-}) \right] \\ =& \frac{\underline{\mu}_{\mathbf{d}} R}{\bar{\mu}_{\mathbf{u}} + r} + \frac{r(\underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}} + r)}{\bar{\mu}_{\mathbf{u}} + r} (W_{t-} J'_{\mathbf{d}}(W_{t-}) - J_{\mathbf{d}}(W_{t-})) \\ \le& \frac{\underline{\mu}_{\mathbf{d}} R}{\bar{\mu}_{\mathbf{u}} + r} + \frac{r(\underline{\mu}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}} + r)}{\bar{\mu}_{\mathbf{u}} + r} \left( W_{t-} \frac{J_{\mathbf{d}}(W_{t-}) - \underline{v}_{\mathbf{d}}}{W_{t-}} - J_{\mathbf{d}}(W_{t-}) \right) \\ =& 0 \ . \end{aligned}$$

where the inequality follows from the concavity of $J_{\mathbf{d}}(w)$ and the last equality follows from equation (4). Q.E.D.

### EC.5.4. Proof of Proposition EC.7

It suffices to show that if (EC.9) is not satisfied, then the following principal's value functions,

$$J_{\mathbf{u}}(w) = \underline{v}_{\mathbf{u}} - w, \text{ and } J_{\mathbf{d}}(w) = \underline{v}_{\mathbf{d}} - w,$$

satisfy the optimality condition $\Phi_t \le 0$, where $\Phi_t$ is defined by (EC.62). If $\theta_t = \mathbf{u}$, then $\nu_t = 0$ and

$$\begin{aligned} \Phi_t &= R - rW_{t-} - \bar{\mu}_{\mathbf{u}}[q_t W_{t-} - (1 - q_t)H_t] - (r + \bar{\mu}_{\mathbf{u}})(\underline{v}_{\mathbf{u}} - W_{t-}) + \bar{\mu}_{\mathbf{u}} q_t \underline{v}_{\mathbf{d}} + \bar{\mu}_{\mathbf{u}}(1 - q_t)(\underline{v}_{\mathbf{d}} - W_{t-} - H_t) \\ &= R - (r + \bar{\mu}_{\mathbf{u}})\underline{v}_{\mathbf{u}} + \bar{\mu}_{\mathbf{u}} \underline{v}_{\mathbf{d}} = 0. \end{aligned}$$

When $\theta_t = \mathbf{d}$ and $\nu_t = 0$, then

$$\begin{aligned}\Phi_t &= -rW_{t-} - \underline{\mu}_{\mathbf{d}}[q_t W_{t-} + (1-q_t)H_t] - (r+\underline{\mu}_{\mathbf{d}})(\underline{v}_{\mathbf{d}} - W_{t-}) + \underline{\mu}_{\mathbf{d}} q_t \underline{v}_{\mathbf{u}} + \underline{\mu}_{\mathbf{d}}(1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t) \\ &= -\left(r+\underline{\mu}_{\mathbf{d}}\right)\underline{v}_{\mathbf{d}} + \underline{\mu}_{\mathbf{d}}\underline{v}_{\mathbf{u}} = 0.\end{aligned}$$

When $\theta_t = \mathbf{d}$ and $\nu_t = 1$, then

$$\begin{aligned}\Phi_t &= -rW_{t-} - c_{\mathbf{d}} - \mu_{\mathbf{d}}[q_t W_{t-} - (1-q_t)H_t] - (r+\mu_{\mathbf{d}})(\underline{v}_{\mathbf{d}} - W_{t-}) + \mu_{\mathbf{d}} q_t \underline{v}_{\mathbf{u}} + \mu_{\mathbf{d}}(1-q_t)(\underline{v}_{\mathbf{u}} - W_{t-} - H_t) \\ &= -c_{\mathbf{d}} - (r+\mu_{\mathbf{d}})\underline{v}_{\mathbf{d}} + \mu_{\mathbf{d}}\underline{v}_{\mathbf{u}} = -c_{\mathbf{d}} - (r+\mu_{\mathbf{d}})\frac{\underline{\mu}_{\mathbf{d}} R}{r(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})} + \mu_{\mathbf{d}}\frac{(r+\underline{\mu}_{\mathbf{d}})R}{r(r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})} \\ &= \frac{\Delta\mu_{\mathbf{d}} R}{r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}} - c_{\mathbf{d}} = \frac{\Delta\mu_{\mathbf{d}}}{r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}}}(R - (r+\underline{\mu}_{\mathbf{d}}+\bar{\mu}_{\mathbf{u}})\beta_{\mathbf{d}}) < 0,\end{aligned}$$

where the inequalities follow from the fact that (EC.16) is not satisfied.     Q.E.D.