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PAPER

Entropic time–energy uncertainty relations: an algebraic approach

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Abstract

We address entropic uncertainty relations between time and energy or, more precisely, between measurements of an observable G and the displacement r of the G-generated evolution $\mathrm{e}^{-\mathrm{i}rG}$. We derive lower bounds on the entropic uncertainty in two frequently considered scenarios, which can be illustrated as two different guessing games in which the role of the guessers are fixed or not. In particular, our bound for the first game improves the previous result by Coles *et al* [Phys. Rev. Lett. **122** 100401 (2019)]. To derive our bounds, we extend a recently proposed novel algebraic method by Gao *et al* [arXiv:1710.10038 [quant-ph]] which was used to derive both strong subadditivity and entropic uncertainty relations for measurements.

1. Introduction

Uncertainty principles are a cornerstone of modern physics [1]. The most famous instantiation is perhaps the Kennard relation [2] $\sigma_x\sigma_p \geqslant \hbar/2$ where σ_x and σ_p are the standard deviations of the measurement of the position and the momentum of a particle respectively. Entropic uncertainty relations, in contrast, offer an operational interpretation of the uncertainty principle, which is often more desirable in applications such as quantum cryptography. The most well-known entropic uncertainty relation, derived by Maassen and Uffink [3], can be interpreted as a guessing game: Alice has the quantum state ρ and can choose whether to measure V or W, Bob wins if he can correctly guess the result of the measurement. Let ρ be the density matrix of a system A and $\mathcal{E}_V$ and $\mathcal{E}_W$ be the measurement quantum channels for the observables V and W, then the uncertainty of Bob's guesses, characterized by the (von Neumann) entropy, satisfies the inequality

$$S(A)_{\mathcal{E}_V(\rho)} + S(A)_{\mathcal{E}_W(\rho)} \geqslant -\log\ \max_{k,j}|\langle v_j|w_k\rangle|^2, \tag{1}$$

where $|v_i\rangle$ and $|w_i\rangle$ are the eigenvectors of V and W and $S(A)_\rho$ is the von Neumann entropy of the state ρ on system A. Equation (1) prevents Bob from perfectly winning this game, provided the right-hand side is non zero, i.e. V and W do not commute. Indeed, if $S(A)_{\mathcal{E}_V(\rho)} = 0$, meaning that he can perfectly guess the measurement result of V, then the inequality implies $S(A)_{\mathcal{E}_W(\rho)} \geqslant -\log\ \max_{k,j}|\langle v_j|w_k\rangle|^2$, and thus Bob will not be able to perfectly guess the measurement result of W.

The entropic uncertainty relation in equation (1) has been further extended to account for the effect of quantum memories [4, 5]: if a quantum memory B is entangled with the original system A, Bob could use it to deduce Alice's measurement outcomes. There are essentially two possible uses of the memory, corresponding to two guessing games. The first game, also referred to as *the tripartite game*, concerns splitting the quantum memory into two parts B_1 and B_2, where B_1 is used for guessing V and B_2 is used for guessing W. Then the following entropic uncertainty relation holds [4, 5]

$$S(A|B_1)_{\mathcal{E}_V(\rho)} + S(A|B_2)_{\mathcal{E}_W(\rho)} \geqslant -\log\ \max_{k,j}|\langle v_j|w_k\rangle|^2, \tag{2}$$

where the measurements are performed only on the system A and $S(A|B)_\rho$ is the quantum conditional entropy of A conditioned on B. On the other hand, the second game regards the memory as a whole and is referred to as *the bipartite game*. In this case, the uncertainty relation becomes

$$S(A|B)_{\mathcal{E}_V(\rho)} + S(A|B)_{\mathcal{E}_W(\rho)} \geqslant -\log \max_{k,j} |\langle v_j|w_k\rangle|^2 + S(A|B)_\rho. \tag{3}$$

In this case, Bob, who keeps the quantum memory, can increase his chance of winning by referring to it. In fact, since the quantum conditional entropy can be negative, Bob can win the game with certainty by using a suitable entangled state for which the right-hand side of equation (3) vanishes.

The two guessing games differ only in whether the memory is split into two parts or not. This difference highlights a subtlety of the uncertainty principle: It is impossible to simultaneously know the values of two noncommuting observables of the same system. On the one hand, since the memory is split into two parts in the tripartite game, Bob can guess both observables at the same time. The fact that the tripartite game cannot be won then matches the uncertainty principle. On the other hand, in each round of the bipartite game Bob can only guess one of the observables. Therefore, even though using a quantum memory can allow him to win the game with certainty, there is no contradiction with the uncertainty principle.

Various extensions of these entropic uncertainty relations with memory have been put forward [see, e.g., references [6–9] and reference [10] for a full survey]. A natural question is whether there is an entropic time–energy uncertainty relation. This is a more subtle situation than relations involving measurements of observables, since an ideal time observable does not exist for finite dimensional systems [11–13]. Possible ways out include defining an approximate time operator [14], or considering the uncertainty of measuring the duration of evolutions, i.e. measuring the state as a quantum clock, instead of directly measuring time.

In this work, we take the latter approach and study the tradeoff between uncertainties of measuring an observable G (e.g. the Hamiltonian of the system) and determining a parameter r of the unitary evolution $\mathrm{e}^{-\mathrm{i}rG}$. Unlike most of the previous works, whose proofs are built on basic properties of quantum entropies and distances, we take a new algebraic approach that makes use of a strong subadditivity on algebras, developed recently by Gao, Junge, and Laracuente [15]. As a result, we obtain entropic uncertainty relations for both of the aforementioned guessing games. Entropic time-uncertainty relations were recently studied in the setting of the tripartite guessing game by Coles *et al* [16]. In comparison, we show that our bound is strictly tighter than their result for von Neumann entropies, though they also study more general Rényi entropies.

The rest of the paper is arranged as follows. In section 2, we define the two guessing games under consideration and state our main results on the entropic uncertainty relation. In section 3, we prepare for the proofs of the uncertainty relations by introducing a few useful results from reference [15]. In section 4, we prove our bounds on the entropic uncertainties. In section 5, we present some numerical examples that show the tightness and advantage of our results. Finally, in section 6, we conclude with a few discussions.

2. Guessing games and entropic uncertainty relations

In this section, we introduce the setting and the main results of our paper. Entropic uncertainty relations arise naturally from guessing games, where players are asked to make guesses on random operations performed by an extra player. Guessing games involve a game operator A and one or multiple guessers, where the operation performed by A is either a measurement of an observable G or a rotation $\rho \mapsto \mathrm{e}^{-\mathrm{i}Gr_k}\rho\mathrm{e}^{\mathrm{i}Gr_k}$ generated by G with r_k being a random number drawn from a fixed finite set $\{r_k\}_{k=1}^{|R|}$.

Now we introduce the first guessing game:

Definition 2.1 (The tripartite guessing game). The game concerns two guessers B_1 and B_2 and runs as follows:

(a) (Setup) three players A, B_1, and B_2 share a quantum state $\rho_{AB_1B_2}$, fix a probability distribution $\{p_k\}_{k=1}^{|R|}$, a generator G acting on A, and a set of rotations $\{r_k\}_{k=1}^{|R|}$.

(b) A tosses a coin to choose between measuring G or applying a rotation $\mathrm{e}^{-\mathrm{i}Gr_k}$.[2]

(c) If A gets a head, she chooses an r_k following the probability distribution $\{p_k\}_{k=1}^{|R|}$ and applies $\mathrm{e}^{-\mathrm{i}Gr_k}$ to her part of ρ. She then sends the rotated state to B_1, with instructions to guess r_k.

[2] Since the rotation does not affect the measurement, we could also say that A always applies a random rotation and then randomly chooses whether to measure G. This version is more easily interpretable if one wants to consider time evolution as the rotation.

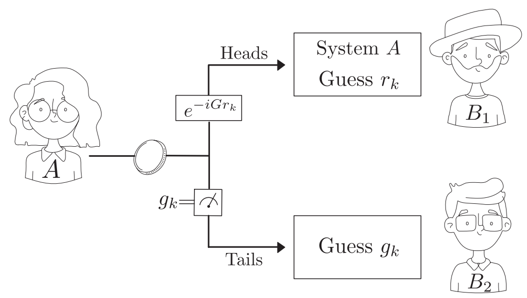


Figure 1. The tripartite guessing game. The above figure illustrates the setting of the tripartite guessing game, where two guessers B_1 and B_2 are assigned different tasks. Depending on the outcome of a coin toss, Alice asks either B_1 to guess a rotation or B_2 to guess a measurement outcome.

(d) If A gets a tail, she measures G on her part of ρ and asks B_2 to guess the measurement outcome.

(e) Accordingly, B_1 or B_2 provides his guess.

A graphical illustration of this game is portrayed in figure 1. To quantify the uncertainty of the guesses in the above game, we use an ancillary Hilbert space $\mathcal{H}_R$ for the random number $\{r_k\}$, which has probability distribution $\{p_k\}$. If A chooses to perform the rotation, the state afterwards is

$$\kappa_{RAB_1B_2} = \sum_{k=1}^{|R|} p_k |r_k\rangle\langle r_k| \otimes \mathrm{e}^{-\mathrm{i}Gr_k} \rho_{AB_1B_2} \mathrm{e}^{\mathrm{i}Gr_k}. \tag{4}$$

If A chooses to measure G, the state afterwards is

$$\omega_{AB_1B_2} = \sum_{k=1}^{|A|} |g_k\rangle\langle g_k| \langle g_k|\rho_{AB_1B_2}|g_k\rangle, \tag{5}$$

where $\{|g_k\rangle\}$ are the eigenstates of G with eigenvalue g_k. The quantity $S(R|AB_1)_\kappa + S(A|B_2)_\omega$ represents the total uncertainty of the game, in the sense that the larger it is, the more difficult it is to guess correctly. Notice that these are entropies of classical random variables conditioned on quantum states, hence they are positive.

Our first result is a lower bound of the total uncertainty, as described in the following theorem.

Theorem 2.2. *The total uncertainty of the tripartite game is lower bounded as*

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geqslant S(R)_\kappa + D(\kappa_{AB_1}\|\omega_{AB_1}) + \max\{0, I(A:B_1)_\omega - I(B_1:B_2)_\rho + S(A|B_1B_2)_\rho\}. \tag{6}$$

The bound is saturated if $\rho_{AB_1B_2}$ is pure or $\rho_{AB_1B_2} = \rho_{AB_1} \otimes \rho_{B_2}$.

Our bound (6) manifests a tradeoff relation between guessing the measurement outcome and guessing the rotation. In particular, it shows that it is impossible for both guesses to be perfect for the same state (unless R is trivial), since the right-hand side of the bound (6) is always positive. If the conditional entropy $S(A|B_2)_\omega$ is low, meaning that B_2 can easily guess the measurement value, then the entropy of the rotation chosen must be large to satisfy the bound, making it hard for B_1 to guess precisely which rotation has been applied.

Note that, in the case $p_k = \frac{1}{|R|}$ for all k, the term $S(R)_\kappa$ is simply $\log|R|$. Clearly, to minimize the uncertainty, B_1 and B_2 want to reduce the last term in the bound (6). From this we can deduce the following conditions for making the uncertainty small:

- B_1 and B_2 need to be as correlated as possible so as to maximize $I(B_1:B_2)_\rho$.
- The system B_1, which is used to guess the rotation, should be as uncorrelated as possible with the measurement result so as to minimize $I(A:B_1)_\omega$.
- A and B_1B_2 should be entangled so that $S(A|B_1B_2)_\rho$ is negative.

The guessing game proposed by Coles *et al* [16] is a special case of the tripartite game presented here. They showed in [16, equation (8)] that when the distribution over R is uniform, the total uncertainty can be bounded as

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geqslant \log\ |R|. \tag{7}$$

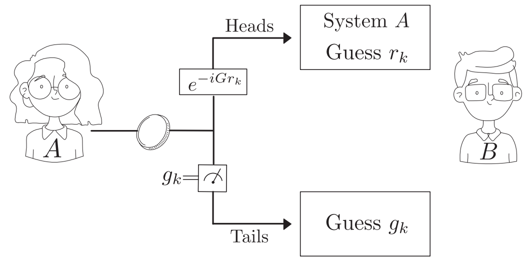


Figure 2. The bipartite guessing game. The above figure illustrates the setting of the bipartite guessing game, where the guesser may be asked to guess either a rotation or a measurement outcome.

Furthermore, for $B_1 = \mathbb{C}$ and $B_2 = B$, they find a stronger bound in [16, equation (E10)]:

$$S(R|A)_\kappa + S(A|B)_\omega \geqslant S(R)_\kappa + D(\kappa_A\|\omega_A), \tag{8}$$

which is tight if ρ^{AB} is pure. It is clear that our bound (6) is tighter since the additional term $\max\{0, I(A:B_1)_\omega - I(B_1:B_2)_\rho + S(A|B_1B_2)_\rho\}$ is positive.

In the tripartite game, the system B is broken into two subsystems B_1 and B_2 and distributed to individual players, whose tasks are fixed. Alternatively, we can consider a variation of the game where B is given to a single player, who may be given either task (to guess the measurement outcome or the rotation).

Definition 2.3 (The bipartite guessing game). The game concerns only one guesser B and runs as follows:

(a) (Setup) two players A and B share a quantum state ρ_{AB}, fix a probability distribution $\{p_k\}_{k=1}^{|R|}$, a generator G acting on A, and a set of rotations $\{r_k\}_{k=1}^{|R|}$.
(b) A tosses a coin to choose between measuring G or applying a rotation $\mathrm{e}^{-\mathrm{i}Gr_k}$.
(c) If A gets a head, she chooses an r_k following the probability distribution $\{p_k\}_{k=1}^{|R|}$ and applies $\mathrm{e}^{-\mathrm{i}Gr_k}$ to her part of ρ. She then sends the rotated state to B, with instructions to guess r_k.
(d) If A gets a tail, she measures G on her part of ρ and asks B to guess the measurement outcome.
(e) B provides his guess.

A graphical illustration of this game is portrayed in figure 2.

In this game the quantity that characterizes the uncertainty is

$$S(R|AB)_\kappa + S(A|B)_\omega, \tag{9}$$

where κ and ω are defined by equations (4) and (5), respectively. Just as the tripartite game, we can bound this total uncertainty as well.

Theorem 2.4. *The total uncertainty for the bipartite game is lower bounded as*

$$S(R|AB)_\kappa + S(A|B)_\omega \geqslant S(R)_\kappa + D(\kappa_A\|\omega_A) + S(A|B)_\rho. \tag{10}$$

The bound is saturated if $\rho_{AB} = \rho_A \otimes \rho_B$ is a product state or if ρ_A is a pure eigenstate of G.

An intriguing distinction between this bound and the bound for the tripartite game (6) is that B may be able to always guess correctly. This is analogous to the bound for the uncertainty principle in the presence of quantum memory [5], in the sense that quantum correlations that make $S(A|B)_\rho$ negative can reduce the bound (10) to zero. To see this, let us consider a simple example in which Alice and Bob hold a qubit each and the two qubits are in the maximally entangled state. Furthermore, take $G = \sigma_z$, $|R| = 2$ and the uniform distribution for the rotations. In this case $\kappa_{AB} = \omega_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. Then clearly the right-hand side is 0 as the relative entropy is 0 and $S(A|B)_\rho = -1$. Moreover one may verify that $S(RAB)_\kappa = 1$ and thus the left-hand side is also 0. Intuitively, in this case the rotations have the same effect of a σ_z measurement, and Bob can apply the same strategy in both cases.

3. Preliminary: a general framework for entropic uncertainty relations

In this section, we introduce part of the main results of reference [15] that will be used in our proof.

3.1. Commuting squares and uncertainty relations

Let M be an algebra of observables and let an $N \subset M$ be subalgebra. For instance, M may be the observables on a bipartite system, and N the observables on just one system. The conditional expectation onto N is the unique surjective CPTP and unital map $\mathcal{E}_N : M \to N$ such that for all $\rho \in M, \sigma \in N$

$$\mathrm{Tr}(\sigma \mathcal{E}_N(\rho)) = \mathrm{Tr}(\sigma\rho). \tag{11}$$

Given a state $\rho \in M$, the *asymmetry measure* of ρ with respect to N is defined as

$$D^N(\rho) := \inf_{\sigma \in s(N)} D(\rho\|\sigma), \tag{12}$$

where $D(\cdot\|\cdot)$ is the relative entropy and $s(N)$ denotes the states on N. When N is the image of a conditional expectation $\mathcal{E}_N$, we have

$$D^N(\rho) = D(\rho\|\mathcal{E}_N(\rho)) = S(N)_{\mathcal{E}_N(\rho)} - S(M)_\rho, \tag{13}$$

where S is the von Neumann entropy. We remark that D^N, albeit not a distance measure, captures the distinction between N and M.

Definition 3.1 (Commuting square). A set of four observable algebras satisfying the inclusions

$$\begin{pmatrix} N & \subset & M \\ \cup & & \cup \\ R & \subset & T \end{pmatrix} \tag{14}$$

is called a commuting square if the conditional expectations satisfy

$$\mathcal{E}_N \circ \mathcal{E}_T = \mathcal{E}_T \circ \mathcal{E}_N = \mathcal{E}_R. \tag{15}$$

The following theorem will be the core of our proof, which says that one entropic uncertainty relation can be identified from each commuting square.

Theorem 3.2. *Let N, M, R, T form a commuting square as in (14). Then for all $\rho \in M$*

$$S(N)_{\mathcal{E}_N(\rho)} + S(T)_{\mathcal{E}_T(\rho)} \geqslant S(M)_\rho + S(R)_{\mathcal{E}_R(\rho)}, \tag{16}$$

which is equivalent to

$$D^N(\rho) + D^T(\rho) \geqslant D^R(\rho). \tag{17}$$

The relation is saturated if and only if there exists a CPTP map $\mathcal{R}$ such that

$$\mathcal{R}(\mathcal{E}_N(\rho)) = \rho \quad \mathcal{R}(\mathcal{E}_R(\rho)) = \mathcal{E}_T(\rho). \tag{18}$$

or equivalently

$$\mathcal{R}(\mathcal{E}_T(\rho)) = \rho \quad \mathcal{R}(\mathcal{E}_R(\rho)) = \mathcal{E}_N(\rho). \tag{19}$$

Equations (16) and (17) are uncertainty relations with respect to a commuting square, which we will use to derive bounds on the time–energy uncertainty.

3.2. Examples of conditional expectations

We provide here some examples of conditional expectations that will be useful later. From now on, Latin uppercase letters will be used to refer to the algebra of Hermitian operators on a corresponding Hilbert space.

3.2.1. Embedding

Let AB be the algebra of Hermitian operators on $\mathcal{H}_A \otimes \mathcal{H}_B$. We want to find a conditional expectation that takes us to the algebra B. One may notice that the partial trace is not a conditional expectation, as it is not unital. To solve this problem, following Example 2.2 in [15], instead of embedding $B \subset AB$ we embed $I_A \otimes B \subset AB$ where $I_A \simeq \mathbb{C}$ is the algebra generated by $\{cI_A : c \in \mathbb{C}\}$. The embedding is done by the map

$$\mathcal{T}_A(\rho_{AB}) = \frac{1}{|A|} I_A \otimes \rho_B, \tag{20}$$

where $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$. The map is clearly unital and CPTP. Let $\sigma = cI_A \otimes \sigma_B \in I_A \otimes B$ and $\rho_{AB} \in AB$, moreover let $\{|a_k\rangle\}_{k=1}^{|A|}$ be a basis of $\mathcal{H}_A$. We have

$$\begin{aligned}\mathrm{Tr}[\sigma\rho_{AB}] &= \mathrm{Tr}[cI_A\otimes\sigma_B\rho_{AB}]\\ &= c\mathrm{Tr}_B\left[\sum_k\sum_j\langle a_k|\left(|a_j\rangle\langle a_j|\otimes\sigma_B\right)\rho_{AB}|a_k\rangle\right]\\ &= c\mathrm{Tr}_B\left[\sum_k\langle a_k|_A\sigma_B\rho_{AB}|a_k\rangle\right] = c\mathrm{Tr}_B[\sigma_B\rho_B].\end{aligned} \tag{21}$$

On the other hand

$$\begin{aligned}\mathrm{Tr}[\sigma\mathcal{T}_A(\rho_{AB})] &= \mathrm{Tr}\left[(cI_A\otimes\sigma_B)\left(\frac{1}{|A|}I_A\otimes\rho_B\right)\right]\\ &= \frac{c}{|A|}\mathrm{Tr}[I_A\otimes\sigma_B\rho_B] = c\mathrm{Tr}_B[\sigma_B\rho_B].\end{aligned} \tag{22}$$

3.2.2. Pinching

Let G be an observable with full support on $\mathcal{H}_A$ and $\{|g_k\rangle\}_{k=1}^{|A|}$ be the eigenbasis of G. The pinching map

$$\mathcal{P}_G : \rho_A \mapsto \sum_{k=1}^{|A|}|g_k\rangle\langle g_k|\langle g_k|\rho_A|g_k\rangle \tag{23}$$

is a conditional expectation onto $\mathrm{span}\{|g_k\rangle\langle g_k|\}_{k=1}^{|A|}$. Notice that this is also an algebra, consisting of all diagonal elements in A, and from now on we denote this kind of subalgebras by $\tilde{A}$.

It is clear that the pinching map $\mathcal{P}_G : A\to\tilde{A}$ is unital and CPTP, and for $\sigma = \sum_{k=1}^{|A|}p_k|g_k\rangle\langle g_k|$ we have

$$\mathrm{Tr}(\sigma\mathcal{P}_G(\rho_A)) = \sum_{k=1}^{|A|}p_k\langle g_k|\rho_A|g_k\rangle \tag{24}$$

and

$$\mathrm{Tr}(\sigma\rho_A) = \sum_{k=1}^{|A|}\mathrm{Tr}(p_k|g_k\rangle\langle g_k|\rho_A) = \sum_{k,j=1}^{|A|}\langle g_j|p_k|g_k\rangle\langle g_k|\rho_A|g_j\rangle = \sum_{k=1}^{|A|}p_k\langle g_k|\rho_A|g_k\rangle. \tag{25}$$

Therefore, $\mathcal{P}_G$ is a conditional expectation on the subalgebra $\tilde{A}$ that is diagonal with respect to the eigenbasis of G.

4. Proof of rotation-measurement uncertainty relations

Theorem 3.2 can be readily applied to obtain an entropic uncertainty relation between noncommutative observables [15]: one can take M to be the (total) von Neumann algebra of AB and take the conditional expectations (11) to be the pinching maps corresponding to the two observables. This approach, however, does not apply immediately to the case of time–energy uncertainty relations, since the rotation generated by time parameters is not a legitimate conditional expectation.

Instead, we find a suitable commuting square of the following structure:

$$\begin{pmatrix}\text{energy} & \subset & \text{total}\\ \cup & & \cup\\ \text{minimum} & \subset & \text{time}\end{pmatrix}. \tag{26}$$

Here 'time' or 'energy' refers to a subalgebra of 'total' containing elements that are diagonal in the time or energy basis respectively. In this case, the time basis is a basis of the classical register R. 'Minimum' is the intersection of 'time' and 'energy' determined by the conditional expectations (11). We then consider a variation of the state κ in equation (4) with different rotations applied coherently in a superposition. A time–energy uncertainty relation is then established by applying theorem 3.2.

Another interesting finding for the time–energy uncertainty is that different choices of the commuting square (26) lead to independent bounds. In fact, for the tripartite game we can find two distinct commuting squares as such. Combining the two obtained bounds yields a stronger bound as given by equation (6).

4.1. The tripartite game

As mentioned, we will find two distinct bounds and combine them.

4.1.1. The first bound

The following proposition, stated and proved for quantum Rényi entropies in reference [16], will be useful.

Proposition 4.1. *Let ρ_M be a state and $\mathcal{E}_N$ be a conditional expectation, then*

$$D^N(\rho) = -S(E|M)_{U\rho U^\dagger}, \tag{27}$$

where U is a Stinespring dilation of $\mathcal{E}_N$ on ME.

Proof. Equation (13) states that

$$D^N(\rho) = S(N)_{\mathcal{E}_N(\rho)} - S(M)_\rho. \tag{28}$$

Clearly $S(M)_{U\rho U^\dagger} = S(N)_{\mathcal{E}_N(\rho)}$ as U is a Stinespring dilation of $\mathcal{E}_N$. Moreover, conjugation by an isometry preserves the eigenvalues, we have $S(M)_\rho = S(ME)_{U\rho U^\dagger}$. Combining both equalities, we have

$$D^N(\rho) = S(M)_{U\rho U^\dagger} - S(ME)_{U\rho U^\dagger} = -S(E|M)_{U\rho U^\dagger}. \tag{29}$$

□

Let $\mathcal{H}_R$ be a register to store the parameter of rotation, namely that, if the state of R is $\sum_k p_k|r_k\rangle\langle r_k|$, Alice will perform the rotation e^{-ir_kG} with probability p_k. Here $\tilde{R}$ is the diagonal subalgebra of R with respect to the observable $\sum_k r_k|r_k\rangle\langle r_k|$, and $\tilde{G}$ is the diagonal subalgebra of A with respect to the observable G. With this convention in mind, let us now consider the following commuting square

$$\begin{pmatrix} \tilde{R}AB_1 & \subset & RAB_1 \\ \cup & & \cup \\ \tilde{R}\tilde{A}B_1 & \subset & R\tilde{A}B_1 \end{pmatrix},$$

where the conditional expectations are the simply corresponding pinching [see equation (23)]. For any state $\rho_{AB_1B_2}$, we define $\phi_{RAB_1} = |\Omega\rangle\langle\Omega|_R \otimes \rho_{AB_1}$ with $|\Omega\rangle = \sum_k \sqrt{p_k}|r_k\rangle$.

Now, let us consider the uncertainty relation of the state

$$\psi_{RAB_1} = \sum_{k,j=1}^{|R|} \sqrt{p_k p_j}|r_k\rangle\langle r_j| \otimes e^{-iGr_k}\rho_{AB_1}e^{iGr_j}, \tag{30}$$

obtained by applying the unitary $U = \sum_{k=1}^{|R|} |r_k\rangle\langle r_k| \otimes e^{-iGr_k}$ to ϕ_{RAB_1}. The conditional expectations result in the states

$$\begin{aligned} \psi_{\tilde{R}AB_1} &= \sum_{k=1}^{|R|} p_k|r_k\rangle\langle r_k| \otimes e^{-iGr_k}\rho_{AB_1}e^{iGr_k} = \kappa_{RAB_1}, \\ \psi^{R\tilde{A}B_1} &= \sum_{k,j=1}^{|R|}\sum_{l=1}^{|A|} \sqrt{p_k p_j}|r_k\rangle\langle r_j| \otimes e^{-ig_l(r_k-r_j)}|g_l\rangle\langle g_l|\langle g_l|\rho_{AB_1}|g_l\rangle \text{, and} \\ \psi^{\tilde{R}\tilde{A}B_1} &= \sum_{k=1}^{|R|} p_k|r_k\rangle\langle r_k| \otimes \sum_{l=1}^{|A|} |g_l\rangle\langle g_l|\langle g_l|\rho_{AB_1}|g_l\rangle = \kappa_R \otimes \omega_{AB_1}. \end{aligned} \tag{31}$$

For a register C and an arbitrary state ρ on it, let $\mathcal{Q}_{C,\rho}$ be the discard and reprepare map

$$\mathcal{Q}_{C,\rho}(\sigma_C) = \rho_C \tag{32}$$

that resets the register's state to ρ. We have

$$\begin{aligned} U\mathcal{Q}_{AB_1,\rho}(\psi_{\tilde{R}\tilde{A}B_1})U^\dagger &= \psi_{\tilde{R}AB_1} \\ U\mathcal{Q}_{AB_1,\rho}(\psi_{R\tilde{A}B_1})U^\dagger &= \psi_{RAB_1}. \end{aligned} \tag{33}$$

Therefore, $\mathcal{R}(\cdot) := U\mathcal{Q}_{AB_1,\rho}(\cdot)U^\dagger$ constitutes a valid recovery map. By theorem 3.2, we have

$$D^{\tilde{R}AB_1}(\psi) + D^{R\tilde{A}B_1}(\psi) = D^{\tilde{R}\tilde{A}B_1}(\psi). \tag{34}$$

The following isometry is a Stinespring dilation on AE of the pinching map on A

$$V = \sum_{k=1}^{|A|} |g_k\rangle_E \otimes |g_k\rangle\langle g_k|_A. \tag{35}$$

Proposition 4.1 applied to the second term of equation (34) yields the term $-S(E|RAB_1)_{V\rho V^\dagger}$. Consider a purification $\rho_{AB_1B_2B'}$ of $\rho_{AB_1B_2}$ (if $\rho_{AB_1B_2}$ is already pure B' is trivial), using the duality of conditional entropy one gets $-S(E|RAB_1)_{V\rho V^\dagger} = S(E|B_2B')_\omega$, with ω the state in equation (5). Since the complementary channel of pinching under the Stinespring dilation V is also the same pinching, which means $\omega_E = \omega_A$, and thus $S(E|B_2B')_\omega = S(A|B_2B')_\omega$. Using equation (13) on the two remaining terms one gets for $\rho_{AB_1B_2}$, we obtain

$$S(\tilde{R}AB_1)_\kappa + S(A|B_2B')_\omega = S(\tilde{R})_\kappa + S(\tilde{A}B_1)_\omega. \tag{36}$$

Abandoning the notation where one keeps track of which subalgebra the state is in for the more standard one and subtracting $S(AB_1)_\kappa$ from both sides, the relation becomes

$$S(R|AB_1)_\kappa + S(A|B_2B')_\omega = S(R)_\kappa + S(AB_1)_\omega - S(AB_1)_\kappa. \tag{37}$$

Since the pinching $\mathcal{P}_G$ (as the conditional expectation) on κ_A yields ω_A, equation (13) implies

$$S(AB_1)_\omega - S(AB_1)_\kappa = D(\kappa_{AB_1}\|\omega_{AB_1}), \tag{38}$$

and thus we can express the entropic uncertainty as

$$S(R|AB_1)_\kappa + S(A|B_2B')_\omega = S(R)_\kappa + D(\kappa_{AB_1}\|\omega_{EB_1}). \tag{39}$$

Finally, using the strong subadditivity $S(A|B_2B')_\omega \leqslant S(A|B_2)_\omega$, we obtain the following bound on the entropic uncertainty

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geqslant S(R)_\kappa + D(\kappa_{AB_1}\|\omega_{EB_1}). \tag{40}$$

From equation (39) it is immediate that the equality holds if and only if $I(A:B'|B_2)_\omega = 0$, which is satisfied when $\rho_{AB_1B_2}$ is pure.

Notice that our bound (40) holds for arbitrary B_1 and B_2, and any arbitrary state of R (i.e. the distribution of the rotation parameter $\{r_k\}$ can be non-uniform). On the other hand, the previous result by Coles *et al* [16], given by equation (7), does not have the second term on the right-hand side of equation (40) and assumes R to have a uniform distribution.

4.1.2. The second bound

Let us now consider an alternative commuting square:

$$\begin{pmatrix} \tilde{R}AB_1I_{B_2} & \subset & RAB_1B_2 \\ \cup & & \cup \\ \tilde{R}\tilde{A}I_{B_1B_2} & \subset & R\tilde{A}I_{B_1}B_2 \end{pmatrix}. \tag{41}$$

We start from the same state as before, namely

$$\psi_{RAB_1B_2} = \sum_{k,j=1}^{|R|} \sqrt{p_kp_j}|r_k\rangle\langle r_j| \otimes \mathrm{e}^{-\mathrm{i}Gr_k}\rho_{AB_1B_2}\mathrm{e}^{\mathrm{i}Gr_j}. \tag{42}$$

For the new commuting square, using the uncertainty relation (16), we get the relation

$$S(RAB_1)_\kappa + S(R\tilde{A}B_2)_\omega \geqslant S(RAB_1B_2)_\psi + S(\tilde{R}\tilde{A})_\omega. \tag{43}$$

Notice that $\psi_{R\tilde{A}B_2} = U\left(|\Omega\rangle\langle\Omega|_R \otimes \sum_{k=1}^{|A|}\langle g_k|\rho_{AB_2}|g_k\rangle|g_k\rangle\langle g_k|\right)U^\dagger$ with $U = \sum_{k=1}^{|R|}|r_k\rangle\langle r_k| \otimes \mathrm{e}^{-\mathrm{i}Gr_k}$, hence $S(R\tilde{A}B_2)_\psi = S(\tilde{A}B_2)_\omega$. Similarly $\psi_{RAB_1B_2} = U\left(|\Omega\rangle\langle\Omega|_R \otimes \rho_{AB_1B_2}\right)U^\dagger$, thus $S(RAB_1B_2) = S(AB_1B_2)$. Moreover $\psi_{\tilde{R}\tilde{A}}$ is a product state. Hence by subtracting $S(A)_\kappa + S(B_1B_2)_\omega$ from both sides and changing the notation like before

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geqslant S(R)_\kappa + S(AB_1B_2)_\rho + S(A)_\omega - S(AB_1)_\kappa - S(B_2)_\rho. \tag{44}$$

To have a better comparison with (40) we can write, using (38)

$$\begin{aligned} S(AB_1B_2)_\rho + S(A)_\omega - S(AB_1)_\kappa - S(B_2)_\rho &= D(\kappa_{AB_1}\|\omega_{AB_1}) + S(AB_1B_2)_\rho - S(B_2)_\rho + S(A)_\omega - S(AB_1)_\omega \\ &= D(\kappa_{AB_1}\|\omega_{AB_1}) + I(A:B_1)_\omega - I(B_1:B_2)_\rho + S(A|B_1B_2)_\rho. \end{aligned} \tag{45}$$

We can combine this with the previous relation and get, as promised

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geqslant S(R)_\kappa + D(\kappa_{AB_1}\|\omega_{AB_1}) + \max\{0, I(A:B_1)_\omega - I(B_1:B_2)_\rho + S(A|B_1B_2)_\rho\}. \tag{46}$$

If $\rho_{AB_1B_2}$ is pure this bound simply reduces to the previous one: as a matter of fact in this case the new term vanishes, since:

$$S(AB_1B_2)_\rho - S(B_2)_\rho + S(A)_\omega - S(AB_1)_\omega = -S(B_1|A)_\omega - S(B_2)_\omega \leqslant 0 \tag{47}$$

because ω_{AB_1} is classical in A. Hence since due to equation (39) the previous bound is saturated by pure states, this one is saturated as well. Otherwise, recall that by theorem 3.2 the relation holds as an equality if there exists a recovery map $\mathcal{R}$ such that

$$\mathcal{R}\left(\mathcal{E}_{RAI_{B_1}B_2}(\rho)\right) = \rho \quad \mathcal{R}\left(\mathcal{E}_{\tilde{R}\tilde{A}I_{B_1B_2}}(\rho)\right) = \mathcal{E}_{\tilde{R}\tilde{A}B_1I_{B_2}}(\rho) \tag{48}$$

hold for this particular $\rho_{RAB_1B_2}$. If $\rho_{AB_1B_2} = \rho_{AB_1} \otimes \rho_{B_2}$ we may define $\mathcal{R}(\cdot) := U\mathcal{Q}_{\tilde{A}B_1}(\cdot)U^\dagger$, where $U = \sum_{k=1}^{|R|} |r_k\rangle\langle r_k| \otimes \mathrm{e}^{-\mathrm{i}Gr_k}$ and $\mathcal{Q}_{\tilde{A}B_1}(\cdot)$ is the discard and prepare map

$$\mathcal{Q}_{AB_1}(\sigma_{AB_1C}) = \rho_{AB_1} \otimes \sigma_C \tag{49}$$

where C is any additional system beyond AB_1. It is straightforward to check that $\mathcal{R}$ indeed satisfies equation (48).

4.1.3. Significance of the bounds

Let us comment on the significance of these bounds for the tripartite game. The right-hand side of equation (46) is always positive, so the relation does in fact pose non trivial bounds on the probability of Bob to win the game, nevertheless it is worth noticing that

$$\kappa_{RA} = U\kappa_R \otimes \rho_A U^\dagger, \tag{50}$$

with $U = \sum_{k=1}^{|R|} |r_k\rangle\langle r_k| \otimes \mathrm{e}^{-\mathrm{i}Gr_k}$, which is unitary. Hence $S(RA)_\kappa = S(R)_\kappa + S(A)_\rho$. The relation in equation (46) reduces to

$$S(AB_1)_\rho + S(AB_2)_\omega \geqslant S(AB_1)_\omega + S(B_2)_\rho + \max\{0, I(A:B_1)_\omega - I(B_1:B_2)_\rho + S(A|B_1B_2)_\rho\}\,. \tag{51}$$

This is not a trivial bound, but it only involves the pinching map and it is not a statement about the rotation twirl. The problem is the artificial conditioning of the entropy $S(RA)_\kappa$. As a matter of fact, in light of equation (50), the non trivial contribution of the state κ is the conditioning of the entropy. In the next section we will obtain a relation for the bipartite game by trying to make the conditioning of the entropy of the state $\kappa_{RAB_1B_2}$ appear naturally in the inequality.

4.2. The bipartite game

In this case, one expects a constraint on the quantity $S(R|AB)_\kappa + S(A|B)_\omega$. To obtain such a relation, let us exploit the property in equation (50) and try to get the term $S(AB)_\kappa$ on the right-hand side naturally. Consider the following commuting square

$$\begin{pmatrix} AI_B & \subset & AB \\ \cup & & \cup \\ \tilde{A}I_B & \subset & \tilde{A}B \end{pmatrix} \tag{52}$$

and start from the state

$$\kappa_{AB} = \sum_{k=1}^{|R|} p_k \mathrm{e}^{-\mathrm{i}Gr_k} \rho_{AB} \mathrm{e}^{\mathrm{i}Gr_k}. \tag{53}$$

The state on $\tilde{A}B$ is just ω_{AB} and the $\log|B|$ terms cancel as always. The relation, keeping the notation $\tilde{A} \to A$, is

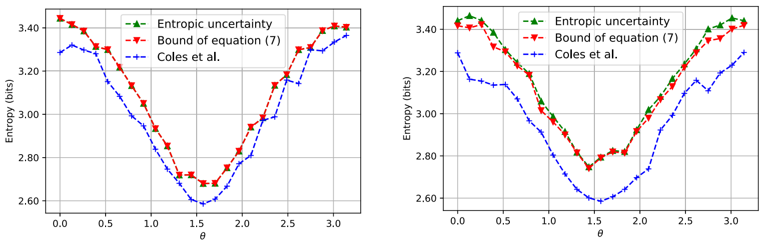
$$S(A)_\kappa + S(AB)_\omega \geqslant S(AB)_\kappa + S(A)_\omega. \tag{54}$$

One can immediately see that this is a non trivial relation involving both the state κ and the state ω. We can now add $S(R)_\kappa + S(AB)_\rho$ on both sides, use equation (50), and subtract $S(B)_\rho$ to get

$$S(R|AB)_\kappa + S(A|B)_\omega \geqslant S(R)_\kappa + S(A|B)_\rho + S(A)_\omega - S(A)_\kappa. \tag{55}$$

Using (38), this can be rewritten as

$$S(R|AB)_\kappa + S(A|B)_\omega \geqslant S(R)_\kappa + D(\kappa_A\|\omega_A) + S(A|B)_\rho. \tag{56}$$



(a) $\rho_{AB_1B_2} = \rho_{AB_1} \otimes \rho_{B_2}$ is a mixed product state.

(b) $\rho_{AB_1B_2}$ is a mixed non-product state.

Figure 3. Comparison of bounds for the tripartite game when B_1 is trivial. The above plots compare the bounds obtained by us [cf equation (6)] and by Coles *et al* in reference [16] for different states $\rho_{AB_1B_2}$. In the case when $\rho_{AB_1B_2} = \rho_{AB_1} \otimes \rho_{B_2}$, our bound (6) is tight whereas the one by Coles *et al* [16, equation (E10)] is not. In the generic case, our bound is not tight but still better than the one in reference [16].

Equality holds if (54) takes equality, and this by theorem 3.2 holds if there exists a recovery map

$$\mathcal{R}(\mathcal{E}_{\tilde{A}I_B}) = \mathcal{E}_{AI_B}(\rho_{AB}) \quad \mathcal{R}(\mathcal{E}_{\tilde{A}B}) = \rho_{AB}. \tag{57}$$

If ρ_{AB} is a product state we may simply take $\mathcal{R}$ to be $\mathcal{Q}_A$, the operation of resetting the state of A to ρ_A just as in section 4.1. If ρ_A is a pure eigenstate of G clearly the recovery map is the identity, hence in both of these cases the bound is saturated.

5. Numerical calculations

Here we present some explicit numerical results as an example of our bounds.

5.1. The tripartite game

Our bound for the tripartite game is given by equation (6), which is saturated when either $\rho_{AB_1B_2}$ is pure or $\rho_{AB_1B_2} = \rho_{AB_1} \otimes \rho_{B_2}$. Let us restrict for the moment to the case $B_2 \simeq B$, $B_1 \simeq \mathbb{C}$, then the bound reduces to

$$S(R|A)_\kappa + S(A|B)_\omega \geqslant S(R)_\kappa + D(\kappa_A \| \omega_A) + \max\{0, S(A|B)_\rho\}. \tag{58}$$

This is to be compared to the following bound obtained in [16]:

$$S(R|A)_\kappa + S(A|B)_\omega \geqslant S(R)_\kappa + D(\kappa_A \| \omega_A). \tag{59}$$

We take $|A| = |B| = 2$, $|R| = 6$ with random angles following a uniform distribution and $G = \sigma_x$. In the following the right and left-hand sides of the bounds are computed and compared for

$$\rho_{AB} = |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|, \tag{60}$$

with $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$, where $\theta \in [0, \pi]$. This is a pure product state. Random noise is added to either $|\psi\rangle\langle\psi|$ or ρ itself to obtain a mixed product state or a mixed non product state respectively. The random noise is obtained by adding a random state produced by the function rand_dm from the Python package QuTiP [17] and rescaling to obtain a trace one matrix. In figure 3 the relevant quantities are plotted for the three cases of a pure product state, a mixed product state and a mixed non product state.

For the tripartite case, where both B_1 and B_2 are nontrivial, equation (6) is to be compared with the one found in [16]

$$S(R|AB_1)_\kappa + S(A|B_2)_\omega \geqslant \log\ |R|. \tag{61}$$

Note that since in these computations the angles follow a uniform distribution, and thus $S(R)_\kappa = \log|R|$. In figure 4 the relevant quantities are plotted taking $|B| = 4$, $|B_1| = |B_2| = 2$, for the state

$$\rho_{AB} = |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| \tag{62}$$

with added random noise. From the plots, it is clear that our bounds outperform the previous ones in reference [16] in both cases.

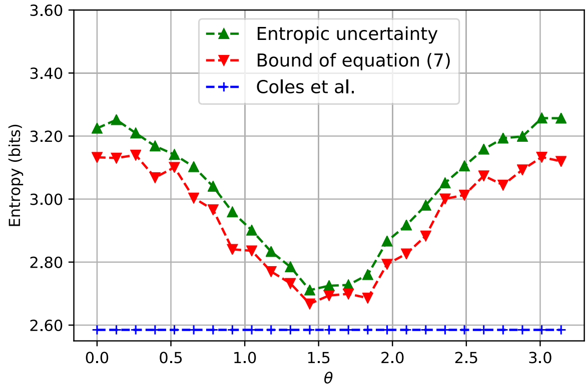


Figure 4. Comparison of bounds for the tripartite game when B_1 is not trivial. The above plot compares the bounds obtained by us [cf equation (6)] and by Coles *et al* [16, equation (8)] for generic $\rho_{AB_1B_2}$. Notice that the bound [16, equation (8)] is simply $\log|R|$ and is thus independent of the state's parameter θ. The plot manifests the gap between the entropic uncertainty and the bound by Coles *et al*, and that our bound is very close to the real uncertainty.

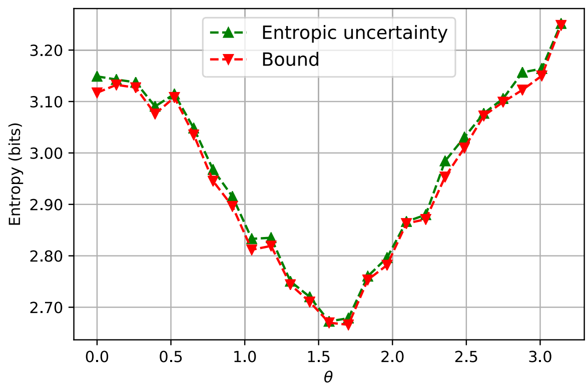


Figure 5. Performance of the entropic uncertainty bound for the bipartite game. In this plot, we examine the tightness of our bound (10) for the bipartite guessing game. It can be seen that our bound is very close to the true value of the uncertainty, even for generic, non-product states.

5.2. The bipartite game

Recall that our bound for the entropic uncertainty in the bipartite game, given by equation (10), is $S(R|AB)_\kappa + S(A|B)_\omega \geqslant S(R)_\kappa + D(\kappa_A\|\omega_A) + S(A|B)_\rho$. It is saturated if ρ_{AB} is a product state or if it is a pure eigenstate of G. In figure 5, the bound is further tested for generic, non-product states generated in the same random way as in section 5.1 for $|A| = |B| = 2$. It can be seen that the bound is still considerably, though not rigorously, tight for generic states.

6. Conclusions

In this work, we utilized the commuting square framework to derive time–energy entropic uncertainty relations based on two different guessing games. Our bound for the tripartite game tightens a previous bound in reference [16], in a way similar to other improvements [18, 19] made to the standard entropic uncertainty bound. Our bounds also strengthen the understanding of time–energy uncertainty, by showing that there is a fundamental difference between the case where the quantum memory is split between two parties and the case where one party holds the whole quantum memory. More precisely, the former case renders a game that is impossible to win, while the latter corresponds to a game that is possible to win but only with quantum memory.

Our work demonstrates the power of the commuting square approach. For time–energy uncertainty, the approach yields a simple and more intuitive proof and, more importantly, a tighter bound. We stress that this approach can also be applied to derive other entropic uncertainties. Several possible generalizations, however, remain open to investigation. One example is how to extend our result to generic Rényi entropies. Some hints have already been given in reference [15], but it might still require a considerable amount of effort to generalize the algebraic approach to this more general setting. Another interesting direction is to extend our approach to the multi-measurement setting [10, 20–22], where Alice is given more options than measuring an observable and performing a unitary. For instance, Alice could choose to implement an evolution on her system or measure one out of two different observables. To achieve this goal, one would in principle have to generalize the algebraic tools in reference [15] to the scenario where the commuting square contains more than two subalgebras. Such a generalization would, in return, lead to more applications of the algebraic tools in quantum information processing.

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